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PRODUCT FORMULAS FOR MODULAR FORMS ON $O(2,n)$
[after R. Borcherds]

by **Maxim KONTSEVICH**

1. INTRODUCTION

1.1. Product formulas

A few years ago, R. Borcherds found a remarkable multiplicative correspondence between classical modular forms with poles at cusps and meromorphic modular forms on complex varieties $SO(n) \times SO(2)\backslash SO(n,2)/\Gamma$, where Γ is an arithmetic subgroup in the real Lie group $SO(n,2)$. He was motivated by generalized Kac-Moody algebras, the Monster group and vertex operator algebras. The first proof of his formulas in completely classical terms (see [3]) was rather indirect and complicated.

In 1995 physicists J. Harvey and G. Moore wrote a paper on string duality where they found a new approach to Borcherds' identities (see [12]). They used divergent integrals, which look formally like integrals in the classical theta correspondence in the theory of automorphic forms. R. Borcherds recently wrote a preprint (see [5]) where he generalized his earlier results using the idea of Harvey and Moore. My exposition is based mainly on this new preprint.

Here is one of Borcherds' theorems:

Theorem. *Let Λ be an even unimodular lattice of signature $(s+1,1)$ where $s = 8, 16, \dots$ and $v_0 \in \Lambda \otimes \mathbf{R}$ be a generic vector of negative norm. Let $F = \sum_{-n_0}^{\infty} c(n)q^n \in \mathbf{Z}((q))$ be a meromorphic modular form of weight $(-s/2)$ for the group $SL(2,\mathbf{Z})$ with poles only at the cusp. Then there is a unique vector $\rho \in \Lambda$ such that the function defined for $v \in \Lambda \otimes \mathbf{C}$ close to $it\,v_0$, $t \gg 1$, by the formula*

$$\Psi(v) = e^{2\pi i(\rho,v)} \prod_{\gamma \in L,\ (\gamma,v_0)>0} \left(1 - e^{2\pi i(\gamma,v)}\right)^{c(-(\gamma,\gamma)/2)}$$

can be analytically continued to a meromorphic modular form of weight $c(0)/2$ for *the group $O(s+2,2;\mathbf{Z})^+$. In particular, the analytic continuation of Ψ satisfies the equation* $\Psi(2v/(v,v)) = \pm\,((v,v)/2)^{c(0)/2}\,\Psi(v)$.

In my exposition I will formulate results only in examples. One reason for this is that what is now known is still far from the complete generality. Another reason is that I want to avoid heavy notations in order not to obscure the logic of the construction.

1.2. An elementary example of a product formula

Product formulas can be considered as statements about formal power series of algebro-geometric origin. The general proof uses analysis: integrals, infinite series and non-holomorphic functions. Here I will show a purely algebraic proof of a simple product formula. Both this formula and the proof are not new. They were discovered by D. Zagier many years ago. Analogous formulas can be found in [11].

We fix notations: $\mathcal{H} = \{\tau \in \mathbf{C} |\ Im(\tau) > 0\}$ denotes the upper-half plane and $j : \mathcal{H} \longrightarrow \mathbf{C}$ is the classical elliptic invariant which identifies the quotient space $\mathcal{H}/SL(2,\mathbf{Z})$ with $\mathcal{M}_1 \simeq \mathbf{C}$, the coarse moduli space of complex elliptic curves. We will compactify it to $\overline{\mathcal{M}}_1 \simeq \mathbf{C}P^1$. Function $q = \exp(2\pi i\tau)$ can be considered as a holomorphic coordinate at a neighborhood of point $j = \infty$. We expand the meromorphic function j on $\mathbf{C}P^1$ in the coordinate q:

$$j(q) = \sum_{n=-1}^{\infty} c(n)q^n = q^{-1} + 744 + \sum_{n=1}^{\infty} c(n)q^n \quad .$$

Theorem. *For $0 < |p|, |q| \ll 1$ one has the equality*

$$j(p) - j(q) = (p^{-1} - q^{-1}) \prod_{k,l=1}^{\infty} (1 - p^k q^l)^{c(kl)}.$$

From this equality follows an infinite sequence of algebraic identities between integer numbers $c(k)$, $k \geq 1$. The first non-trivial identity is

$$c(4) = c(3) + \frac{c(1)^2 - c(1)}{2}, \quad 20245856256 = 864299970 + \frac{196884^2 - 196884}{2} .$$

For each integer $n \geq 1$ there is an algebraic curve $C_n \subset \mathbf{C} \times \mathbf{C}$, the graph of the Hecke correspondence. In coordinates (q_1, q_2) at the neighborhood of the point $(\infty, \infty) \in \mathbf{C}P^1 \times \mathbf{C}P^1$ this curve is again algebraic and its branches are given by equations $q_1^{d_1} = q_2^{d_2}$ where $d_1 d_2 = n$, $d_1, d_2 \geq 1$. The Hecke operator T_n is defined in the usual way using the correspondence C_n. It acts on meromorphic functions on $\mathbf{C}P^1$, on meromorphic 1-forms (=modular forms of weight 2), etc.

The main object will be a meromorphic differential 2-form on $\mathbf{C}P^1 \times \mathbf{C}P^1$

$$\Omega = d_{along\ j_1} d_{along\ j_2} (\log (j_1 - j_2)) = \frac{1}{(j_1 - j_2)^2} dj_1 \wedge dj_2$$

where (j_1, j_2) are coordinates on $\mathbf{C}P^1 \times \mathbf{C}P^1$.

Let us write $j(q_1) - j(q_2)$ as an infinite product $(q_1^{-1} - q_2^{-1}) \times \prod_{k,l=1}^{\infty} (1 - q_1^k q_2^l)^{b(k,l)}$. We want to prove that $b(k,l) = c(kl)$. Using symbols $b(k,l)$ we can write an explicit formula for the q-expansion of Ω:

$$\Omega = \frac{1}{(q_1 - q_2)^2} dq_1 \wedge dq_2 + \sum_{k,l=1}^{\infty} b(k,l) \frac{kl\, q_1^{k-1} q_2^{l-1}}{(1 - q_1^k q_2^l)^2} dq_1 \wedge dq_2 \quad .$$

Now we calculate the following double residue for $N, M \geq 1$:

$$Res_{q_2=0}\, Res_{q_1=0} \left(j(q_1) j(q_2) \left[T_N^{(1)} \circ T_M^{(2)} (\Omega) \right] \right) \quad .$$

This expression is equal to 0 because $Res_{q_1=0}(\ldots)$ is a meromorphic 1-form on $\mathbf{C}P^1$ with pole only at $j_2 = \infty$. We replace Ω by the sum as above. In the first term

$$Res_{q_2=0}\, Res_{q_1=0} \left(j(q_1) j(q_2) \left[T_N^{(1)} \circ T_M^{(2)} \left(\frac{1}{(q_1 - q_2)^2} dq_1 \wedge dq_2 \right) \right] \right)$$

we can substitute q_1^{-1} for $j(q_1)$ because of the regularity at $q_1 = 0$ of all other factors for generic q_2. Thus the first term can be expressed linearly in numbers $c(n)$. The second term is equal to

$$Res_{q_2=0}\, Res_{q_1=0} \left(j(q_1) j(q_2) \sum_{k,l=1}^{\infty} (\ldots) \right) = Res_{q_2=0}\, Res_{q_1=0} \left(q_1^{-1} q_2^{-1} \sum_{k,l=1}^{\infty} (\ldots) \right)$$

because of the regularity at zero of the double sum. This term can be expressed linearly in numbers $b(k,l)$. I leave to the reader the rest of the calculation.

2. STANDARD FACTS ABOUT AUTOMORPHIC FORMS

2.1. Definition of automorphic forms

Let G be a connected unimodular Lie group, K a maximal compact subgroup, and Γ a discrete subgroup of G of finite covolume. Let us fix a homomorphism

$\chi : \mathcal{Z}(U(\mathbf{g}))\longrightarrow\mathbf{C}$ from the center of the universal enveloping algebra of the Lie algebra $\mathbf{g} := Lie(G)$ to $\mathbf{C}$. Automorphic forms are complex-valued C^∞-functions on G/Γ which are K-finite and annulated by a finite power of the ideal $Ker(\chi)$. A more general definition is obtained if one considers not just functions but sections of a local system associated with a finite-dimensional representation $\rho : \Gamma\longrightarrow GL(N,\mathbf{C})$. Any automorphic form is automatically real analytic because it satisfies an elliptic differential equation with real analytic coefficients.

Usually people consider functions satisfying certain growth conditions at cusps, i.e. they consider l^2-integrable functions or functions with polynomial growth at infinity. In the classical case of $G = SL(2,\mathbf{R})$ and $\Gamma = SL(2,\mathbf{Z})$ automorphic forms include (anti)-holomorphic modular forms of weights $k = 4, 6, \ldots$ and Maass wave forms (eigenfunctions of the Laplace operator on $\mathcal{H}/\Gamma = K\backslash G/\Gamma$). The standard growth condition can be formulated for holomorphic modular forms in terms of q-expansion as the absence of terms $c(k)q^k$ with $k < 0$. One of reasons to ignore automorphic forms with exponential growth at cusps is that the algebra of Hecke operators acts "freely" on such forms.

2.2. Theta correspondence

Theta correspondence transforms automorphic forms from one Lie group G_1 to another Lie group G_2, where $G_1 \times G_2$ is a subgroup of the symplectic linear group (see [14]). The typical example is $G_1 = Sp(V_1)$ and $G_2 = SO(V_2)$ where $(V_1, (,)_1)$ is a symplectic real vector space and $(V_2, (,)_2)$ is a real vector space with a non-degenerate symmetric bilinear form. The tensor product $V = V_1 \otimes V_2$ carries the natural symplectic structure $(,)_1 \otimes (,)_2$.

We denote by $W = W(V)$ the Hilbert space of the Weil representation of the double covering $\widetilde{Sp(V)}$ of the symplectic group $Sp(V)$. The space W can be naturally identified with the space of l^2-integrable functions on any Lagrangian subspace of V. Thus one can speak about the nuclear space $W^{-\infty}$ consisting of distributions of moderate growth. Restricting the Weil representation of $\widetilde{Sp(V)}$ to $\widetilde{Sp(V_1)} \times \widetilde{SO(V_2)}$ one gets a partially defined correspondence between projective representations of G_1 and G_2. R. Howe observed that this is a partial bijection in many cases.

Let $\Lambda_1 \subset V_1$ and $\Lambda_2 \subset V_2$ be integral lattices. Then $\Lambda := \Lambda_1 \otimes \Lambda_2$ is an integral lattice in V. Denote by $\Gamma_1, \Gamma_2, \Gamma$ arithmetic subgroups of G_1, G_2, $G = Sp(V)$ consisting of automorphisms of these lattices. The space of invariants $(W^{-\infty})^\Gamma$ is finite-dimensional and consists of certain theta functions. Theta correspondence is given by an integral operator from G_1/Γ_1 to G_2/Γ_2 with the kernel equal to a theta function. In the next section we will consider an important example.

2.3. Siegel theta function

Let (V_1, Λ_1) be $(\mathbf{R}^2, \mathbf{Z}^2)$ with the standard symplectic form and Λ_2 be an even unimodular lattice of signature (n_+, n_-). We denote by Gr the set of orthogonal decompositions of $V_2 := \Lambda_2 \otimes \mathbf{R}$ into the sum $V_+ \oplus V_-$ of positive definite and negative definite subspaces. Gr can be considered as an open subset of the Grassmanian of n_--dimensional subspaces in V_2. If $p \in Gr$ is such a decomposition we denote by p_+, p_- projectors onto V_+, V_- respectively. The Siegel theta function (see [16]) is the restriction of the standard theta function for $Sp(V_1 \otimes V_2)$ to the symmetric subspace $\mathcal{H} \times Gr \subset Sp(V_1 \otimes V_2)/U(n_+ + n_-)$. The explicit formula for it is

$$\Theta(\tau, p) = \sum_{\lambda \in \Lambda_2} \exp\left(2\pi i \left(\frac{(p_+(\lambda), p_+(\lambda))}{2}\tau + \frac{(p_-(\lambda), p_-(\lambda))}{2}\overline{\tau}\right)\right) =$$

$$= \sum_{\lambda \in \Lambda_2} q^{\frac{(\lambda,\lambda)}{2}} |q|^{-(p_-(\lambda), p_-(\lambda))} \quad .$$

This function is invariant under the action of $\Gamma_2 = Aut(\Lambda_2)$ on Gr. It transforms under the action of $\Gamma_1 = SL(2, \mathbf{Z})$ on $\tau \in \mathcal{H}$ as

$$\Theta\left(\frac{a\tau + b}{c\tau + d}, p\right) = \pm(c\tau + d)^{n_+/2}(c\overline{\tau} + d)^{n_-/2}\Theta(\tau, p) \quad .$$

If F is a holomorphic modular form for $\Gamma_1 = SL(2, \mathbf{Z})$ of weight $\frac{n_- - n_+}{2}$, or a Maass form for $n_- = n_+$, then the theta transform of F is defined as

$$\Phi(p) = \int_{\mathcal{H}/PSL(2,\mathbf{Z})} \Theta(\tau, p) F(\tau) y^{\frac{n_-}{2}} \frac{dxdy}{y^2} \quad ,$$

where $\tau = x + iy,\ x, y, \in \mathbf{R}$. This integral converges for parabolic F.

The image of theta transform satisfies differential equations. Namely, there is a homomorphism $\alpha : \mathcal{Z}(U(\mathbf{g}_2)) \longrightarrow \mathcal{Z}(U(\mathbf{g}_1))$ such that for any vector v in the Weil representation $W(V_1 \otimes V_2)$ and any $z \in \mathcal{Z}(U(\mathbf{g}_2))$ one has $z(v) = (\alpha(z))(v)$. If we apply it to the theta function we get the formula for the annulator of Φ in $\mathcal{Z}(U(\mathbf{g}_2))$.

The Siegel theta function also appears in string theory where it is the partition function of the torus $\mathbf{C}/(\mathbf{Z} + \tau\mathbf{Z})$ in the Narain model associated with the indefinite lattice Λ_2 and the orthogonal decomposition p.

3. BORCHERDS-HARVEY-MOORE CONSTRUCTION

3.1. Classical modular forms with poles at cusps

The main idea of Borcherds-Harvey-Moore construction is a formal application of theta correspondence to modular forms for arithmetic subgroups in $SL(2,\mathbf{R})$ with at most exponential growth at the cusps. In holomorphic case and for $\Gamma_1 = SL(2,\mathbf{Z})$ any such form is meromorphic on $\mathcal{H}/\mathbf{Z} \sqcup \{\infty\}$ with only pole at the cusp. It can be presented as $\Delta(\tau)^{-k} F_0(\tau)$ where $k \geq 0$, $\Delta = q\prod_{n=1}^{\infty}(1-q^n)^{24}$, and F_0 is a parabolic holomorphic modular form. Unlike in the classical theory, holomorphic modular forms with poles at cusps can have negative weights. In the case of Maass forms for any $\lambda \in \mathbf{C}$ there is an infinite-dimensional vector space of solutions of the equation $\Delta F = \lambda F$ on $\mathcal{H}/\Gamma_1$ with exponential growth at the cusp.

There are also other automorphic forms like $E_2'(\tau) := 1 - 24\sum_{n=1}^{\infty} \frac{n\,q^n}{1-q^n} - \frac{3}{\pi y}$, real analytic Eisenstein series, Siegel theta functions, etc.

All these forms have the following common property: there exists $M \geq 0$ such that for any $N \geq 0$ the form can be expanded in a neighborhood of the cusp as

$$o(y^{-N}) + \sum_{m:|m|<M} e^{2\pi(imx+|m|y)} \left(\sum_{j\in\ finite\ \ set} c_{m,j} (\log(y))^{k_{m,j}} y^{\sigma_{m,j}} + \epsilon_m(y) \right)$$

where $k_{m,j} \in \mathbf{Z}_{\geq 0}$, $m \in \mathbf{Z}$, $c_{m,j}, \sigma_{m,j} \in \mathbf{C}$ and $\epsilon_m(y) = o(y^{-N})$ depends on y only.

3.2. Regularization of divergent integrals

Let us assume that F is a holomorphic modular form of weight $(n_- - n_+)/2$ with poles at cusps. We want to make sense of the divergent integral

$$\Phi(p) = \int_{\mathcal{H}/PSL(2,\mathbf{Z})} \Theta(\tau,p) F(\tau)\, y^{\frac{n_-}{2}} \frac{dxdy}{y^2} \ .$$

After the expansion of $\Theta(\tau,p)$ and $F(\tau)$ at $y = \operatorname{Im}\tau \longrightarrow +\infty$ there are only finitely many divergent terms of the form

$$\int_{x\in[0,1],\ y\geq const} \exp\left(2\pi i m x + 2\pi|m|y - Ly\right) y^{n_-/2-2}\ dxdy \ ,$$

where L is a non-negative real-valued function on Gr.

If $m \neq 0$ then we define the regularized value of this integral to be 0. It is the natural choice if we perform the integration along the variable x first. If $m = 0$ and $L > 0$ then the integral is absolutely convergent. The trouble arises only when $m = L = 0$. In this case we can subtract the divergent term y^k from the integral in domain $y > y_0$. The result will be a function of y_0.

In the more general case for non-holomorphic forms there are finitely many divergent components of the product $F(\tau)\Theta(\tau, p)$ with frequency $m = 0$ along coordinate x. All these components are of type $\left(\log(y)\right)^k y^\sigma$. One can multiply F by y^{-s}, or (better) by a real analytic Eisenstein series. Then we can assign a value to the integral for $Re(s) \gg 0$ and continue it to a meromorphic function for all $s \in \mathbf{C}$. The regularized integral can be defined as the constant term of the Laurent expansion at $s = 0$.

3.3. Automorphic forms with singularities at locally homogeneous submanifolds

Let us see what kind of divergences our integral has for holomorphic form $F = \sum_n c(n)q^n$. First of all, if the $c(0) \neq 0$ then the constant term in the series for Θ corresponding to the vector $\lambda = 0$ produces troubles. This problem is independent of the point p in the Grassmanian, and we can resolve it in one way or another. The result is that we still can define F modulo an additive constant.

Other divergent terms appear when there is a *non-zero* lattice vector $\lambda \in \Lambda_2$ such that λ belongs to the the positive subspace V_+ and has a special length. Namely, a term $q^{-(\lambda,\lambda)/2}$ should be present in the q-expansion of F.

Thus we see that the singular set of Φ in $X = Gr/\Gamma_2$ consists of a finite union of certain totally geodesic submanifolds of type $X' = K'\backslash G'/\Gamma'$. The same fact holds for non-holomorphic modular forms Φ admitting an asymptotic expansion at infinity as in 3.1. Also, one can write explicitly the types of singularities of Φ, i.e. functions Φ' defined at a neighborhood of X' such that $\Phi - \Phi'$ can be continued to a real-analytic function. These functions Φ' are finite linear combinations of functions $x \mapsto (\log(dist(x, X')))^k (dist(x, X'))^\sigma$ where $dist(x, X')$ is the distance between x and X' in a natural metric.

The function Φ on the domain of definition satisfies differential equations. If $c(0) = 0$ then these will be homogeneous linear differential equations $z(\Phi) = 0$ for some $z \in \mathcal{Z}U(\mathbf{g}_2)$ (see the end of 2.3). The divergent term $c(0)q^0$ produces certain universal r.h.s. for these equations.

In the case $n_+ = 1$ submanifolds X' are locally hyperplanes in the hyperbolic space. In the case $n_+ = 2$ they are complex hypersurfaces. The same is true for $G_2 = SU(N,1)$. If $G_2 = SP(2g,\mathbf{R})$ and we consider $G_1 = PSL(2,\mathbf{R})$ as the orthogonal group $SO(2,1)$) then subvarieties X' are complex subvarieties of codimension g.

If $(G_2,\Gamma_2) = (SL(2,\mathbf{R}), SL(2,\mathbf{Z}))$ then the forms Φ on $\overline{\mathcal{M}}_1$ have singularities at Heegner points, i.e. moduli of elliptic curves with complex multiplication, or equivalently, points with the coordinate τ in an imaginary quadratic field, $\tau = x + iy$, where $x \in \mathbf{Q}$ and $y^2 \in \mathbf{Q}$.

3.4. Fourier expansions at cusps

In theta correspondence one can write an expansion of Φ at cusps via Fourier coefficients of F. The usual trick (Rankin-Selberg method) consists in replacing of the integral over the fundamental domain of $SL(2,\mathbf{Z})$ by an integral over the fundamental domain of $\mathbf{Z}$ in $\mathcal{H}$. In the case of divergent integrals one should be cautious when interchanging infinite sums and integrals.

Let $\lambda_0 \in \Lambda_2$ be a primitive null-vector, $(\lambda_0,\lambda_0) = 0$. Such vector always exists for undefinite lattices of sufficiently large rank, including all even unimodular lattices. We define smaller lattice $\tilde{\Lambda}$ as $\lambda_0^{\perp}/\mathbf{Z}\lambda_0$. It is easy to see that any orthogonal decomposition p of $V_2 = \Lambda_2 \otimes \mathbf{R}$ defines an orthogonal decomposition $\tilde{p}$ of $\tilde{V} = \tilde{\Lambda} \otimes \mathbf{R}$ using natural isomorphism between $\tilde{V}$ and $(\mathbf{R}p_+(\lambda_0) + \mathbf{R}p_-(\lambda_0))^{\perp}$.

Using Poisson summation formula in direction $\mathbf{Z}\lambda_0$ one can rewrite $\Theta(\tau,p)$ as

$$\Theta(\tau,p) = \frac{1}{\sqrt{2y(p_+(\lambda_0),p_+(\lambda_0))}} \sum_{\lambda' \in \Lambda_2/\mathbf{Z}\lambda_0} \sum_{l \in \mathbf{Z}} \exp(\ldots)$$

where $\exp(\ldots)$ is the exponent of an explicit algebraic expression.

Let us choose an additional lattice vector λ_1 such that $(\lambda_0,\lambda_1) = 1$. Then we can embed $\tilde{\Lambda}$ in Λ_2 as $(\mathbf{Z}\lambda_0 + \mathbf{Z}\lambda_1)^{\perp}$. Moreover, we can parametrize $\Lambda_2/\mathbf{Z}\lambda_0$ by $\tilde{\Lambda} \times \{k\,\lambda_1 |\, k \in \mathbf{Z}\}$. The total sum becomes a sum over $(k,l) \in \mathbf{Z}^2$ of certain twisted theta series associated with the lattice $\tilde{\Lambda}$ and the orthogonal decomposition $\tilde{p}$. The total formula is quite cumbersome.

Main Identity. *Let F be a bounded measurable function on $\mathcal{H}$ satisfying the equation*

$$F\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^{\frac{n_- - n_+}{2}} F(\tau) \ .$$

Then the following identity holds:

$$\int_{\mathcal{H}/SL(2,\mathbf{Z})} \Theta(\tau,p)F(\tau)y^{\frac{n_-}{2}}\frac{dxdy}{y^2} = \frac{1}{\sqrt{2\epsilon}}\int_{\mathcal{H}/SL(2,\mathbf{Z})} \Theta(\tau,\tilde{p})F(\tau)y^{\frac{n_- -1}{2}}\frac{dxdy}{y^2} +$$

$$+\frac{2}{\sqrt{2\epsilon}}\sum_{n>0}\int_{\mathcal{H}/\mathbf{Z}} F(\tau)\exp\left(-\frac{\pi n^2}{2y\,\epsilon}\right)\sum_{\tilde{\lambda}\in\tilde{\Lambda}} e^{2\pi i n(\tilde{\lambda},\mu)}q^{(\tilde{\lambda},\tilde{\lambda})/2}|q|^{-(\tilde{p}_-(\tilde{\lambda}),\tilde{p}_-(\tilde{\lambda}))}y^{\frac{n_- -1}{2}}\frac{dxdy}{y^2}\ .$$

Here $\epsilon = (p_+(\lambda_0), p_+(\lambda_0)) > 0$ and

$$\mu = (\lambda_1 + (p_+(\lambda_0) - p_-(\lambda_0))/2\epsilon)\ mod\,\mathbf{R}\lambda_0 \in {\lambda_0}^{\perp}/\mathbf{R}\lambda_0 = \tilde{\Lambda}\otimes\mathbf{R}\ .$$

Correspondence $p \mapsto (\tilde{p}, \epsilon, \mu)$ can be considered as a local parametrization of Gr.

This identity follows from the formula for $\Theta(\tau, p)$ as a sum over pairs of integers (k, l). The first term comes from the term corresponding to $k = l = 0$. Any other pair of integers (k, l) can be presented as (nc, nd) where $n > 0$ and c and d are coprime. We identify in the usual way $SL(2,\mathbf{Z})/\left\{\begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix}\Big|\, n\in\mathbf{Z}\right\}$ with the set of primitive vectors (c, d) in $\mathbf{Z}^2$ and rewrite the sum over non-zero pairs (k, l) as the sum over $n > 0$ and over the set of copies in $\mathcal{H}/\mathbf{Z} = \{(x + iy|\, 0 < x \leq 1,\ \ 0 < y < +\infty\}$ of the classical fundamental domain of $SL(2,\mathbf{Z})$ in $\mathcal{H}$.

Now let us see what happens for functions F which admit an asymptotic expansion at the cusp as in 3.1. The divergence of the sum above as $y \longrightarrow \infty$ is of the same form as one of the original integral for Φ. In the integral over $\mathcal{H}/SL(2,\mathbf{Z})$ one might expect a priori divergences near cusps on $\mathbf{Q} \subset \mathbf{R}$. In fact, the function F has exponential growth at these points. Nevertheless the integral near $\mathbf{R}$ is absolutely convergent because of the factor $\exp(-\pi/2\epsilon y)$ which makes the total integrand small enough, as $\epsilon \ll 1$. Thus the exchange of the order of the sum and of the integral is justified for small $\epsilon = (p_+(\lambda_0), p_+(\lambda_0))$.

One can calculate explicitly integrals over $\mathcal{H}/\mathbf{Z}$ corresponding to individual terms in q-expansion of F. It reduces to classical integrals for Bessel functions

$$\int_{y>0}\exp(-\beta/y - \alpha y)y^{\nu-1}dy = 2(\beta/\alpha)^{\nu/2}K_\nu(2\sqrt{\alpha\beta})\ .$$

3.5. Hyperbolic case

Let us consider the case of hyperbolic even unimodular lattices, $n_- = 1$ and $n_+ > 1$. As in the previous section, we pick a primitive null-vector $\lambda_0 \in \Lambda_2$. The additional vector λ_1 such that $(\lambda_0, \lambda_1) = 1$ is chosen now among null-vectors. Lattice $\tilde{\Lambda} \simeq (\mathbf{Z}\lambda_0 + \mathbf{Z}\lambda_1)^{\perp}$ is considered as a sublattice of Λ. Thus we have a decompositon $\Lambda_2 = \mathbf{Z}\lambda_0 + \mathbf{Z}\lambda_1 + \tilde{\Lambda}$.

We identify the space Gr with the (half of the) hyperboloid

$$H = \{v \in \Lambda_2 \otimes \mathbf{R} |\ (v,v) = -1,\ \ (v, \lambda_0) > 0\}\ .$$

Projector p_- corresponding to v is the orthogonal projector to 1-dimensional space $\mathbf{R}\,v$. In terms of parameters $(\epsilon, \mu) \in \mathbf{R}_+ \times (\tilde{\Lambda} \otimes \mathbf{R})$ from the previous section we have

$$v = v(\epsilon, \mu) = \frac{-\frac{1}{\sqrt{\epsilon}} - \sqrt{\epsilon}(\mu,\mu)}{2}\lambda_0 + \sqrt{\epsilon}\lambda_1 + \sqrt{\epsilon}\mu\ .$$

Parameter $\tilde{p}$ does not vary because $\tilde{\Lambda}$ is positive definite and $\tilde{Gr}$ is a one point set.

Let $F = \sum_n c(n) q^n \in \mathbf{C}((q))$ be a holomorphic modular form of weight $\frac{1-n_+}{2}$.

Theorem. *Theta transform Φ of F is locally the restriction of a continuous piecewise linear function on $\tilde{\Lambda} \otimes \mathbf{R}$ to the hyperboloid H.*

The rest of this section is devoted to the proof of this Theorem. Function Φ has the following expansion as $\epsilon \longrightarrow 0$:

$$\Phi = \Phi(\epsilon, \mu) = \frac{1}{\sqrt{2}\,\epsilon} \int\limits_{\mathcal{H}/SL(2,\mathbf{Z})} \Theta(\tau, \tilde{p}) F(\tau) \frac{dxdy}{y^2} +$$

$$+ \frac{2}{\sqrt{2}\,\epsilon} \sum_{n>0} \int\limits_{\mathcal{H}/\mathbf{Z}} F(\tau) \exp\left(-\frac{\pi n^2}{2y\,\epsilon}\right) \sum_{\tilde{\lambda} \in \tilde{\Lambda}} e^{2\pi i n (\tilde{\lambda}, \mu)} q^{(\tilde{\lambda}, \tilde{\lambda})/2} \frac{dxdy}{y^2}\ .$$

Notice that each integral in this formula can be unambigously regularized using rules from 3.2. We denote terms in this formula by $\Phi_{(1)}$ and $\Phi_{(2)}$.

Term $\Phi_{(1)}$ is proportional to $(v, \lambda_0)^{-1}$ because $(v, \lambda_0) = \sqrt{\epsilon}$.

After the expansion of F into the series we see that in the term $\Phi_{(2)}$ we have to calculate integrals

$$\int\limits_{\mathcal{H}/\mathbf{Z}} q^{m + \frac{(\tilde{\lambda}, \tilde{\lambda})}{2}} \exp\left(-\frac{\pi n^2}{2y\,\epsilon}\right) \frac{dxdy}{y^2}\ .$$

If $m + \frac{(\tilde{\lambda}, \tilde{\lambda})}{2} = 0$ then this integral is equal to $2\epsilon/\pi n^2$, otherwise it vanishes. Thus we see that the second term is

$$\Phi_{(2)} = \frac{4\pi\sqrt{\epsilon}}{\sqrt{2}} \sum_{\tilde{\lambda} \in \tilde{\Lambda}} c(-(\tilde{\lambda}, \tilde{\lambda})/2) \sum_{n>0} \frac{e^{2\pi i n (\tilde{\lambda}, \mu)}}{n^2}\ .$$

Every vector $\tilde{\lambda}$ appears in this formula together with the opposite vector $-\tilde{\lambda}$. It implies that we can replace in the formula from above exponent by the cosine. Now we use the formula

$$\sum_{n>0} \frac{\cos(2\pi n x)}{n^2} = \pi^2 \left(x^2 + \alpha(x)x + \frac{1}{6} \right), \quad x \in \mathbf{R}$$

where $\alpha(x)$ is a locally constant function of x: $\alpha(x) = -2n - 1$ for $n \le x < n+1$.

Finally, we get a formula for Φ as a finite sum:

$$\Phi(\epsilon, \mu) = \Phi_{(1)} + \frac{\pi}{\sqrt{2}} \sum_{\tilde{\lambda} \in \tilde{\Lambda}} c(-(\tilde{\lambda}, \tilde{\lambda})/2) \cdot 4\sqrt{\epsilon} \left\{ (\tilde{\lambda}, \mu)^2 + \alpha((\tilde{\lambda}, \mu))(\tilde{\lambda}, \mu) + \frac{1}{6} \right\} .$$

Terms (locally) proportional to $\sqrt{\epsilon}(\tilde{\lambda}, \mu)$ are restrictions of linear functions $v = v(\epsilon, \mu) \mapsto const(v, \tilde{\lambda})$. Terms proportional to $\sqrt{\epsilon}$ are restrictions of linear functions $v \mapsto const(v, \lambda_0)$. We claim that the rest is also the restriction of linear function (proportional to $v \mapsto (v, \lambda_1)$ in fact).

Using the fact that $\Phi_{(1)}(v) = const\,(v, v)/(v, \lambda_0)$ we see that

$$\Phi = \Phi(v) = (\text{piecewise linear function of } v) + \frac{1}{(v, \lambda_0)}(\text{quadratic polynomial of } v) .$$

Applying the following automorphism of Λ_2:

$$\lambda_0 \mapsto \lambda_1, \ \lambda_1 \mapsto \lambda_0, \ \tilde{\lambda} \mapsto \tilde{\lambda} \ \text{for} \ \tilde{\lambda} \in \tilde{\Lambda}$$

we see that

$$\Phi(v) = (\text{piecewise linear function of } v) + \frac{1}{(v, \lambda_1)}(\text{quadratic polynomial of } v) .$$

Comparing two expressions for $\Phi(v)$ as above we conclude that Φ is a piecewise linear function of v.

This finishes the proof of the Theorem of this section. R. Borcherds calculated theta transform in a more general situation and obtained that Φ is the restriction of a piecewise polynomial function on the hyperboloid H.

3.6. Product formulas for meromorphic forms

Now we consider the case when $n_- = 2$ and F is a holomorphic modular form with pole at the cusp.

As in the previous section we fix decomposition $\Lambda_2 = \mathbf{Z}\lambda_0 + \mathbf{Z}\lambda_1 + \tilde{\Lambda}$ where $\tilde{\Lambda}$ is now a hyperbolic lattice. The space Gr is convenient to parametrize by vectors $v \in \tilde{\Lambda} \otimes \mathbf{C}$ such that $(Im(v), Im(v)) < 0$. Projector p in $\Lambda_2 \otimes \mathbf{R}$ and corresponding parameters $(\tilde{p}, \epsilon, \mu)$ from the Section 3.4 are given by the following formulas:

$$p_-(\Lambda_2 \otimes \mathbf{R}) = \mathbf{R} \cdot Re(u) + \mathbf{R} \cdot Im(u) \subset \Lambda_2 \otimes \mathbf{R} \quad \text{where} \quad u = -\frac{(v,v)}{2}\lambda_0 + \lambda_1 + v,$$

$$\tilde{p}_-(\tilde{\Lambda} \otimes \mathbf{R}) = \mathbf{R} \cdot Im(v),$$

$$\epsilon = \frac{-1}{(Im(v), Im(v))}.$$

$$\mu = -Re(v) \ .$$

The integral over $\mathcal{H}/SL(2,\mathbf{Z})$ in the first term of the formula for the Fourier expansion at cusps (Section 3.4) was evaluated in the pervious section. The result is that the first term has the form

$$\Phi_{(1)} = \Phi_{(1)}(v) = (W(v), Im(v))$$

where $W(v)$ is a locally constant function on Gr with values in $\tilde{\Lambda} \otimes \mathbf{C}$.

Now we consider $\Phi_{(2)}$, the sum of integrals over $\mathcal{H}/\mathbf{Z}$.

The contribution of terms with $\tilde{\lambda} = 0$ and $n > 0$ is a divergent sum $const + 2\,c(0)\sum_{n>0} 1/n$. Nevertheless, if we use some regularization procedure, we obtain $2\,c(0)\,\sum_{n>0} \epsilon^s/n^{2s+1}$ as $s \longrightarrow 0$. An easy calculation shows that the regularized value is $const + c(0)\log(\epsilon)$.

The contribution of the term corresponding to $\tilde{\lambda} \neq 0$ and $n > 0$ is equal to

$$2\,c(-(\tilde{\lambda},\tilde{\lambda})/2)\frac{1}{n}\,e^{2\pi i n(\tilde{\lambda}, -Re(v))}\exp\left(-2\pi n\sqrt{\frac{-(\tilde{p}_-(\tilde{\lambda}), \tilde{p}_-(\tilde{\lambda}))}{\epsilon}}\right) \quad .$$

Here we use the classical formula $K_{-1/2}(z) = \sqrt{\pi/2z} \cdot \exp(-z)$.

Elementary calculations show that $\sqrt{\frac{-\left(\tilde{p}_-(\tilde{\lambda}), \tilde{p}_-(\tilde{\lambda})\right)}{\epsilon}} = |(\tilde{\lambda}, Im(v))|$. The sum over n and two opposite vectors $\pm\tilde{\lambda}$ gives

$$-4\,c(-(\tilde{\lambda},\tilde{\lambda})/2)\log|1 - \exp(2\pi i(\tilde{\lambda}_+, v)|$$

where $\tilde{\lambda}_+$ is the one of two vectors $(\tilde{\lambda}, -\tilde{\lambda})$ which has positive scalar product with $Im(v)$.

The resulting formula for Φ is (up to an additive constant)

$$(W(v), Im(v)) + c(0)\log(\epsilon) - 4 \sum_{\tilde{\lambda}:\, (\tilde{\lambda}, Im(v))>0} c(-(\tilde{\lambda},\tilde{\lambda})/2) \log|1-\exp(2\pi i(\tilde{\lambda}_+, v)| \ .$$

Notice that all terms here except $c(0)\log(\epsilon)$ are locally sums of holomorphic and anti-holomorphic functions of v.

Let us now assume that the coefficients $c(n)$ of F are integers. Denote by $\mathcal{L}$ an equivariant complex line bundle over Gr whose total space is the complex cone

$$\{w \in \Lambda_2 \otimes \mathbf{C} |\, (w,w) = 0,\ \big(Im(w), Im(w)\big) < 0\} \ .$$

Projection $\mathcal{L} \longrightarrow Gr$ is $w \mapsto u = w/(w, \lambda_0)$. Bundle $\mathcal{L}$ carries invariant hermitean scalar product $\|w\| := \sqrt{-(w,\overline{w})/2}$. We claim that there exists a meromorphic section Ψ of $(\mathcal{L})^{\otimes(c(0)/2)}$ such that

$$\log\|\Psi\| = -\Phi/4 \ .$$

Locally, it follows from the expression for Φ from above and from the identity

$$\epsilon = \frac{|(w,\lambda_0)|^2}{\|w\|^2} \ .$$

Globally, we use the information about singularities from 3.3. In general, Ψ is not Γ_2-equivariant and it gives a section on Gr/Γ_2 of $\mathcal{L}^{\otimes(c(0)/2)}$ twisted with a unitary character of $\Gamma_2/[\Gamma_2, \Gamma_2]$. In this way one obtains a proof of the Theorem from 1.1.

R. Borcherds proved more general product formulas for congruence subgroup in $SL(2,\mathbf{Z})$, non-unimodular lattices, and proposed to consider the case $G_2 = SU(N,1)$. He also developed the formalism for generalized theta functions associated with harmonic polynomials.

4. EXPLICIT EXAMPLES

I will present only 3 examples.

The simplest example of the product formula for the group $SO(1,2)$ is completely trivial: $\eta(q) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)^1$ where all the exponents 1 are coefficients of q^{n^2} of a form of weight 1/2, namely of the theta function $1/2 + \sum_{n=1}^{\infty} q^{n^2}$.

An example for $SO(2,2)$ is the product formula for $j(p) - j(q)$ in 1.2.

The next example for the group $SO(3,2)$ is also very beautiful:

$$\sum_{m,n\in\mathbf{Z}} (-1)^{m+n} p^{m^2} q^{n^2} r^{mn} = \prod_{a+b+c>0} \left(\frac{1-p^a q^c r^b}{1+p^a q^c r^b}\right)^{f(ac-b^2)}$$

where $\sum f(n)q^n = 1/(\sum_n (-1)^n q^{n^2}) = 1 + 2q + 4q^2 + 8q^3 + 14q^4 + \ldots$.

5. CONNECTIONS WITH OTHER PARTS OF MATHEMATICS

5.1. Generalized Kac-Moody algebras

There are many nice examples of so-called generalized Kac-Moody algebras constructed by R. Borcherds and later by V. Gritsenko and V. Nikulin. These Lie superalgebras are graded by a lattice, and the generating function for the dimensions of homogeneous components is essentially an automorphic form for $O(n,2)$. The product formula is the Weyl-Kac-Borcherds denominator identity. The Weyl group for these algebras is often an arithmetic group generated by reflections. Also the Monster group appears as an automorphism group.

5.2. K3 surfaces, Mirror symmetry, string duality

Let Λ be an even hyperbolic sublattice of (unique) even unimodular lattice $\Lambda_{3,19}$ of signature $(3,19)$. We define $\mathcal{M}_\Lambda$ as the moduli space of algebraic $K3$-surfaces X such that $\Lambda \subset Pic(X) \subset H^2(X,\mathbf{Z}) \simeq \Lambda_{3,19}$. By the classification theorem for $K3$-surfaces we see that $\mathcal{M}_\Lambda$ is of the type where Borcherds' products are defined. The image of $\mathcal{M}_\Lambda$ under the period map is

$$\{w \in \Lambda^\perp \otimes \mathbf{C} |\ (w,w)=0,\ (w,\overline{w})<0;\ \forall\lambda \in \Lambda_{3,19}\cap\Lambda^\perp\cap w^\perp \quad (\lambda,\lambda)\neq -2\}/\mathbf{C}^\times \ .$$

Thus Borcherds' results mean that for some Λ the standard line bundle over $\mathcal{M}_\Lambda$ is a torsion element in $Pic(\mathcal{M}_\Lambda)$. One of such lattices is the one-dimensional lattice $\Lambda = \mathbf{Z}\lambda,\ (\lambda,\lambda)=2$. In general, product formulas produce linear relations between certain divisors in Shimura varieties.

Also one can consider the moduli space of Riemannian metrics on 4-dimensional manifolds, which are hyperkaehler metrics on $K3$-surfaces. This space is locally modeled by $SO(19,3)/SO(19)\times SO(3)$. Borcherds' construction gives a certain real-analytic function on it. Presumably, it is related to the regularized determinant of the Laplace operator (see [15]).

Gritsenko-Nikulin, Jorgenson-Todorov and Harvey-Moore (see [9,13,15]) made conjectures about the relation between Kac-Moody algebras, $K3$-surfaces, Borcherds' product formulas and mirror symmetry. One expects that certain numbers of curves of various genera on a generic element of the family $\mathcal{M}_\Lambda$ coincide with exponents in a product formula associated with the dual family where Λ is equal to the lattice of transcedental cycles on a generic element.

In fact, J. Harvey and G. Moore tried to find a conceptual construction of the generalized Kac-Moody algebra associated with the $K3$-surface X. Conjecturally, it is the direct sum of all cohomology groups of all moduli spaces of stable coherent sheaves on X. The Lie bracket is given by a correspondence in the cube of the total (disconnected) moduli space. This correspondence is expected to consist of triples of sheaves from all possible short exact sequences.

The idea of the Harvey-Moore integral arose from a new duality in string theory relating elliptic curves on one manifold to numbers of all curves of all genera on another manifold. Thus the integral over the moduli space of elliptic curves appeared. In some cases we get in the formula for Φ an infinite sum of 3-logarithm functions as in the usual mirror symmetry. R. Dijkgraaf, G. Moore, E. Verlinde and H. Verlinde proposed a Borcherds' type identity involving elliptic genera, see [7].

5.3. Hyperbolic case and Donaldson invariants

It is well-known that in Donaldson theory 4-dimensional manifolds X with the $b_+ = 1$ are very special. The Donaldson invariant is a piecewise polynomial function on the cone $\{x \mid x \cdot x < 0\}$ in the hyperbolic space $H^2(X, \mathbf{R})$. R. Borcherds observed that in certain cases (like $\mathbf{C}P^2$ blown up at 9 points) the Donaldson invariant coincides with one of the functions Φ given by the theta correspondence.

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