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Hyperbolic manifolds according to Thurston and Jørgensen

Séminaire N. Bourbaki, 1981, exp. nº 546, p. 40-53

<http://www.numdam.org/item?id=SB_1979-1980__22__40_0>

Séminaire BOURBAKI — 546-01

32e année, 1979/80, n° 546 — Novembre 1979

HYPERBOLIC MANIFOLDS

ACCORDING TO THURSTON AND JØRGENSEN

by Michael GROMOV

0. Preliminaries

Surfaces of constant negative curvature

Consider a complete connected orientable 2-dimensional Riemannian manifold V with curvature -1. The most important single invariant of V is its volume (or area) $\mathrm{Vol}(V)$. We are interested now in the case when this volume is finite.

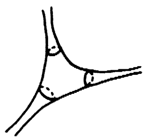
According to the Gauss-Bonnet theorem the condition $\mathrm{Vol}(V) < \infty$ implies that $\mathrm{Vol}(V) = -2\pi\chi(V)$, where χ denotes the Euler characteristic.

It follows that the possible values of $\mathrm{Vol}(V)$ are 2π, 4π, 6π, ...

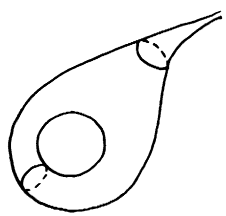
It is also well known that there are only finitely many topologically different V's with a given volume. More precisely, when $\mathrm{Vol}(V) = 2\pi k$ and k is odd there are $\frac{k+3}{2}$ possibilities. When k is even there are $\frac{k}{2}+2$ topological types of V's with $\mathrm{Vol}(V) = 2\pi k$.

Examples

For $k = 1$ we have

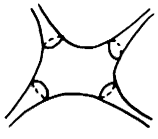


sphere with three punctures

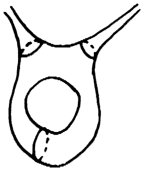


torus with one puncture

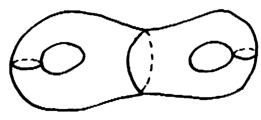
For $k = 2$ we have



sphere with four punctures

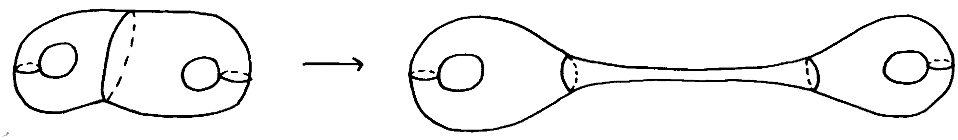


torus with two punctures



closed surface of genus two

Notice that each surface (with the only exeption of the sphere minus three points) supports a continuum of complete metrics with curvature -1 . The following picture exhibits a typical deformation of a hyperbolic metric on the closed surface of genus two



For the manifolds V with $\dim(V) \geq 4$ the volume function $V \longmapsto \mathrm{Vol}(V)$ has the same essential properties as for $\dim = 2$, but, as it was discovered by Thurston and Jørgensen, the manifolds of dimension three display quite different amazing features.

Manifolds of dimensions 4 , 5 , ...

We use the words hyperbolic manifold for a complete Riemannian manifold with constant sectional curvature -1 .

Wang's Finiteness Theorem.- *If $n \geq 4$ then for each real x there are only finitely many isometry classes of n-dimensional hyperbolic manifolds V with $\mathrm{Vol}(V) \leq x$.* (see [W]).

Remarks.- Wang's result in [W] is applicable to almost all locally symmetric spaces.

The number of V's with $\mathrm{Vol}(V) \leq x$ can be effectively estimated (by something like $x \exp(\exp(\exp(n+x)))$) but one has no realistic upper bound.

Wang's theorem implies that for a fixed $n \neq 3$ the values of $\mathrm{Vol}(V)$ form a discrete set on the real line. When n is even the Gauss-Bonnet theorem says more

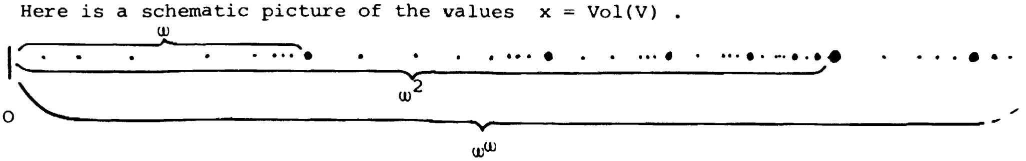
$$\mathrm{Vol}(V) = C_n \chi(V) ,$$

where C_n is a universal constant.

1. Three dimensional manifolds

Thurston's theorem.- *The values of the function $V \longmapsto \mathrm{Vol}(V)$, where V runs over all 3-dimensional hyperbolic manifolds with $\mathrm{Vol}(V) < \infty$, form a closed non-discrete set on the real line. This set is well ordered and its ordinal type is ω^ω . The function $V \longmapsto \mathrm{Vol}(V)$ is finite to one, i.e. there are only finitely many V's with a given volume*

Here is a schematic picture of the values $x = \mathrm{Vol}(V)$.



Thurston's theorem says that there is a manifold with the smallest volume x_1. Then there is the next smallest volume x_2, and so forth. The sequence $x_1 < x_2 < x_3 < \dots$ has a limit point x_ω. We shall see that the number x_ω represents the smallest volume of a complete non-compact manifold. The next smallest volume of a complete non-compact manifold is $x_{2\omega}$. The point x_{ω^2} corresponds to the first manifold with two cusps (see the definition in section 2) and so forth.

The second statement of Thurston's theorem says that for each $x = x_j$, $j = 1, 2, \dots, \omega, \dots$ the number $N = N(x)$ of different V's with $\mathrm{Vol}(V) = x$ is finite. (It is clear that the function $N(x_j)$ is unbounded, because each V has many non-isometric finite coverings of a fixed degree).

We shall present below only the basic ideas of Thurston's proof. The details can be found in chapters 5 and 6 of his lectures [T].

Let us mention two unresolved problems.

Is the function $N(x_j)$ locally bounded ?

Can some of the numbers x_i/x_j be irrational.

2. The Margulis-Jørgensen decomposition

Let V be a hyperbolic manifold. Denote by $\ell(v)$, $v \in V$, the length of the shortest geodesic loop based at v.

Fix a positive number ε and look at the ε-ball around $v \in V$. When $\varepsilon < \frac{1}{2}\ell(v)$, the geometry of this ball is standard ; this ball is isometric to the ε-ball in the hyperbolic space H^n, $n = \dim(V)$. (Recall, that H^n is the complete simply connected manifold with curvature -1. The universal covering of every hyperbolic manifold is isometric to H^n).

The Kazhdan-Margulis theorem (see [K-M]) implies the existence of an universal constant $\mu = \mu_n > 0$, such that each ε-ball in V with $\varepsilon \le \mu$ has more or less standard geometry, even at a point $v \in V$ with $\ell(v) \le 2\varepsilon$. We shall discuss here only the 3-dimensional case.

Kazhdan-Margulis theorem (special case).- *There exists a positive number μ such that for each orientable hyperbolic manifold V and each $v \in V$ the loops based at v of length $\le 2\mu$ generate in $\pi_1(V,v)$ a free Abelian subgroup of rank at most two.* (See [K-M], [T]).

This theorem shows that each ε-ball $\varepsilon < \mu$ in V is isometric to an ε-ball in a hyperbolic space H^3 or in a hyperbolic manifold with fundamental group $\mathbb{Z}$ or $\mathbb{Z} \oplus \mathbb{Z}$. These manifolds can be explicitely described as follows.

Cusps

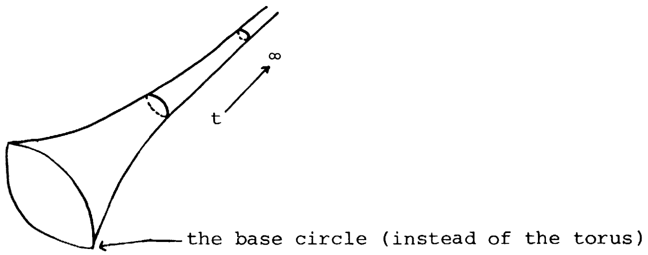
Take a flat two torus T and let ds^2 denote its (flat) metric. The product $T \times \mathbb{R}^1$ with the metric $e^{-r}ds^2 + dr^2$ is called the double infinite cusp $C = C_T$

based on T .

It is easy to see that C is a hyperbolic manifold with $\pi_1 = \mathbb{Z}+\mathbb{Z}$ and that every hyperbolic 3-manifold with $\pi_1 = \mathbb{Z}+\mathbb{Z}$ is isometric to a double infinite cusp.

The manifold $C^+ = T \times [0,\infty) \subset C$ is called the cusp based on T . This manifold has the boundary $T = T \times 0$ and the isometry type of C^+ is uniquely determined by (the isometry type of) T .

Here is the picture of a cusp for $n = 2$



Notice, that the double infinite cusps have infinite volume but the cusps have finite volumes.

Tubes

Consider a 3-dimensional hyperbolic manifold with $\pi_1 = \mathbb{Z}$. It is not hard to see that there are only two possibilities.

a) Our manifold has no closed geodesics. In this case it is isometric to an infinite cyclic covering of a double infinite cusp.

b) The manifold has a closed geodesic. In this case we call this manifold an infinite tube.

An infinite tube $\mathcal{D}$, clearly, has a unique simple closed geodesic $\gamma \subset \mathcal{D}$ which is called the axial geodesic.

Let us introduce a (finite) tube $\mathcal{D}_r \subset \mathcal{D}$, $r \geq 0$, as the set of the points $v \in \mathcal{D}$ with $\mathrm{dist}(v,\gamma) \leq r$.

The boundary $T = \partial\mathcal{D}_r$ is the topological torus and the induced metric in T is, clearly, flat. The torus T containes a simple closed geodesic $\tau \in T$ which is contractible in $\mathcal{D}_r$. This geodesic is uniquely determined, up to rotations of T , by the homomorphism $\mathbb{Z}+\mathbb{Z} = \pi_1(T) \longrightarrow \pi_1(\mathcal{D}_r) = \mathbb{Z}$. Notice that the length of τ is equal to the length of the radius r circle in the hyperbolic plane. It follows that $r \sim \log(\mathrm{length}(\tau))$ when this length is large.

It is not hard to see that the isometry type of $\mathcal{D}_r$ is uniquely determined by (T,τ) . One can also show that for each flat torus T and a simple closed geodesic $\tau \subset T$ there exists a tube $\mathcal{D}_r$ with $\partial\mathcal{D}_r \subset T$ such that τ is contractible in $\mathcal{D}_r$. This tube is contracted as follows. Take a geodesic $\tilde{\gamma} \in H^3$ and the r-neighbourhood $\tilde{\mathcal{D}}_r$ around $\tilde{\gamma}$. The number r is chosen such that the length of the hyperbolic radius r circle equals length(τ) . The boundary $\tilde{T} = \partial \tilde{\mathcal{D}}_r$ is the flat cylinder $S^1 \times \mathbb{R}^1$ with length(S^1) = length(τ) . It follows that there is an isometric $\mathbb{Z}$-action on $\tilde{T}$ such that $\tilde{T}/\mathbb{Z} = T$. This action uniquely extends to $\tilde{\mathcal{D}}_r$ and we get $\mathcal{D}_r$ as $\tilde{\mathcal{D}}_r/\mathbb{Z}$.

We call such a $\mathcal{D}_r$ <u>a tube based on</u> (T,τ) .

When a manifold is isometric to a tube or a cusp, the function $\ell = \ell(v)$ (= length of the shortest non-contractible loop at v) has a very simple structure. When V is a cusp $T \times \mathbb{R}^1$ the value $\ell(v) = \ell(t,r)$ depends only on $r \in \mathbb{R}^1$ and ℓ is a strictly increasing function of $-r$. When $r \longrightarrow -\infty$ the function $\ell(r)$ is about $|2r|$ and when $r \longrightarrow \infty$ we have $\ell(r) \approx e^{-r}$.

When V is an infinite tube with the axial geodesic γ the function $\ell(v)$ depends only on $r(v) = \mathrm{dist}(v,\gamma)$ and ℓ is a strictly increasing function of $r \in [0,\infty)$. When $r = 0$ we have $\ell(r) = \mathrm{length}(\gamma)$ and for $r \longrightarrow \infty$ we have $\ell(r) \approx 2r$.

In the case of a cusp the function ℓ has no critical values and all levels $\ell^{-1}(x)$ are tori. When V is a tube there is one critical value $\lambda = \mathrm{length}(\gamma)$ and all levels $\ell^{-1}(x)$, $x > \lambda$ are tori.

Using these remarks and the Kazhdan-Margulis theorem we obtain a rather complete picture of the geometry of V at the points where ℓ is small.

<u>Decomposition theorem</u>.- <u>Let</u> V <u>be an orientable three dimensional hyperbolic manifold of finite volume and let</u> $0 < \varepsilon < \frac{1}{2}\mu$, <u>where</u> μ <u>denotes the Kazhdan-Margulis constant</u>. <u>Then the set</u> $\ell^{-1}[0,\varepsilon] \subset V$ <u>consists of finitely many (if any) components and each of these components is isometric to a cusp or to a tube</u>.

This theorem works as follows. Fix a positive $\varepsilon < \frac{1}{2}\mu$ such that V containes no closed geodesic of length ε . In this case $V_{(0,\varepsilon]}$ and $V_{[\varepsilon,\infty)}$ are bounded by tori.

The ε-neighbourhoods (in V) of the points $v \in V_{[\varepsilon,\infty)}$ are isometric to the hyperbolic balls. It follows that $V_{[\varepsilon,\infty)}$ can be covered by $N = \mathrm{const}_\varepsilon \mathrm{Vol}(V)$ balls of radius ε .

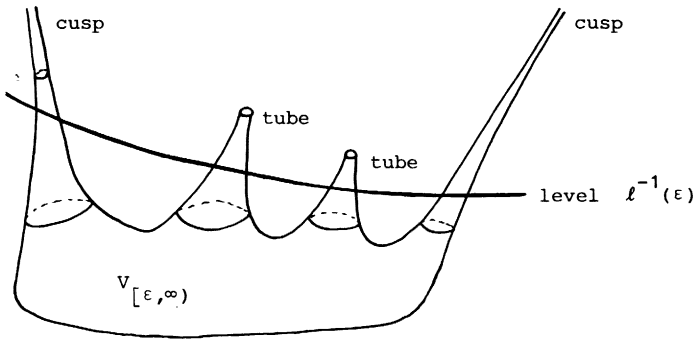
We assume, as usual, that V is connected. Clearly, the set $V_{[\varepsilon,\infty)}$ is also connected and hence, its diameter is bounded from above by $2N = 2\,\mathrm{const}_\varepsilon \mathrm{Vol}(V)$.

The manifold $V_{(0,\varepsilon]}$ has a more complicated local geometry than $V_{[\varepsilon,\infty)}$, but, globally, it is a rather standard object. In particular, one can see that the number of the components of $V_{(0,\varepsilon]}$ does not exceed const Vol(V) , where "const" is of the order μ^{-1} .

It follows that V containes at most const Vol(V) of closed geodesics of length $\leq \frac{1}{2}\mu$ (they serve as the axial geodesics of the tubes in $V_{(0,\varepsilon]}$) and when ε is less than the length of the shortest of these geodesics, the manifold

$V_{(0,\varepsilon]}$ consists only of cusps and their number does not depend on ε .

Here is a schematic picture



One of the immediate corollaries of the decomposition theorem is the following

V is diffeomorphic to the interior of a compact manifold bounded by tori and hence, there are only countably many topologically different V's .

By the Mostow theorem (see [M]) the fundamental group $\pi_1(V)$ determines V uniquely, up to isometry. Thus, there are only countably many isometry types of V's .

3. Convergence of manifolds

For two metric spaces X , Y and a map $f : X \longrightarrow Y$ we set

$$L(f) = \sup_{\substack{x_1,x_2 \in X \\ x_1 \neq x_2}} \left| \log \frac{\operatorname{dist}(x_1,x_2)}{\operatorname{dist}(f(x_1),f(x_2))} \right| .$$

Consider a sequence of metric spaces X^i , $i = 1 , 2 , \ldots$, with base points $x_i \in X$. We say that the sequence (X^i,x_i) converges to (Y,y) if for arbitrary numbers $\varepsilon > 0$, $r > 0$, there is a number j , such that for each $i \geq j$ there exists a map f from the radius r ball $B \subset X^i$ around x_i into Y such that

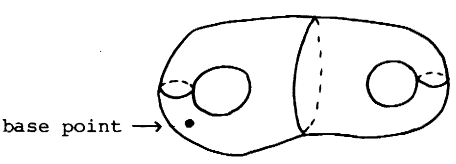
a) $f(x_i) = y$

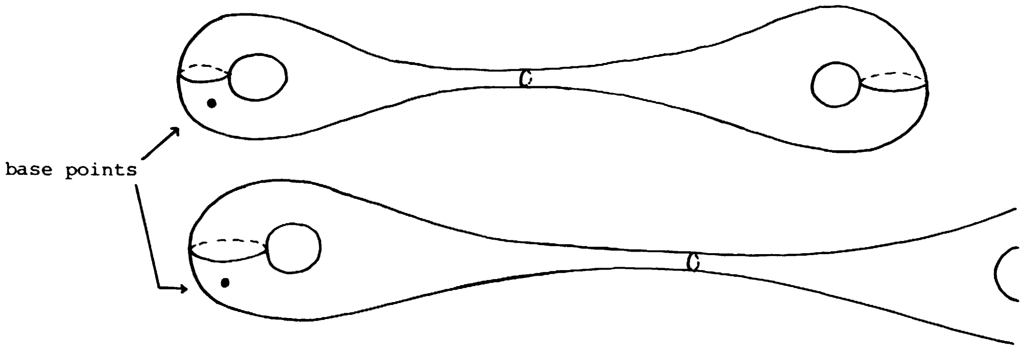
b) the image $f(B) \subset Y$ contains the ball of radius $r - \varepsilon$ around $y \in Y$

c) $L(f) \leq \varepsilon$.

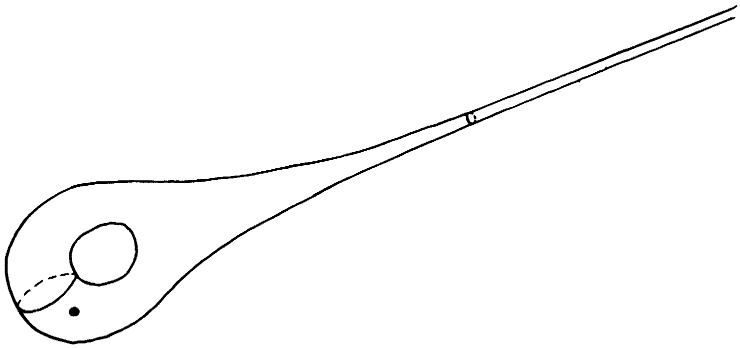
Example

Look at the following sequence of compact hyperbolic surfaces of genus two





It is clear, that the limit is the punctured torus.



In this limit process we have lost one half of our surface. This happend because the set $V_{[\mu,\infty)}$ was disconnected, and there is a more refined notion of convergence, which takes into accent all components of $V_{[\mu,\infty)}$. On the other hand, for the 3-dimensional hyperbolic manifolds this complication does not show up.

(Jørgensen).- Let (V^i, v_i) , $i = 1, 2, \ldots$, be a convergent sequence of 3-dimensional hyperbolic manifolds with $\sup_i \operatorname{Vol}(V^i) < \infty$. Then the limit space V is a hyperbolic 3-manifold with $\operatorname{Vol}(V) = \lim_{i\to\infty} \operatorname{Vol}(V^i)$.

Proof.- The decomposition theorem implies that for a sufficiently small $\varepsilon > 0$ the sets $V^i_{[\varepsilon,\infty)} \subset V^i$ are connected and $v_i \in V^i_{[\varepsilon,\infty)}$. Using the decomposition theorem one can also see that if $\varepsilon_i \longrightarrow 0$, then $\operatorname{Vol}(V^i_{(0,\varepsilon_i]}) \longrightarrow 0$. Since $\operatorname{diameter}(V^i_{[\varepsilon,\infty)}) \leq \operatorname{const}_\varepsilon$, we have $\lim_i \operatorname{Vol}(V^i) = \operatorname{Vol}(V)$.

Recall a simple general fact

(Chabauty).- Let (V^i, v_i) , $i = 1, 2, \ldots$, be a sequence of hyperbolic manifolds of a fixed dimensions. If the geodesic loops at v_i satisfy $\inf_i \ell(v_i) > 0$, then our sequence has a convergent subsequence. (See [C], [W]).

Remark.- This compactness criterion holds for a sequence of arbitrary complete

Riemannian manifolds with sectional curvatures pinched between two constants.

Corollary (Jørgensen).- The values of $\mathrm{Vol}(V)$ form a closed set in $\mathbb{R}_+$ when V runs over all 3-dimensional hyperbolic manifolds of finite volume.

Proof.- According to the decomposition theorem each V has a point $v \in V$ with $\ell(v) > \mu$. It follows that each sequence of V's has a convergent subsequence and by the Jørgensen theorem the function $V \mapsto \mathrm{Vol}(V)$ is continuous. Q.E.D.

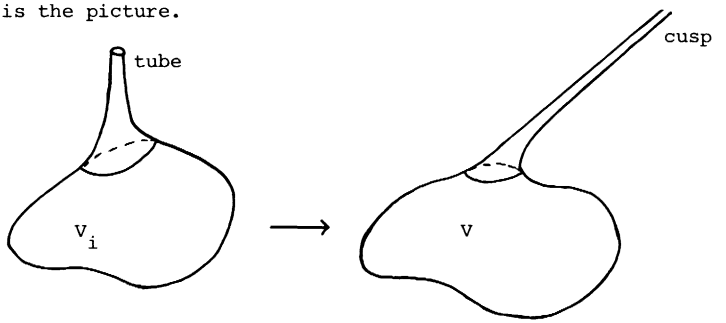
4. Opening and closing the cusps

Let V^i be a convergent sequence (relative to some choices of the base points) of hyperbolic 3-manifolds with $\sup_i \mathrm{Vol}(V^i) < \infty$, and let V denote the limit manifold.

It follows from the decomposition theorem that for each $\varepsilon > 0$ and for each sufficiently large i (i.e. $i \geq j(\varepsilon)$) the sets $V^i_{[\varepsilon,\infty)}$ are diffeomorphic to $V_{[\varepsilon,\infty)}$.

When ε is small (and i is large) the sets $V_{(0,\varepsilon]}$ have the same number of components as $V_{(0,\varepsilon]}$. We can also assume (by making ε smaller if necessary) that all components in $V_{(0,\varepsilon]}$ are cusps. It follows, that the only possible change in topology, when we pass to the limit $V_i \to V$ is the turning of the tubes of $V^i_{(0,\varepsilon]}$ into cusps in $V_{(0,\varepsilon]}$.

Here is the picture.



The formal statement is the following

The cusp opening theorem (Jørgensen).- Take a sequence of hyperbolic orientable 3-manifolds with uniformely bounded volumes. Then there is a subsequence V^i, which converges to a V, and a positive sequence $\varepsilon_i \xrightarrow[i\to\infty]{} 0$ such that each V^i has p cusps and q simple closed geodesics of length $\leq \varepsilon_i$ with p and q independent of i. The manifold V has $p+q$ cusps and it is diffeomorphic to each of the V^i's minus these q geodesics.

In the picture above we had $p = 0$, $q = 1$.

It is worth mentioning that a tube minus the axial geodesic is diffeomorphic to a cusp. Moreover, one can easily see that if C is the cusp based on a flat torus T and $\tau_i \subset T$ is a sequence of simple closed geodesics with $\text{length}(\tau_i) \underset{i \to \infty}{\longrightarrow} \infty$, then the sequence of tubes $\mathcal{D}_i$ based on (T, τ_i) converges to C. (The base points are taken at the boundaries of these tubes).

It is not clear, however, why the limit process $\mathcal{D}_i \longrightarrow C$ can occur whithin complete (i.e. without boundary) manifolds with $\text{Vol} < \infty$. In other words it could have happend that in theorem above q is always zero, and hence all V^i are themselves diffeomorphic (and by Mostow's theorem isometric) to V.

The second problem is the behaviour of the volumes $\text{Vol}(V^i)$ when $V^i \longrightarrow V$. Even when V is a non-trivial limit of V^i's (i.e. V^i's are not isometric to V) the volumes $\text{Vol}(V^i)$ and $\text{Vol}(V)$ can be, a priori, equal. In such a case we would have a discrete set of the volumes.

Both problem are resolved by the following remarkable theorems of Thurston.

A. <u>Closing the cusps</u>.- <u>Let</u> V <u>be a complete orientable manifold with</u> $\text{Vol}(V) < \infty$ <u>which has</u> $p+q$ <u>cusps</u>. <u>Then there is a sequence</u> $V^i \underset{i \to \infty}{\longrightarrow} V$ <u>as above</u>, <u>such that each</u> V^i <u>has exactly</u> p <u>cusps and</u> q <u>short geodesics</u>.

This theorem implies that V is a " $(p+q)$-fold" limit. In particular, V can be represented by a limit of compact manifolds.

B. <u>The volume limit theorem</u>.- <u>Let</u> $V^i \longrightarrow V$ <u>be a sequence as in the cusp opening theorem and let</u> $q > 0$. <u>Then</u> $\text{Vol}(V) > \text{Vol}(V^i)$, $i = 1, 2, \ldots,$.

This theorem implies that the set of the values $\text{Vol}(V)$ is well ordered, because each convergent sequence V^i with $\text{Vol}(V^1) \geq \text{Vol}(V^2) \geq \ldots$ must stabilize. This also shows that the function $V \longmapsto \text{Vol}(V)$ is finite to one, and by using theorem A we see that the set of the values $\text{Vol}(V)$ is of the type ω^ω. Thus Thurston's theorem of section 1 is reduced to A and B.

Theorem B is a special case of the following.

B'. <u>Thurston's rigidity theorem</u>.- <u>Let</u> V <u>be a hyperbolic manifold with</u> $\text{Vol}(V) < \infty$ <u>and let</u> V' <u>denotes the manifold which is obtained from a complete hyperbolic manifold of finite volume by deleting some disjoint simple closed geodesics</u>. <u>Let</u> $f : V \longrightarrow V'$ <u>be a proper map of positive degree</u> d. <u>Then</u> $\text{Vol}(V) \geq d\,\text{Vol}(V')$ <u>and the equality holds iff</u> f <u>is homotopic to an isometric covering of degree</u> d (<u>i.e.</u> V <u>is isometric to a</u> d-<u>sheeted covering of</u> V'). <u>In particular the equality</u> $\text{Vol}(V) = d\,\text{Vol}(V')$ <u>implies that</u> V' <u>was complete</u> (<u>and no geodesics were deleted</u>).

Let us explain how $B' \Rightarrow B$. We take for V' a manifold V^i minus q short geodesics. We know that V' is diffeomorphic to V, and so we have our f of degree 1. If $q > 0$ theorem B' implies that $\text{Vol}(V) > \text{Vol}(V') = \text{Vol}(V^i)$.

Let us show that B' implies the Mostow rigidity theorem.

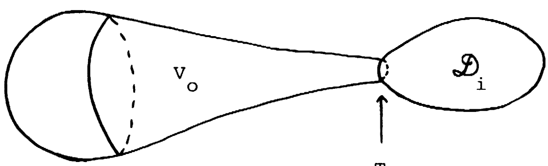
If V and V' are complete hyperbolic manifolds with isomorphic fundamental groups one has proper maps $V \longrightarrow V'$ and $V' \longrightarrow V$ of degree 1 . We can assume that $\mathrm{Vol}(V) \leq \mathrm{Vol}(V')$ (otherweise, we interplace V and V'), and B' says that the map $V \longrightarrow V'$ is homotopic to an isometry.

5. The cusp closing theorem

We shall discuss only the simplest case of a V with one cusp, when the cusp closing theorem simply says that V is a limit of compact hyperbolic manifold.

By the decomposition theorem V can be divided into a compact piece $V_o = V_{[\varepsilon,\infty)}$ bounded by the flat torus $T = V_\varepsilon = \ell^{-1}(\varepsilon)$ and the cusp $C = V_{(0,\varepsilon]}$ based on T .

We know that C is a limit of tubes based on (T,τ_i) with $\mathrm{length}(\tau_i) \xrightarrow[i\to\infty]{} \infty$. If we replace the cusp C by a tube $\mathcal{D}_i$ based on (T,τ_i) we get a sequence $\tilde{V}^i$ of compact manifolds such that $\tilde{V}^i \xrightarrow[i\to\infty]{} V$. The manifolds $\tilde{V}^i$ are not hyperbolic : the natural metrics in $\tilde{V}^i$ have curvature -1 outside of T but at T these metrics are singular, as in the following picture.



Since $\mathcal{D}_i \longrightarrow C$ this singularity is getting "smaller and smaller" as $i \longrightarrow \infty$. In order to eliminate this singularity and to make V_o and some of $\mathcal{D}_i$ fit at T one must construct appropriate deformations of the hyperbolic metrics in V_o and in $\mathcal{D}_i$.

It is not difficult to visualize all possible hyperbolic deformations of a tube because we have an explicit description of <u>all</u> hyperbolic tubes. Thus we are left with a (much more serious) problem of constructing a non-trivial deformation of V_o .

The hyperbolic metric in V_o is essentially determined by the holonomy representation of the fundamental group $\Gamma = \pi_1(V_o) = \pi_1(V)$ in the group of the isometries of the hyperbolic space covering V . Observe, that this representation (and hence, the underlying hyperbolic manifold) may have no non-trivial deformations. For example, there is no deformations when V is compact and $\dim(V) \geq 3$ or when V has finite volume and $\dim(V) \geq 4$. The last rigidity property (this is a special case of A. Weil's rigidity theorem) plays the crucial role in Wang's finiteness theorem stated in § 0.

Let us return to our 3-dimensional case. Notice first that the group of the orientation preserving isometries of H^3 is identical with the group of the conformal transformations of the sphere S^2 i.e. with the group $PSL(2,\mathbb{C})$, that is a complex algebraic group of the complex dimension 3 , and the representations $\Gamma \longrightarrow PSL(2,\mathbb{C})$ form a complex algebraic variety.

When Γ admits a presentation with k generators and ℓ relations the space of the representations $\Gamma \longrightarrow PSL(2,\mathbb{C})$ is of complex dimension at least $3(k-\ell)$ and the space of small non-trivial deformations of a given representation is of dimension at least $3(k-\ell-1)$, because we have to factor out only the trivial deformations, i.e. the conjugations in $PSL(2,\mathbb{C})$.

When V is a non-compact hyperbolic 3-manifold with finite volume its Euler characteristic is zero and the minimal cell decomposition has k cells of dimension 1 and $k-1$ of 2-cells. It follows that $\Gamma = \pi_1(V)$ can be presented by k generators with $k-1$ relations and the crude estimate from above gives no deformations. However, by using a rearrangement of the relations in Γ , Thurston shows that the space of the non-trivial deformations of the representation $\Gamma \longrightarrow PSL(2,\mathbb{C})$ has positive dimension (in general, this dimension is not less than the number of the cusps of V).

In the case of only one cusp, the Mostow rigidity theorem implies that the restriction of the deformed representation of the group $\mathbb{Z}\oplus\mathbb{Z} = \pi_1(T)$ can not be injective and discrete.

It follows, that there exist arbitrary small deformed representations $\mathbb{Z}+\mathbb{Z} \longrightarrow PSL(2,\mathbb{C})$ which factor as $\mathbb{Z}\oplus\mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow PSL(2,\mathbb{C})$ where the last representation is generated by a fixed point free isometry of H^3 , which has an invariant geodesic. It is not hard to see that the image Γ' of the corresponding representation $\Gamma \longrightarrow PSL(2,\mathbb{C})$ is a discrete cocompact group without torsion and the corresponding manifold $V' = H^3/\Gamma'$ is obtained from V by replacing C by a tube. A slightly more careful argument allows us to represent each $\tilde{V}^i$ from above (i is assumed to be sufficiently large) in this form, i.e. to equip it with a hyperbolic structure which is, automatically, close to the original singular metric in $\tilde{V}^i$.

The details, and the generalization to several cusps can be found in chapter 5 of Thurston's lectures [T].

6. Thurston's rigidity theorem

Recall, that the hyperbolic space H^n is projectively isomorphic to the open Euclidean ball $B^n \subset \mathbb{R}^n$ i.e. there is a diffeomorphism $H^n \longrightarrow B^n$ which sends geodesics from H^n onto straight segments. A set $\Delta \subset H^n$ is called a straight simplex if the corresponding $\tilde{\Delta} \subset B^n$ is a usual Euclidean simplex.

The following elementary fact plays the crucial role in Thurston's argument.

<u>For each</u> k , $2 \leq k \leq n$, <u>the hyperbolic volume of a</u> k-<u>dimensional straight</u>

simplex $\Delta \subset H^n$ is bounded by a constant C_k .

When $k = 2$ the maximal simplex $\Delta \subset H^n$ has all three vertices at infinity, i.e. the corresponding $\tilde{\Delta} \subset B^n$ has the vertices at the boundary $S^{n-1} = \partial B^n$. The volume (i.e. area) of Δ is π , i.e. $C_2 = \pi$. Notice that all straight 2-dimensional simplices with vertices at infinity are isometric.

The maximal 3-dimensional simplex Δ also has the vertices at infinity. Milnor showed (see chapter 7 in [T]) that this simplex is regular, i.e. the corresponding $\tilde{\Delta}$ is projectively equivalent to a regular Euclidean simplex with vertices at S^{n-1} . The volume C_3 of this simplex is given by $C_3 = \frac{3}{2} \sum_{i=1}^{\infty} \frac{1}{i^2} \sin(\frac{2\pi i}{3}) \approx 1.0149$ (see [T]).

When $k \geq 4$ the maximal simplices are also regular simplices with vertices at infinity. This is a recent result of Haagerup and Mankholm (see [H-M]).

Let us emphasize that there is a principal difference between the cases $k = 2$ and $k \geq 3$. All ideal (i.e. with the vertices at infinity) 2-simplices are regular and have the same volume, but when $k \geq 3$ the volumes of the ideal simplices vary between 0 and C_k . The last fact is intimately related to the rigidity phenomena in dimensions ≥ 3 .

Notice also that C_k have the following asymptotics when $k \longrightarrow \infty$

$$C_k \sim \frac{\sqrt{k}}{k!} e .$$

(see [H-M]).

A map from an Euclidean simplex s into H^n is called straight if it sends this simplex homeomorphically onto a straight simplex in H^n .

A map from s into a hyperbolic manifold V is called straight if its lifting to the universal covering ($= H^n$) is straight.

A map from a simplicial polyhedron into V is called straight if the restriction of this map to each simplex is straight.

The following fact is obvious.

Let K be an m-dimensional simplicial polyhedron and let V be an n-dimensional hyperbolic manifold with $n \geq m$. Then every continuous map $K \longrightarrow V$ is homotopic to a straight map.

The following result can be viewed as a crude version of Thurston's rigidity theorem.

Thurston's mapping theorem.- Let M be a closed oriented n-dimensional manifold. There exists a constant $C = C(M)$ such that for an arbitrary oriented n-dimensional hyperbolic manifold and for an arbitrary continuous map $f : M \longrightarrow V$ one has

$$|\deg(f)| \leq \frac{C}{\mathrm{Vol}(V)}$$

Proof.- Fix a triangulation of M and assume that f is straight relative to this triangulation. Let $s_1,\dots,s_j$ denote the n-dimensional simplices of this triangulation. We have

$$|\deg(f)| \le (\mathrm{Vol}(V))^{-1} \sum_{i=1}^{j} \mathrm{Vol}(f(s_i)) \le \frac{C_n j}{\mathrm{Vol}(V)} \qquad \text{Q.E.D.}$$

<u>Corollary.- Let</u> V <u>be a compact orientable hyperbolic manifold</u>. <u>Then an arbitrary continuous map</u> $f : V \longrightarrow V$ <u>satisfies</u>

$$|\deg(f)| \le 1$$

<u>Proof</u>.- If $|\deg(f)| \ge 2$ then the iterates of f have arbitrary large degrees.

Let us explain how these ideas can be applied to Thurston's rigidity theorem.

Take two compact oriented hyperbolic manifolds V and V' and triangulate V into straight simplices. Let $s_1,\dots,s_j$ be the n-dimensional simplices of this triangulation and let $v_i = C_n(1-\varepsilon_i)$, $i = 1,\dots,j$, denote the volumes of these simplices.

For a straight map $f : V \longrightarrow V'$ we have

$$|\deg(f)| \le (\mathrm{Vol}(V'))^{-1} \sum_{i=1}^{j} v_i\left(\frac{C_n}{v_i}\right) = (\mathrm{Vol}(V'))^{-1} \sum_{i=1}^{j} \frac{v_i}{1-\varepsilon_i}$$

Let $\varepsilon = \max_i \varepsilon_i$. Then we have

$$(1-\varepsilon)\deg(f) \le (\mathrm{Vol}(V'))^{-1} \sum_{1}^{j} v_i = \frac{\mathrm{Vol}(V)}{\mathrm{Vol}(V')}$$

If we could make $\varepsilon = 0$ we would get $\deg(f) \le \frac{\mathrm{Vol}(V)}{\mathrm{Vol}(V')}$.

Unfortunately, there is no usual triangulation consisting of infinite regular simplices (they are the only ones with volume C_n), but one can use instead some ideal triangulations.

Denote by S the set of all ideal (i.e. with the vertices at infinity) n-dimensional simplices in the universal covering $\tilde{V} = H^n$ and let $R \subset S$ denote the set of the regular simplices. One views the set R as an ideal triangulation of V .

Denote by S' and R' the corresponding sets of simplices associated to V' .

One can show by using Furstenberg's boundary construction (see [F]) or by a direct geometric argument as in [T], that the map f induces a measurable (i.e. the pullbacks of the Borel sets are measurable) map $\tilde{f} : R \longrightarrow S'$. Using the inequality $\mathrm{Vol}(\tilde{f}(s)) \le C_n = \mathrm{Vol}(s)$, $s \in R$, one can show that $\mathrm{Vol}(V') \le d^{-1}\mathrm{Vol}(V)$, $d = \deg(f)$, and that the equality holds iff $\tilde{f}$ sends almost all R into $R' \subset S'$. When $n \ge 3$ (and hence $S' \ne R'$) this property of $\tilde{f}$ implies that the map f is homotopic to an isometric covering. See chapter 6 of [T] for the actual proof which is valid for the noncompact manifolds with finite volume.

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