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A. TOGNOLI

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ALGEBRAIC APPROXIMATION OF MANIFOLDS AND SPACES

by A. TOGNOLI

Introduction

We can state the two following informal problems :

I - given a compact C^∞ manifold M is it possible to induce on M a real algebraic structure M_a such that the geometry of M_a can be described by algebraic elements ? (for example, such that any $\alpha \in H_p(M_a, \mathbb{Z}_2)$ can be represented by an algebraic cycle).

II - characterize the topological spaces that are homeomorphic to a singular real algebraic variety.

The main progress in the study of these problems can be summarized as follows : Seifert studied problem I in the case of the complete intersections (1936), Whitney showed (in the analytic case) that problem I could be treated also in the general case (1936), Nash gave a partial solution of the problem using the Whitney methods in the real algebraic case (1952), Wallace demonstrated that any compact manifold, which is a boundary, has an algebraic structure (1957), we proved that any compact C^∞ manifold has algebraic structure (1973).

Now we have also some information about particular algebraic structures, in which a part of the geometric invariants of M_a are algebraic (see remark 2, 3 of section g), but a lot of questions are yet open in the area of problem I .

Problem II was studied by Kuiper who proved that any 8-dimensional, P.L. manifold, has an algebraic structure (1968). Kuiper's method was refined by Akbulut (1977). Finally Akbulut and King gave a beautiful and complete characterisation of the topology of the real algebraic isolated singularities (1978).

If the singularities are not isolated very little is known. All the results contained in this exposition, except remark 1 of section g, appeared before (at least as preprint).

a. Definitions and preliminary results

In the following by algebraic variety we shall mean : real, affine reduced algebraic variety $(V, \mathcal{O}_V)$ in the sense of F.A.C.. The sheaf $\mathcal{O}_V$ is often omitted in the notations because we consider only reduced structures.

The morphisms of algebraic varieties are called algebraic (or regular) maps.

An algebraic variety V of $\mathbb{R}^n$ is called regular in x if, near x, V is described by polynomial equations $f_1 = \dots = f_q = 0$, $q = n - \dim V$, and $(df_1)_x \dots (df_q)_x$ are linearly independent.

V is called regular (or a manifold) if it is regular at any point.

For any $n,q \in \mathbb{N}$ we shall denote by $G_{n,q}$ the Grassmann manifold of the q-dimensional linear subspaces of $\mathbb{R}^n$. Let us denote by $\gamma_{n,q} = \{(a,b) \in G_{n,q} \times \mathbb{R}^n \mid a \ni b\} \xrightarrow{\omega} G_{n,q}$ the tautological bundle.

$G_{n,q}$ and $\gamma_{n,q}$ shall be considered with the usual projective (and hence affine) structure.

Let now $d' : G_{n,q} \times G_{n,q} \longrightarrow \mathbb{R}$ be a metric inducing the usual topology and d the euclidean metric of $\mathbb{R}^n$.

DEFINITION 1.- Let W be a closed C^∞ submanifold of $\mathbb{R}^n$, V a C^∞ submanifold of W and X a subset of V.
Given $\varepsilon > 0$, we shall say that the C^∞ submanifold V' of W is an ε-approximation of V, relative to X, in W, if there is a diffeomorphism $h : V \longrightarrow V'$ such that :

(i) $d(x,h(x)) < \varepsilon$, $x \in V$

(ii) $d'(T_{V_x}, T_{V'_{h(x)}}) < \varepsilon$, $x \in V$ where T_{V_x}, $T_{V'_{h(x)}}$ are the tangent varieties to V, V' in x and $h(x)$

(iii) $X = X'$ and $h_{|X} = \mathrm{id}$.

DEFINITION 2.- In the above situation if we replace the condition (iii) by the weaker one :

(iv) $h : X \longrightarrow X'$ is a homeomorphism

we shall say that the pair (V',X') is an ε-approximation of (V,X) in W.

In the above definitions (V',X') shall be called a (regular) algebraic ε-approximation if V', X' are (regular) algebraic varieties.

We shall say that (V,X) has algebraic approximation if for any $\varepsilon > 0$ it has an ε-approximation.

A similar terminology shall be used for approximation of maps.

DEFINITION 3.- An algebraic subvariety X of $\mathbb{R}^n$ is called quasi-regular if

(i) for any $x \in X$, the ideal $I_{X,x}$ of the germs of analytic functions vanishing on X_x is generated by $I_X = \{P \in \mathbb{R}[X_1 \dots X_n] \mid P|_X \equiv 0\}$.

DEFINITION 4.- Let V be a C^∞ manifold and $\{S_i\}_{i=1,\dots,q}$ a finite family of closed submanifolds.

We shall say that the $\{S_i\}$ are in <u>general position</u> if for any subset $i_o,\dots,i_t$ of $1,\dots,q$ we have that S_{i_o} cuts transversally $\bigcap_{j=1}^{t} S_{i_j}$ and for any $x \in (\bigcap_{j=0}^{p} S_{i_j}) \cap (\bigcap_{j=p+1}^{t} S_{i_j})$ the germ of $\bigcap_{j=0}^{t} S_{i_j}$ is transverse to the germ of $\bigcap_{j=p+1}^{t} S_{i_j}$ in the tangent space of the union.

The definition 3 is justified by the following

THEOREM 1.- <u>Let</u> X <u>be a compact, quasi-regular algebraic subset of the</u> (Zariski) <u>open set</u> U <u>of</u> $\mathbb{R}^n$.

<u>Let</u> $f : U \longrightarrow \mathbb{R}$ <u>be a</u> C^∞ <u>function such that</u> $f|_X$ <u>is regular</u> (polynomial).

<u>Then</u> f <u>can be approximated, in the usual</u> C^∞ <u>topology, by regular</u> (polynomial) <u>functions</u> f_λ <u>such that</u> $f_\lambda|_X = f|_X$. <u>If</u> X <u>and</u> $f|_X$ <u>are defined on the subfield</u> K <u>of</u> $\mathbb{R}$ <u>then we can chose the</u> f_λ <u>defined on</u> K .

Theorem 1 is proved in [16].

<u>Remark</u> 1.- It is easy to prove that X is quasi-regular if, and only if, it is coherent and for any $x \in X$ the analytic complexification of X coincides with the germ induced by the algebraic complexification of X .

From the above criterion we deduce that X is quasi-regular if and only if the property is true for any irreducible component of X . It follows that any finite union of regular algebraic varieties is quasi-regular.

<u>Remark</u> 2.- By a result of Malgrange and ideal of $C^\infty(U)$ that is generated by analytic functions is closed ; hence the hypothesis " X quasi-regular" is necessary to obtain the result of theorem 1.

DEFINITION 5.- Let $(V, \mathcal{O}_V)$ be an algebraic variety and $F \longrightarrow V$ an algebraic vector bundle (where F is an abstract real algebraic variety). F is called <u>strongly algebraic</u> if there exists a regular map $\varphi : V \longrightarrow G_{n,q}$ such that $F = \varphi^*(\gamma_{n,q})$, where $\gamma_{n,q}$ is the universal bundle.

In general a coherent sheaf of $\mathcal{O}_V$ modules $\mathcal{F}$ is called A-<u>coherent</u> if there exists an exact sequence

$$\mathcal{O}_V^p \longrightarrow \mathcal{O}_V^q \longrightarrow \mathcal{F} \longrightarrow 0$$

DEFINITION 6.- Let V be a C^∞ manifold (or an algebraic manifold) and $F \longrightarrow V$ a C^∞ (or strongly algebraic) vector bundle. A C^∞ (algebraic) submanifold S of V is called a <u>weak</u> C^∞ (algebraic) <u>complete intersection</u> in V (respect to F) if there exists a C^∞ (algebraic) section $\gamma : V \longrightarrow F$ such that :

(i) $S = \{x \in V \mid \gamma(x) = 0\}$ and γ is transverse to the zero section.

If F is the trivial bundle S is a complete intersection.

Let V be a C^∞ manifold and $\alpha \in H_p(V,\mathbb{Z}_2)$; it is known (see [15]) that there exists a C^∞ compact manifold M and a C^∞ map $\varphi : M \longrightarrow V$ such that

(1) $$\alpha = \varphi_*(\text{fundamental class of } M).$$

DEFINITION 7.- Let V be an algebraic variety, $\alpha \in H_p(V,\mathbb{Z}_2)$ is called algebraic if it is possible to find a regular connected algebraic variety M and a regular map $\varphi : M \longrightarrow V$ such that (1) holds.

If $H_p(V,\mathbb{Z}_2)$ has algebraic generators then it is called algebraic. If any $H_p(V,\mathbb{Z}_2)$ is algebraic we shall say that the homology of V is algebraic.

Let V be a C^∞ manifold, we shall denote by $\mathfrak{N}_q(V)$ the q-bordism group of the classes of non-oriented maps $\psi : M \longrightarrow V$, $q = \dim M$.

DEFINITION 8.- Let V be an algebraic variety, $\alpha \in \mathfrak{N}_q(V)$ shall be called algebraic if the class α has an algebraic representative $\psi : M \longrightarrow V$ (i.e. M is regular algebraic and ψ an algebraic map).

We shall use a similar terminology for $\mathfrak{N}_q(V)$ and $\mathfrak{N}_*(V) = \oplus \mathfrak{N}_q(V)$.

b. Strongly algebraic vector bundles

Let us consider the polynomial $\varphi(x_1 \ldots x_n) = x_1^2(x_1 - 1)^2 + \sum_{i=2}^{n} x_i^2$ $n \geq 2$, it is easy to prove that φ is irreducible and $\{\varphi = 0\} = \{0,\ldots,0\} \cup \{1,0,\ldots,0\}$. Let $a = \{0,\ldots,0\}$, $b = \{1,0,\ldots,0\}$, $U_1 = \mathbb{R}^n - a$, $U_2 = \mathbb{R}^n - b$.

In [16] it is proved that :

(i) the cocycle $\frac{1}{\varphi} : U_1 \cap U_2 \longrightarrow \mathbb{R}$ defines a non-zero element of $H^1(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n})$

(ii) the line bundle $F \to \mathbb{R}^n$ defined by $\frac{1}{\varphi}$ is not trivial (considered as algebraic vector bundle) and it is not strongly algebraic

(iii) the global algebraic sections of F do not generate the fiber of F at any point.

To avoid the above pathologic examples we gave the notion of strongly algebraic vector bundle. The results contained in this section shall be used in the next section to reduce the approximation problem to a simpler one. The following proposition, (see [5]), is useful to handle the definition of strongly algebraic vector bundle :

PROPOSITION 1.- Let V be an algebraic vatiety and $F \longrightarrow V$ an algebraic vector bundle of rank k , where F is an algebraic abstract variety. The following conditions are equivalent :

(i) F is stongly algebraic

(ii) there exists a regular embedding of fiber bundles $F \longrightarrow V \times \mathbb{R}^n$ $n > \dim V + k$

(iii) the sheaf $\mathcal{F}$ of algebraic sections of F is A-coherent

(iv) the abstract variety F is affine

(v) F is an algebraic subbundle of $V \times R^n$ and F has a strongly algebraic complement $F^{\perp}$.

Proof.- (i) $\Rightarrow$ (iv) . Let $g : V \longrightarrow G_{n,k}$ be the regular map such that $F = g^*(\gamma_{n,k})$. By construction we have $g^*(\gamma_{n,k}) = \{(x,y) \in V \times \gamma_{n,k} \mid g(x) = \omega(y)\}$ hence $g^*(\gamma_{n,k})$ is an affine variety.

(iii) $\Rightarrow$ (ii) . By hypothesis we have an exact sequence of sheaves :

$$(1) \qquad \mathcal{O}_V^p \longrightarrow \mathcal{O}_V^q \longrightarrow \mathcal{F} \longrightarrow 0$$

Using the duality of Grauert-Grothendieck the sequence (1) gives an exact sequence of bundles :

$$(2) \qquad 0 \longrightarrow F \longrightarrow V \times R^q$$

this proves the thesis.

(iv) $\Leftrightarrow$ (iii) is proved in [11].

(ii) $\Rightarrow$ (v) . The embedding $j : F \longrightarrow V \times R^n$ defines two maps : $g : V \longrightarrow G_{n,k}$, $g' : V \longrightarrow G_{n,n-k}$ given by : $g(x) = j(\pi^{-1}(x))$, $g'(x) = g(x)^{\perp}$.

Clearly $F = g^*(\gamma_{n,k})$, $F^{\perp} = g'^*(\gamma_{n,n-k})$, hence it is enough to prove that g and g' are regular maps.

To verify that g and g' are regular it is enough to remember the following facts :

(1) the problem is local, hence F can be considered trivial

(2) the embedding $F \longrightarrow V \times R^n$ is algebraic

(3) we go from g to g' using the Gramm-Schmidt process and this is algebraic.

(v) $\Rightarrow$ (i) . It is clear from the construction given in (ii) $\Rightarrow$ (v) .

The proposition is now proved.

PROPOSITION 2.- Let V be a compact algebraic variety and $F \longrightarrow V$ a strongly algebraic vector bundle. Let $X \subset V$ be a quasi-regular algebraic set and $\gamma : V \longrightarrow F$ a C^{∞} section such that $\gamma_{|X}$ is algebraic.

In these hypotheses γ can be approximated, in the usual C^{∞} topology, by algebraic sections $\gamma_{\lambda} : V \longrightarrow F$ such that : $\gamma_{\lambda|X} = \gamma_{|X}$.

Proof.- From the condition (v) of proposition 1 we know that F has n algebraic sections $\gamma_1,\dots,\gamma_n$ such that $\gamma_1(x),\dots,\gamma_n(x)$ generate the fiber for any $x \in V$. Let $\mathcal{F}$ be the sheaf of the germs of algebraic sections of F , we have that $\mathcal{F}$ is A-coherent and hence there is an exact sequence $\mathcal{O}^n_{V|X} \longrightarrow \mathcal{F} \longrightarrow 0$.

The functor Γ of the global sections is exact ([16] pag. 43), hence we have : $\gamma(x) = \sum_i \alpha_i(x)\gamma_i(x)$, $x \in X$, where α_i are regular functions on X .

Again from the exactness of Γ we may suppose there exists $\beta_i \in \Gamma(\mathcal{O}_V)$ such that $\beta_{i|X} = \alpha_i$.

Let us consider the section $\beta = \gamma - \sum_i \beta_i \gamma_i$.
To prove the proposition it is enough to approximate β by β_λ algebraic such that $\beta_{\lambda|X} \equiv 0$.
Locally we can write $\beta = \sum_i \delta_i \gamma_i$, where δ_i are C^∞ functions, such that $\delta_{i|X} \equiv 0$.
By a partition of unity we may suppose to have globally $\beta = \sum_i \delta_i \gamma_i$.
We can now apply theorem 1 to approximate the δ_i by regular functions δ_i^λ such that : $\delta^\lambda_{i|X} = 0$.
Clearly $\beta_\lambda = \sum \delta_i^\lambda \gamma_i$ approximates β and proves the proposition.

COROLLARY 1.- <u>Let</u> V <u>be a compact algebraic variety and let</u> $F_i \longrightarrow V$, $i = 1, 2$ <u>a couple of strongly algebraic vector bundles</u>. <u>If</u> F_1 <u>is topologically isomorphic to</u> F_2 <u>then there exists an algebraic isomorphism of vector bundles</u> $\psi : F_1 \longrightarrow F_2$.

<u>Proof</u>.- It is enough to verify that we can apply the result of proposition 2 to the vector bundle $\mathrm{Hom}(F_1,F_2) \overset{\mathrm{def}}{=} H$. From the condition (iii), of proposition 1, it follows that H is a strongly algebraic vector bundle and hence the corollary is proved.

<u>Remark</u> 1.- The category of strongly algebraic vector bundles on X is dual to the category of locally free A-coherent sheaves. We recall that the category of the A-coherent sheaves on V is isomorphic to the category of coherent sheaves on Spec $\Gamma(\mathcal{O}_V)$ (see [16]).

c. <u>A reduction of the approximation problem</u>

Let $\mathbb{R}^n \supset V \supset M \supset X$ where V is an algebraic compact subvariety of $\mathbb{R}^n$, M a closed C^∞ submanifold of V and X a closed set. We have

<u>Problem</u> 1.- When has M algebraic approximation in V , relatively to X ?

Problem 1 seems very difficult and we are able to give some answers only in the case : M is a weak complete intersection and X is a quasi-regular algebraic set. Some results are contained in remark 4 of section g.

We may consider the simpler

<u>Problem</u> 2.- When has M algebraic approximation in $\mathbb{R}^n$, relatively to X ?

We shall see that, in general, problem 2 has not a positive answer, also in the case X is a regular algebraic subvariety. But we shall prove that the <u>pair</u> (M,X) <u>has algebraic approximation in</u> $\mathbb{R}^n$ <u>if</u> $n > 2 \dim M$ <u>and</u> X <u>is a finite union of</u> C^∞ <u>submanifolds in general position</u>.

In the remaining of this section, following [6], we shall study a reduction of problem 1.

We consider the above situation $\mathbb{R}^n \supset V \supset M \supset X$, $v = \dim V$, $m = \dim M$.

Let $\varphi_M : V \longrightarrow G_{n,v-m}$ be a C^∞ map such that : $T_{V_x} = \varphi_M(x) \oplus T_{M_x}$, where T_{V_x} , T_{M_x} are the tangent varieties to V , M .

Let $\rho_V : U_V \longrightarrow V$ be a retraction of a tubular neighbourhood of V in R^n onto V and : $\varphi_M^*(\gamma_{n,v-m}) = \{(x,y,z) \in M \times G_{n,v-m} \times R^n \mid \varphi_M(x) = y , z \in y\}$. The map $p' : \varphi_M^*(\gamma_{n,v-m}) \longrightarrow R^n$ defined by $p'(x,y,z) = x+y$ gives, by the implicit function theorem, a C^∞ isomorphism $q = \rho_V \circ p' : U_{Z_o} \longrightarrow U_M^V$ between a neighbourhood U_{Z_o} of the zero section Z_o of $\varphi_M^*(\gamma_{n,v-m})$ and a neighbourhood U_M^V of M in V . The vector bundle $\gamma_{n,v-m} \longrightarrow G_{n,v-m}$ has a strongly algebraic structure, hence there exists a strongly algebraic vector bundle F' such that $\gamma_{n,v-m} \oplus F' \approx G_{n,v-m} \times R^n$ where the isomorphism is algebraic.

It follows that we have two algebraic maps : $\gamma_{n,v-m} \xrightarrow{i} G_{n,v-m} \times R^n \xrightarrow{p} \gamma_{n,v-m}$ such that $p \circ i = id$ and i and p respect the fibers.

We can summarize the situation in the following commutative diagram :

$$\begin{tikzcd}
G_{n,v-m} & & & G_{n,v-m} \times R^n \arrow[lll, "\pi"'] \arrow[dll, "\pi'"'] \arrow[d, shift left, "p"] \\
& R^n & \varphi_M^*(\gamma_{n,v-m}) \arrow[r, "\varphi_M^*"] \arrow[d] & \gamma_{n,v-m} \arrow[u, shift left, "i"] \arrow[d] \\
& U_M^V \arrow[uul, "\gamma"] \arrow[u, "\gamma'"] \arrow[ur, "q^{-1}"] & M \arrow[l, hook'] \arrow[r, "\varphi_M"] & G_{n,v-m}
\end{tikzcd}$$

where π , π' are the natural projections and $\gamma = \pi \circ i \circ \varphi_M^* \circ q^{-1}$, $\gamma' = \pi' \circ i \circ \varphi_M^* \circ q^{-1}$.

Now we ramark that :

a) q^{-1} is a diffeomorphism, φ_M^* a bundle map, hence $\sigma = \varphi_M^* \circ q^{-1} : U_M^V \longrightarrow \gamma_{n,v-m}$ is a map transverse to the zero section Z of $\gamma_{n,v-m}$ and we have $M = \sigma^{-1}(Z)$

b) $\gamma_{n,v-m}$ is a strongly algebraic vector bundle, hence if we approximate the map σ by a regular map σ_o , $\sigma_o^{-1}(Z)$ shall be an algebraic approximation of M in U_M^V .

c) If $\sigma_o : U_M^V \longrightarrow \gamma_{n,v-m}$ is an algebraic approximation of σ such that $\sigma_o|_X = \sigma|_X$ then the approximation $M_o = \sigma_o^{-1}(Z)$ is relative to X .

d) To approximate σ it is enough to approximate $i \circ \sigma$ (if β approximates $i \circ \sigma$ then $p \circ \beta$ approximates σ). To approximate $i \circ \sigma$ is equivalent to the problem of the approximation of $\gamma : U_M^V \longrightarrow G_{n,v-m}$, $\gamma' : U_M^V \longrightarrow R^n$. If we wish an approximation relative to X it is enough to approximate γ , γ' by $\tilde{\gamma}$, $\tilde{\gamma}'$ such that $\tilde{\gamma}'|_X \equiv 0$.

e) γ' is a scalar function, hence, by theorem 1, can be approximated if X is quasi-regular.

So the main problem is to approximate γ.

DEFINITION 9.- In the above situation γ and γ' are called the <u>equations</u> of M in U.

The above definition is justified by the relation :
$M = \{x \in U_M^V \mid \hat{\gamma}(x)(\gamma'(x)) = 0\}$, where $\hat{\gamma}(x)$ is the linear projection of R^n onto $\gamma(x)$.

So we have obtained the following :

THEOREM 2.- *Let us consider $R^n \supset V \supset M \supset X$ where V is a compact algebraic subvariety of R^n, M is a C^∞ submanifold of V and X a subset of M. A sufficient condition to have algebraic approximation of M, relatively to X in an open neighbourhood U_M^V of M in V, is the possibility of approximating a pair of C^∞ equations γ, γ' by algebraic equations $\tilde{\gamma}$, $\tilde{\gamma}'$ such that $\tilde{\gamma}'|_X \equiv 0$.*

<u>Remark</u> 2.- Theorem 2 proves that problem 1, in the category of real analytic manifolds or compact Nash manifolds can always be solved if X is a coherent subset (see [6]).

d. Some examples

From the reduction theorem of section c we are induced to study the following problem : let $\varphi : V \longrightarrow W$ be a C^∞ map where V and W are algebraic varieties. When can φ be approximated by algebraic maps ?

A very important particular case of the above question is $W = G_{n,q}$. To study some examples of maps $\varphi : V \longrightarrow W$ that have no algebraic approximation it is useful to state :

PROPOSITION 3.- *Let V be a regular algebraic variety and $\varphi : V \longrightarrow R^t$ an algebraic map such that : $\varphi : V \longrightarrow \varphi(V) \subset R^t$ is an analytic isomorphism. Under these hypotheses there exists an algebraic subvariety $\hat{V}$ of R^t such that : $\hat{V} \supset \varphi(V)$, $\hat{V} - \varphi(V)$ is contained in the singular set of $\hat{V}$. If $\varphi : V \longrightarrow W$ is a regular, surjective map between two algebraic, regular, irreducible varieties of the same dimension, then the degree, modulo 2, of φ is constant.*

For the proof see [5]. In [4], [5] are contained the following examples :

<u>Example</u> 1.- Let $V_1 = \{(x,y,t) \in P_2(R) \mid y^2t - x^3 + xt^2 = 0\}$ be the projective cubic considered as affine variety ($P_n(R)$ is isomorphic to a closed algebraic subvariety of R^N ([16])). Let us denote by $V_1 = V_1' \cup V_1''$ the topological decomposition of V_1 in

connected components and by V_2 the unitary circle. Let $\varphi : V_2 \longrightarrow V_1$ a C^∞ map such that $\varphi : V_2 \longrightarrow V_1'$ is a diffeomorphism.

From proposition 3 we deduce that on V_2, considered as C^∞ manifold, doesn't exist an algebraic structure V_2^a, such that φ can be approximated by a regular map $\psi : V_2^a \longrightarrow V_1$.

Example 2.- Let $V_1 = V_1' \cup V_1''$ as before and let $\varphi : V_1 \longrightarrow V_1$ be a C^∞ map such that $\deg \varphi|_{V_1'} = 1$, $\deg \varphi|_{V_1''} = 0$. From proposition 3 we deduce that on V_1, considered as a C^∞ manifold, doesn't exists an algebraic irreducible structure V_1^a such that there is a regular map $\psi : V_1^a \longrightarrow V_1^a$ homotopic to φ.

Example 3.- Let $V_1 = V_1' \cup V_1''$ be as before and $\varphi : V_1 \longrightarrow V_1$ a C^∞ map such that $\varphi(V_1) \subset V_1'$ and $\varphi : V_1' \longrightarrow V_1'$ is a diffeomorphism.

Again from proposition 3 it is clear that the graph $\Gamma_\varphi \subset V_1 \times V_1$ of φ has no algebraic approximation in $V_1 \times V_1$.

Example 4.- Let $\Gamma_\varphi \subset V_1 \times V_1$ be as in example 3 and let $F_\varphi \longrightarrow V_1 \times V_1$ the C^∞ line bundle associated to the divisor Γ_φ. From propositions 2 and 3 it follows that F_φ has no strongly algebraic structure.

The examples show that a C^∞ map $\varphi : V \longrightarrow W$ between algebraic, compact, regular varieties has, in general, no algebraic approximation, even if we change the algebraic stucture on V or $W = G_{n,q}$.

In the next section we shall prove that, given $\varphi : V \longrightarrow W$, if the bordism class of φ is algebraic, then V has an algebraic structure V^a such that $\varphi : V^a \longrightarrow W$ has algebraic approximation.

For this reason it is important to know when $\mathfrak{N}_*(W)$ is algebraic. The following two lemmes can be useful in this direction.

Lemma 1.- *Let* V *be a compact algebraic variety and let us denote by* $\mathfrak{N}_q(V)$ *the unoriented* q-*bordism group.*

If $H_q(V,\mathbb{Z}_2)$ *is algebraic then* $\mathfrak{N}_q(V)$ *is algebraic.*

Lemma 2.- *Let* $G_{n,p}$ *be the Grassmann manifold with the usual algebraic structure, then for any* $t \in \mathbb{N}$, $H_t(G_{n,p},\mathbb{Z}_2)$ *is algebraic.*

To prove lemma 1 it is enough to remember the geometrical interpretation of the isomorphism $\mu : \mathfrak{N}_q(V) \longrightarrow H_q(V,\mathbb{Z}_2)$ defined in [9]. To prove lemma 2 it is enough to remark that Schubert's cycles are algebraic.

The details are in [4] and [5].

e. The approximation theorem

The results of this section are contained in [6].

We wish to prove the following :

THEOREM 3.- Let M be a compact C^∞ submanifold of $\mathbb{R}^n$, $n \geq 2m+1$, $m = \dim M$ and $X \subset M$ a quasi-regular algebraic subset of $\mathbb{R}^n$. Let $\varphi : M \to G = \prod_{i=1}^{q} G_{s_i,n_i} \times H$ be a C^∞ map of M into a product of Grassmann manifolds multiplied by a regular compact algebraic variety H such that $H_*(H,\mathbb{Z}_2)$ is algebraic. Let us suppose $\varphi_{|X}$ is algebraic. Let $\varphi_o : X \to G_{n,n-m}$ be the map $\varphi_o(x)$ = linear variety orthogonal to the tangent space T_{M_x}, and let us suppose φ_o has algebraic approximation. In these hypotheses, for any $\varepsilon > 0$, there exists an algebraic ε-approximation $h : M \to M'$ of M such that :

(i) M' is a regular algebraic subvariety of $\mathbb{R}^n$

(ii) $M' \supset X$ and there is a regular map $\tilde{\varphi} : M' \to G$ such that $\tilde{\varphi}_{|X} = \varphi_{|X}$, $\tilde{\varphi} \circ h$ is an ε-approximation of φ.

The theorem shall be proved by several lemmas.

We start by the :

DEFINITION 10.- Let M be a C^∞ submanifold of $\mathbb{R}^n$ and U a relatively compact open set of M. We shall say that U has Nash proper approximation if for any $\varepsilon > 0$ there exists an algebraic variety $M' \subset \mathbb{R}^n$, a neighbourhood D_U of $\bar{U}$ in $\mathbb{R}^n$ and a diffeomorphism $h : U \to U' \subset M'$ such that : U' is open in M', $U' \subset D_U$, the points of $M' \cap D_U$ are regular and h is an ε-approximation.

Lemma 3.- Let $V \subset \mathbb{R}^n$, $W \subset \mathbb{R}^m$ be algebraic subvarieties of $\mathbb{R}^n$, $\mathbb{R}^m$ and $X \subset V$ be a quasi-regular algebraic compact subset of V. Let $\varphi : V \to W$ be a C^∞ map such that $\varphi_{|X}$ is algebraic and $U \subset\subset V$, $U \supset X$ be an open set such that any point of U and of $\varphi(\bar{U})$ is regular.

In these hypotheses U has a Nash proper approximation $U' \subset \mathbb{R}^n \times \mathbb{R}^m$ such that :

(i) $U' \supset X$

(ii) there is an algebraic map $\varphi' : U' \to W$ such that $\varphi'_{|X} = \varphi_{|X}$

(iii) for any $\varepsilon > 0$, U', φ' can be constructed in such a way that $\varphi' \circ \pi$ is an ε-approximation of φ, where $\pi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is the natural projection.

Proof.- By the theorem 1 we can find a regular map $\psi : \mathbb{R}^n \to \mathbb{R}^m$ such that $\psi_{|X} = \varphi_{|X}$ and ψ approximates φ on the closure $\bar{U}$ of a relatively compact neighbourhood U of X in V.

Let $p : W \to \mathbb{R}^{m^2}$ be defined by $p(x)$ = matrix giving the projection of $\mathbb{R}^m$ the tangent space to W in X. It is known (see [16]) that p is algebraic.

Let now $V' = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^m \mid x \in V\ ,\ \psi(x)+y \in W\ ,\ p(\psi(x)+y)y = 0\}$. Clearly

V' is an algebraic variety, $V' \supset X$, the map $\varphi'(x,y) = \psi(x) + y$ is algebraic and $\varphi'|_X = \varphi|_X$.

If $\psi|_U$ is near enough to $\varphi|_U$, by transversality reasons V' near U is regular and $\pi : \pi^{-1}(U) \cap V' \longrightarrow U$ is a Nash isomorphism (if $\psi(\bar{U})$ is contained in a tubular neighbourhood of $\varphi(\bar{U})$ there is only one "nearest point" to $\psi(x)$ in W).

It is also clear that when ψ approaches φ the map φ' approaches φ. The lemma is now proved.

<u>Lemma</u> 4.- <u>Let</u> $G = \prod_{i=1}^{q} G_{n_i,s_i} \times H$ <u>be as in theorem</u> 3 <u>and</u> M <u>a compact</u> C^∞ <u>manifold</u>. <u>Let</u> $\varphi : M \longrightarrow G$ <u>be a</u> C^∞ <u>map</u> ; <u>then there exists a</u> C^∞, <u>compact manifold</u> W, $W \subset \mathbb{R}^n$, <u>with boundary</u> ∂W <u>and a</u> C^∞ <u>map</u> $\psi : W \longrightarrow G$ <u>such that</u> :

(i) $\partial W = M \cup M'$ and M' is a regular algebraic variety

(ii) $\psi|_{M'}$ is an algebraic map, $\psi|_M = \varphi$.

<u>Proof</u>.- By Künneth formula and lemma 2, $H_*(G,\mathbb{Z}_2)$ is algebraic, hence lemma 4 is a consequence of lemma 1.

<u>Lemma</u> 5.- <u>Let</u> $B \longrightarrow X$ <u>be a</u> C^∞ <u>bundle</u>, <u>where</u> B <u>and</u> X <u>are paracompact</u> C^∞ <u>manifold</u>. <u>Let</u> $S \subset X$ <u>be a closed set and</u> $\gamma : X \longrightarrow B$ <u>a</u> C^∞ <u>section</u>. <u>If the</u> C^∞ <u>section</u> $\gamma' : S \longrightarrow B$ <u>approaches enough</u> $\gamma|_S$, <u>then there exists a</u> C^∞ <u>section</u> $\tilde{\gamma} : X \longrightarrow B$ <u>such that</u> $\tilde{\gamma}|_S = \gamma'$ <u>and</u> $\tilde{\gamma}$ <u>approaches</u> γ.

The above result is a consequence of the fact that the problem of extending a section is a homotopy problem (see [6] for references).

<u>Proof of theorem</u> 3

Let $\psi : W \longrightarrow G$ be the cobordism of $\varphi : M \longrightarrow G$, as defined in the lemma 4. Now we remark that if we embed canonically $\mathbb{R}^n \hookrightarrow \mathbb{R}^n \times \mathbb{R}^p$, any algebraic map $\alpha : X \longrightarrow G_{n,n-m}$ defines an algebraic map $\hat{\alpha} : X \longrightarrow G_{n+p,n+p-m}$ given by $\hat{\alpha}(x) = \alpha(x) \oplus \mathbb{R}^p$, where $\mathbb{R}^p = \{0\} \times \mathbb{R}^p \subset \mathbb{R}^{n+p}$. The same argument can be used for any map $\beta : M \longrightarrow G_{n,n-q}$, so we may consider W embedded in any $\mathbb{R}^{n+p}$. Later we shall suppose $n > 2m + 3$ and, in the end, we shall project the approximating manifold in $\mathbb{R}^{2m+1}$.

Let now T be the double of W ; by standard arguments (see [16]) we may suppose T is realized in $\mathbb{R}^{n+1}$ in such a way that :

(i) $T \cap \{x_{n+1} = 0\} = M \cup M'$ where x_{n+1} is the last coordinate of $\mathbb{R}^{n+1}$

(ii) T near $x_{n+1} = 0$ is a product (there is $\varepsilon > 0$ such that $(\mathbb{R}^n \times]-\varepsilon,+\varepsilon[) \cap T = (M \cup M') \times]-\varepsilon,+\varepsilon[$).

(iii) the map ψ has a C^∞ extension $\tilde{\psi} : T \longrightarrow G$.

Let now $\varphi_T : T \longrightarrow G_{n+1,n-m}$ be defined by :

$\varphi_T(x)$ = linear subvariety orthogonal to the tangent space to T in X. From condition (ii) we deduce that $\varphi_{T|M'}$ is algebraic and, from the hypotheses of the theorem, we know that $\varphi_{T|M}$ has an algebraic approximation $\varphi' : X \longrightarrow G_{n,n-m} \subset G_{n+1,n-m}$. If φ' is near to $\varphi_{T|X}$, in the sense of lemma 5, then there is a C^∞ map $\varphi_o : T \longrightarrow G_{n+1,n-m}$, such that $\varphi_{o|X\cup M'}$ is algebraic and φ_o approximates φ_T .

Let now $\varphi_o \times \widetilde{\Psi} = \rho : T \longrightarrow G_{n+1,n-m} \times G \subset R^p$ and $\widetilde{\rho} : U_T \longrightarrow G_{n+1,n-m} \times G$ a C^∞ extension of ρ to a relatively compact tubular neighbourhood U_T of T in R^{n+1} . Let us apply lemma 3 to the variety R^{n+1} and the map $\widetilde{\rho}$, we find an algebraic subvariety $D' \subset R^{n+1} \times R^p$ such that :

I. D' has an open set U' , such that any point of U' is regular and $\pi : U' \longrightarrow U$ is a Nash isomorphism

II. there exists a regular map $\widetilde{\rho}' : U' \longrightarrow G_{n+1,n-m} \times G$ such that $\widetilde{\rho}'$ approximates $\widetilde{\rho} \circ \pi$

III. $U' \supset X \cup M'$ and $\widetilde{\rho}_{|X\cup M'} = \widetilde{\rho}'_{|X\cup M'}$

Using the map $\pi' \circ \widetilde{\rho}'$, where π' is the projection : $G_{n+1,n-m} \times G \xrightarrow{\pi'} G_{n+1,n-m}$, we find the equations of a Nash proper approximation T' of T such that : $T' \subset D'$, $T' \cap \{x_{n+1} = 0\} \supset X \cup M'$, T' cuts tranversely $\{x_{n+1} = 0\}$. (For the construction of the equations see remark d) of the section c).

Let $T'' = T' \cup S$ be the smaller algebraic subvariety of R^{n+p+1} containing T' and $h : R^{n+p+1} \longrightarrow R$ a C^∞ map such that :

1) h is near enough to X_{n+1} in a neighbourhood of T' ,
($X_{n+1} : (x_1 \dots x_{n+1}) \longrightarrow x_{n+1}$)

2) $h_{|X\cup M'} = 0$

3) $h(x) = 1$ on the complement of a compact set K , such that $K \cap S = \emptyset$

Clearly h defines a C^∞ map $h' : S^{n+p+1} \longrightarrow R$, where the sphere is considered as the compactification of R^{n+p+1} .

So we may approximate h' by a regular function $h'' : S^{n+p+1} \longrightarrow R$ such that :

1)' h'' is near enough to X_{n+1} in a neighbourhood of T'

2)' $h''_{|X\cup M'} = 0$

3)' $h''(x) > \frac{1}{2}$ if $x \in R^{n+p+1} - K$.

Let now $T' \cap \{h'' = 0\} = \widetilde{M} \cup M'$; we have that $\widetilde{M}$ approximates M and $\widetilde{M} \supset X$. The theorem is proved.

<u>Lemma</u> 6.- <u>Let</u> $\{S_i\}_{i=1,..,q}$ <u>be a family of regular algebraic subvarieties of</u> R^n <u>and let us suppose that for any</u> i , K <u>and</u> $x \in S_i \cap S_K$ <u>the germ of</u> S_i <u>is transverse to the germ of</u> S_K <u>in the Zariski tangent space of</u> $(S_i)_x \cup (S_K)_x$. <u>In these hypotheses if</u> $\varphi : \bigcup_{i=1}^{q} S_i \longrightarrow R$ <u>is a continuous function such that</u> $\varphi_{|S_i}$ <u>is a regular function,</u> <u>for any</u> i , <u>then</u> φ <u>is a regular function</u>.

Proof.- The problem is local, then we may suppose S_i transverse to any S_K. Let now $X = X_1 \cup X_2$ be the union of two algebraic subvarieties of $\mathbb{R}^n$. Let us suppose that the ideal $I_{X_1 \cap X_2}$, of the polynomials zero on $X_1 \cap X_2$, is generated by the ideals I_{X_1}, I_{X_2} defined by X_1 and X_2. Under this hypothesis we wish to prove that any continuous function $\psi : X \longrightarrow \mathbb{R}$, such that $\psi|_{X_i}$, $i = 1, 2$ are regular, is, actually, a regular function.

Let ψ_1, ψ_2 be two regular functions that extend $\psi|_{X_1}$ and $\psi|_{X_2}$ to a Zariski open set of $\mathbb{R}^n$. We have $\psi_1 - \psi_2 \in I_{X_1 \cap X_2}$ hence, by the above condition, we have $\psi_1 - \psi_2 = f_1 - f_2$, $f_i \in I_{X_i}$, $i = 1, 2$.

Then the regular function $\psi_1 - f_1 = \psi_2 - f_2$ extends ψ and this proves that ψ is a regular function.

Now we remark that S_1 and S_2 satisfy the above condition, hence $\varphi|_{S_1 \cup S_2}$ is a regular function.

Now S_3 cuts transversely S_1 and S_2, hence $I_{S_3 \cap (S_1 \cup S_2)}$ is generated by I_{S_3} and $I_{S_1 \cup S_2}$, so $\varphi|_{S_1 \cup S_2 \cup S_3}$ is a regular function.

After a finite number of steps we prove that φ is a regular function.

We have :

PROPOSITION 4.- Let M be a compact C^∞ submanifold of $\mathbb{R}^n$, $n \geq 2 \dim M + 1$ and $\{S_i\}_{i=1,..,q}$ be a family of compact C^∞ submanifolds of M in general position. Then the pair $(M, \bigcup_{i=1}^q S_i)$ has algebraic approximation $(M', \bigcup_{i=1}^q S'_i)$ in $\mathbb{R}^n$ and we may suppose M', S'_i regular and that $(M', \bigcup_{i=1}^q S'_i)$ is isotopic to $(M, \bigcup_{i=1}^q S_i)$.

Proof.- Let us consider the finite family of all subvarieties $S_{i_1,\dots,i_t} = S_{i_1} \cap \dots \cap S_{i_t} \overset{\text{def}}{=} S_I$ where $I = (i_1,\dots,i_t)$. Let us consider for any pair of subsets I', I'' of $1,\dots,q$, $I' \subset I''$, the injection $S_{I''} \hookrightarrow S_{I'}$ and let us denote by $\gamma_{I'I''} : S_{I''} \longrightarrow G_{n,i}$ the equations of $S_{I''}$ in $S_{I'}$. For any $I \subset (1,\dots,q)$ let us consider the family of the maps $\gamma_{I'I} : S_I \longrightarrow G_{n,i}$, $I' \subset I$ and all the restrictions $\beta_{I^\circ I''}$ of the maps $\gamma_{I^\circ I''}$, $I^\circ \subset I$, $I'' \subset I^\circ$. Finally let $\psi_I : S_I \longrightarrow \Pi G_{n,i}$ be the product of all the maps $\gamma_{I'I}$ and $\beta_{I^\circ I''}$ just described.

Let now $I^q = (1,\dots,q)$ and S_{I^q} be the corresponding C^∞ subvariety. Let $S_{I^q} \xrightarrow{h} S'_{I^q}$ be an algebraic approximation of S_{I^q} such that :

(i) $\psi_{I^q} \circ h^{-1}$ has an algebraic approximation ψ'_{I^q}

(ii) there exists a C^∞ pair $(M^\circ, \bigcup_{i=1}^q S^\circ_i)$ in $\mathbb{R}^n$ isotopic to $(M, \bigcup_{i=1}^q S_i)$

(iii) $\bigcap_i S_i^\circ = S'_{I^q}$ and the S_i° are near enough to the S_i .

To obtain the above result we apply theorem 3 to $M = S_{I^q}$, $X = \emptyset$ and by standard arguments we construct an isotopy of $\mathbb{R}^n$ into itself sending S_{I^q} onto S'_{I^q} .
Let now denote $S^i_{I^{q-1}} = \bigcap_{j\neq i} S_j^\circ$; we have : $S^i_{I^{q-1}} \cap S^K_{I^{q-1}} = S'_{I^q}$. Let now apply theorem 3 to any pair $(S^i_{I^{q-1}}, S'_{I^q})$, starting from $(S^1_{I^{q-1}}, S'_{I^q})$ ... ; after any step we construct an isotopy that doesn't move the manifolds that we have approximated and gives a new C^∞ situation that contains the modified manifolds. After the approximation of the $S^i_{I^{q-1}}$ ... we can approach the $S^{i,K}_{I^{q-2}} \stackrel{\text{def}}{=} \bigcap_{j\neq i,K} S_j$ without changing the $S^i_{I^{q-1}}$ ... We ramark that theorem 3 can be applied because the approximations of the ψ_I , that are regular on any S_I , just approached, are regular also on the union of these manifolds (see lemma 6). In this way, after a finite number of steps, we end the construction.

<u>Remark</u> 1.- Let H be a compact regular algebraic variety such that $H_*(H,\mathbb{Z}_2)$ is algebraic and $\varphi : M \longrightarrow H$ a C^∞ map. Now let M be as in proposition 4, then the proof we have given shows that we can construct the algebraic approximations $(M', \bigcup_i S'_i)$ in such a way that there exists a regular map $\varphi' : M' \longrightarrow H$ that approximates φ .

f. The S. Akbulut and C. King result

We wish to prove the following result stated by S. Akbulut and C. King in [2].

THEOREM 4.- <u>Let</u> V <u>be a</u> C^∞ <u>compact manifold and</u> T <u>a triangulation of</u> V . <u>Let</u> $K_i \subset V$, $i = 1,\dots,p$ <u>be a family of closed disjoint subpolyhedra of</u> T <u>and</u> V_K <u>the quotient space of</u> V <u>obtained by the equivalence relation</u> : $x \sim y$ <u>if and only if</u> $x = y$ <u>or</u> $\{x\} \cup \{y\} \subset K_i$, $i = 1,\dots,p$. <u>Under these hypotheses</u> V_K <u>is homeomorphic to a real algebraic variety having, at most,</u> p <u>singular points</u>. <u>Moreover, any such algebraic variety has this form</u>.

The Hironaka desingularisation theorem proves that any algebraic variety, with p isolated singularities, has the above form.

To prove the converse we need some lemmas that are contained in [2] :

<u>Lemma</u> 7.- <u>Let</u> $V \subset \mathbb{R}^n$ <u>be an algebraic variety and</u> $S \subset V$ <u>a compact algebraic subvariety</u>. <u>Then there exists an affine variety</u> V_S <u>and a regular map</u> $\varphi : V \longrightarrow V_S$ <u>such that</u> :

(i) $\varphi : V - S \longrightarrow V_S - \varphi(S)$ is an isomorphism

(ii) $\varphi(S)$ is a point of V_S .

The lemma is proved by the following remarks :

I. Let us suppose $\mathbb{R}^n$ embedded in $P_n(\mathbb{R})$ and let S be the locus of zeros of a polynomial of degree d . Using the Veronese embedding of degree d , $P_n(\mathbb{R}) \longrightarrow P_N(\mathbb{R})$, we may suppose $i(S)$ is the part of $i(V)$ contained in one hyperplane of $P_N(\mathbb{R}) \supset i(V)$.

II. The stereographic projection $\mathbb{R}^N \longrightarrow S^N$ has a natural algebraic extension $p : P_N(\mathbb{R}) \longrightarrow S^N$ such that the image of $P_N(\mathbb{R}) - \mathbb{R}^N$ is one pole of S . The variety $p(i(V)) = V_S$ satisfies the required property. To verify the assertion we recall that if the sphere S has equation $\sum_{i=1}^N x_i^2 + (x_{N+1} + 1)^2 = 1$, and we project $x_{N+1} = -2$ on S^N from the origin, the map has the equation :
$p(x_1,\dots,x_N, -2) = \left(\frac{4}{\sum_1^N x_i^2 + 1} x_1 , \dots , \frac{4}{\sum_1^N x_i^2 + 1} x_N , \frac{-8}{\sum_1^N x_i^2} + 1\right)$ and hence has a regular extension $p' : P_N(\mathbb{R}) \longrightarrow S^N$.

The proof of lemma 7 given in [2] is slightly different.

<u>Lemma</u> 8.- <u>Let</u> W <u>be a compact connected</u> C^∞ <u>manifold with boundary</u> ∂W . <u>Then there is a family</u> $\{\widetilde{E}_\sigma\}_{\sigma=1,\dots,s}$ <u>of closed embedded discs in</u> $W - \partial W$ <u>such that</u> :

1) the boundaries $S_\sigma = \partial\widetilde{E}_\sigma$ of the discs are a family of C^∞ submanifolds of W in general position

2) there is a family $\{D_\alpha\}_{\alpha=1,\dots,q}$ of closed embedded discs of $W - \partial W$ such that $W - \bigcup_{\alpha=1}^q D_\alpha$ is a regular neighbourhood of $\bigcup_{\sigma=1}^s S_\sigma$ in W .

<u>Proof</u>.- Let us fix a triangulation τ of W , such that the interior of any simplex σ is an open set in a C^∞ , locally closed, submanifold V_σ of W . Moreover we shall suppose $V_\sigma \supset \bar{\sigma}$. Such a triangulation shall be called smooth.

Now we suppose that τ is so refined that the union U of all the simplices that intersect ∂W is a regular neighbourhood of ∂W . Let now K the subpolyedron of all the simplices that do not touch ∂W .

We wish to construct a family of "simplicial" closed discs $\{\widetilde{E}_\sigma\}_{\sigma=1,\dots,s}$ such that :

(i) the E_i are in one - one correspondence with the simplices σ_i of K (simplices of any dimension)

(ii) $\bigcup_{i=1}^s E_i$ is a regular neighbourhood of K

(iii) the boundaries ∂E_i of the E_i are in general position (as subpolyhedra)

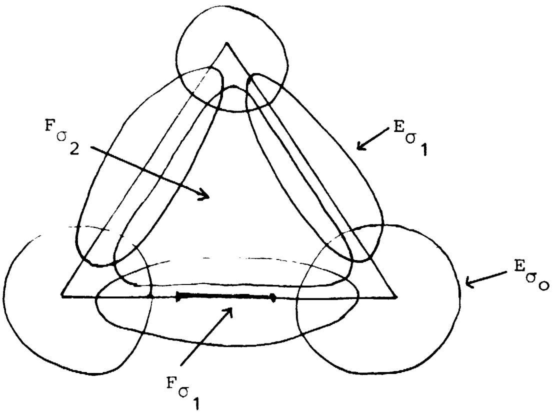
(iv) the closure of any connected component of $\bigcup_{i=1}^s E_i - \bigcup_{i=1}^s \partial E_i$ is a topological closed disc.

Let σ_p be a simplex of dimension p of K and F_{σ_p} the union of all simplices, of a $p+1$-th barycentric smooth subdivision of σ_p, wich do not touch the boundary $\partial\sigma_p$.

Let E_{σ_p} be the union of all simplices, of a $p+2$-th barycentric smooth subdivision of K, that intersect F_{σ_p}.

If $p = 2$ the situation is illustrated by fig. 1.

figure 1



It is a direct verification that the "simplicial" disc E_σ satisfy the required properties.

Let now $\tilde{E}_\sigma$ be a family of C^∞ discs, obtained from E_σ, smoothing the corners. We shall suppose the $\tilde{E}_\sigma$ approximate enough the E_σ and the properties (i), (ii), (iii), (iv) are satisfied for the family $\tilde{E}_\sigma$.

Let now $\{A_\alpha\}_{\alpha=1,\dots,q}$ be the set of the connected components of $\bigcup_{\sigma=1}^{s} E_\sigma - \bigcup_{\sigma=1}^{s} \partial\tilde{E}_\sigma$ and for any A_α let D_α be a closed smooth disc embedded in A_α. Clearly $\bigcup_{\sigma=1}^{s} \tilde{E}_\sigma - \bigcup_{\alpha=1}^{q} D_\alpha$ is a regular neighbourhood of $\bigcup_{\sigma=1}^{s} \partial\tilde{E}_\sigma$ (because any $\bar{A}_\alpha$ is homeomorphic to a closed disc).

The lemma is now proved because W is a regular neighbourhood of $\bigcup_{\sigma=1}^{s} \tilde{E}_\sigma$.

Lemma 9.- Let W be a connected compact manifold of dimension n and ∂W the boundary. There exists a compact manifold V such that :

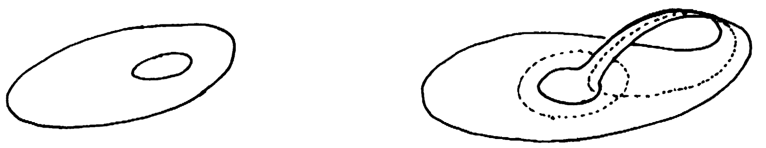
(i) $\partial V = \partial W$

(ii) V is a regular neighbourhood of $T = (\bigcup_{i=1}^{s} S_i^{n-1}) \cup (\bigcup_{j=1}^{q} S_j^1)$, where the S_i^{n-1}, S_j^1, are diffeomorphic to the $n-1$ and 1 spheres. Moreover the manifolds of the family S_i^{n-1}, S_j^1 are in general position.

Proof.- From lemma 8 we know that there exists two families of embedded closed discs $\{\tilde{E}_\sigma\}_{\sigma=1,\dots,s}$, $\{D_\alpha\}_{\alpha=1,\dots,q}$ in $W - \partial W$ such that : $W - \bigcup_{\alpha=1}^{q} D_\alpha$ is a regular neighbourhood of $S = \bigcup_{\sigma=1}^{s} \partial\tilde{E}_\sigma$. Moreover, from the construction, we know that the $\partial\tilde{E}_\sigma$ are diffeomorphic to S^{n-1} and they are in general position.

Now to prove the lemma we construct a new manifold that has the same boundary (up to diffeomorphism) of W and that can be deformed on T , for some family $\{S^1_j\}$ of embedded circles. We remark that the boundary of $W - \bigcup_{\alpha=1}^{q} D_\alpha$ is the union of ∂W and of disjoint S^{n-1} spheres. Then in our construction we must kill the new boundary $\bigcup_{\alpha=1}^{q} \partial D_\alpha$. Let us consider the simple situation $q = 1$, W = a disc . In this case the new manifold W' that we construct is illustrated, in dimension 2, by figure 2,below :

<u>figure</u> 2

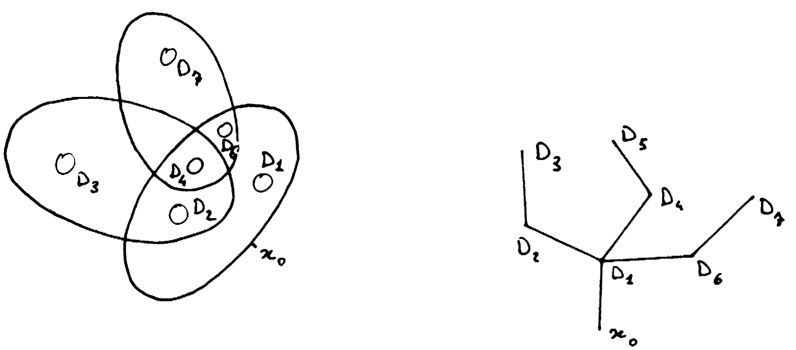


The boundary of W' is diffeomorphic to ∂W (a circle in our picture) and W' can be continuously retracted on $S^{n-1} \cup S^1$. It is not difficult to see that the above construction gives us the desired result for any dimension n . This construction shall be called the "iron" construction.

Let now consider the general case.

Let us fix a point $x_o \in \partial W$ and let us construct a tree, starting from x_o . The tree has a vertex for any connected component of $W - S$ and a one simplex joining two vertices if the closure of the corresponding connected components has non-empty intersection (see fig. 3).

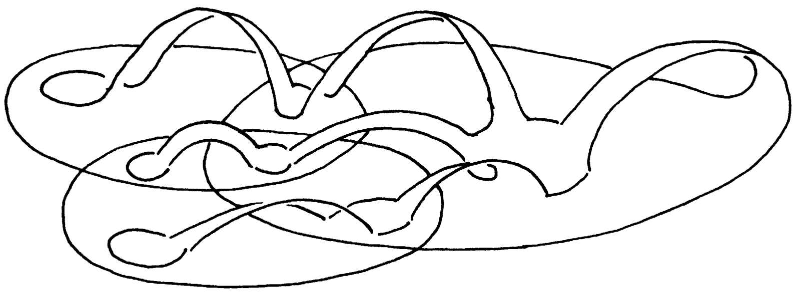
<u>figure</u> 3



Clearly the tree can be constructed in many different ways ; we shall suppose, only, that all the D_i are reached from the tree, and this is possible because W is connected.

to end the proof of the lemma it is now enough to construct a ribbon along the tree having an "iron" at any vertex (see fig. 4).

<u>figure</u> 4



It is now easy to realize that the manifold W'', of fig. 4, has a boundary diffeomorphic to ∂W (if all the corners are smoothed). Moreover W'' is a regular neighbourhood of $(\bigcup_\sigma \partial\widetilde{E}_\sigma) \cup (\bigcup_j S^1_j)$ where we have one S^1_j for any 1-simplex of the tree.

The lemma is proved.

<u>Proof of theorem</u> 4 (see [2]).- To collapse K_i is equivalent to collapse a regular neighbourhood W_i of K_i.

W_i is a C^∞ manifold and ∂W_i is its boundary ; we shall suppose $W_i \cap W_K = \emptyset$, $i, K = 1,\dots,p$, $i \neq K$.

Lemmas 8 and 9 prove that ∂W_i is the boundary of a compact C^∞ manifold U_i such that U_i is a regular neighbourhood of a finite family $S^i = \bigcup_j S^i_j$ of compact C^∞ manifolds in general position.

Let V_U be the C^∞ manifold obtained gluing $V - \bigcup_{i=1}^{p} W_i$ and $\bigcup_{i=1}^{p} U_i$ along $\bigcup_i \partial W_i$.

Now to collapse the K in V is equivalent to collapse the U_i in V_U and hence to collapse the S^i in V_U. By proposition 4 the pair $(V_U, \bigcup_i S^i)$ has an algebraic approximation and, by lemma 7, the quotient space obtained collapsing the S^i is an algebraic variety, with at most p singular points.

The theorem is proved.

<u>Remark</u> 1 (see [2]).- Let $V \subset \mathbb{R}^n$ be an algebraic variety ; lemma 7 proves that the "one point compactification" of V is homeomorphic to a (compact) algebraic variety.

So we have : a topological space X is homeomorphic to an algebraic variety if and only if this is true for the one point compactification of X. From this remark and theorem 4 we can deduce characterisations of the topology of non-compact algebraic varieties.

Now we prove (see [6]) that any pure dimensional, compact, real analytic set, having isolated singularities, is homeomorphic to an algebraic variety.

PROPOSITION 5.- Let $\{S^i_{x_i}\}_{i=1,..,q}$ be a family of germs of real analytic sets, of pure dimension n, having x_i as isolated singularity.

There exists a compact, connected, real algebraic variety V and q points $y_1, \dots, y_q$ of V such that :

(i) $V - \bigcup_{i=1}^{q} y_i$ is a regular algebraic variety of dimension n

(ii) the germ of V at y_i is homeomorphic to $S^i_{x_i}$, $i = 1,\dots,q$

Proof.- Let U^i be a realisation of $S^i_{x_i}$ and let us suppose x'_i be the only singular point of U^i . From the Hironaka desingularisation theorem there exists a proper map $p_i : \hat{U}^i \longrightarrow U^i$ such that $\hat{U}^i$ is an analytic manifold, $p_i(\hat{U}^i) = U^i$, $p^i : \hat{U}^i - p_i^{-1}(x'_i) \longrightarrow U^i - \{x_i\}$ is an analytic isomorphism. Let D be a compact neighbourhood of $p_i^{-1}(x'_i)$ in $\hat{U}^i$ such that D_i is a connected manifold with boundary ∂D_i . It is known (Lojasiewicz) that the pair $(D_i , p_i^{-1}(x'_i))$ has a triangulation.

Let now $\hat{D}_i$ be the double of D_i and W the connected union of the $\hat{D}_i$, $i = 1,\dots,q$. Let us suppose W be constructed in such a way that the "tubes" joining the components $\hat{D}_i$ do not touch the set $\cup\, p_i^{-1}(x'_i)$.

The manifold W is compact and connected. Theorem 4 proves that if we collapse to a point each set $p_i^{-1}(x'_i)$ the quotient space is homeomorphic to an algebraic variety. The proposition is proved.

g. Final remarks

In this section we shall give some refinements of the results given before.

Remark 1.- Let M be a C^∞ compact submanifold of $\mathbb{R}^n$, $m = \dim M$. In the approximation process (of theorem 3, for example) we construct a regular algebraic approximation $M' \subset \mathbb{R}^n \times \mathbb{R}^p$ of M such that the projection $\pi(M') \overset{\text{def}}{=} M'' \subset \mathbb{R}^n$ is a Nash approximation of M and locally $\pi_{|M'}$ is an isomorphism. From proposition 3 we know that there exists an algebraic subvariety $\hat{M}''$ of $\mathbb{R}^n$ such that $\hat{M}'' \supset M''$, $S = \hat{M}'' - M''$ is contained in the singular set of $\hat{M}''$. Let us denote by S' the singular set of $\hat{M}''$ and let $S'' = S' \cap M''$.

Let us consider the projection $\tilde{\pi} : \mathbb{C}^n \times \mathbb{C}^p \longrightarrow \mathbb{C}^n$ and the complexification $\tilde{M}'$, $\tilde{\hat{M}}''$ of M' , $\hat{M}''$.

It is clear that any point $x \in S''$ is the image, under $\tilde{\pi}_{|M'}$, of at least 3 points of $\tilde{M}'$, in fact we have in $\tilde{\pi}^{-1}(x) \cap \tilde{M}'$ one real point and, at least, a couple of complex points. It follows that S'' is contained in the image, under $\tilde{\pi}$, of the triple points set $\Sigma_3(\tilde{\pi}_{|\tilde{M}'})$ of $\tilde{\pi}_{|\tilde{M}'}$.

It is known (see [13], propositions 7.2, 7.4) that, if $\tilde{\pi}$ is generic, we have $\dim \Sigma_3(\tilde{\pi}|_{\tilde{M}'}) = 3m - 2n$.

We deduce that, if $n > \frac{3}{2}m$, the set S'' is empty.

Let $n > \frac{3}{2}m$ and, eventually after a little change of π, let us suppose $S'' = \emptyset$. There exists a polynomial P such that $P|_{\hat{M}'' - M''} = 0$ and $P(x) \neq 0$ if $x \in M''$ (we have $\hat{M}'' - M'' = \operatorname{sing} \hat{M}''$ because locally $\pi|_{M'}$ is an isomorphism).

Let now $Q : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a polynomial such that $Q|_{\hat{M}'' - M''} = 0$, Q approximates the function 1 in a neighbourhood of M''. (Clearly we can find a C^∞ function $Q' = P \cdot h$, $h(x) \neq 0$, $\forall x$, having the above property. If we approximate h by a polynomial function h' we obtain the desired Q).

If we consider the embedding $\mathbb{R}^n - \{Q = 0\} \longrightarrow \mathbb{R}^{n+1}$ given by

$(x_1, \ldots, x_n) \xmapsto{j} (x_1, \ldots, x_n, \frac{1}{Q(x_1 \ldots x_n)})$ we find an algebraic approximation of M in $\mathbb{R}^{n+1}$ (it is $j(\pi(M'))$).

We may summarize : _Let_ M _be a_ C^∞ _compact_, m-_dimensional subvariety of_ $\mathbb{R}^n$ _and let us suppose_ $n > \frac{3}{2}m$.

Under these hypotheses M _has in_ $\mathbb{R}^{n+1} \supset \mathbb{R}^n \times \{0\} = \mathbb{R}^n$ _algebraic approximation_ M'.

Moreover we have the same result, even if we require that M' satisfies the extra conditions of theorem 3 or of proposition 4.

Remark 2.- Let M be an algebraic variety, we shall call M _totally algebraic_ if any topological vector bundle $F \longrightarrow M$ has a strongly algebraic structure.

In theorem 3 (and in proposition 4) we can suppose that the algebraic approximation M' is totally algebraic.

To prove this it is enough to show that, given M, there is a finite number of maps $\varphi_i : M \longrightarrow G_{n_i, q_i}$ such that if we approximate φ_i by regular maps $\varphi'_i : M' \longrightarrow G_{n_i, q_i}$ then all the vector bundles on M' have a strongly algebraic structure.

This last fact ensues from the remarks :

(i) F has a strongly algebraic structure if and only if there exists $p \in \mathbb{N}$ such that $F \oplus M' \times \mathbb{R}^p$ has this property

(ii) the Grothendieck ring $K(M)$ has a finite number of generators.

The details are contained in [5].

The notion of totally algebraic variety is not satisfactory ; for example, if M and N are totally algebraic, $M \times N$, in general, is not (see [7]).

In the case of line bundles we have : a sufficient condition to ensure that any line bundle $F \longrightarrow M \times N$ has a strongly algebraic structure is : any irreducible component of M, N is connected and all the line bundles on M, N have a strongly algebraic structure (see [7]).

A family of examples of totally algebraic varieties is given by the suspensions (see [5]). In this case the totally algebraic structure is a consequence of a topological property. For example any regular algebraic variety diffeomorphic to a sphere is totally algebraic.

Remark 3.- It seems very interesting (see section d) to know if any C^{∞} compact manifold M has an algebraic structure M_a such that the homology $H_*(M_a,\mathbb{Z}_2)$ is algebraic. Up to now we can prove the following (see [7]) :

THEOREM.- Let M be a compact C^{∞} manifold, then M has an algebraic structure M_a such that :

(i) M_a is totally algebraic

(ii) all $\alpha \in H_p(M_a,\mathbb{Z}_2)$ that are represented by C^{∞} smooth cycles are also represented by regular algebraic subvarieties

(iii) any $\alpha \in H_p(M_a,\mathbb{Z}_2)$ that is dual to a Stiefel-Whitney class of some vector bundle is algebraic.

Remark 4.- Proposition 2 proves that, if M is compact, totally algebraic then any C^{∞} weakly complete intersection S of M can be approximated, in M, by a weak algebraic complete intersection S'.

The case of complete intersection is, up to now, the only case in which we are able to give a positive answer to problem 1 of section c.

A necessary and sufficient condition to the fact : S is a weak C^{∞} complete intersection is given in [7].

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A. TOGNOLI, Istituto Leonida Tonelli
via Bonaroti 2
PISA - Italie