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par

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§ 1. Introduction

The theory of differentiable dynamical systems investigates the orbit structures of one parameter groups - discrete or continuous - of diffeomorphisms. For many applications, the principal questions are :

What is the asymptotic behavior of typical orbits ? (theory of attractors)
How does this behavior change as the generator of the group changes continuously ? (bifurcation theory)

This report is an introduction to what is known about these questions in the case, which ought to be the simplest possible, where the space on which the transformations act is a compact interval. Even in this simple context, complicated behavior is not only possible but inevitable. There is, however, relatively simple theory which accounts for much of the complexity. This is a subject with the charm of concreteness ; all the phenomena to be described occur in the one-parameter family of mappings

$$x \longmapsto 1 - \mu x^2 \quad ,$$

on $[-1,1]$, where the parameter μ is in $[0,2]$. Although limits of time and space prevent elaboration on this point, it should be mentioned that much of what has been learned about one-dimensional transformations has direct application to higher dimensional dynamical systems.

We will be discussing mappings of a compact interval (not a circle) to itself which are continuously differentiable everywhere (not just piecewise continuously differentiable). This rules out immediately the interesting subject of expanding mappings. Monotone mappings generate uninteresting dynamical systems ; every orbit converges either to a fixed point or to a periodic orbit of period two. We will therefore discuss mappings which are not monotone and hence not invertible. In fact, we will concentrate exclusively on iterates of mappings ψ with a single critical point x_c . By reversing the orientation of the interval if necessary, we can assume that x_c is a maximum. A mapping of an interval into itself with a single maximum and no other critical point will be said to be unimodal. If $\psi(x_c) \leq x_c$, then the range of ψ is contained in the subinterval where ψ is increasing, which effectively puts us back in the uninteresting monotone case ; we will therefore usually assume that $\psi(x_c) > x_c$. The example mentioned above, $x \longmapsto 1-\mu x^2$, $\mu \in (0,2]$, has all the desired properties.

As already indicated, we want to study the behavior of typical orbits for these mappings. This is not the same as determining the structure of all orbits. An example may clarify the distinction. It is not difficult to analyze completely the orbit structure of the mapping $x \longmapsto 1-1.76x^2$ of $[-1,1]$ into itself. It has a closed invariant Cantor set on which it is conjugate to a Markov shift, has periodic orbits of all periods, etc. All this sounds intimidatingly complicated. On the other hand, it can be checked by direct computation that this mapping has an attracting periodic orbit of period 3, and it then follows from a general theorem to be stated shortly that all orbits save only a set of Lebesgue measure zero converge asymptotically to this periodic attractor. (The invariant Cantor set is contained in the Lebesgue null set of exceptional orbits). For applications, what is relevant is that essentially all orbits are asymptotically periodic ; not the existence of exceptional orbits which are more complicated.

This report will be organized around a certain number of phenomena which have mostly been discovered by numerical experiments. The next section will be devoted to a description of the principal phenomena ; the announcement of the plan for the remainder of the report will be postponed until after this description.

Bibliographic Note : The literature on smooth transformations of intervals is vast and heterogeneous. It would be entirely beyond my capacity to prepare either a comprehensive bibliography or a carefully selective one. The references provided for specific results are intended only as indications of a few places to look for more details.

There are two general references which are particularly useful. A monograph on transformations of intervals by P. Collet and J.-P. Eckmann [2] has just appeared. This monograph gives a systematic discussion, with detailed proofs, of most of the results discussed in this report (and a great deal more) ; it is certainly the best general source available on the theory. The second reference is a review by the biologist R. May [8] . This article was enormously influential in bringing the subject of smooth transformations of intervals to the attention of a wide audience. It is recommended as an excellent and colorful description of the phenomena, but is now out of date on a number of theoretical issues.

§ 2. Phenomenology

The methods used to study smooth transformations of intervals are by and large, elementary, and the theory could have been developed long ago if anyone had suspected that there was anything worth studying. In actual fact, the main phenomena were discovered through numerical experimentation, and the theory has been developed to account for the observations. In this respect, computers have played a crucial role in its development.

I will describe in this section the observed behavior of the family $1-\mu x^2$ as the parameter μ is varied from 0 to 2. Very similar behavior can be seen in other families, such as $x \longmapsto \mu \sin(\pi x)$ on $[0,1]$ as μ is varied from $\frac{1}{2}$ to 1. To start with, one can imagine performing a very simple experiment : Fix the transformation ; choose an initial point "at random" ; iterate ; and see what comes out. Then choose another initial point and see if something similar happens. In the description which follows, no particular effort will be made to distinguish between results of such experiments, plausible extrapolations therefrom, and theoretical results ; the objective is to present a picture. The remainder of this report will be concerned, precisely, with the extent to which this picture can be explained mathematically.

For small μ (up to 1/2) the mapping $1-\mu x^2$ is contractive on $[-1,1]$ and hence every orbit converges to the unique fixed point (whose location can easily be computed explicitly). The fixed point continues to attract all orbits until $\mu = \mu_0 = .75$ when it ceases to be attracting : the slope of the mapping at the fixed point passes through -1. At this value of μ an attracting orbit of period 2 appears through the occurrence of a <u>period doubling bifurcation</u>. As long as the period 2 orbit remains attracting, it attracts essentially all orbits, but at $\mu = \mu_1 = 1.25$ it ceases to be attracting and undergoes a period doubling bifurcation producing an attracting orbit of period 4. The period 4 orbit in turn undergoes a period doubling bifurcation at $\mu = \mu_2 = 1.368...$, and so on through periods 8, 16, 32,...

If we denote by μ_n the parameter value at which the period 2^n orbit loses stability and bifurcates into an orbit of period 2^{n+1}, the sequence of μ_n's converges to a limit $\mu_\infty = 1.401...$. We will use the term <u>periodic regime</u> to denote an interval in parameter space throughout which a one parameter family ψ_μ has an attracting periodic orbit of a given (fixed) period. Thus, the above description may be summarized by the statement that the parameter interval $(0,\mu_\infty)$ is composed of a sequence of periodic regimes with periods 2^n , $n = 0,1,2,\dots$

In one respect, the observed periodic behavior is unexpectedly simple. If, for a particular parameter value μ , the orbit under $1-\mu x^2$ of one randomly chosen initial point is asymptotically periodic, then every randomly chosen initial point gives an orbit converging asymptotically to that same periodic orbit. A parameter value μ thus gives unambiguously either periodic or aperiodic behavior, and, if it is periodic, the period is uniquely determined.

What happens for μ between μ_∞ (the accumulation point of period doublings) and 2 (the largest parameter value for which the interval is mapped into itself) is complicated. The dominant feature is that, for most parameter values in this range, $1-\mu x^2$ seems to be aperiodic, i.e., typical orbits are not asymptotically periodic. The word "most" is used here in an informal quantitative sense ; there are definitely periodic regimes imbedded in the interval $(\mu_\infty,2)$ and occupying a non-zero fraction of its length. For example, for $1.75 < \mu < 1.7685\dots$, $1-\mu x^2$ has an attracting orbit of period 3 , which at $\mu = 1.7685\dots$ undergoes a period-doubling bifurcation to an attracting orbit of period 6 which bifurcates to period 12 and so on through all periods of the form $2^n\cdot 3$. Close examination reveals a great many of these imbedded periodic regimes, each accompanied by its cascade of period doublings, and it even appears that the periodic regimes, taken together, form a dense subset of $(\mu_\infty,2)$ (Density of the periodic regimes is, however, one aspect of the picture which has so far eluded proof.) Moreover, different families always seem to display the same sequence of periodic regimes, i.e., the same periods in the same order. Including only periods up to 7 , the observed universal sequence of periodic regimes is

$$1,2,4,6,7,5,7,3,6,7,5,7,6,7,4,7,6,7,5,7,6,7 .$$

(The 1,2,4 at the beginning comes from the initial sequence of period doublings.) The universality of this sequence was discovered by Metropolis, Stein, and Stein [9]

In spite of the occurence of infinitely many periodic regimes, extensive and careful computations performed by E.N. Lorenz [7] strongly suggest that the set of μ's for which $1-\mu x^2$ is aperiodic has Lebesgue measure which is not only non-zero but even quite a large fraction of the total length of $(\mu_\infty,2)$.

We come now to what is perhaps the most surprising discovery about one-parameter families of smooth unimodal transformations. We have noted that, as μ is increased from 0 , the family of mappings $1-\mu x^2$ undergoes a sequence of period-doubling bifurcations, and we have denoted the parameter value where the bifurcation from period 2^n to period 2^{n+1} occurs by μ_n and the limit of the μ_n by μ_∞ . Precise computation of the μ_n's shows that

$$\mu_\infty - \mu_n \sim \text{const} \times (4.6692\ldots)^{-n} \quad .$$

The multiplicative constant can be changed by a smooth reparametrization and so is unlikely to have any special significance, but M. Feigenbaum [4] discovered that the asymptotic ratio 4.6692... appears to be universal, i.e., is the same for a number of families given by quite different formules*. It is also the same for different cascades of period doublings within a given family. For example, if we temporarily let $\bar{\mu}_0$ denote the parameter value at which $1-\mu x^2$ undergoes a period doubling bifurcation from period 3 to period 6 , $\bar{\mu}_n$ the value at which the bifurcation from period $3\cdot 2^n$ to $3\cdot 2^{n+1}$ occurs in the subsequent cascade, and $\bar{\mu}_\infty$ the limit of the $\bar{\mu}_n$'s then

$$\bar{\mu}_\infty - \bar{\mu}_n \sim \overline{\text{const}}\cdot(4.6692\ldots)^{-n} \quad .$$

The accumulation of period doubling bifurcations has other universal features as well, but we will postpone their description until we discuss the theory of these phenomena.

The plan is now the following : In Section 3 we discuss a technical condition which is very convenient in singling out well-behaved transformations and which accounts, in particular, for the unambiguous dichotomy between periodic and aperiodic behavior in the family $1-\mu x^2$. In Section 4 we introduce briefly the period doubling bifurcation. Section 5 contains a sketch of the combinatorial theory which accounts for the universality of the sequence of periodic regimes. In Section 6 we introduce some technical tools needed in the analysis of the accumulation of period doublings, and, in Section 7 , we apply these tools to explain the universality of the rate of accumulation.

It has not been possible to discuss in this report the prevalence of aperiodic behavior in the parameter range $(\mu_\infty, 2)$ and the analysis of the behavior of typical orbits in the aperiodic case. There are a number of important and deep results bearing on these questions, for which we refer to Section III.2 of the monograph of Collet and Eckmann [2] .

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* Universality of the rate of convergence of period doublings was also discovered by P. Coullet and J. Tresser [12].

§ 3. Negative Schwarzian

A continuously differentiable function ψ of a real variable will be said to have negative Schwarzian if $1/\sqrt{|\psi'(x)|}$ is convex on each interval when $\psi'(x)$ does not vanish. The terminology comes from the fact that, if ψ is three times continuously differentiable, the condition is equivalent to negativity of the Schwarzian derivative

$$S\psi(x) = \frac{\psi'''(x)}{\psi'(x)} - \frac{3}{2}\left(\frac{\psi''(x)}{\psi'(x)}\right)^2 .$$

The negative Schwarzian condition has had phenomenal success as an economical way of singling out a class of well-behaved mappings. It was introduced into the subject by D. Singer [11], who showed that a unimodal mapping with negative Schwarzian can have no more than one attracting periodic orbit.

Most deep results about smooth transformations of intervals require that the transformation in question have negative Schwarzian. Many simple examples, including the mappings of the form $1-\mu x^2$, have this property. We will find it convenient, therefore, to assume that all mappings to be discussed henceforth have negative Schwarzian.

The following theorem due to J. Guckenheimer [6] and M. Misiurewicz [10] well illustrates the effectiveness of the negative Schwarzian condition :

Theorem : If a unimodal mapping with negative Schwarzian has an attracting periodic orbit, the set of orbits which do not converge to the attracting periodic orbit has Lebesgue measure zero.

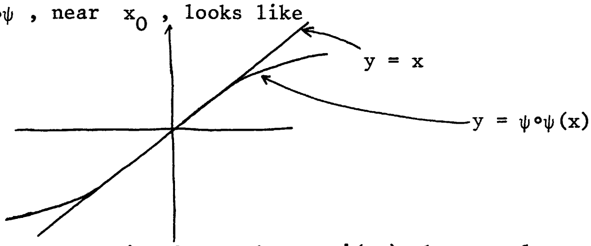
This explains why, for a mapping of the form $1-\mu x^2$, one either always sees periodic behavior or always sees aperiodic behavior.

We will not develop the theory of the negative Schwarzian condition systematically, but, in the hope of making it seem slightly less mysterious, we will give a brief indication of how it is used. Its strength is that it is preserved by composition : If ψ_1 and ψ_2 have negative Schwarzian, then so does $\psi_1 \circ \psi_2$. This can be proved in the C^3 case by a computation which is straightforward if not particularly illuminating. The most common use of the condition is via the remark that, if ψ has negative Schwarzian, then for any n, since $1/\sqrt{|(\psi^n)'(x)|}$ is convex, it cannot have a local maximum, i.e., the only local minima of $|(\psi^n)'(x)|$ are critical points of $\psi^n(x)$.

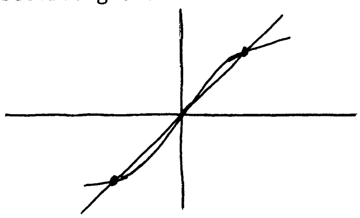
§ 4. The period doubling bifurcation

Consider a smooth function ψ of a real variable x with a fixed point x_0. We should think of ψ and x_0 as depending on a parameter which we can, for this section, suppress from the notation. The fixed point will be attracting if $|\psi'(x_0)| < 1$; repelling if the opposite inequality holds. What happens when $\psi'(x_0)$ passes through -1 ?

Let us look at what happens to $\psi\circ\psi$ when $\psi'(x_0)$ is exactly equal to -1 . Then $(\psi\circ\psi)'(x_0) = +1$, and a simple calculation shows that $(\psi\circ\psi)''(x_0)=0$. The third derivative, on the other hand, has no special tendancy to vanish, and we will assume that it does not do so. In general, the third derivative can have either sign, but <u>it must be negative if</u> ψ <u>has negative Schwarzian</u>. Thus, the graph of $\psi\circ\psi$, near x_0 , looks like



Now perturb ψ slightly so that $\psi'(x_0)$ becomes less than -1 . Then $(\psi\circ\psi)'(x_0)$ becomes greater than $+1$ and the picture becomes :



$\psi\circ\psi$ has acquired a new pair of attracting fixed points, indicated by the dark dots in the sketch. Since ψ itself has negative derivative at x_0 , it must interchange these two points, i.e., they must constitute an attracting orbit of period 2 for ψ .

Appearance of an attracting orbit of period 2 will thus occur when an attracting fixed point of a mapping with negative Schwarzian loses attractivity by having the derivative at the fixed point pass through -1. Applying this argument to ψ^p (which again has negative Schwarzian) shows that an analogous period doubling will occur when an attracting orbit of period p becomes repelling provided that ψ^p is orientation reversing at the points of the orbit.

The period doubling bifurcation is a simple and illuminating construct and provides the basis for understanding quite a lot about the sequence of periodic regimes for one-parameter families of transformations. (See, e.g., Guckenheimer [5]) . Nevertheless, more global considerations are needed at some points in the analysis. We will develop in the next section a slightly different approach, emphasizing these global considerations, which leads very quickly to a general understanding of the necessary occurrence of the various periodic regimes but gives less information about how the transition from one to another takes place.

§ 5. The universal sequence of periodic regimes

The proof that rather general one parameter families of smooth unimodal mappings have many periodic regimes proceeds by combining a topological argument to prove the existence of periodic orbits with the simple remark that a periodic orbit which passes sufficiently near to a critical point must be attracting. We will first give a precise version of this latter remark.

Let ψ be a smooth mapping of an interval into itself and x_0 a periodic point for ψ with period p . The orbit of x_0 will be attracting if $|(\psi^p)'(x_0)| < 1$. If the orbit of x_0 contains a critical point for ψ , then x_0 is a critical point of ψ^p and the criterion for attractivity is certainly satisfied. A smooth mapping of an interval to itself which has a critical point which is periodic with period p will be said to be <u>superstable of period</u> p . If ψ_μ is a family of mappings depending on a parameter, and if ψ_{μ_0} is superstable of period p , then μ_0 lies in a periodic regime of period p . To prove the existence of periodic regimes in a family, it suffices therefore to find superstable elements.

A complete analysis of the occurrence of superstable elements in one-parameter families requires a long and laborious combinatorial argument. Rather than try to outline this argument, we will prove some simple partial results to illustrate the reasoning ; then summarize schematically one version of the main result. Complete details may be found in Collet and Eckmann [2] .

To prove the existence of superstable elements in a family, it is necessary to assume something about the family. We will say that a one parameter family of unimodal mappings is <u>full</u> if it contains both

- a member which sends the whole interval to the right of the critical point
- a member which maps the interval <u>onto</u> itself in a 2-to-1 fashion.

Our standard example $1-\mu x^2$, mapping $[-1,1]$ into itself for $0 \leq \mu \leq 2$, is a full family ; for $\mu < 1$, every point is mapped to the right of the critical point 0 , while for $\mu = 2$ the mapping is 2-to-1 and surjective.

We are going to show that each full family has infinitely many superstable members. The notation will be as follows : the interval on which our mappings are defined will be denoted by $[x_{min}, x_{max}]$ and the critical point of ψ_μ by x_c (which may vary - continuously - with μ). We assume, as always, that $\psi_\mu(x_c) > x_c$. The parameter interval will be taken to be $[0,1]$ and we will assume

$$\psi_0(x) > x_c \qquad \text{for all } x \in [x_{min}, x_{max}]$$

$$\psi_1 \quad \text{is 2-to-1 and surjective.}$$

We introduce the sequence of functions

$$f_p(\mu) = \psi_\mu^p(x_c) \qquad .$$

Finding a value of μ at which ψ_μ is superstable of period p is equivalent to finding a solution of the equation

$$f_p(\mu) = x_c \qquad ,$$

which is not simultaneously a solution of

$$f_q(\mu) = x_c$$

for any divisor q of p . The following sequence of simple remarks proves the existence of many solutions :

1. $f_p(0) > x_c$ for all p since $f_p(0) = \psi_0(f_{p-1}(0))$ and $\psi_0(x) > x_c$ for all x.

2. $f_p(1) = x_{min} < x_c$ for all $p = 2,3,4,\ldots$. To see this, recall that ψ_1 is 2-to-1 and onto, from which it follows readily that $\psi_1(x_c) = x_{max}$; $\psi_1(x_{max}) = \psi_1(x_{min}) = x_{min}$. Thus, $\psi_1^p(x_c) = f_p(1) = x_{min}$ for $p = 2,3,\ldots$. Remarks 1. and 2. already imply that each equation

$$f_p(\mu) = x_c$$

has at least one solution and thus that there is a superstable element of each prime period

3. If $f_{p-1}(\mu) = x_c$ then $f_p(\mu) > x_c$, since $f_p(\mu) = \psi_\mu(f_{p-1}(\mu)) = \psi_\mu(x_c) > x_c$.

4. The largest solution of $f_p(\mu) = x_c$ is larger than any solution of $f_{p-1}(\mu) = x_c$ and hence (by induction) than any solution of $f_q(\mu) = x_c$ for any $q < p$. This follows from 2. and 3. ; if $\overline{\mu}$ is the largest solution of $f_{p-1}(\overline{\mu}) = x_c$ then, by 3. $f_p(\overline{\mu}) > x_c$, whereas, by 2., $f_p(1) < x_c$; hence, $f_p(\mu) = x_c$ has a solution between $\overline{\mu}$ and 1 .

From 4. it follows that there is at least one superstable element of each period $p \geq 2$.

5. If $f_2(\mu) < x_c$ and $f_{p-2}(\mu) = x_c$, then $f_p(\mu) < x_c$, since $f_p(\mu) = \psi_\mu^2(f_{p-2}(\mu)) = \psi_\mu^2(x_c) < x_c$.

6. If $f_{p-1}(\mu_1) = x_c$ and $f_{p-2}(\mu_2) = x_c$, and if μ_2 is larger than the largest solution of $f_2(\mu) = x_c$, then there is a solution of $f_p(\mu) = x_c$ between μ_1 and μ_2 . This follows at once from 3. and 5.

Repeated application of 4. and 6. shows that every full family contains at least one superstable element of period 2,3, and 4 ; at least 2 of period 5 , at least 3 of periodic 6 , etc., in the order 2,3,5,6,4,6,5,6 . It does not, however, give all of the superstable elements which can be shown to be exist in all full families ; we have missed, for example, one period 4 , one period 5 , two period 6's, etc.

Starting from another point of view, it is possible to give a more comprehensive description of the sequence of unavoidable periodic regimes, although it remains difficult to enumerate them explicitly. We associate with any superstable ψ of period p a sequence of p-1 symbols "L" or "R" ; the i-th symbol is L or R according as $\psi^i(x_c)$ is to the left or the right of x_c . We will say that the sequence represents ψ , and that a finite sequence of L's and R's is realizable if it represents some superstable unimodal ψ . Under our assumptions, realizable sequences begin with R , but not all sequences beginning with R are realizable. (For example, no sequence beginning RR is realizable).

It turns out to be possible to give

- a combinatorial criterion which is necessary and sufficient for realizability
- an explicit linear ordering on the set of realizable sequences such that any continuous one-parameter family moves through the ordered set of possible superstable behaviors without jumps, or, more precisely, such that

If ψ_μ is a continuous family of unimodal mappings, if ψ_{μ_1} and ψ_{μ_2}

are superstable, and if $\underline{A}$ is a realizable sequence lying between, in the ordering on realizable sequences, the sequences representing ψ_{μ_1} and ψ_{μ_2}, then there is a μ_3 between μ_1 and μ_2 such that ψ_{μ_3} is superstable and represented by $\underline{A}$.

Thus, the order in which kinds of superstable behavior occur in continuous families is strongly constrained, rather in the way that the order in which a continuous real-valued function takes on rational values is constrained. Furthermore, any full family traverses the full range of kinds of superstable behavior, i.e., every sequence which is realizable at all is realized in every full family (and in particular in the family $1-\mu x^2$). Numerical computations suggest that the family $1-\mu x^2$ traverses the ordered set of kinds of superstable behavior monotonically, but this has not been proved for this (or any other) family.

Since the criterion for realizability and the definition of the ordering can easily be stated explicitly, we give them here. We first define a linear ordering on the set of <u>all</u> finite sequences of L's and R's which, restricted to the subset of realizable sequences, gives the desired ordering. The criterion for realizability will then be stated in terms of the ordering.

The order is a twisted lexicographic order. Let $\underline{A} = (a_1,\dots,a_n)$ and $\underline{B} = (b_1,\dots,b_m)$ be two distinct finite sequences of L's and R's, and let i-1 be the length of their longest common initial segment. Thus $a_1 = b_1,\dots,a_{i-1} = b_{i-1}$, and either $\underline{A}$ or $\underline{B}$ has length i-1 or $a_i \neq b_i$. We define $\underline{A} < \underline{B}$ to mean either

i) $a_i = L$ or $b_i = R$ or both, if there are an even number of R's in $a_1,\dots,a_{i-1}$

or

ii) $a_i = R$ or $b_i = L$ or both, if there are an odd number of R's in $a_1,\dots,a_{i-1}$.

Then :

A sequence is realizable if and only if it majorizes, in this order, each of its terminal sequences.

Thus, for example, to check whether RLRLL is realizable or not, it is only necessary to check the four relations :

$$\text{LRLL} < \text{RLRLL} \ ; \ \text{RLL} < \text{RLRLL} \ ; \ \text{LL} < \text{RLRLL} \ ; \ \text{L} < \text{RLRLL} \quad .$$

(As it happens, the second is false so the sequence is not realizable).

§ 6. The doubling operator

In this section we develop yet another approach to period doubling adapted to the analysis of the accumulation of repeated doublings. We assume, in this and the following section, that co-ordinates can be and have been chosen so that

The interval on which our transformations act is $[-1,1]$; each transformation has its critical point at zero, which it maps to one ; each transformation is even.

Consider, now, such a transformation which is superstable of period 2, i.e., such that $\psi^2(0) = 0$. Since 0 is a critical point for ψ , ψ^2 is contractive at 0 and so it is possible to find a small closed interval J_0 about 0 which is mapped by ψ^2 into its own interior. If we let $J_1 = \psi(J_0)$, so J_1 is a small closed interval ending at 1 , then ψ sends J_0 onto J_1 and J_1 back into J_0 . Because $\psi^2(J_0)$ is contained in the interior of J_0 , this picture persists if ψ is perturbed slightly.

We will examine what happens when ψ is perturbed in such a way that $\psi^2(0)$ moves to the left of zero, i.e., so that $a \equiv -\psi(1)$ becomes slightly positive. The smallest interval symmetric about 0 which could be mapped into itself by ψ^2 is then $[-a,a]$, and a slightly more careful version of the argument in the preceding paragraph shows that this interval is indeed mapped to itself provided that ψ is sufficiently near to a fixed superstable mapping. The corresponding J_1 is the interval $[b,1]$ where $b = \psi(a)$. As ψ has a single maximum in J_0 and is monotonically decreasing on J_1 , $\psi\circ\psi$ has a single critical point in J_0 which is a minimum. In other words : Except for a reversal of orientation and a change of scale, $\psi\circ\psi$ restricted to $[-a,a]$ looks just like one of the transformation we have been studying. To bring it into standard form we make the change of variables $x \longrightarrow -ax$; in terms of the new variable the restricted $\psi\circ\psi$ becomes

$$-\frac{1}{a}\,\psi\circ\psi(-ax)$$

which is unimodal, maps $[-1,1]$ to itself, and is correctly normalized. We denote the rescaled $\psi\circ\psi$ by $T\psi$, and we call T the <u>doubling operator</u>. The idea to be developed in this and the following section is that the investigation of the doubling operator as a nonlinear mapping of the space of transformations into itself is a powerful way to obtain information about one parameter families of transformations.

A careful definition of the domain of the doubling operator includes the specification that the intervals $[-a,a]$ and $[b,1]$ which are interchanged by ψ are non-overlapping. Since $T\psi$ is simply $\psi\circ\psi$ restricted to $[-a,a]$, re-expressed in more convenient coordinates, it is easy to deduce properties of ψ from properties of $T\psi$. For example : If $T\psi$ has an attracting orbit of period p , then $\psi\circ\psi$ has an attracting orbit of period p in $[-a,a]$ which constitutes half of an attracting orbit of period 2p for ψ itself, the other half lying in $[b,1]$. Similarly, as $T\psi$ undergoes a bifurcation from period p to period 2p , ψ itself undergoes a bifurcation from period 2p to period 4p .

The first important fact about the doubling transformation is : Let ψ_μ be a full family of unimodal transformations defined for μ in the parameter interval M_0 . Then there is a subinterval M_1 such that the family $T\psi_\mu$, $\mu \in M_1$ is again full. The proof of this result is quite simple and most of the ingredients have already been given. The details may be found in Section 3 of Collet, Eckmann, Lanford [3] . It has, as a consequence, that the full sequence of unavoidable periodic regimes occurs for $T\psi_\mu$ as μ runs through the parameter subinterval M_1 and hence <u>occurs for ψ_μ itself with all periods doubled</u>. Furthermore, the above result can be applied recursively : there is a decreasing sequence of parameter intervals $M_1 \supset M_2 \supset M_3 \supset \ldots$ such that the family $T^j\psi_\mu$, $\mu \in M_j$ is full. As μ runs through M_j ; each family ψ_μ runs through a sequence of periodic regimes with periods 2^j times the periods of the sequence of periodic regimes unavoidable in full families. Since every full family contains, in particular, a periodic regime of period 2, this shows immediately that every full family has a sequence of periodic regimes of periods 2,4,8,16,32,.. . It is also easy to see that, once due allowance is made for the possibility that the family is not monotone, this sequence of regimes occurs essentially in order and essentially at the beginning of the sequence of unavoidable periodic regimes. (These facts can also be proved combinatorially, using the machinery described in the preceding section).

§ 7. Feigenbaum theory

We are now prepared to take up the questions of the universality of the rate of accumulation of period doubling bifurcations. A unimodal transformation sending the critical point 0 to 1 has a unique fixed point in $[0,1]$; we will denote this fixed point by x_0 . Let Σ_0 denote the set of unimodal transformations such that $\psi'(x_0) = -1$; Σ_0 is a smooth surface of codimension one in the space of all unimodal transformations. As we have seen, a one-parameter family of mappings which crosses this surface in the right direction undergoes a period-doubling bifurcation from period 1 to period 2 , i.e. Σ_0 is the bifurcation surface for this period-doubling bifurcation. If $\mathcal{T}\psi \in \Sigma_0$, then $\mathcal{T}\psi$ is undergoing a bifurcation from period 1 to period 2 , so ψ is undergoing a bifurcation from period 2 to period 4 . In other words, the pre-image of Σ_0 under $\mathcal{T}$, which we denote by Σ_1 , is the bifurcation surface for the bifurcation from period 2 to period 4 . In general, we write

$$\Sigma_{n+1} \quad \text{for} \quad \mathcal{T}^{-1}\,\Sigma_n \quad , \; n = 1,2,\ldots$$

then Σ_n is the bifurcation surface for the bifurcation from period 2^n to period 2^{n+1} . To understand the accumulation of period doubling bifurcations, we want to see what the surfaces Σ_n do for large n .

M. Feigenbaum, who discovered the universal rate of accumulation of period doublings, also proposed a very elegant explanation for this phenomenon*. His explanation took the form of a set of conjectures about the doubling operator together with a simple geometrical argument deriving the universality of the rate of accumulation of period doublings from these conjectures. Feigenbaum did not prove his conjectures, although he did perform extensive and careful numerical verifications which strongly supported them. We will take up later the status of their proof, but first we state the conjectures (stripped of technical qualifications) and discuss their consequences.

The conjectures are :

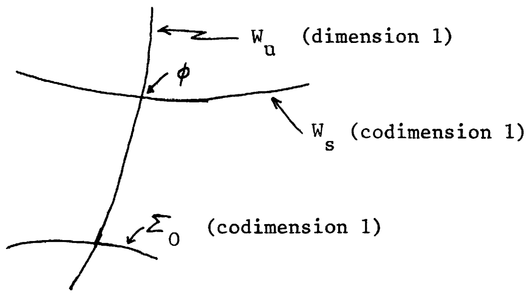
1. $\mathcal{T}$ has a fixed point ϕ .
2. The derivative $D\mathcal{T}(\phi)$ of $\mathcal{T}$ at ϕ is hyperbolic with one-dimensional unstable subspace. (In other words, $D\mathcal{T}(\phi)$ has a simple eigenvalue δ with modulus greater than one and the remainder of its spectrum is contained in the open unit disk.) Moreover, the large eigenvalue δ is real and positive.

It follows from 2. and standard invariant manifold theory that $\mathcal{T}$ has, at ϕ , a stable manifold W_s of codimension one and an unstable manifold W_u of dimension one.

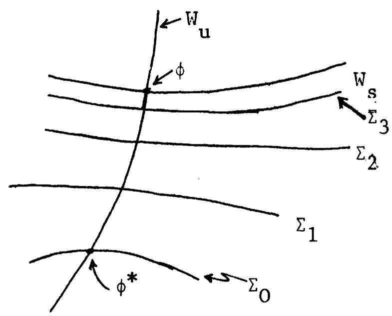
3. The unstable manifold crosses the codimension-one surface Σ_0 (the bifurcation surface for the bifurcation from period 1 to period 2) transversally. The crossing point will be denoted by ϕ^* .

The geometry of the space of transformations can thus be represented by the following sketch :

* A similar proposal was made by P. Coullet and J. Tresser [12].



We now add to the sketch the surfaces $\Sigma_1, \Sigma_2, \ldots$, which are given recursively by $\Sigma_{n+1} = T^{-1} \Sigma_n$. Since T contracts in the W_s direction and expands in the W_u direction, these surfaces must look like :



The surfaces Σ_n converge to W_s . Furthermore, it is not hard to see that the separation between Σ_n and W_s is asymptotic for large n to δ^{-n} times an n-independent function of position on W_s (where δ still denotes the expanding eigenvalue of $DT(\phi)$.) What is essential here is that the rate of convergence of Σ_n to W_s is independent of position on W_s .

In terms of this geometry, the universal rate of accumulation of period doublings is easy to understand. A parametrized family ψ_μ of transformations may be regarded as a curve in the above sketch. Assume that this curve crosses W_s with non-zero transverse velocity. (As the stable manifold has codimension one, crossing it is at least not exceptional. How likely crossing is will depend on how far the stable manifold extends). The curve must then cross the Σ_n for all sufficiently large n , in order. Since we have already argued that Σ_n is the bifurcation surface for bifurcation from period 2^n to 2^{n+1} , the parameter value at which ψ_μ crosses Σ_n is μ_n (and the parameter value at which ψ_μ crosses W_s is μ_∞). In view of the way the Σ_n converge to W_s , it is immediate that

$$\mu_\infty - \mu \sim \text{const. } \delta^{-n} \quad .$$

Thus, any one-parameter family of mappings which crosses W_s with non-zero transverse velocity undergoes a cascade of period doublings with asymptotic ratio

$$\lim_{n\to\infty} \frac{\mu_\infty - \mu_n}{\mu_\infty - \mu_{n+1}} = \delta \quad .$$

The empirical asymptotic ratio 4.6692... found by calculating the μ_n's for the family $1 - \mu x^2$ or another similar family is thereby identified with the large eigenvalue of the derivative of T at the fixed point ϕ . Both the empirical asymptotic ratio and the eigenvalue δ can be computed numerically, and agree to at least twelve digits.

Feigenbaum's argument implies a number of other regularities in the accumulation of period doublings. For example, let $\tilde{\Sigma}_0$ denote the codimension-one surface of transformations ψ which are superstable of period 3 . If $\tilde{\Sigma}_0$ also intersects W_u transversally then the surfaces $\tilde{\Sigma}_n = T^{-n}\tilde{\Sigma}_0$ also converge geometrically with asymptotic ratio δ . If $\tilde{\mu}_n$ is the value of μ near μ_∞ where ψ_μ crosses $\tilde{\Sigma}_n$ (defined for sufficiently large n), then, again

$$\tilde{\mu}_n - \mu_\infty \sim \text{const.} \times \delta^{-n} \quad .$$

To interpret this result, note that, as $\psi_{\tilde{\mu}_n} \in \tilde{\Sigma}_n$, $T^n\psi_{\mu_n} \in \tilde{\Sigma}_0$, i.e. $T^n\psi_{\mu_n}$ is superstable of period 3 ,i.e. $\psi_{\tilde{\mu}_n}$ is superstable of period $2^n \cdot 3$. Thus, μ_∞, which is approached on one side by a sequence of periodic regimes of period 2^n, is approached on the other by a sequence of periodic regimes of period 3.2^n. There are in fact infinitely many other such "descending cascades" of periodic regimes with periods of the form $2^n \cdot p$, all converging to μ_∞ geometrically with asymptotic ratio δ .

To end this - nonexhaustive - list of universal features of one-parameter families at period-doubling accumulation points implied by the Feigenbaum conjectures , note that, since T is contracting in the W_s direction, we have

$$\lim_{n\to\infty} T^n\psi_{\mu_\infty} = \phi$$

$$\lim_{n\to\infty} T^n\psi_{\mu_n} = \phi^* \quad \text{(the point where } W_u \text{ crosses } \Sigma_0) \quad .$$

In particular, the limits on the left are independent of the choice of one-parameter family ψ_μ provided only that this family crosses W_s transversally.

The Feigenbaum conjectures are, then, completely successful in accounting for a number of phenomena which have not been explained otherwise. To what extent can they be proved ? It is in fact not hard to see that they cannot be true without some qualifications about the regularity of the mappings involved. If, for example, ψ_{μ_∞} is smooth except at 0 but has the form $1-\text{const.}|x|^{1+\varepsilon} + o(|x|^{1+\varepsilon})$ near 0 , then $T^n\psi_{\mu_\infty}$ has this same form near 0 for all n and so cannot be expected to converge to a fixed point ϕ with a different behavior at 0 . This heuristic argument is borne out by numerical investigations which show that the rate of accumulation of period doublings for the families $1-\mu|x|^{1+\varepsilon}$ varies with ε . Universality can hold only within classes of unimodal transformations with the same regularity at the critical point. There is not a single set of Feigenbaum conjectures, but many, depending on choice of function space.

One way to proceed is to consider the space of all unimodal mappings which can be written in the form

$$\psi(x) = f(|x|^{1+\varepsilon})$$

with f analytic in a complex neighborhood of $[0,1]$ and with $f'(0) < 0$. The spaces for different ε's are disjoint, and each is mapped into itself by the doubling operator T . The Feigenbaum conjectures can hence be investigated in any one of these spaces. Evidently, the value $\varepsilon = 1$ is of particular importance as each corresponding ψ is analytic and has a non-degenerate critical point. The case which has proved to be easiest to treat, however, is that of small positive ε . In that limit, a complete proof has been given by Collet Eckmann, and myself [3] using a perturbation analysis.

The situation for $\varepsilon = 1$ is less satisfactory, although there is not much question about the correctness of the conjectures. A proof of the first of them has been given by Campanino, Epstein and Ruelle [1] , and I have now essentially completed a programm which, unless something unexpectedly goes wrong in the verification of the last estimates, should prove the second conjecture and a weakened but adequate version of the third. The proofs - especially my own - are however laborious and not very enlightening ; it really seems as though something crucial is being overlooked.

Let us discuss only the first conjecture. What is asserted is the existence of a solution ϕ to the functional equation

$$\phi(x) = -\frac{1}{\lambda}\,\phi(\phi(\lambda x)) \quad , \qquad \lambda = -\phi(1) \qquad .$$

The solution ϕ is to be defined, unimodal, even, and fairly smooth on [-1,1] , and to satisfy :

$$\phi(0) = 1 \quad ; \quad \phi(1) < 0 \quad ; \phi(\lambda) \geq \lambda \ ; \quad \phi''(0) < 0 \quad .$$

Almost nothing is known - or even plausibly conjectured - about uniqueness, except that it is comparatively easy to produce a great many (irrelevant) solutions which are once but not twice continuously differentiable. It is quite possible that there is only one twice differentiable solution. The solution which has been found is actually analytic in quite a large domain in the complex plane, containing the whole real and imaginary axes ; its MacLaurin expansion is

$$\phi(x) = 1 - 1.5276\ldots\ x^2 + .1048\ldots\ x^4 + .0267x^6 + \ldots$$

It turns out to be remarkably easy to find, with the aid of a computer, polynomials which are excellent approximate solutions to the functional equation. My investigation uses a completely straightforward approach : The computer is used to verify, with strict error estimates, enough detailed properties of T and DT in an extremely small neighborhood of an explicit approximate solution to show that a variant of Newton's method converges to a nearby exact solution.

Campanino, Epstein, and Ruelle use a much more sophisticated scheme to search for a solution, but still need to make extensive numerical and algebraic computations to localize the solution in a small region of function space before they can apply their procedure. In both cases, the proofs rest on long and relatively blind computations which could perfectly well, so far as one can see without actually doing them, have come out differently. It think it is fair to say that, although we know <u>that</u> a solution exists, we don't at all understand <u>why</u> it must exist. In view of the simplicity of the functional equation, this seems a most unsatisfactory state of affairs.

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