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TRAVAUX DE LAUMON

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To A. Grothendieck, on his 60'th birthday

In this exposé we will try to explain Laumon's "principle of stationary phase" for the ℓ-adic Fourier Transform; where it comes from, what it is, and what it's good for.

Background: The Formalism of Fourier Transform

Let us begin by recalling the classical Fourier Transform in the case of a finite abelian group G, written additively, with Pontryagin dual group $G^\vee$. For a function f on G, its Fourier Transform FT(f) is the function on $G^\vee$ defined as $\chi \mapsto \Sigma_G f(x)\chi(x)$. By means of the canonical isomorphism $G \approx (G^\vee)^\vee$ defined by $x(\chi) := \chi(x)$, we have the inversion formula $FT(FT(f))(x) = \#(G)f(-x)$. Traditionally we think of the functions f in question as being complex-valued, but we can just as well allow values in any subfield of $\mathbb{C}$ (or indeed in any field in which $\#G$ is invertible) which contains all roots of unity of order dividing the exponent of G.

The particular case of interest to us is this: k is a finite field, E is a finite dimensional k-vector space, $E^\vee$ is the dual vector space, $\langle x,y\rangle$ is the canonical pairing of E with $E^\vee$, and $\psi: (k,+) \to \mathbb{C}^\times$ is a nontrivial additive character of k. Then the pairing $\psi(\langle x,y\rangle)$ makes E and $E^\vee$ into Pontryagin duals, and we can speak of the Fourier Transform $FT_\psi(f)$ of a function f on E, defined as $y \mapsto \Sigma_E f(x)\psi(\langle x,y\rangle)$.

The simplest example of this situation is when E = k; then we can identify $E^\vee$ with k in such a way that $\langle x,y\rangle$ is just xy, and we have $FT_\psi(f)(y) = \Sigma_E f(x)\psi(xy)$. In the case of the prime field $\mathbb{F}_p$, a common choice of ψ is $x \mapsto \exp(2\pi ix/p)$, and with this choice the formula for FT(f) becomes $y \mapsto \Sigma_E f(x)\exp(2\pi ixy/p)$.

Let us briefly recall one sort of complex-valued function on $\mathbb{F}_p$ one is interested in Fourier Transforming. Given a polynomial in several variables $P(x_1, \dots, x_n) \in \mathbb{Z}[x_1, \dots, x_n]$, one is interested in the question of for which N the equation $P(x) = N$ has integral solutions, and "how many" it has. In studying this question, one is led to the analogous "mod p" question, namely how many $\mathbb{F}_p$ solutions the equation $P(x) = N$ has for each $N \in \mathbb{F}_p$. If we let f be the $\mathbb{Z}$-valued function on $\mathbb{F}_p$ defined by $f(N) :=$ the number of $\mathbb{F}_p$ solutions of $P(x) = N$, then FT(f) is the function on $\mathbb{F}_p$ defined by the exponential sums

$$y \mapsto \Sigma_{x_1, \dots, x_n} \exp(2\pi i y P(x_1, \dots, x_n)/p).$$

It is the systematic consideration of such exponential sums that, with the advent of ℓ-adic cohomology, leads inexorably to the notion of the ℓ-adic Fourier Transform.

At this point, it is necessary to summarize, albeit crudely, those aspects of Grothendieck's ℓ-adic cohomology theory which are indispensable to the development of our story. For technical reasons, it is necessary to choose a prime number ℓ, and an algebraic closure $\overline{\mathbb{Q}}_\ell$ of the field $\mathbb{Q}_\ell$ of ℓ-adic numbers, and to think about $\overline{\mathbb{Q}}_\ell$-valued (rather than $\mathbb{C}$-valued) functions in our discussion of the classical Fourier transform on finite abelian groups.

Let X be a scheme of finite type over $\mathbb{Z}[1/\ell]$. Then for any finite field k, the set X(k) of k-valued points of X is a finite set. If X is connected, then once we pick some geometric point ξ of X as a base point, we can speak of the Grothendieck pro-finite fundamental group $\pi_1 := \pi_1(X, \xi)$. A **lisse** $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{F}$ on a connected X is nothing other than a continuous representation of π_1 on a finite-dimensional $\overline{\mathbb{Q}}_\ell$-space which is definable over a locally compact subfield of $\overline{\mathbb{Q}}_\ell$. We will admit as a "black box" the more general concept of a constructible $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{F}$(a "sheaf", if no confusion is overly likely) on X, and more generally the concept of an object K (a "complex") in the corresponding derived category $D^b{}_c(X, \overline{\mathbb{Q}}_\ell)$. The key point for us is that, for every finite field k, a sheaf $\mathcal{F}$on X gives rise to a $\overline{\mathbb{Q}}_\ell$-valued function $\mathrm{Trace}_{k,\mathcal{F}}$ on the finite set X(k), its "trace function", which is defined as follows. A point x in X(k) may be viewed tautologically as a morphism $\varphi_{x,k}$: Spec(k) $\rightarrow$ X; the pullback of $\mathcal{F}$ by this morphism is then a sheaf on Spec(k), i.e., a continuous representation of $\mathrm{Gal}(\overline{k}/k)$ on a finite-dimensional $\overline{\mathbb{Q}}_\ell$-space which is definable over a locally compact subfield. If we denote by F_k the **inverse** of the standard generator $x \mapsto x^q$, $q = \#k$, of $\mathrm{Gal}(\overline{k}/k)$, then we may form the trace of F_k acting on $(\varphi_{x,k})^*(\mathcal{F})$, and we define $\mathrm{Trace}_{k,\mathcal{F}}$ to be the function on X(k) defined by

$$\mathrm{Trace}_{k,\mathcal{F}}(x) := \mathrm{Trace}(F_k \mid (\varphi_{x,k})^*(\mathcal{F})\ ).$$

(If $\mathcal{F}$ is lisse on a connected X, then we can speak of the Frobenius conjugacy class $F_{x,k}$ in $\pi_1(X, \xi)$, namely the image of F_k under the map of π_1's induced by $\varphi_{x,k}$, and $\mathrm{Trace}_{k,\mathcal{F}}(x)$ is just the trace of this class $F_{x,k}$ in the representation of π_1 which $\mathcal{F}$ "is".)
In a similar vein, given a complex K, it has finitely many nonvanishing cohomology sheaves $\mathcal{H}^i := \mathcal{H}^i(K)$, and we define

$$\mathrm{Trace}_{k,K} := \Sigma\, (-1)^i \mathrm{Trace}_{k,\mathcal{H}^i}.$$

It is tautologous that the trace function of a direct sum (respectively of a tensor product) of complexes is the sum (resp. the product) of the trace functions of the individual terms. In particular, the map

$$D^b{}_c(X, \overline{\mathbb{Q}}_\ell) \to \Pi_k \,\mathrm{Maps}(X(k), \overline{\mathbb{Q}}_\ell)$$

$$K \mapsto \text{its trace functions } \mathrm{Trace}_{k,K}\text{, for every finite field } k$$

factors through the Grothendieck group, and it results from Čebotarev that on the Grothendieck group this map is in fact injective.

Now suppose that we are given a morphism φ: $X \to Y$ of schemes of finite type over $\mathbb{Z}$. For every finite field k, φ induces a map φ_k: $X(k) \to Y(k)$ of finite sets, which in turn induces maps of pullback $(\varphi_k)^*$ and summation (integration) over the fibres $(\varphi_k)_!$ on the spaces of functions on these two finite sets:

$$(\varphi_k)^* : f \text{ on } Y(k) \mapsto \text{ the fct. } f\circ\varphi_k \text{ on } X(k),$$

$$(\varphi_k)_! : f \text{ on } X(k) \mapsto \text{ the fct. } y \mapsto \Sigma\, f(x), \text{ sum over } x \text{ in } (\varphi_k)^{-1}(y).$$

It is essentially tautologous that for K a complex on Y, the pulled back complex has its trace function given by the pullback of the trace function of K:

$$\mathrm{Trace}_{k,\, \varphi^*(K)} = (\varphi_k)^* (\mathrm{Trace}_{k,K}).$$

It is a deep fact, namely Grothendieck's Lefschetz Trace Formula ([Gr]), that for a complex K on X, its lower shriek direct image $R\varphi_!K$ on Y has its trace function given by

$$\mathrm{Trace}_{k,\, R\varphi_!(K)} = (\varphi_k)_! (\mathrm{Trace}_{k,K}).$$

Before discussing Laumon's work on the ℓ-adic Fourier Transform, it is convenient to recall how Grothendieck's formalism allows us to discuss the general concept of an "integral transform" in ℓ-adic cohomology. The situation is this. We are given a scheme S of finite type over $\mathbb{Z}$ on which ℓ is invertible, two S-schemes X and Y of finite type, and an ℓ-adic sheaf $\mathcal{F}$ on the fibred product $X \times_S Y$. The idea is that $\mathcal{F}$ is to play the role of the kernel function for our integral transform. Given a complex K on X, we define the complex $T_{\mathcal{F},!}(K)$ on Y to be $Rpr_{2!}(pr_1{}^*(K)\otimes\mathcal{F})$. In view of the Lefschetz Trace formula, the trace function of $T_{\mathcal{F},!}(K)$ on Y(k) at a point y, which lies over a given point s in S(k), is given by

$$y\mapsto\Sigma \mathrm{Trace}_{k,K}(x)\mathrm{Trace}_{k,\mathcal{F}}(x,y),$$

the sum over all points x in X(k) which lie over the given s in S(k).

In other words, the trace function of $T_{\mathcal{F},!}(K)$ on Y(k) is obtained from the trace function of K on X(k) by applying to it the integral transform defined by the trace function of $\mathcal{F}$ on the fibred product $X(k)\times_{S(k)}Y(k)$

In a similar vein, we could define another version $T_{\mathcal{F},*}$ of this integral transform by defining $T_{\mathcal{F},*}(K)$ on Y to be $Rpr_{2*}(pr_1{}^*(K)\otimes\mathcal{F})$. There is a natural "forget supports" map of functors from $T_{\mathcal{F},!}$ to $T_{\mathcal{F},*}$, which in general has no particular reason to have any particular property.

With this in mind, we fix a finite field k of characteristic $p \neq \ell$, an integer $n \geq 1$, the standard affine space $\mathbb{A}^n$ over k, with coordinates $x_1,\ldots, x_n$, and the dual affine space, with coordinates $y_1,\ldots,y_n$. We also fix a nontrivial $\overline{\mathbb{Q}}_\ell$-valued additive character ψ of k. As Hasse [Ha] first pointed out, pushing out the Artin-Schreier covering (the Lang isogeny $z \mapsto z - z^q$, q:=Card(k), for $\mathbb{G}_a$ over k) of $\mathbb{A}^1$ by $\mathbb{A}^1$ over k by the additive character ψ of k gives a lisse rank one ℓ-adic sheaf $\mathcal{L}_\psi$ on $\mathbb{A}^1$ whose trace function is ψ itself on k, and $\psi\circ\mathrm{Trace}_{k'/k}$ on finite extensions k' of k. The pullback by any morphism $f : Z \to \mathbb{A}^1$ of the sheaf $\mathcal{L}_\psi$ is denoted $\mathcal{L}_{\psi(f)}$; in particular we have $\mathcal{L}_{\psi(\Sigma x_i y_i)}$ on $\mathbb{A}^n\times\mathbb{A}^n$, which we take as our kernel for defining the two versions $FT_{\psi,!}$ and $FT_{\psi,*}$ of Fourier Transform. For any k-scheme S, pulling back the entire situation to $(\mathbb{A}^n\times\mathbb{A}^n)_S$ gives us $FT_{\psi,!,S}$ and $FT_{\psi,*,S}$ from $(\mathbb{A}^n)_S$ to the dual $(\mathbb{A}^n)_S$.

Up to this point the discussion has been purely formal, granted the Grothendieck formalism; we have done what we must (and what Deligne did, cf. [De-3]) to define a notion (namely $FT_!$) of FT for complexes on $\mathbb{A}^n$ over a finite field which induces the classical FT on their trace functions and observed in passing that $FT_!$ has a "lower star" analogue FT_* which has no visible trace properties. The "miracle" of FT is the following

Theorem 1.(Verdier) Suppose that S is of finite type over an extension field of k. Then the canonical "forget supports" morphism $\alpha : FT_{\psi,!,S} \to FT_{\psi,*,S}$ is an isomorphism.

Verdier's proof, never published, may be sketched as follows. It is easy to show that $FT_{\psi,!,S}$ is essentially involutive up to a Tate twist and a degree shift; $FT_{\psi,!,S} \circ FT_{\bar{\psi},!,S}[2n](n)$ is the identity. The same holds with ψ replaced by the inverse character $\bar{\psi}$. The dual of $FT_{\bar{\psi},!,S}$ is $FT_{\psi,*,S}$, so by duality $FT_{\psi,*,S}$ satisfies the same involutivity as $FT_{\psi,!,S}$.

Fix a choice of $\sqrt{q}$, q := Card(k), so we can Tate twist by (1/2). Let us denote by A (resp. $\bar{A}$) the functor $FT_{\psi,!,S}[n](n/2)$ (resp. $FT_{\bar{\psi},!,S}[n](n/2)$) by B (resp. $\bar{B}$) the functor $FT_{\psi,*,S}[n](n/2)$ (resp. $FT_{\bar{\psi},*,S}[n](n/2)$). Consider the morphisms of functors

$$A\circ\bar{A}\circ A \xrightarrow{①} B\circ\bar{A}\circ A \xrightarrow{②} B\circ\bar{B}\circ A \xrightarrow{③} B\circ\bar{B}\circ B.$$

If we admit that both of the composites

$$A\circ\bar{A} \to B\circ\bar{A} \to B\circ\bar{B}$$
$$\bar{A}\circ A \to \bar{B}\circ A \to \bar{B}\circ B$$

are the identity endomorphism of the identity functor, then ②∘① is id_A, and ③∘② is id_B. Since ① and ③ are each $\alpha[n](n/2)$, we see that α is invertible with inverse ② $[-n](-n/2)$. QED

This remarkable result, taken together with Deligne's Weil II results ([De-2]), functions as a very effective "black box"(i.e., one needn't know anything other than the statements) in the theory of exponential sums in several variables, but in view of the genesis of FT in that theory, it is not at all surprising that this should be the case. However, we will not speak here of these applications (cf. [Br], [Ka-La], [Ka-3]) at all, but will rather focus on the beautiful structure Laumon has found in the one-dimensional (n = 1) FT over S = k itself, which he calls the "principle of stationary phase", and on two applications he has given of it, neither of which at first sight has anything to do with Fourier Transform of any sort.

The Principle of Stationary Phase

Recall (cf. [Hor], 7.7) that the classical principle of stationary phase in its simplest form applies to integrals of the form $\int \varphi(x)\exp(itf(x))dx$ taken over $\mathbb{R}^n$, where φ is a $\mathcal{C}^\infty$ function of compact support, f is a $\mathcal{C}^\infty$ function, and t is a real parameter. It asserts that if the derivative grad(f) does not vanish at any point in Supp(φ), then the integral, as a function of t, is rapidly decreasing at ∞;this implies that if f has finitely many critical points in Supp(φ),then asymptotically for t tending to ∞ this integral is a finite sum of contributions, one from each critical point of f in Supp(φ).

In the p-adic case as well there is a principle of stationary phase, which is even simpler than in real case in the sense that in sufficiently nice cases it gives **exact** rather than asymptotic formulas. Suppose we are given a smooth $\mathbb{Z}_p$-scheme V of finite type and of relative dimension

n, and a function f on V (i.e., f is a $\mathbb{Z}_p$-morphism from V to the affine line $\mathbb{A}^1$). Let us suppose that f is a "Morse function" in the following sense: the subscheme D of V defined by the vanishing of grad(f) is finite etale over $\mathbb{Z}_p$ (e.g., V the variety $x_1x_2.....x_{n+1} = 1$ in $\mathbb{A}^{n+1}$, f the function $x_1 + x_2 +... ...+ x_{n+1}$, and $n+1$ prime to p). Consider the integral

$$\int \exp(2\pi i t f(x))dx$$

over $V(\mathbb{Z}_p)$, for t in $\mathbb{Q}_p$ tending to ∞ p-adically. [The function φ of the real case is here taken as the characteristic function of $V(\mathbb{Z}_p)$, and the role of $\mathbb{R}^n$ is played by $V(\mathbb{Q}_p)$.] By this integral we mean the following: if $\mathrm{ord}_p(t) = -m < 0$, then $\int$ is the sum $(1/p)^{nm}\sum \exp(2\pi i t f(x))$, the sum over x in $V(\mathbb{Z}/p^m\mathbb{Z})$, where the complex-valued function $z \mapsto \exp(2\pi i z)$ on $\mathbb{Q}_p$ is defined by viewing $\mathbb{Q}_p/\mathbb{Z}_p$ as lying in $\mathbb{Q}/\mathbb{Z}$ as its p-primary torsion subgroup. One sees easily that if $D(\mathbb{Z}_p)$ is empty, then for $m \geq 2$, the $\int$ **vanishes**, and that in general for $m \geq 2$ the $\int$ is a sum of extremely simple local terms, one for each of the finitely many points x_{crit} of $D(\mathbb{Z}_p)$. For even m the local term is the value at x_{crit} of $(1/\sqrt{p})^{nm}\exp(2\pi i t f(x))$, whereas in the case of odd $m \geq 3$, the local term is this value multiplied by the normalized multidimensional Gauss sum (a fourth root of unity!)

$$(1/\sqrt{p})^{n}\sum \exp(2\pi i p^{m-1} t H(z)), \text{ sum over } z \text{ in } (\mathbb{Z}/p\mathbb{Z})^n$$

for the quadratic function $H(z)$ given by the Hessian mod p of f at the critical point x_{crit}). Unfortunately(?), the situation for $m = 1$ is not so simple as this.

How are we to interpret the principle of stationary phase for the one-variable ℓ-adic FT? The integral $\int \varphi(x)\exp(itf(x))dx$ involves a $\varphi(x)$ of compact support, but this is present only as a convergence factor. Omitting it, the finite field analogue of the integral is the sum $\Sigma_x \psi(tf(x))$, over x in the finite field k. This sum we can rewrite as the sum $\Sigma_x\, \psi(tx)(\#\{y \text{ in } k \text{ with } f(y) = x\})$, and we recognize this as $\Sigma_x\, \psi(tx)\mathrm{Trace}_{k,f_!\overline{\mathbb{Q}}_\ell}(x)$, the FT of the trace function of the direct image sheaf $\mathcal{F} := f_!\overline{\mathbb{Q}}_\ell$, or what is the same,.we recognize this sum as the trace at time t of the complex $\mathrm{FT}(\mathcal{F})$. For a "reasonable" polynomial f (e.g., one whose degree is prime to the characteristic of k, or more generally one which as a map of $\mathbb{A}^1$ to itself is generically etale), the **critical values of f** are precisely the **points of $\mathbb{A}^1$ where $\mathcal{F}$ is not lisse.**

In order to to pursue this chain of thought, it will be convenient to recall the concrete "galois" description of an ℓ-adic sheaf on a curve.

Let k be a perfect field, X a proper smooth geometrically connected curve over k, K the function field k(X) of X, K^{sep} a separable closure of K, Gal := Gal(K^{sep}/K). For each closed point x of X, viewed as a discrete valuation of K/k, we fix a place $\tilde{x}$ of K^{sep} lying over it, and we denote by $I(x) \subset D(x) \subset$ Gal the inertia and decomposition subgroups of Gal attached to the choice $\tilde{x}$. In terms of this data, a constructible ℓ-adic sheaf $\mathcal{G}$ on a nonvoid open set U in X is a continuous ℓ-adic representation $\mathcal{G}_{\bar{\eta}}$ of Gal which is unramified almost everywhere (I(x) acts trivially for all but finitely many closed points x), together with the giving for each x in U of a continuous representation $\mathcal{G}_x$ of D(x)/I(x) and of a D(x)-equivariant ("specialization") map $sp_x : \mathcal{G}_x \to \mathcal{G}_{\bar{\eta}}$ which is an isomorphism for almost all x. One says that $\mathcal{G}$ is lisse at x in U if sp_x is an isomorphism. If the map sp_x is injective for all x in U, one says that $\mathcal{G}$ has no punctual sections. If U is open then $H_c^0(U\otimes\bar{k}, \mathcal{G})$ vanishes if and only if $\mathcal{G}$ has no punctual sections; indeed as Gal($\bar{k}$/k)-module

$$H_c^0(U\otimes\bar{k}, \mathcal{G}) = \bigoplus_{x \text{ in } U} \operatorname{Ind}_{k(x)}^{k} (\operatorname{Ker}(sp_x : \mathcal{G}_x \to \mathcal{G}_{\bar{\eta}})),$$

where $\operatorname{Ind}_{k(x)}^{k}$ is induction from $D(x)/I(x) \approx \operatorname{Gal}(\bar{k}/k(x))$ to Gal($\bar{k}$/k). If the map sp_x defines an isomorphism of $\mathcal{G}_x$ with the inertial invariants $(\mathcal{G}_{\bar{\eta}})^{I(x)}$ for all x in U, one says that $\mathcal{G}$ is extended by direct image from its open set of lisseness. The difference in dimensions dim $\mathcal{G}_{\bar{\eta}}$ - dim $\mathcal{G}_x$ is called $\operatorname{drop}_x(\mathcal{G})$.

For purposes of analogy with the $\mathcal{C}^\infty$ case, it may be useful to think of "$\mathcal{G}_{\bar{\eta}}$ as D(x)-representation" as being its "asymptotic expansion in a punctured disc around x, modulo functions which are both $\mathcal{C}^\infty$ in the entire disc and which vanish at x to all orders" and of the extra data $sp_x : \mathcal{G}_x \to \mathcal{G}_{\bar{\eta}}$ as being an attempted approximation of $\mathcal{G}$ in an entire (undeleted) neighborhood near x by a constant.

Given $\mathcal{G}$ on U and a closed point x of X , we denote "$\mathcal{G}_{\bar{\eta}}$ as D(x)-representation" by $\mathcal{G}(x)$. It is important to notice that $\mathcal{G}(x)$ is meaningful for **every** closed point in X, while $\mathcal{G}_x$ is meaningful only for points x in U. To avoid confusion with Tate twist, we will denote the latter in boldface : e.g., $\mathcal{G}(\mathbf{-1})$.

For later use, we recall the notion of the "slopes" or "breaks" of a finite-dimensional ℓ-adic representation M of an inertia group I(x) (cf. [Ka-2] and [Se-2]). Recall that I := I(x) carries a decreasing "upper numbering" filtration by closed normal subgroups $I^{(r)}$, r in $\mathbb{R}_{\geq 0}$, with $I^{(0)}$ = I itself, and the wild inertia subgroup P is the closure of

$\cup_{r>0}\ I^{(r)}$. One knows that M has a canonical "break decomposition" M = $\oplus_{r\geq 0} M\langle r\rangle$, such that $M\langle 0\rangle = M^P$ and such that for r>0, $(M\langle r\rangle)^{I^{(r)}} = 0$ while for any s > r, $I^{(s)}$ acts trivially on $M\langle r\rangle$. It is useful to think of breaks as analogous to orders of exponential growth. We say that r is a break of M if $M\langle r\rangle$ is nonzero, and that its multiplicity is $\dim M\langle r\rangle$. According to the Hasse-Arf theorem, for any r with $M\langle r\rangle \neq 0$, the product $r\times\dim M\langle r\rangle$ lies in $\mathbb{Z}$. The sum $\Sigma_{r\geq 0}\ r\times\dim M\langle r\rangle$ is the Swan conductor of M as I(x)-representation, denoted Sw(M) or $Sw_x(M)$. For $\mathcal{F}$ a lisse sheaf on a smooth connected curve U over an algebraically closed field of characteristic $\neq \ell$, with complete nonsingular model X, Grothendieck's Euler-Poincaré formula (cf. [Ra]) is

$\chi_c(U, \mathcal{F}) := \Sigma_i(-1)^i\dim H_c{}^i(U, \mathcal{F}) = \operatorname{rank}(\mathcal{F})\chi(U) - \Sigma_{x \text{ in } X-U} Sw_x(\mathcal{F})$,

where $\chi(U)$ is the topological Euler characteristic $2 - 2g - \#(X - U)$.

With these preliminaries out of the way, we can now return to explaining what the "principle of stationary phase" is to mean for the ℓ-adic Fourier Transform.

If we take $FT(\mathcal{F})(\infty)$ as the "asymptotic development at ∞" of $FT(\mathcal{F})$, then the principle of stationary phase should be the statement that the $D(\infty)$-representation $FT(\mathcal{F})(\infty)$ is the direct sum of terms, one for each point of nonlisseness of $\mathcal{F}$ on $\mathbb{A}^1$ and (possibly) one for the point ∞, with the understanding that the "local term" attached to such a point x is to depend functorially on the data

$$\begin{cases} sp_x : \mathcal{F}_{\bar{x}} \to \mathcal{F}(x), \text{ if } x \text{ in } \mathbb{A}^1 \\ \mathcal{F}(\infty) \text{ alone, if } x = \infty. \end{cases}$$

For this to make sense, we need to know that $FT_\psi[1]$ (the [1] just means to shift all degrees by 1; $\mathcal{H}^{i-1}(K[1]) = \mathcal{H}^i(K)$) carries sheaves to sheaves, or nearly does so. For any sheaf $\mathcal{F}$ on $\mathbb{A}^1$, we define its "naive Fourier Transform" sheaf $NFT_\psi(\mathcal{F})$ by

$$NFT_\psi(\mathcal{F}) := \mathcal{H}^1(FT_\psi(\mathcal{F})) := R^1pr_{2!}(pr_1{}^*\mathcal{F}\otimes\mathcal{L}_{\psi(xy)}).$$

Theorem 2. Suppose that $\mathcal{F}$ is an ℓ-adic sheaf on $\mathbb{A}^1$ over k such that $\mathcal{F}$ has no punctual sections. Then

(0) The sheaves $\mathcal{H}^iFT_\psi(\mathcal{F})$ vanish for $i \neq 1,2$ and $\mathcal{H}^2$ is punctual.

(1) $FT_\psi(\mathcal{F})[1]$ is a single sheaf (necessarily $NFT_\psi(\mathcal{F})$) if and only if

$$H_c{}^2(\mathbb{A}^1\otimes\bar{k}, \mathcal{F}\otimes\mathcal{L}_{\psi(ax)}) = 0 \text{ for all a in } \bar{k}.$$

(2) The sheaf $NFT_\psi(\mathcal{F})$ has no punctual sections.

(3) There exists a largest nonempty open set U of $\mathbb{A}^1$ on which $FT_\psi(\mathcal{F})$ is lisse in the sense that on U, $FT_\psi(\mathcal{F})[1]$ coincides with $NFT_\psi(\mathcal{F})$ **and**

$\mathrm{NFT}_\psi(\mathcal{F})$ *is lisse on* U. *A point* t *in* $\bar{k}$ *lies in* U *if and only if* $\mathcal{F}\otimes\mathcal{L}_{\psi(tx)}$ *as* $I(\infty)$-*representation has all its breaks* ≥ 1, *or equivalently if and only if the integer-valued function on the dual* $\mathbb{A}^1$, $y \mapsto \mathrm{Sw}_\infty(\mathcal{F}\otimes\mathcal{L}_{\psi(yx)})$,

attains its maximum at t. *The rank of* $\mathrm{NFT}(\mathcal{F})$ *on* U *is*

$$\sum_{\text{breaks } \lambda > 1 \text{ of } \mathcal{F}(\infty), \text{ with mult.}} (\lambda - 1) \quad + \quad \sum_{\text{closed points } x \text{ in } \mathbb{A}^1} \deg(x)\{\mathrm{Swan}_x(\mathcal{F}) + \mathrm{drop}_x(\mathcal{F})\}.$$

(4) If $\mathcal{F}$ *is the extension by direct image from a nonvoid open set* U *of a lisse sheaf which is geometrically irreducible (i.e.,irreducible as a representation of* $\pi_1(U\otimes\bar{k}, \xi)$) *and not geometrically isomorphic to* $\mathcal{L}_{\psi(tx)}$ *for any* t *in* $\bar{k}$, *then* $\mathrm{FT}_\psi(\mathcal{F})[1]$ *is the single sheaf* $\mathrm{NFT}_\psi(\mathcal{F})$, *and* $\mathrm{NFT}_\psi(\mathcal{F})$ *satisfies the same conditions as* $\mathcal{F}$.

Proof. (0) and (1) are straightforward consequences of the fact that $\mathcal{L}_{\psi(ax)}$ is lisse on $\mathbb{A}^1$ with $\mathrm{Sw}_\infty(\mathcal{L}_{\psi(ax)}) = 1$ for $a\neq 0$. The vanishing of $\mathcal{H}^0\mathrm{FT}\mathcal{F}$ is equivalent to $\mathcal{F}$'s having no punctual sections. Part (2) now follows from the fact that, by Fourier inversion, $\mathrm{FT}_{\bar{\psi}}(\mathrm{FT}_\psi(\mathcal{F}))$ is a single sheaf placed in degree 2, and the $H_c{}^0$ interpretation of punctual sections. Part (3) follows from (2) and the Euler-Poincaré formula. Part (4), due originally to Brylinski [Br], is another manifestation of Fourier inversion.QED

Example In (1) above, if $\mathcal{F}(\infty)$ has all breaks < 1 (e.g., if $\mathcal{F}(\infty)$ is tame)), then $U = \mathbb{G}_m$. We will see below how $\mathrm{FT}_\psi(\mathcal{F})$ looks at 0 a bit further on.

Remark A complex K on $\mathbb{A}^1$ is called **m-perverse** if $\mathcal{H}^i = 0$ for $i \neq m, m-1$, $\mathcal{H}^m$ is punctual and $\mathcal{H}^{m-1}$ has no punctual sections. A restatement of parts (0) and (2) above is to say that FT[1] preserves m-perversity. This is the point of view taken by Laumon.

For $\mathcal{F}$ a sheaf with no punctual sections, we will often write $\mathrm{FT}_\psi(\mathcal{F})(\infty)$ for the $D(\infty)$-representation $\mathrm{NFT}_\psi(\mathcal{F})(\infty)$, i.e., for what is strictly speaking $(\mathrm{FT}_\psi(\mathcal{F})[1])(\infty)$. (This shift of [1] causes no end of bookkeeping trouble; Laumon's solution is to **define** his FT to be what we call here FT[1].)

We are at last able to state the principle of stationary phase for the one-variable ℓ-adic Fourier Transform over a perfect field k:

Theorem of ℓ-adic Stationary Phase 3. (Laumon) For each closed point t in $\mathbb{A}^1 \cup \infty$, there is an exact functor

$$FT_\psi loc(t,\infty): (\ell\text{-adic } D(t)\text{-rep's}) \to (\ell\text{-adic } D(\infty)\text{-rep's})$$

such that if $\mathcal{F}$ is an ℓ-adic sheaf on $\mathbb{A}^1$ which is the extension by zero of a lisse sheaf on a nonvoid open set $\mathbb{A}^1$ - S, there is a canonical direct sum decomposition of $NFT_\psi(\mathcal{F})(\infty)$ as $D(\infty)$-representation

$$NFT_\psi(\mathcal{F})(\infty) = \bigoplus_{s \text{ in } S\cup\infty} FT_\psi loc(s,\infty)(\mathcal{F}(s)).$$

Local Consequences : Detailed Study of Local Fourier Transform

Once we know the **existence** of such a decomposition into local terms, we can deduce properties of the individual local terms $FT_\psi loc(t,\infty)$ by the ℓ-adic analogue of a "partition of unity" argument, feeding in global $\mathcal{F}$'s which have given local behavior. This sort of "global-to-local" argument, which has a long and honorable history in number theory, is considerably simplified in our case by a systematic use of the "canonical extension":

Lemma 4.(cf. [Ka-1]) Given an ℓ-adic representation M of D(0), there exists a lisse $\mathcal{F}$ on $\mathbb{G}_m$ together with an isomorphism $\mathcal{F}(0) \approx M$ of D(0)-representations, such that $\mathcal{F}(\infty)$ is **tame**. Moreover, I(0) and the geometric monodromy group $\pi_1(\mathbb{G}_m \otimes \bar{k}, \bar{\eta})$ have the same image in $\mathrm{Aut}(\mathcal{F}_{\bar{\eta}})$.

An $\mathcal{F}$ as above is called a canonical extension. By feeding in (the extensions by zero to $\mathbb{A}^1$ of) canonical $\mathcal{F}$'s and their additive translates (to move the "bad" singularity from 0 to an arbitrary rational point s) one gets a complete analysis of the local terms, modulo understanding $FT_\psi loc(\infty,\infty)(M)$ for M a **tame** $D(\infty)$-representation. Fortunately, this is no obstruction:

Lemma 5. For M a tame $D(\infty)$-representation, $FT_\psi loc(\infty,\infty)(M) = 0$, and for N a tame (resp. I(0)-unipotent) D(0)-representation, $FTloc(0,\infty)(N)$ is a tame (resp. I(0)-unipotent) $D(\infty)$-representation of the same rank.

Proof. The question is geometric, so we may assume k algebraically closed. By devissage on M, we reduce to the case where M is $\mathcal{F}(\infty)$, for $\mathcal{F}$ either $\bar{\mathbb{Q}}_\ell$ (lisse on $\mathbb{A}^1$), or a Kummer sheaf $\mathcal{L}_{\chi(x)}$ (lisse on $\mathbb{G}_m$ extended by zero) with χ a nontrivial character of $\pi_1(\mathbb{G}_m)^{\text{tame}}$. Since $FT(\bar{\mathbb{Q}}_\ell)$ is the delta function supported at the origin, while $\bar{\mathbb{Q}}_\ell$ is lisse on $\mathbb{A}^1$, we have

$$FT_\psi loc(\infty,\infty)(\bar{\mathbb{Q}}_\ell) = FT(\bar{\mathbb{Q}}_\ell)(\infty) = 0,$$

and hence $FT_\psi loc(\infty,\infty)(M) = 0$ for any unramified M.

For $\mathcal{L}_{\chi(x)}$, we have $FT(\mathcal{L}_{\chi(x)})[1] = \mathcal{L}_{\overline{\chi}(y)}$ for $\overline{\chi}$ the inverse character, whence

$$(*) \qquad FT_\psi loc(0,\infty)(\mathcal{L}_{\chi(x)}) \oplus FT_\psi loc(\infty,\infty)(\mathcal{L}_{\chi(x)}))$$
$$= FT(\mathcal{L}_{\chi(x)})(\infty) = \mathcal{L}_{\overline{\chi}(y)} \text{ has rank one.}$$

Therefore **either** $FT_\psi loc(\infty,\infty)(\mathcal{L}_{\chi(x)})$ **or** $FT_\psi loc(0,\infty)(\mathcal{L}_{\chi(x)})$ vanishes, and the other is of rank one. A similar consideration of $\mathcal{L}_{\overline{\chi}(x-1)}$, whose FT[1] is $\mathcal{L}_{\psi(y)} \otimes \mathcal{L}_{\chi(y)}$, shows that **either** $FT_\psi loc(\infty,\infty)(\mathcal{L}_{\overline{\chi}(x-1)})$ **or** $FT_\psi loc(1,\infty)(\mathcal{L}_{\overline{\chi}(x-1)})$ vanishes, and the other is of rank one. Now consider the sheaf $\mathcal{F} := \mathcal{L}_{\chi(x/x-1)}$ on $\mathbb{G}_m - \{1\}$, extended by zero.. It is **lisse** at ∞, I(0)-isomorphic to $\mathcal{L}_{\chi(x)}$ at 0, I(1)-isomorphic to $\mathcal{L}_{\overline{\chi}(x-1)}$ at 1, and its FT[1] is lisse of rank two on $\mathbb{G}_m$. Because $\mathcal{F}$ is lisse at ∞, the stationary phase decomposition of $FT(\mathcal{F})(\infty)$ has no (∞,∞) contribution, whence

$$FT_\psi(\mathcal{F})(\infty) = FT_\psi loc(0,\infty)(\mathcal{L}_{\chi(x)}) \oplus FT_\psi loc(1,\infty)(\mathcal{L}_{\overline{\chi}(x-1)}).$$

Counting ranks, we see that both $FT_\psi loc(0,\infty)(\mathcal{L}_{\chi(x)})$ and $FT_\psi loc(1,\infty)(\mathcal{L}_{\overline{\chi}(x-1)})$ must be of rank one, whence the desired vanishing of $FT_\psi loc(\infty,\infty)(\mathcal{L}_{\chi(x)})$.

We now see from $(*)$ that for χ nontrivial

$$FT_\psi loc(0,\infty)(\mathcal{L}_{\chi(x)}) = FT(\mathcal{L}_{\chi(x)})(\infty) = \mathcal{L}_{\overline{\chi}(y)},$$

(a direct calculation shows that this is also true if χ is trivial) and this proves the second assertion. QED

Proposition 6.

(1) For any D(0)-representation M,

$$\dim FT_\psi loc(0,\infty)(M) = Sw_0(M) + \dim(M).$$

(2) $FT_\psi loc(0,\infty)(M)$ as $I(\infty)$-representation has all breaks < 1.

(3) If $\mathcal{G}$ is a sheaf on $\mathbb{A}^1$ with no punctual sections, then $FT_\psi(\mathcal{G})$ is lisse on $\mathbb{G}_m$ if and only if $\mathcal{G}$ has no ∞-break 1.

Proof. (1)Let $\mathcal{F}$ be the canonical extension of M, extended by zero to $\mathbb{A}^1$. Then its $FTloc(\infty,\infty)$ vanishes, and so stationary phase gives

$$FT(\mathcal{F})(\infty) \approx FT_\psi loc(0,\infty)(M),$$

and by Thm.2,(3), $FT(\mathcal{F})$ is lisse on $\mathbb{G}_m$ of rank $Sw_0(M) + \dim(M)$.

(2) We already know this in case M is tame, so it suffices to treat the case when M has all breaks >0. For such an M, $FT(\mathcal{F})$ is the single sheaf $\mathcal{G} := NFT_\psi(\mathcal{F})$, which is lisse on $\mathbb{G}_m$ and has $drop_0(\mathcal{G}) = \dim(M)$. By Fourier inversion, $NFT_{\overline{\psi}}(\mathcal{G})$ is (a Tate twist of) $\mathcal{F}$. Using Thm.2 (3) to compute the generic rank of $NFT_{\overline{\psi}}(\mathcal{G})$, which we know to be $\dim(M)$, we see that $\mathcal{G}$ has all its ∞-breaks ≤ 1. Once we know this, the fact that

$NFT_{\bar\psi}(\mathcal{G})$ is lisse on $\mathbb{G}_m$ reduces us to (3). For such $\mathcal{G}$, $\mathcal{G}\otimes\mathcal{L}_{\psi(ax)}$ has all ∞-breaks ≥ 1 for all $a\neq 0$. Therefore if we denote by N the part $\mathcal{G}(\infty)\langle 1\rangle$ of break 1, we obtain a $D(\infty)$-representation with the property that $N\otimes\mathcal{L}_{\psi(ax)}$ has all ∞-breaks $=1$ for **all** a. Now reverse the roles of 0 and ∞ in the canonical extension to produce a "canonical extension of N", i.e., a lisse sheaf $\mathcal{K}$ on $\mathbb{G}_m$, extended by zero, which is tame at zero and whose $D(\infty)$-representation is N. Then $FT(\mathcal{K})$ is lisse on $\mathbb{A}^1$ of rank dim(N), and playing with Fourier inversion and the formula for generic rank shows that $N=0$ (cf. [Ka-2], 8.5.7). QED

Corollary 7. <u>For any</u> $D(\infty)$-<u>representation</u> N, $FTloc(\infty,\infty)(N) = 0$ <u>if and only if</u> N <u>has all its breaks</u> ≤ 1.

Proof. As above, let $\mathcal{K}$ be a canonical extension of N. Let λ_i be the dimN breaks of N, counting multiplicity. By stationary phase, $FTloc(\infty,\infty)(N)=0$ if and only if the generic rank of $FT(\mathcal{K})$, namely $\Sigma_i \max(\lambda_i, 1)$ is equal to $\dim FTloc(0,\infty)(\mathcal{K}(0))$, namely dimN. QED

Theorem 8. <u>If</u> N <u>is a nonzero</u> $D(\infty)$-<u>representation with all breaks</u> > 1, <u>then</u> $FT_\psi loc(\infty,\infty)(N)$ <u>is nonzero, it has all breaks</u> >1, <u>and</u>

$$FT_{\bar\psi} loc(\infty,\infty)\circ FT_\psi loc(\infty,\infty)(N) \approx N(-\mathbf{1}).$$

<u>The functor</u> $FT_\psi loc(\infty,\infty)$ <u>is an autoequivalence of the category of</u> $D(\infty)$-<u>representations with all breaks</u> >1; in <u>particular it carries irreducibles to irreducibles</u>.

Proof. For N with all ∞-breaks > 1, and $\mathcal{K}$ its canonical extension as above, $FT_\psi(\mathcal{K})$ is lisse on $\mathbb{A}^1$. Applying stationary phase to its $FT_{\bar\psi}$, we find

$$N(-\mathbf{1}) \approx FT_{\bar\psi} loc(\infty,\infty)(FT_\psi(\mathcal{K})(\infty)),$$

and applying stationary phase to the inner term gives

$$\begin{aligned} FT_\psi(\mathcal{K})(\infty) &\approx FT_\psi loc(0,\infty)(\mathcal{K}(0)) \oplus FT_\psi loc(\infty,\infty)(\mathcal{K}(\infty)) \\ &= \quad (\text{tame}) \quad \oplus FT_\psi loc(\infty,\infty)(N), \end{aligned}$$

whence the asserted involutivity, since $FT_{\bar\psi} loc(\infty,\infty)$ kills (tame).

The rest is formal; let $FT_\psi loc(\infty,\infty)(N) = A_{\leq 1} \oplus A_{>1}$ be its decomposition into (breaks ≤ 1) $\oplus$ (breaks >1). In view of Cor.7, the involutivity gives, up to a Tate twist, $N \approx FT_{\bar\psi} loc(\infty,\infty)(A_{>1})$. Applying $FT_\psi loc(\infty,\infty)$ to this gives, up to a Tate twist, $FT_\psi loc(\infty,\infty)(N) \approx A_{>1}$ has all breaks > 1, as asserted. QED

Remark 9 Since irreducibles have a single break (with some multiplicity), one can calculate the effect of $FT_\psi loc(\infty,\infty)$ on breaks just from its known effect on Swans and ranks. The conclusion is that if N has break $(a+b)/a$ with multiplicity a, then $FT_\psi loc(\infty,\infty)(N)$ has break $(a+b)/b$ with multiplicity b.

In order to understand the situation at 0, we also need the following **Theorem** 10. (Laumon) <u>There is an exact functor</u>

$$FT_\psi loc(\infty,0) : (\ell\text{-adic } D(\infty)\text{-rep's}) \quad\rightarrow\quad (\ell\text{-adic } D(0)\text{-rep's})$$

<u>such that if</u> $\mathcal{F}$ <u>is an</u> ℓ-<u>adic sheaf on</u> $\mathbb{A}^1$ <u>with no punctual sections, there is a four term exact sequence of</u> $D(0)$-<u>representations</u>

$$0\rightarrow H_c^1(\mathbb{A}^1\otimes\bar{k},\ \mathcal{F})\rightarrow NFT(\mathcal{F})(0)\rightarrow FT_\psi loc(\infty,0)(\mathcal{F}(\infty))\rightarrow H_c^2(\mathbb{A}^1\otimes\bar{k},\ \mathcal{F})\rightarrow 0.$$

As above, once we admit the **existence** of such a functor, we can use the canonical extension to unveil its properties. For example:
Lemma 11. <u>For</u> $\mathcal{F}(\infty)$ <u>unramified, we have</u>

$$FT_\psi loc(\infty,0)(\mathcal{F}(\infty)) \approx \mathcal{F}(\infty)(-\mathbf{1}).$$

Proof If $\mathcal{F}(\infty)$ is unramified, there exists a geometrically constant $\mathcal{G}$ on $\mathbb{A}^1$ with $\mathcal{G}(\infty) \approx \mathcal{F}(\infty)$. The NFT of such a $\mathcal{G}$ vanishes, and $H_c^2(\mathbb{A}^1\otimes\bar{k},\ \mathcal{G}) \approx H_c^2(\mathbb{A}^1\otimes\bar{k},\ \bar{\mathbb{Q}}_\ell)\otimes\mathcal{G}(\infty) \approx \mathcal{G}(\infty)(-\mathbf{1}) \approx \mathcal{F}(\infty)(-\mathbf{1})$.QED

Proposition 12. (1) <u>If</u> N <u>is a</u> $D(\infty)$-<u>representation with all breaks</u> ≥ 1, <u>then</u> $FT_\psi loc(\infty,0)(N) = 0$. (2) <u>If</u> N <u>is a tame</u> (<u>resp.</u> $I(\infty)$-<u>unipotent</u>) $D(\infty)$-<u>representation, then</u> $FT_\psi loc(\infty,0)(N)$ <u>is a tame</u> (<u>resp</u>. $I(0)$-<u>unipotent</u>) $D(0)$-<u>representation of the same rank</u>.

Proof. (1)Let $\mathcal{G}$ be the canonical extension of N, extended by zero to $\mathbb{A}^1$. Then $FT_\psi(\mathcal{G})$ is, near zero, a single lisse sheaf $NFT_\psi(\mathcal{F})$ in degree one, so the result is immediate from the four term exact sequence of Thm.10. (2). One reduces to the case $k=\bar{k}$, $N=\mathcal{L}_\chi(\infty)$, in which case the same exact sequence shows that then $FT_\psi loc(\infty,0)(N) = \mathcal{L}_{\bar{\chi}}(0)$. QED

Theorem 13. <u>Let</u> M <u>be a</u> $D(0)$-<u>representation with</u> $M^{I(0)} = 0$, <u>and</u> N <u>a</u> $D(\infty)$-<u>representation with all breaks < 1 and</u> $N^{I(\infty)} = 0$. <u>Then we have</u>

$$M(-\mathbf{1}) \approx FT_{\bar{\psi}} loc(\infty,0)\circ FT_\psi loc(0,\infty)(M),$$

$$N(-\mathbf{1}) \approx FT_\psi loc(0,\infty)\circ FT_{\bar{\psi}} loc(\infty,0)(N).$$

<u>The functor</u> $FT_\psi loc(0,\infty)$ <u>is an equivalences of categories from</u> ($D(0)$-rep's M with $M^{I(0)} = 0$) <u>to</u> ($D(\infty)$-rep's N with all breaks < 1 and $N^{I(\infty)} = 0$), <u>with quasi-inverse</u> $FT_{\bar{\psi}} loc(\infty,0)(\mathbf{1})$. <u>It sends irreducibles to irreducibles, and</u> (M's of break a/b with multiplicity b) <u>to</u> (N's of break a/(a+b) with multiplicity a+b).

Proof. Let us begin with an M as above, and denote by $\mathcal{F}$ its canonical extension to $\mathbb{G}_m$, extended by zero to $\mathbb{A}^1$. Then $FT_\psi\mathcal{F}[1]$ is a single sheaf $\mathcal{H}$, and as $\mathcal{F}(\infty)$ is tame, stationary phase gives $\mathcal{H}(\infty) \approx FT_\psi loc(0,\infty)(M)$. Applying Thm. 10's exact sequence to $\mathcal{H}$ now gives the first isomorphism.

This isomorphism shows in turn that $FT_\psi loc(0,\infty)(M)$ has no nonzero $I(\infty)$-invariants, thanks to Prop.12 (2). Now start with an N at ∞ as above, and denote by $\mathcal{G}$ its canonical extension to $\mathbb{G}_m$, extended by zero. Again $FT_{\bar\psi}(\mathcal{G})[1]$ is a single sheaf $\mathcal{H}$, lisse on $\mathbb{G}_m$ because N has all its breaks < 1, but $\mathcal{H}_0$ is possibly nonzero. But we have a short exact sequence on $\mathbb{A}^1$

$$0 \to j_!j^*\mathcal{H} \to \mathcal{H} \to \{\mathcal{H}_0\}_{\text{conc. at } 0} \to 0, \qquad j : \mathbb{G}_m \to \mathbb{A}^1 \text{ the inclusion.}$$

Taking its FT gives a short exact sequence

$$0 \to (\text{the constant sheaf } \mathcal{H}_0) \to FT_\psi(j_!j^*\mathcal{H})[1] \to \mathcal{G}(-\mathbf{1}) \to 0.$$

Now stationary phase gives $\mathcal{H}(\infty) \approx FT_{\bar\psi} loc(0,\infty)(N) \oplus FT_{\bar\psi} loc(\infty,\infty)(N)$, but the second term vanishes because N has all breaks < 1 (Cor.7), and hence (Prop.6) $\mathcal{H}(\infty)$ has all breaks < 1. This in turn implies that $FT_\psi(j_!j^*\mathcal{H})[1](\infty) \approx FT_\psi loc(0,\infty)(\mathcal{H}(0))$; the above exact sequence at ∞ gives

$$0 \to (\text{the trivial } D(\infty)\text{-rep. } \mathcal{H}_0) \to FT_\psi loc(0,\infty)(\mathcal{H}(0)) \to \mathcal{G}(\infty)(-\mathbf{1}) \to 0.$$

Independently of all this, Thm.10 applied to $\mathcal{G}$ gives

$$0 \to \mathcal{H}_0 \to \mathcal{H}(0) \to FT_{\bar\psi} loc(\infty,0)(N) \to 0,$$

which in turn yields under $FT_\psi loc(0,\infty)$ an exact sequence

$$0 \to (\text{the trivial } D(\infty)\text{-rep. } \mathcal{H}_0) \to FT_\psi loc(0,\infty)(\mathcal{H}(0)) \to$$
$$\to FT_\psi loc(0,\infty) \circ FT_{\bar\psi} loc(\infty,0)(N) \to 0,$$

which gives the second isomorphism upon comparison with the earlier resolution of $\mathcal{G}(\infty)(-\mathbf{1})$. By Lemma 5, we now see that $FT_{\bar\psi} loc(\infty,0)(N)$ has no nonzero $I(0)$-invariants. The slope formulas follow formally (cf. Rmk. 9). QED

In some ways, the two results Thm.8 and Thm.13 are the most provocative and least understood part of all that has come out of Laumon's stationary phase insight so far, since they furnish remarkable and unexpected transformations of the spaces of ℓ-adic representations of equal-characteristic decomposition groups. It would be of great interest to "really" understand what is going on here to the point of being able to do something similar in the case of mixed characterisic.

Consequences for Determinants

Let us now explain a rather special-sounding consequence of the above theory, which will turn out to have important consequences.

Suppose that $\mathcal{F}$ is lisse on $\mathbb{G}_m$ over a finite field k, extended by zero to $\mathbb{A}^1$, and that $\mathcal{F}(\infty)$ is unramified. Then on $\mathbb{G}_m$, $FT(\mathcal{F})[1]$ is a lisse sheaf $NFT(\mathcal{F})$; its $D(\infty)$ representation is a **single** local contribution

$FT_\psi loc(0,\infty)(\mathcal{F}(0))$ with all ∞-breaks < 1, and its $D(0)$-representation is virtually equal to

$$H_c^1(\mathbb{G}_m \otimes \bar{k}, \mathcal{F}) - H_c^2(\mathbb{G}_m \otimes \bar{k}, \mathcal{F}) + \mathcal{F}(\infty)(-\mathbf{1}).$$

Therefore $\det NFT(\mathcal{F})$ is lisse on $\mathbb{G}_m$; it is unramified at the origin (because $NFT(\mathcal{F})$ is itself unipotent at the origin, being an extension of two trivial $I(0)$-representations) and it is tamely ramified at ∞ (by Hasse-Arf, since all ∞-breaks of $NFT(\mathcal{F})$ are < 1). Because $\mathbb{A}^1 \otimes \bar{k}$ is tamely simply connected, $\det NFT(\mathcal{F})$ is **unramified** at ∞, so its extension by direct image to $\mathbb{P}^1$, say $\mathcal{D}$, is lisse on $\mathbb{P}^1$. Because $\mathbb{P}^1 \otimes \bar{k}$ is simply connected, $\mathcal{D}$ is **geometrically constant** of rank one, necessarily of the form $\alpha^{\deg}$ for some ℓ-adic unit α. This means that for any closed point x in $\mathbb{P}^1$, and any element F_x in $D(x)$ lifting Frobenius, F_x acts on $\mathcal{D}(x)$ as $\alpha^{\deg(x)}$. Comparing ∞ and zero, we get two evaluations of α, so a strange-looking equality

$$(14) \qquad \det(F_\infty \mid FT_\psi loc(0,\infty)(\mathcal{F}(0))) = q^{\mathrm{rank}(\mathcal{F})} \det(F_\infty | \mathcal{F}(\infty)) \times$$
$$\times [\det(F_k \mid H_c^1(\mathbb{G}_m \otimes \bar{k}, \mathcal{F}))/\det(F_k \mid H_c^2(\mathbb{G}_m \otimes \bar{k}, \mathcal{F}))]$$

giving $\det(Frob \mid H_c^*(\mathcal{F}))$ in terms of $\mathcal{F}(0)$ and $\mathcal{F}(\infty)$.

Slightly more generally, suppose that $\mathcal{F}$ is lisse on $\mathbb{G}_m$, extended by zero to $\mathbb{A}^1$, and that $\mathcal{F}(\infty)$ has all breaks < 1 (e.g.,is tame). Then on $\mathbb{G}_m$, $FT(\mathcal{F})[1]$ is a lisse sheaf $NFT(\mathcal{F})$; its $D(\infty)$ representation is a **single** local contribution $FT_\psi loc(0,\infty)(\mathcal{F}(0))$ all of whose ∞-breaks are < 1, and whose $D(0)$-representation is virtually equal to

$$H_c^1(\mathbb{G}_m \otimes \bar{k}, \mathcal{F}) - H_c^2(\mathbb{G}_m \otimes \bar{k}, \mathcal{F}) + FT_\psi loc(\infty,0)(\mathcal{F}(\infty)).$$

By the reciprocity law of global class field theory for the field $k(x)$, if we denote by F_0 and F_∞ arbitrary elements of $D(0)$ and $D(\infty)$ respectively whose images in the abelianizations correspond via local reciprocity to the particular parameters x and $1/x$ respectively, then for any lisse rank one $\mathcal{D}$ on $\mathbb{G}_m$ we have

$$\det(F_\infty \mid \mathcal{D}(\infty)) = \det(F_0 \mid \mathcal{D}(0)),$$

whence a stationary phase determinant formula

$$(15) \quad \det(F_\infty \mid FT_\psi loc(0,\infty)(\mathcal{F}(0))) = \det(F_0 \mid FT_\psi loc(\infty,0)(\mathcal{F}(\infty))) \times$$
$$\times [\det(F_k \mid H_c^1(\mathbb{G}_m \otimes \bar{k}, \mathcal{F}))/\det(F_k \mid H_c^2(\mathbb{G}_m \otimes \bar{k}, \mathcal{F}))].$$

These determinant formulas will be the key to the product formula for local constants.

First Application: Product Formula for "Global Constants"

Let us briefly recall the problem. Given a smooth geometrically connected open curve U over a finite field k of characteristic p having q elements, X/k its complete nonsingular model, of genus g, $j: U \to X$ the inclusion, an $\ell \neq p$, and a lisse ℓ-adic sheaf $\mathcal{F}$ on U, one attaches to $\mathcal{F}$ its L-function $L(U/k,\mathcal{F}; T)$ in $\overline{\mathbb{Q}}_\ell[[T]]$, defined by the usual Euler product, which by Grothendieck's Lefschetz Trace Formula is in fact a rational function of T, namely

$L(U/k,\mathcal{F}; T) = \det(1 - TF_k | H_c^1(U\otimes\overline{k}, \mathcal{F})/\det(1 - TF_k | H_c^2(U\otimes\overline{k}, \mathcal{F}).$

We also define

$L_*(U/k,\mathcal{F}; T) := \det(1 - TF_k | H^1(U\otimes\overline{k}, \mathcal{F}))/\det(1 - TF_k | H^0(U\otimes\overline{k}, \mathcal{F})).$

(In terms of L-functions of constructible complexes on X, $L(U/k,\mathcal{F}; T)$ is the L-function of $j_!\mathcal{F}$ on X, while $L_*(U/k,\mathcal{F}; T)$ is that of $Rj_*\mathcal{F}$ on X.) If we denote by $\mathcal{F}^\vee$ the linear dual to $\mathcal{F}$ (i.e., the contragredient representation of π_1), then by Poincaré duality we have a functional equation of the form

$$L(U/k,\mathcal{F}; T) = \varepsilon(\mathcal{F})\times T^{-\chi(U\otimes\overline{k},\, \mathcal{F})}\times L_*(U/k,\mathcal{F}^\vee; 1/qT)$$

where the coefficient $\varepsilon(\mathcal{F})$ of $T^{-\chi(U\otimes\overline{k},\, \mathcal{F})}$, called the "global constant" ("Artin root number" in the old terminology), is given by

$\varepsilon(\mathcal{F}) = \det(-F_k \mid H_c^1(U\otimes\overline{k}, \mathcal{F}))/\det(- F_k | H_c^2(U\otimes\overline{k}, \mathcal{F})).$

The concrete problem is this: once U is small enough that there exist a meromorphic one-form ω on X with neither zero nor pole in U, give a "product formula" for $\varepsilon(\mathcal{F})q^{(g-1)\mathrm{rank}(\mathcal{F})}$ as the product over the closed points x of X - U of local constants $\varepsilon_x(\mathcal{F}(x), \omega)$ which satisfy a reasonable list of axioms (cf. [De-1], [Ta-2]). It was proven up to sign by Dwork (whose axioms were too strong, cf. [Dw] and [Ta-2],pp. 101-4) , then on the nose by Langlands(1968,unpublished), then reproven by a global-to-local method by Deligne ([De-1]), that there exists a unique theory of local constants satisfying the axioms, **and** that for $\mathcal{F}$'s whose geometric monodromy is finite, the product of these local constants is the global constant. In this case of geometrically finite monodromy, one uses Brauer's theorem to reduce to the **abelian** case, where the existence of local constants and the product formula is classical ([Ta-1], last page). Deligne's argument also worked for mod ℓ representations, and gave a mod ℓ product formula; from this Deligne deduced that the product formula held for any ℓ-adic $\mathcal{F}$ which is part of a **compatible system** of ℓ-adic representations for an **infinite** set of primes ℓ. Deligne also treated the case of an ℓ-adic $\mathcal{F}$ which has only tame ramification (1980, unpublished, but cf. [De-4]).

However, the case of a general ℓ-adic representation remained mysterious; no one knew an a priori reason why the global constant should be **any** sort of product of local terms, although that it should be so was suggested by the "Langlands program", and turns out to have important consequences for it (cf. [Lau-2], 3.1.3 3.1.5, 3.2.2).

Here is a sketch of how Laumon's principle of stationary phase solves this problem.

Lemma 16. Let k be a perfect field of characteristic $p > 0$, and U a smooth, geometrically connected curve over k. There exists a nonempty open set V in U and a finite etale map $V \to \mathbb{A}^1$ over k.

Proof. This analogue of Belyi's beautiful "three point" theorem [Be] over $\overline{\mathbb{Q}}$ may be proven as follows. We may assume by shrinking that U is affine, with lots of points at ∞. Taking a function f on X with simple poles at ∞ and no poles in U, we get a finite flat map of X to $\mathbb{P}^1$ which is etale over ∞ in $\mathbb{P}^1$, hence finite etale over some dense open set $\mathbb{A}^1 - S$ in $\mathbb{A}^1$. For $U = \mathbb{A}^1 - S$, let G be the finite (!) additive subgroup of $\overline{k}$ generated by the elements of $S(\overline{k})$. Shrink U to $\mathbb{A}^1 - G$; the quotient group map $\mathbb{A}^1 \to \mathbb{A}^1/G \approx \mathbb{A}^1$ is finite etale, and it makes $\mathbb{A}^1 - G$ finite etale over $\mathbb{G}_m$. Finally, we make $\mathbb{G}_m$ a finite etale cover of $\mathbb{A}^1$ by the function $x \mapsto x^p + 1/x$. QED

Using this lemma and the compatibility of local and global constants with induction (direct image), we reduce first to the case of an $\mathcal{F}$ which is lisse on $\mathbb{A}^1$, then by inverting $(x \mapsto 1/x)$, we reduce to the case of an $\mathcal{F}$ on $\mathbb{A}^1$ which is lisse on $\mathbb{G}_m$ extended by zero, and such that $\mathcal{F}(\infty)$ is unramified. For such an $\mathcal{F}$, the desired product formula is

$$\varepsilon(\mathcal{F}) = q^{\mathrm{rank}(\mathcal{F})} \times \varepsilon_0(\mathcal{F}(0), dx) \times \varepsilon_\infty(\mathcal{F}(\infty), dx).$$

The stationary phase determinant formula (14) unscrews to give

$$\varepsilon(\mathcal{F}) = q^{\mathrm{rank}(\mathcal{F})} \times \det(-F_\infty \mid FT_\psi \mathrm{loc}(0,\infty)(\mathcal{F}(0)) \times$$
$$\times \left(1 \Big/ q^{2\mathrm{rank}(\mathcal{F})} \times \det(-F_\infty \mid \mathcal{F}(\infty))\right)$$

for **any** choice of F_∞ in $D(\infty)$ lifting Frobenius.

The only problem left is to identify the corresponding terms in the two formulas. That the ∞-terms coincide is an easy consequence of the axioms for local constants, and the fact that $\mathcal{F}(\infty)$ is unramified.

It remains only to show that for any $D(0)$-representation M, there is **some** element F_∞ in $D(\infty)$ such that

$$(**) \qquad \varepsilon_0(M, dx) = \det(-F_\infty \mid FT_\psi \mathrm{loc}(0,\infty)(M)).$$

Theorem 17. (Laumon) For any $D(0)$-representation M, $(**)$ holds with F_∞ any element of $D(\infty)$ whose image in $D(\infty)^{ab}$ corresponds by local class field theory to the choice of uniformizing parameter $1/x$ in the ∞-adic completion of $k(x)$.

Proof. If M is tame, this is done by a reduction to the case of an $\mathcal{L}_\chi$ with χ of finite order, and then by a direct calculation. We will explain how to reduce to this case.

By devissage, we reduce to the case when M is D(0)-irreducible, in which case I(0) acts through a finite group (by Grothendieck's "local monodromy theorem"([Se-Ta], App.). Now pass to the canonical extension $\mathcal{G}$ of M. Then $\mathcal{G}$ has finite geometric monodromy, so by Deligne we know that the product formula holds for it:

$$\varepsilon(\mathcal{G}) \;=\; q^{\mathrm{rank}(\mathcal{G})} \times \varepsilon_0(\mathcal{G}(0), dx) \times \varepsilon_\infty(\mathcal{G}(\infty), dx).$$

Because $\mathcal{G}$ is tame at ∞, we may apply to it the stationary phase determinant formula (15), which unscrews to

$$\varepsilon(\mathcal{G}) \;=\; q^{\mathrm{rank}(\mathcal{G})} \times \det(-F_\infty | \, FT_\psi loc(0,\infty)(\mathcal{F}(0)) \times$$
$$\times \left(1 \;\Big/\; q^{\mathrm{rank}(\mathcal{G})} \times \det(-F_0 \, | FT_\psi loc(\infty,0)(\mathcal{G}(\infty))) \right).$$

Comparing these two formulas for $\varepsilon(\mathcal{G})$, it now suffices to show the equality of their ∞-terms. This is a problem about tame $D(\infty)$-representations. We now repeat the above comparison argument, using the canonical extension (switching the roles of 0 and ∞) of $\mathcal{G}(\infty)$ to a lisse $\mathcal{H}$ on $\mathbb{G}_m$ which is tame at 0, to reduce to showing that (**) holds for the **tame** representation $\mathcal{H}(0)$. QED

Second Application:Construction of the Artin Representation in Equal Characteristic

Fix a finite quotient G of the inertia group I(0) at the origin on $\mathbb{A}^1$ over an algebraically closed field of characteristic $p > 0$. Recall that for any finite-dimensional characteristic-zero representation M of G, its Artin conductor Artin(M) is the nonnegative integer defined by

$$\mathrm{Artin}(M) := \mathrm{Sw}_0(M) + \dim M - \dim M^G.$$

It vanishes if and only if M is a trivial representation of G. Here is how it occurs in Fourier Transform.

We may restrict the functor $FTloc(0,\infty)$ to the category of G-representations, and, forgetting the $D(\infty)$-structure, view it as landing in (fin. dim. $\overline{\mathbb{Q}}_\ell$-spaces). Since we know that

$$\dim FTloc(0,\infty)(M) = \mathrm{Sw}_0(M) + \dim M,$$

it follows by the exactness of $FTloc(0,\infty)$ that for the quotient M/M^G we have

$$\dim FTloc(0,\infty)(M/M^{I(0)}) = \mathrm{Sw}_0(M) + \dim M - \dim M^G$$
$$= \mathrm{Artin}(M), \text{ the Artin conductor of M.}$$

Since the functorM $\mapsto M/M^G$ is exact (remember G is **finite**), the composite functor

$$M \mapsto M/M^G \mapsto \mathrm{FTloc}(0,\infty)(M/M^G)$$

is a $\overline{\mathbb{Q}}_\ell$-linear exact functor

(left $\overline{\mathbb{Q}}_\ell[G]$ modules of finite dimension) $\to$ (fin. dim. $\overline{\mathbb{Q}}_\ell$-spaces).

Any such functor T is necessarily of the form

$$M \mapsto A \otimes_{\overline{\mathbb{Q}}_\ell[G]} M$$

for some projective right $\overline{\mathbb{Q}}_\ell[G]$ module A; moreover, this A can only be the result of applying the functor T to the group-ring $\overline{\mathbb{Q}}_\ell[G]$, where the left module structure on the group-ring is used up in viewing the group-ring as being in the domain of T, defined on left modules, and the right structure is used to endow $T(\overline{\mathbb{Q}}_\ell[G])$ with a structure of right module, thanks to the fact that T is a functor. Therefore we see that

$$A = T(\overline{\mathbb{Q}}_\ell[G]) := \mathrm{FTloc}(0,\infty)(\overline{\mathbb{Q}}_\ell[G]/\overline{\mathbb{Q}}_\ell)$$

is a projective right group-ring module with the property that

$$\dim(A \otimes_{\overline{\mathbb{Q}}_\ell[G]} M) = \mathrm{Artin}(M)$$

for every left G-module M. This means precisely that A is the Artin representation of G. Thus we obtain a geometric construction of the Artin representation in equal characteristic. In the case of unequal characteristic, there is still no a priori construction of either the Artin or the Swan representation!

Sketch of the ingredients in and actual proof of Laumon's principle of stationary phase

The first ingredient we must discuss will seem at first rather abstruse, but its relevance will soon become clear. This is the notion of UBC (universal base change) data. We are given a finite local ring R whose order is invertible on a scheme S, a morphism $f : X \to S$ of finite presentation, and an R-flat sheaf $\mathcal{F}$ of R-modules on X. We say that the data $(f, \mathcal{F})$, best viewed horizontally

$$\mathcal{F} \text{ on } X \xrightarrow{\quad f \quad} S,$$

is BC (base change) if for every quasi-compact and quasi-separated S-scheme $g : Y \to S$, and any R-sheaf $\mathcal{G}$ on Y, when we form the cartesian square

$$\begin{array}{ccc} X \times_S Y & \xrightarrow{\quad \mathfrak{f} \quad} & \mathcal{G} \text{ on } Y \\ \downarrow \mathfrak{g} & & \downarrow g \\ \mathcal{F} \text{ on } X & \xrightarrow[\quad f \quad]{} & S \end{array}$$

the natural change of base morphism on X

$$\mathcal{F}\otimes f^*Rg_*\mathcal{G} \to R\mathbf{g}_*((\mathbf{g}^*(\mathcal{F})\otimes(\mathbf{f}^*\mathcal{G}))$$

is an isomorphism. We say that the data (f, $\mathcal{F}$) is UBC (universally BC) if after any change of base S' → S, the morphism $f_{S'} : X_{S'} \to S'$ with the pulled back sheaf $\mathcal{F}_{S'}$ on $X_{S'}$ is BC.

The main (hard) result on UBC that we need is the "smooth base change" theorem ([SGA 4], Exp.XV), which says that if f is smooth and if $\mathcal{F}$ is a lisse sheaf of free R-modules, then the data (f, $\mathcal{F}$) is UBC. We will also make use of the following six facts:

UBC(0). if f is finite, surjective and radicial, and if $\mathcal{F}$ is a lisse sheaf of free R-modules, then the data (f, $\mathcal{F}$) is UBC.

UBC(1). if (f, $\mathcal{F}$) is UBC, and if $\mathcal{L}$ is a direct factor of $\mathcal{F}$, then (f,$\mathcal{L}$) is UBC.

UBC(2). if (f, $\mathcal{F}$) is UBC, and if $j : U \hookrightarrow S$ is the inclusion of an open set with complementary closed set Z such that $\mathcal{F} \mid f^{-1}(Z)$ is zero, then for any $\mathcal{L}$ on S such that $\mathcal{L} \mid U$ is a lisse sheaf of free R-modules, (f, $\mathcal{F}\otimes f^*\mathcal{L}$) is UBC.

UBC(3). is (f, $\mathcal{F}$) is UBC, and if $\mathcal{G}$ is a lisse sheaf of free R-modules on X, then (f, $\mathcal{G}\otimes\mathcal{F}$) is UBC.

UBC(4). if f can be factored as $f = g\circ h$, and if (g, R) and (h, R) are each UBC, then so is (f, R).

UBC(5). if (f, $\mathcal{F}$) is UBC, and if f can be factored as $f = g\circ h$ with h **finite**, then (g, $h_*\mathcal{F}$) is UBC.

Of these, 0, 1, 2, 3, 4 are elementary, while 5 uses the proper base change theorem for the morphism h.

[A note for the experts: What we call here UBC is in fact equivalent to what Illusie calls "universal strong local acyclicity" in SGA4½, where, just as in the original SGA4, the emphasis in the definitions is on the local acyclicity, and the UBC property is **deduced**. However, since it is the UBC property which is used in the arguments which will follow, we have chosen to emphasize **it** rather than the local conditions to which it is equivalent.]

The first application of UBC is to the comparison of the ! and * versions of an integral transform. Let S be a scheme, R a finite ring whose order is invertible on S, X and Y two S-schemes of finite presentation, $\mathcal{F}$ an R-flat sheaf of R-modules on $X\times_S Y$, $T_{\mathcal{F},*}$ and $T_{\mathcal{F},!}$ the integral transforms defined by $\mathcal{F}$.

Lemma 18. Let $j: X \hookrightarrow \overline{X}$ be an open immersion of X into a proper S-scheme $\overline{X}$. Consider the morphism $\overline{pr_1}: \overline{X}\times_S Y \to \overline{X}$, and endow its source with $(j\times id)_!\mathcal{F}$. If this data is BC, then the "forget supports" map of functors from $T_{\mathcal{F},!}$ to $T_{\mathcal{F},*}$ is an isomorphism.

Proof. For any R-sheaf $\mathcal{G}$ on X, simply "test" the BC property against the situation

$$\begin{array}{c} \mathcal{G} \text{ on } X \\ \big\downarrow j \\ \overline{X}. \end{array}$$

This means considering

$$\begin{tikzcd} X \times_S Y \arrow[r, "pr_1"] \arrow[d, "(j \times id)"] & X \ \ \mathcal{G} \text{ on} \arrow[d, "j"'] \\ (j \times id)_! \mathcal{F} \text{ on } \overline{X} \times_S Y \arrow[r, "\overline{pr}_1"'] & \overline{X}. \end{tikzcd}$$

By BC, we have

$$\begin{array}{ccc} (j \times id)_! \mathcal{F} \otimes \overline{pr}_1{}^* Rj_* \mathcal{G} & \xrightarrow{\sim} & R(j \times id)_*(\mathcal{F} \otimes (pr_1{}^* \mathcal{G})), \\ \big\downarrow \wr & & \\ R(j \times id)_!(\mathcal{F} \otimes (pr_1{}^* \mathcal{G})) & & \end{array}$$

and composing with $R(\overline{pr}_2)_! = R(\overline{pr}_2)_*$ for the projection $\overline{pr}_2$ of $\overline{X} \times_S Y$ onto Y gives the desired isomorphism from $T_{\mathcal{F},!}(\mathcal{G})$ to $T_{\mathcal{F},*}(\mathcal{G})$.QED

We now explain Laumon's fundamental UBC result on Fourier Transform, which gives a "geometric" proof of Verdier's theorem that $FT_! \approx FT_*$ by the mechanism of the above lemma.

Theorem 19.(Laumon) Let k be a finite field of characteristic p with q elements, R a finite local ring whose residue field contains p distinct p'th roots of unity, ψ: (k, +) → $R^\times$ a nontrivial additive character. Denote by $\mathbb{A}^1$ the standard affine line over k, and by $j : \mathbb{A}^1 \to \mathbb{P}^1$ its standard compactification. Consider the sheaf $\mathcal{L}_{\psi(xy)}$ on $\mathbb{A}^1 \times_k \mathbb{A}^1$, and its extension by zero $(j \times id)_!(\mathcal{L}_{\psi(xy)})$ to $\mathbb{P}^1 \times_k \mathbb{A}^1$. The data

$$(j \times id)_!(\mathcal{L}_{\psi(xy)}) \quad \text{on} \quad \mathbb{P}^1 \times_k \mathbb{A}^1 \xrightarrow{\ \overline{pr}_1\ } \mathbb{P}^1$$

is UBC.

Proof. The question is Zariski local on $\mathbb{P}^1$, and over the open $\mathbb{A}^1$ we are UBC by usual smooth base change. So we may work over any affine open neighborhood of ∞. Inverting x, our situation becomes the following: U is $\mathbb{A}^1$ with coordinate z (= 1/x), $j : U - \{0\} \to U$ is the inclusion, and we are looking at

$$(j \times id)_!(\mathcal{L}_{\psi(y/z)}) \quad \text{on} \quad U\times_k\mathbb{A}^1 \xrightarrow{pr_1:(z,y) \mapsto z} U.$$

Suppose we already know that for $U\times_k\mathbb{G}_m \to U$, this situation is UBC for $(j \times id)_!(\mathcal{L}_{\psi(y/z)})$. Then by property UBC(2) above, this same situation is UBC for $(j \times id)_!(\mathcal{L}_{\psi(y/z)})\otimes pr_1{}^*(j_!\mathcal{L}_{\psi(1/z)})$, which is isomorphic by the translation $y \mapsto y-1$ to the situation for our original $(j \times id)_!(\mathcal{L}_{\psi(y/z)})$ and $U\times_k(\mathbb{A}^1 - \{1\}) \to U$. As the UBC property is Zariski local on the source, this would conclude the proof.

Over $U\times_k\mathbb{G}_m$, we can make a change of variable $(t,y):= (z/y, y)$, in terms of which our situation becomes

$$(j_!\mathcal{L}_{\psi(1/t)})\boxtimes R \text{ on } \quad U\times_k\mathbb{G}_m \xrightarrow{(t,\, y) \mapsto ty} U.$$

Now consider the completed Artin-Schreier covering

$$\bar{\pi} : \mathbb{P}^1 \to \mathbb{P}^1,\ w \mapsto t:= 1/(w^q - w);$$

its restriction to the affine t line U is a covering

$$\pi : \mathbb{P}^1 - \mathbb{F}_q \to U$$

which is finite etale galois outside $t=0$ with group $\mathbb{F}_q$, and fully ramified over $t=0$.

Therefore we have a direct sum decomposition of sheaves on U

$$\pi_*R = R \oplus \bigoplus_{\text{nontriv } \psi} (j_!\mathcal{L}_{\psi(1/t)}),$$

and by proper base change for the finite morphism π, we can take the product of π with $\mathbb{G}_m$ to obtain a direct sum decomposition of sheaves on $U\times_k\mathbb{G}_m$

$$(\pi\times id)_*R = R \oplus \bigoplus_{\text{nontriv } \psi} (j_!\mathcal{L}_{\psi(1/t)}))\boxtimes R.$$

So by UBC(1) and UBC(5) we are reduced to showing that the map

$$(\mathbb{P}^1 - \mathbb{F}_q) \times_k \mathbb{G}_m \to U \ ; \quad (w, y) \mapsto y/(w^q - w)$$

endowed with the constant sheaf R is UBC. Introducing yet another change of variable $v := 1/w$, this becomes the morphism

$$(\mathbb{A}^1 - \mu_{q-1}) \times_k \mathbb{G}_m \to U; \qquad (v, y) \mapsto yv^q/(1 - v^{q-1}).$$

In terms of the new coordinates $(v, s) := (v, y/(1 - v^{q-1}))$, this becomes

$$(v, s) \mapsto sv^q,$$

which is the composite of the finite surjective radicial endomorphism

$$(v, s) \mapsto (v^q, s)$$

and of the smooth morphism

$$(\mathbb{A}^1 - \mu_{q-1}) \times_k \mathbb{G}_m \rightarrow U \ ; \quad (v, s) \mapsto vs,$$

both of which are UBC. QED

Notice that the initial data in Laumon's theorem, namely the sheaf $\mathcal{L}_{\psi(xy)}$ on $\mathbb{A}^1 \times_k \mathbb{A}^1$ is symmetric in x and y. Therefore by symmetry we also have the following theorem, whose significance for stationary phase will soon become clear.

Symmetric Theorem 20.(Laumon) Let k be a finite field of characteristic p with q elements, R a finite local ring whose residue field contains p distinct p'th roots of unity, ψ: (k, +) $\rightarrow R^\times$ a nontrivial additive character. Denote by $\mathbb{A}^1$ the standard affine line over k, and by $j : \mathbb{A}^1 \rightarrow \mathbb{P}^1$ its standard compactification. Consider the sheaf $\mathcal{L}_{\psi(xy)}$ on $\mathbb{A}^1 \times_k \mathbb{A}^1$, and its extension by zero $(\mathrm{id} \times j)_!(\mathcal{L}_{\psi(xy)})$ to $\mathbb{A}^1 \times_k \mathbb{P}^1$. The data

$$(\mathrm{id} \times j)_!(\mathcal{L}_{\psi(xy)}) \quad \text{on} \quad \mathbb{A}^1 \times_k \mathbb{P}^1 \xrightarrow{\ pr_2\ } \mathbb{P}^1$$

is UBC.

In order to explain the relevance of this result to stationary phase, it is first necessary to recall the theory of vanishing cycles. Let us consider the following geometric situation: S is the spec of a strictly henselian discrete valuation ring A (e.g., complete, and with separably closed residue field), and $f : X \rightarrow S$ is an S-scheme. We denote by s the closed point of S, and we denote by $\bar{\eta}$ a geometric point lying over the generic point of S. (i.e., the spec of a separably closed overfield of the fraction field of A). We suppose given a torsion sheaf $\mathcal{F}$ on X.

$$\begin{array}{ccccc} & & \mathcal{F} & & \\ & i & \text{on} & j & \\ X_s & \hookrightarrow & X & \longleftarrow & X_{\bar{\eta}} \\ \downarrow & & \downarrow f & & \downarrow \\ & i_0 & & j_0 & \\ s & \hookrightarrow & S & \longleftarrow & \bar{\eta} \end{array}$$

The object is to compare the cohomologies of the two fibres X_s and $X_{\bar{\eta}}$ with coefficients in (the pullbacks $i^*\mathcal{F} := \mathcal{F}_s$ and $j^*\mathcal{F} := \mathcal{F}_{\bar{\eta}}$ of) $\mathcal{F}$, at least when f is proper. By the (derived category version of the) Leray spectral sequence, we have

$$H^i(X_{\bar{\eta}}, \mathcal{F}_{\bar{\eta}}) \cong H^i(X, Rj_*(\mathcal{F}_{\bar{\eta}})) \cong H^i(S, Rf_*Rj_*(\mathcal{F}_{\bar{\eta}})).$$

The strict henselianity of S yields

$$H^i(S\,,\,Rf_*Rj_*(\mathcal{F}_{\bar\eta})) \cong H^i(s\,,\,i_0{}^*Rf_*Rj_*(\mathcal{F}_{\bar\eta})),$$

and, **if f is proper**, then proper base change for f yields

$$H^i(s\,,\,i_0{}^*Rf_*Rj_*(\mathcal{F}_{\bar\eta})) \cong H^i(X_s\,,\,i^*Rj_*(\mathcal{F}_{\bar\eta})).$$

The complex $i^*Rj_*(\mathcal{F}_{\bar\eta})$ on X_s is denoted $R\Psi(\mathcal{F})$. Its grand merit is to be a complex on X_s which, when f is proper, calculates the cohomology of $X_{\bar\eta}$. There is a natural adjunction map

$$i^*\mathcal{F} \to R\Psi(\mathcal{F}),$$

whose mapping cone is denoted $R\Phi(\mathcal{F})$. Thus we have a "distinguished triangle"

$$i^*\mathcal{F} \to R\Psi(\mathcal{F}) \to R\Phi(\mathcal{F}) \xrightarrow{+1}$$

on X_s, and the corresponding long exact sequence of cohomology groups on X_s is, for f proper,

$$\ldots \to H^i(X_s\,,\,\mathcal{F}_s) \to H^i(X_{\bar\eta}\,,\,\mathcal{F}_{\bar\eta}) \to H^i(X_s\,,\,R\Phi(\mathcal{F})) \to \ldots$$

The complex $R\Phi(\mathcal{F})$ on X_s is called the complex of vanishing cycles. Its global cohomology groups are, for f proper, the obstructions to the isomorphy of the specialization maps

$$H^i(X_s\,,\,\mathcal{F}_s) \to H^i(X_{\bar\eta}\,,\,\mathcal{F}_{\bar\eta}).$$

The stalks of the sheaves $R^i\Psi(\mathcal{F})$ and $R^i\Phi(\mathcal{F})$ at a geometric point x of X_s have the following purely local description. Denote by $X_{(x)}$ the strict henselization of X at x, and by $(X_{(x)})_{\bar\eta}$ its fibre over $\bar\eta$. Then we have

$$R^i\Psi(\mathcal{F})_x = H^i((X_{(x)})_{\bar\eta}, \mathcal{F}) \text{ for all } i$$

$$R^i\Phi(\mathcal{F})_x = R^i\Psi(\mathcal{F})_x \text{ for } i \geq 1, = 0 \text{ for } i \leq -2$$

and a four term exact sequence

$$0 \to R^{-1}\Phi(\mathcal{F})_x \to \mathcal{F}_x \to H^0((X_{(x)})_{\bar\eta}, \mathcal{F}) \to R^0\Phi(\mathcal{F})_x \to 0.$$

Lemma 21. <u>Suppose that</u> R <u>is a finite local ring of order invertible on</u> S, <u>that</u> $\mathcal{F}$ <u>is an</u> R-<u>sheaf, and that</u> $U \subset X$ <u>is an open set such that the data</u> $(f \mid U, \mathcal{F} \mid U)$ <u>is</u> BC. <u>Then the restriction to</u> U_s <u>of</u> $R\Phi(\mathcal{F})$ <u>vanishes.</u>

Proof. Simply "test" the BC property by the S-scheme $j_0 : \bar\eta \to S$, with $\bar\eta$ endowed with the constant sheaf R. The cartesian diagram in question is

$$\begin{array}{ccc} U_{\bar{\eta}} & \longrightarrow & \begin{matrix} R \\ \text{on} \\ \bar{\eta} \end{matrix} \\ \downarrow j & & j_0 \downarrow \\ \mathcal{F} \text{ on } U & \xrightarrow[f \mid U]{} & S \end{array}$$

and the natural change of base isomorphism on U given by BC is

$$\mathcal{F} \otimes (f \mid U)^*(Rj_{0*}R) \;\cong\; R\, j_*(\mathcal{F}_{\bar{\eta}}).$$

Since S is strictly henselian, $i_0{}^*Rj_{0*}R$ is just the constant sheaf R on s, and so pulling back this isomorphism to U_s gives us that the natural map $i^*\mathcal{F} \to R\Psi(\mathcal{F})$ is an isomorphism over U_s, as asserted.QED

Suppose now in addition that f is proper and that, denoting by Z the complement $Z := X - U$, Z_s consists of a finite set of closed points of X_s. Then the cohomology sheaves $R^i\Phi(\mathcal{F})$ of $R\Phi(\mathcal{F})$ are skyscraper sheaves supported in Z_s, and the long exact cohomology sequence of vanishing cycles looks like

$$\ldots \to H^i(X_s, \mathcal{F}_s) \to H^i(X_{\bar{\eta}}, \mathcal{F}_{\bar{\eta}}) \to \bigoplus_{z \text{ in } Z_s} (R^i\Phi(\mathcal{F}))_z \to \ldots.$$

If in addition there is an integer N such that

$$(22)\quad \left\{ \begin{array}{l} \text{a. } H^i(X_{\bar{\eta}}, \mathcal{F}_{\bar{\eta}}) = 0 \text{ for } i \neq N, \\[1ex] \text{b. } H^N(X_s, \mathcal{F}_s) \to H^N(X_{\bar{\eta}}, \mathcal{F}_{\bar{\eta}}) \text{ is injective (i.e., } R^Nf_*\mathcal{F} \text{ has} \\ \text{no punctual sections),} \\[1ex] \text{c. } H^i(X_s, \mathcal{F}_s) = 0 \text{ for } i \neq N, N+1 \end{array} \right.$$

then

$$R^i\Phi(\mathcal{F}) \text{ vanishes for } i \neq N,$$

and we have a four term exact sequence

$$(23)\quad 0 \to H^N(X_s, \mathcal{F}_s) \to H^N(X_{\bar{\eta}}, \mathcal{F}_{\bar{\eta}}) \to \bigoplus_{z \text{ in } Z_s} (R^N\Phi(\mathcal{F}))_z \to$$
$$\to H^{N+1}(X_s, \mathcal{F}_s) \to 0.$$

When in addition $\mathcal{F}$ **vanishes** on the entire special fibre. X_s, the situation becomes

$$(24)\begin{cases} H^N(X_{\bar{\eta}}, \mathcal{F}_{\bar{\eta}}) \cong \bigoplus_{z \text{ in } Z_s} (R^N\Phi(\mathcal{F}))_z \\ H^i(X_{\bar{\eta}}, \mathcal{F}_{\bar{\eta}}) = 0 \quad \text{for } i \neq N. \end{cases}$$

With this background, we can now sketch the proofs of the "fine" versions of Laumon's stationary phase theorem (Thm.3) and his Thm.10, namely with finite coefficients R (a finite local ring whose residue field contains p distinct p'th roots of unity) rather than $\overline{\mathbb{Q}}_\ell$. We fix a nontrivial additive character ψ of $\mathbb{F}_p$ with values in $R^\times$.

Suppose that $\mathcal{F}$ is a sheaf of R-modules on $\mathbb{P}^1$ over a perfect field k of characteristic p, such that $\mathcal{F}$ has no punctual sections, vanishes at ∞, and on a nonvoid open set $\mathbb{A}^1 - S$ is a lisse sheaf of finite free R-modules. To $\mathcal{F}$ we functorially associate the sheaf $\mathcal{K}_\mathcal{F}$ on $\mathbb{P}^1 \times_k \mathbb{P}^1$ defined by

$$\mathcal{K}_\mathcal{F} := (\mathrm{pr}_1{}^*\mathcal{F})\otimes(\mathcal{L}_{\psi(xy)} \text{ on } \mathbb{A}^1 \times_k \mathbb{A}^1, \text{ ext. by } 0).$$

Notice that for any rational point (a,b) of $\mathbb{P}^1 \times_k \mathbb{P}^1$, the restriction of $\mathcal{K}_\mathcal{F}$ to the henselization of $\mathbb{P}^1 \times_k \mathbb{P}^1$ at (a,b) depends only on the restriction of $\mathcal{F}$ to the henselization of $\mathbb{P}^1$ at the rational point a.

Consider the proper morphism

$$\begin{array}{c} \mathbb{P}^1 \times_k \mathbb{P}^1 \text{ with } \mathcal{K}_\mathcal{F} \\ \downarrow \mathrm{pr}_2 \\ \mathbb{P}^1. \end{array}$$

Since $\mathrm{pr}_1{}^*\mathcal{F}$ is lisse on $(\mathbb{P}^1 - (S \cup \{\infty\})) \times_k \mathbb{P}^1$, this situation is UBC on the open set $(\mathbb{P}^1 - (S \cup \{\infty\})) \times_k \mathbb{P}^1$, by the symmetric version of Laumon's theorem (Thm.20).

The key point is this: if we denote $j: \mathbb{A}^1 \to \mathbb{P}^1$ the inclusion, then from (proper base change and) the very definition of $\mathrm{FT}_{\psi,!}$, we see that $j_!\mathrm{FT}_{\psi,!}(j^*\mathcal{F})$ is precisely $R(\mathrm{pr}_2)_*\mathcal{K}_\mathcal{F}$. By what we already know (Thm.2) about FT, $R^i(\mathrm{pr}_2)_*\mathcal{K}_\mathcal{F}$ vanishes for $i \neq 1, 2$, $R^1(\mathrm{pr}_2)_*\mathcal{K}_\mathcal{F}$ has no punctual sections, and $R^2(\mathrm{pr}_2)_*\mathcal{K}_\mathcal{F}$ is punctual. Therefore we are in the N = 1 situation considered above (22) whenever we strictly localize on the target $\mathbb{P}^1$ at any closed point. Over the point ∞ of $\mathbb{P}^1$ we have in addition that $\mathcal{K}_\mathcal{F}$ vanishes identically on the fibre over ∞.

Localizing at ∞, the above isomorphism (24) is none other than the desired stationary phase decomposition of $\mathrm{NFT}_\psi((j^*\mathcal{F}))(\infty)$; the functors

$FT_\psi loc(a, \infty)$ are "just" $\mathcal{F} \mapsto$ the stalks of the punctual (and only non-identically vanishing) sheaf of vanishing cycles $R^1\Phi(\mathcal{K}_\mathcal{F})$, for the localization over ∞ of our morphism. Concretely, for $a \in S \cup \{\infty\}$,

$$FT_\psi loc(a, \infty)(\mathcal{F}(a)) := H^1(((\mathbb{P}^1 \times_k \mathbb{P}^1)_{(a,\infty)})_{\bar{\eta}}, \mathcal{K}_\mathcal{F})$$

for any $\mathcal{F}$ with $\mathcal{F}_a = 0$.

When we localize over 0, it results easily from Deligne's theorem on semicontinuity for Swan conductors (cf. [Lau-1]) that of the finitely many points $(a,0)$, $a \in S \cup \{\infty\}$, over which we are possibly not UBC, only the point $(\infty, 0)$ can have $R^1\Phi(\mathcal{K}_\mathcal{F}) \neq 0$.(Alternatively, we could apply Deligne's Cor. 2.16 [Thm. Finitude, SGA4½] to $K=\mathcal{F}$ on $X=\mathbb{A}^1$ to show this vanishing.) The functor $FT_\psi loc(\infty,0)$ is

$$FT_\psi loc(\infty,0)(\mathcal{F}(\infty)) := H^1(((\mathbb{P}^1 \times_k \mathbb{P}^1)_{(\infty,0)})_{\bar{\eta}}, \mathcal{K}_\mathcal{F}),$$

and the exact sequence of Thm.10 is the vanishing cycle exact sequence (23) above.

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