PUBLICATIONS MATHÉMATIQUES DE L'I.H.É.S.

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Publications mathématiques de l'I.H.É.S., tome 12 (1962), p. 5-68

<http://www.numdam.org/item?id=PMIHES_1962__12__5_0>

ON THE ZETA FUNCTION OF A HYPERSURFACE [1]

By BERNARD DWORK

This article is concerned with the further development of the methods of p-adic analysis used in an earlier article [1] to study the zeta function of an algebraic variety defined over a finite field. These methods are applied to the zeta function of a non-singular hypersurface $\mathfrak{H}$ of degree d in projective n-space of characteristic p defined over the field of q elements. According to the conjectures of Weil [3] the zeta function of $\mathfrak{H}$ is of the form

$$(1) \qquad \zeta(\mathfrak{H}, t) = \mathrm{P}(t)^{(-1)^n} \Big/ \prod_{i=0}^{n-1} (1 - q^i t)$$

where P is a polynomial of degree $d^{-1}\{(d-1)^{n+1} + (-1)^{n+1}(d-1)\}$, (here $n \geq 0$, $d \geq 1$, for a discussion of the trivial cases $n = 0, 1$ see § 4 b below). It is well known that this is the case for plane curves and for special hypersurfaces, [3]. We verify (Theorem 4.4 and Corollary) this part of the Weil conjecture provided $1 = (2, p, d)$, that is provided either p or d is odd.

In our theory the natural object is not the hypersurface alone, but rather the hypersurface together with a given choice of coordinate axes $X_1, X_2, \ldots, X_{n+1}$. If for each (non-empty) subset, A, of the set $S = \{1, 2, \ldots, n+1\}$ we let $\mathfrak{H}_A$ be the hypersurface (in lower dimension if $A \neq S$) obtained by intersecting $\mathfrak{H}$ with the hyperplanes $\{X_i = 0\}_{i \notin A}$, then writing equation (1) for $\mathfrak{H}_A$, we define a rational function P_A by setting

$$(2) \qquad \zeta(\mathfrak{H}_A, t) = P_A(t)^{(-1)^{m(A)}} (1 - q^{m(A)} t) \Big/ \prod_{i=0}^{m(A)} (1 - q^i t),$$

where $1 + m(A)$ is the number of elements in A. If $\mathfrak{H}_A$ is non-singular for each subset A of S and if the Weil conjectures were known to be true then we could conclude that P_A is a polynomial for each subset A.

Our investigation rests upon the fact that without any hypothesis of non-singularity we have

$$\textbf{(4.33)} \qquad \chi_F^{\delta^{n+1}}(t) = (1 - t) \prod_A P_A(qt),$$

[1] This work was partially supported by National Science Foundation Grant Number G7030 and U.S. Army, Office of Ordnance Research Grant Number DA-ORD-31-124-61-G95.

the product on the right being over all subsets A of S and χ_F is the characteristic series of the infinite matrix [2] associated with the transformation $\alpha = \psi \circ F$ introduced in our previous article [1] and studied in some detail in § 2 below. We recall that $\chi_F^S(t) = \chi_F(t)/\chi_F(qt)$ and the fundamental fact in our proof of the rationality of the zeta function is that χ_F is an entire function on Ω, the completion of the algebraic closure of $\mathbf{Q}'$, the field of rational p-adic numbers.

In § 2 we develop the spectral theory of the transformation α and show that the zeros of χ_F can be explained in terms of primary subspaces precisely as in the theory of endomorphisms of finite dimensional vector spaces. In this theory it is natural to restrict our attention to a certain class of subspaces $L(b)$ (indexed by real numbers, b) of the ring of power series in several variables with coefficients in Ω. The definition of $L(b)$ is given in § 2, for the present we need only mention that if $b' > b$, then $L(b') \subset L(b)$.

An examination of (4.33) shows that if the right side is a polynomial and if θ^{-1} is a zero of that polynomial of multiplicity m then $(\theta q^j)^{-1}$ must be a zero of χ_F of multiplicity $m\binom{n+j}{n}$. This is « explained » by the existence of differential operators $D_1, \ldots, D_{n+1}$ satisfying

$$\alpha \circ D_i = q D_i \circ \alpha \tag{4.35}$$

The space $L(b)\Big/\sum_{i=1}^{n+1} D_i L(b)$ is studied in § 3 *d* (in a slightly broader setting than required for the geometric application), for $1/(p-1) \leq b \leq p/(p-1)$, the main results being Lemmas 3.6, 3.10, 3.11. This is applied in § 4 to show that if $\mathfrak{H}_A$ is non-singular for each subset A of S then the right side of (4.33) is a polynomial of predicted degree and is the characteristic polynomial of $\bar{\alpha}$, the endomorphism of $L(b)/\Sigma D_i L(b)$ obtained from α by passage to quotients. (Theorems 4.1, 4.2, 4.3) We emphasize that this result is valid for all p (including $p = 2$).

The main complication in our theory lies in the demonstration (Theorem 4.4 and corollary) that if $1 = (2, p, d)$ then $P_S(tq)$ is the characteristic polynomial of $\bar{\alpha}^S$, the restriction of $\bar{\alpha}$ to the subspace of $L(b)/\Sigma D_i L(b)$ consisting of the image of $L^S(b)$ under the natural map, $L^S(b)$ being the set of all power series in $L(b)$ which are divisible by $X_1 X_2 \ldots X_{n+1}$. This result is of course based on the study (§ 3 *e*) of the action of the differential operators on $L^S(b)$. This study is straightforward for $p \nmid d$ but for $p \mid d$ the main results are shown to be valid only for special differential operators.

We must now explain that for a particular hypersurface we have many choices for the operator α (see § 4 *a* below) but once α is chosen the differential operators satisfying (4.35) are fixed. With a simple choice of α the eigenvector spaces lies in $L\left(\frac{p-1}{p}\right)$ while for a more complicated choice of α the eigenvector space is known to lie in $L\left(\frac{p}{p-1}\right)$. The special differential operators referred to above in connection with the case $p \mid d$ are those which correspond to the simple choice of α for which the eigenvector space lies in $L\left(\frac{p-1}{p}\right)$. Unfortunately $\frac{p-1}{p}<\frac{1}{p-1}$ if $p=2$ (and fortunately, only in that case). Thus for $p=2$, if $2\,|\,d$ we cannot apply the results of § 3 *e* to determine the action of the special differential operators on $L(1/2)$.

Finally (§ 4 *c*) using an argument suggested by J. Igusa, we show that our conclusions concerning $P=P_S$ remain valid without the hypothesis that $\mathfrak{H}_A$ is non-singular for each choice of A.

This completes the sketch of our theory. We believe that our methods can be extended to give similar results for complete intersections. We note that the Weil conjectures for non-singular hypersurfaces also assert that the polynomial P in equation (1) has the factorization $P(t)=\prod_i(1-\theta_i t)$ such that

$$|\theta_i|=q^{(n-1)/2} \text{ for each } i \text{ (Riemann Hypothesis)}$$
$$\theta_i\to q^{n-1}/\theta_i \text{ is a permutation of the } \theta_i \text{ (functional equation).}$$

We make no comment concerning these further conjectures.

In fulfillment of an earlier promise we have included (§ 1) a treatment of some basic function theoretic properties of power series in one variable with coefficients in Ω.

It does not appear convenient to give a complete table of symbols. We note only that throughout this paper, $\mathbf{Z}$ is the ring of integers, $\mathbf{Z}_+$ is the set of non-negative integers and $\mathbf{R}$ is the field of real numbers.

§ 1. P-adic Holomorphic Functions.

Let Ω be an algebraically closed field complete under a rank one valuation $x\to\operatorname{ord} x$. This valuation is a homomorphism of the multiplicative group, Ω^*, of Ω into the additive group of real numbers and is extended to the zero element of Ω by setting $\operatorname{ord} 0=+\infty$. Furthermore $\operatorname{ord}(x+y)\leq\operatorname{Min}(\operatorname{ord} x, \operatorname{ord} y)$ for each pair of elements x, y in Ω and the value group, $\mathfrak{G}$, of Ω (i.e., the image of Ω^* under the mapping $x\to\operatorname{ord} x$) contains the rational numbers.

For each real number b, let

$$\Gamma_b=\{x\in\Omega\,|\operatorname{ord} x=b\}$$
$$U_b=\{x\in\Omega\,|\operatorname{ord} x>b\}$$
$$C_b=\{x\in\Omega\,|\operatorname{ord} x\geq b\}.$$

As is well known, Ω is totally disconnected, and each of these sets are both open and closed. However by analogy with the classical theory it may be useful to refer to the set C_b (resp: U_b) as the closed (resp: open) disk of additive radius b.

$U_{-\infty}$ will be understood to denote Ω and clearly Γ_b is empty if b does not lie in the value group of Ω. We further note that $U_{b'}$ (resp: $C_{b'}$) is a proper subset of U_b (resp: C_b) if $b'>b$. If $b\notin\mathfrak{G}$ then $U_b=C_b$.

The power series in one variable with coefficients in Ω,

$$F(t)=\sum_{s=0}^{\infty} A_s t^s \tag{1.1}$$

will be viewed as an Ω valued function on the maximal subset of Ω in which the series converges. (This is to be interpreted as a remark concerning notation, the power series and the associated function cannot be identified unless (cf. Lemma 1.2 below) the series converges on some disk, U_b, $b>\infty$). It is well known that F converges at $x\in\Omega$ if and only if $\lim_{s\to\infty} A_s x^s=0$. An obvious consequence may be stated :

Lemma **1.1.** — F *converges in* C_b *if and only if*

$$\lim_{j\to\infty} (\text{ord } A_j+jb)=\infty, \tag{1.2}$$

provided $b\in\mathfrak{G}$. *The series converges in* U_b *if and only if*

$$\liminf_{j\to\infty} (\text{ord } A_j)/j\geq -b. \tag{1.3}$$

We may now prove the analogue of Cauchy's inequality as well as the analogue of the maximum principle for closed disks.

Lemma **1.2.** — *If* F *converges on* C_b *and* $b\in\mathfrak{G}$ *then*

$$\operatorname*{Min}_{x\in\Gamma_b} \text{ord } F(x)=\operatorname*{Min}_{0\leqslant j<\infty} (\text{ord } A_j+jb) \tag{1.4}$$

Furthermore

$$\operatorname*{Min}_{x\in\Gamma_b} \text{ord } F(x)=\operatorname*{Min}_{x\in C_b} \text{ord } F(x).$$

Proof. — Since Γ_b is not compact it is not immediately obvious that ord $F(x)$ assumes a minimum value at some point of Γ_b. However the existence of the right side of (1.4) is an immediate consequence of Lemma 1.1. Let $M=\text{Min }(\text{ord } A_j+jb)$, then ord $(A_j x^j)\geq M$ for all $x\in\Gamma_b$ and hence ord $F(t)\geq M$ on Γ_b. Let S be the set of all $j\in\mathbf{Z}_+$ such that ord $A_j+jb=M$. By definition S is not empty and Lemma 1 shows that S is finite. Let $g(t)=\sum_{j\in S} A_j t^j$, $f(t)=F(t)-g(t)$. Lemma 1 also shows that there exists $\varepsilon>0$ such that ord $A_j+jb\geq M+\varepsilon$ for each $j\notin S$. Hence ord $f(t)\geq M+\varepsilon$ everywhere on Γ_b. Let $\pi\in\Gamma_b$, $\pi'\in\Gamma_M$ and let $g_1(t)=g(\pi t)/\pi'$. Let B_j be the coefficient of t^j in g_1. For $j\in S$, ord $B_j=\text{ord } A_j+jb-M=0$. Thus the coefficients of g_1 are integral and the image of g_1 in the residue class field of Ω is non-trivial. Since the residue class field is infinite there exists a unit x in Ω such that ord $g_1(x)=0$. This shows that ord $g(\pi x)=M$. However $\pi x\in\Gamma_b$ and hence ord $F(t)$ assumes the value M on Γ_b. This shows that the left side of (1.4) exists and is equal to the right side. The assertion concerning C_b follows from the obvious fact that for $b'>b$, we have ord $A_j+jb'\geq$ ord A_j+jb for each $j\in\mathbf{Z}_+$ and hence $\operatorname*{Min}_{x\in\Gamma_{b'}} \text{ord } F(x)\geq\operatorname*{Min}_{x\in\Gamma_b} \text{ord } F(x)$, which implies the assertion of the lemma.

As in [1], the ring of power series in one variable, t, with coefficients in Ω, $\Omega\{t\}$, is given the structure of a complete topological ring by letting the subgroups $\{C_b\{t\}+t^m\Omega\{t\}\}_{b\in\mathbf{R},\, m\in\mathbf{Z}_+}$ constitute a basis of the neighborhoods of zero. This topology will be referred to as the weak topology of $\Omega\{t\}$. It may also be described as the topology of coefficientwise convergence.

We now obtain an elementary, but useful relation between convergence in the weak topology and uniform convergence in the function theoretic sense.

Lemma **1.3**. — *Let* $f_1, f_2, \ldots,$ *be a sequence of elements of* $\Omega\{t\}$, *each converging in* C_b, $b\in\mathfrak{G}$.

(i) *If the sequence converges uniformly on* C_b *to a function* F *then*

a) *The sequence is uniformly bounded on* C_b.

b) *The sequence converges in the weak topology to* $f\in\Omega\{t\}$ *which itself converges on* C_b *and* $f(x)=F(x)$ *for all* $x\in C_b$.

(ii) *Conversely, if*

a) *the sequence is uniformly bounded on* C_b,

b) *the sequence converges in the weak topology to* $f\in\Omega\{t\}$

then f *converges in* U_b *and for each* $\varepsilon>0$ *the sequence converges uniformly to* f *on* $C_{b+\varepsilon}$.

Proof. — Let $f_i(t)=\sum_{j=0}^{\infty} A_{i,j}t^j$ for $i=1, 2, \ldots$

(i) Since the sequence converges uniformly on C_b and since, by Lemma 1.2, f_1 is bounded on C_b, we may conclude that the sequence is uniformly bounded on C_b. By hypothesis, given $N>0$ there exists $n\in\mathbf{Z}_+$ such that $\operatorname{ord}(f_i(t)-f_{i'}(t))\geq N$ for all $t\in C_b$ and all $i, i'>n$. Hence by Lemma 1.2, for $i, i'>n$ and for all $j\in\mathbf{Z}_+$

$$(1.5)\qquad \operatorname{ord}(A_{i,j}-A_{i',j})\geq N-jb.$$

For fixed j, (5) shows that $\{A_{i,j}\}_{i=1,2,\ldots}$ is a Cauchy sequence and hence converges to an element A_j of Ω. It now follows from (1.5), letting $i'\to\infty$ that for $i>n$ and all $j\in\mathbf{Z}_+$

$$(1.6)\qquad \operatorname{ord}(A_{i,j}-A_j)\geq N-jb.$$

Let $f(t)=\sum_{j=0}^{\infty} A_jt^j$. If f does not converge on C_b then we may suppose N chosen such that $\operatorname{ord} A_j+jb<N$ for all j in some infinite subset, T, of $\mathbf{Z}_+$. Let i be a fixed integer, $i>n$. Since f_i converges in C_b, we know that $\operatorname{ord} A_{i,j}+jb>N$ for all $j\in\mathbf{Z}_+-T'$ where T′ is a finite (possibly empty) subset of $\mathbf{Z}_+$. For $j\in T-T'$, $\operatorname{ord} A_{i,j}>\operatorname{ord} A_j$, which together with (1.6) shows that $\operatorname{ord} A_j+jb\geq N$. Hence $T-T'$ must be empty, a contradiction, which shows that f converges on C_b. Lemma 1.2, together with equation (1.6), shows that for $i>n$, $\operatorname{ord}(f_i(t)-f(t))\geq N$ everywhere on C_b. In particular for fixed $t\in C_b$, letting $N\to\infty$ we conclude that $f(t)=\lim_{i\to\infty}f_i(t)=F(t)$. This completes the proof of (i).

(ii) By hypothesis the sequence is uniformly bounded on C_b and hence by Lemma 1.2 there exists a real number, M, such that

$$\text{(1.7)} \qquad \text{ord}\, A_{i,j} + jb \geq M$$

for all $i, j \in \mathbf{Z}_+$. Furthermore, writing $f = \sum_{j=0}^{\infty} A_j t^j$, we know that for each $j \in \mathbf{Z}_+$, $\lim_{j\to\infty} A_{i,j} = A_j$. For each $j \in \mathbf{Z}_+$, therefore, there exists i (depending on j) such that $\text{ord}\,(A_{i,j} - A_j) \geq M - jb$. Hence by comparison with equation (1.7) we may conclude that

$$\text{(1.8)} \qquad \text{ord}\, A_j + jb \geq M$$

for all $j \in \mathbf{Z}_+$. This shows that f converges in U_b. Now let ε be a real number, $\varepsilon > 0$. Given a real number N, let $j_0 \in \mathbf{Z}_+$ be chosen such that $j_0\varepsilon + M > N$. Then by (1.7) and (1.8) we have

$$\text{ord}\, A_j + j(b+\varepsilon) > N, \qquad \text{ord}\, A_{i,j} + j(b+\varepsilon) > N$$

for all $i \in \mathbf{Z}_+$ and all $j > j_0$. Hence $\text{ord}\ (A_{i,j} - A_j) + j(b+\varepsilon) > N$ for all $j > j_0$, $i \in \mathbf{Z}_+$, while since $\lim_{i\to\infty} A_{i,j} = A_j$, we may conclude that there exists $n \in \mathbf{Z}_+$ such that $\text{ord}\,(A_{i,j} - A_j) + j(b+\varepsilon) > N$ for all $j \leq j_0$, $i > n$. Hence for $i > n$, $j \in \mathbf{Z}_+$, $\text{ord}\,(A_{i,j} - A_j) + j(b+\varepsilon) > N$ and hence by Lemma 1.2, $\text{ord}\,(f_i(t) - f(t)) > N$ everywhere on C_b, which shows that the sequence converges uniformly to f on $C_{b+\varepsilon}$. This completes the proof of the lemma.

With $F(t)$ as in equation (1.1) we define the j^{th} derivative of F (for $j \in \mathbf{Z}_+$) to be the power series $F^{(j)}(t) = \sum_{s=j}^{\infty} s(s-1)\ldots(s-j+1)A_s t^{s-j}$ and let $F^{[j]}(t) = \sum_{s=0}^{\infty} \binom{s}{j} A_s t^{s-j}$ where $\binom{s}{j}$ denotes the binomial coefficient of t^j in the polynomial $(1+t)^s$. Clearly $F^{(j)} = j!\,F^{[j]}$, the notation $F^{[j]}$ being convenient if the characteristic of Ω is not zero.

We now prove an analogue of Taylor's theorem.

Lemma **1.4.** — *If* $F \in \Omega\{t\}$ *converges in* C_b, $(b \in \mathfrak{G})$ *then*

(i) F *is a continuous function on* C_b *and is the uniform limit of its partial sums.*

(ii) $F^{[j]}$ *converges in* C_b *for each* $j \in \mathbf{Z}_+$.

(iii) *For fixed* $x \in C_b$, *the polynomials* $P_n(t) = \sum_{j=0}^{n} F^{[j]}(x)(t-x)^j$ $(n = 1, 2, \ldots)$ *converge weakly in* $\Omega\{t\}$ *to* $F(t)$. *The element* $L(Y) = \sum_{j=0}^{\infty} F^{[j]}(x) Y^j \in \Omega\{Y\}$ *converges for all* $Y \in C_b$ *and* $F(t) = L(t-x)$ *for each* $t \in C_b$.

Proof. — (i) In the notation of equation (1.1), we conclude from (1.2) that given $N > 0$, there exists $n \in \mathbf{Z}_+$ such that $\text{ord}\ A_j + jb > N$ for all $j > n$. Hence by Lemma 1.2, $\text{ord}\,(F(t) - \sum_{j=0}^{\infty} A_j t^j) > N$ everywhere on C_b. Hence F is the uniform limit on C_b of its partial sums and thus continuity of F follows from the continuity of polynomials.

Assertion (ii) is a direct consequence of Lemma 1.1.

(iii) For $j \in \mathbf{Z}_+$, let $M_j = \operatorname*{Min}_{s \geq j} (\operatorname{ord} A_s + sb)$. Since F converges on C_b, Lemma 1.1 shows that $M_j \to \infty$ as $j \to \infty$. Lemma 1.2 shows that for $x \in C_b$,

$$\operatorname{ord} F^{[j]}(x) \geq \operatorname*{Min}_{s \geq j} \{\operatorname{ord} \tbinom{s}{j} + \operatorname{ord} A_s + (s-j)b\}.$$

Hence

$$\operatorname*{Min}_{x \in C_b} \operatorname{ord} F^{[j]}(x) \geq M_j - jb, \ M_{j+1} \geq M_j. \tag{1.9}$$

Hence by Lemma 1.1, the series L(Y) converges for all $y \in C_b$ and hence by part (i), $P_n(t)$ converges uniformly to $L(t-x)$ on C_b (as $n \to \infty$). Thus in view of part (i) of Lemma 1.3, the proof is completed if we can show that $P_n(t)$ converges weakly to $F(t)$ as $n \to \infty$. Let $P_n(t) = \sum_{s=0}^{n} A_{n,s} t^s$. We must show for fixed s that $\lim_{n \to \infty} A_{n,s} = A_s$. From the definitions

$$A_{n,s} = \sum_{j=0}^{n} F^{[j]}(x) \tbinom{s}{j} (-x)^{j-s}. \tag{1.10}$$

We now write $F = F_n + G_n$, where $F_n(t) = \sum_{j=0}^{n} A_j t^j$. Clearly $A_{n,s} = A'_{n,s} + A''_{n,s}$, where $A'_{n,s}$ (resp. $A''_{n,s}$) is given by the right side of (1.10) upon replacing F by F_n (resp. G_n). Since Taylor's theorem is formally true for polynomials, $A'_{n,s} = A_s$ for $s \leq n$, $A'_{n,s} = 0$ for $s > n$. On the other hand for all $j \in \mathbf{Z}_+$, $\operatorname{ord}(G_n^{[j]}(x)) \geq M_n - jb$ and hence $\operatorname{ord} A''_{n,s} \geq M_n - sb$. Hence for $n \geq s$, $\operatorname{ord}(A_s - A_{n,s}) = \operatorname{ord} A''_{n,s} \geq M_n - sb \to \infty$ as $n \to \infty$. This completes the proof of the lemma.

We can now give some equivalent definitions of the multiplicity of a zero of a power series.

Lemma **1.5.** — *If* F *converges in* C_b, $m \in \mathbf{Z}_+$ *and* $x \in C_b$ *then the following statements are equivalent*

α) $\lim_{t \to x} F(t)/(t-x)^m$ *exists.*

β) $F^{[i]}(x) = 0$ *for* $i = 0, 1, \ldots, m-1$.

γ) *The formal power series,* $F(t)(1-t/x)^{-m}$ *converges in* C_b *if* $x \neq 0$ *while if* $x = 0$, t^m *divides* $F(t)$ *in* $\Omega\{t\}$.

Proof. — By Lemma 1.4 for $t \in C_b$, $t \neq x$, we have

$$F(t)/(t-x)^m = \sum_{i=0}^{m-1} F^{[i]}(x)/(t-x)^{m-i} + \sum_{i=m}^{\infty} F^{[i]}(x)(t-x)^{i-m}.$$

Hence, by the continuity of power series, the limit exists if and only if (β) is true. Thus (α) and (β) are equivalent. If $x = 0$ then (β) and (γ) are clearly equivalent. Hence we may suppose that $x \neq 0$. Let $f \in \Omega\{t\}$, $f(t)(1-t/x)^m = F(t)$. Since the rules of multiplication of formal power series and of convergent power series (in the function theoretic sense) are the same, it follows that if f converges in C_b then as a function, $f(t) = F(t)/(1-t/x)^m$ for all $t \in C_b - \{x\}$. The continuity of convergent power series now shows that (γ) implies (α). To complete the proof we show that (β) implies (γ). It follows from (β) and Lemma 1.4 that in the weak topology $F(t) = \lim_{n\to\infty} \sum_{j=m}^{n} F^{[j]}(x)(t-x)^j$ and hence in that topology, $F(t)(1-t/x)^{-m} = (-x)^m \lim_{n\to\infty} \sum_{j=m}^{n} F^{[j]}(x)(t-x)^{j-m}$. The coefficient B_s of t^s is clearly $B_s = \sum_{j=m}^{\infty} F^{[j]}(x)\binom{j-m}{s}(-x)^{j-s}$ so that by (1.9),

$$\operatorname{ord} B_s \geq \operatorname*{Min}_{j\geq s}\{M_j - sb\}.$$

Thus $\operatorname{ord} B_s + sb \geq M_s$ and since $M_s \to \infty$ with s, this shows that $F(t)(1-t/x)^{-m}$ converges in C_b.

If F converges in C_b, $x \in C_b$, we say that x is a zero of multiplicity $m \geq 0$ if $F^{[i]}(x) = 0$ for $i = 0, 1, \ldots, m-1$, while $F^{[m]}(x) \neq 0$. In particular if H converges in C_b, $x \neq 0$, $H(x) \neq 0$ and $F(t) = (1-t/x)^m H(t)$ then x is a zero of F of multiplicity m.

Let F be an element of $\Omega\{t\}$ which converges in U_b for some $b < \infty$ (i.e., the domain of convergence of F is not the origin). We assume with no loss in generality that $F \in 1 + t\Omega\{t\}$. In the notation of equation (1.1), the *Newton polygon* of F is the convex closure in $\mathbf{R}\times\mathbf{R}$ (=two dimensional Euclidean space with general point (X, Y)) of the positive half of the Y axis and the points $(j, \operatorname{ord} A_j)$, $j = 0, 1, \ldots,$ it being recalled that $\operatorname{ord} A_j = +\infty$ if $A_j = 0$. The Newton polygon will have a second vertical side of infinite extent if F is a polynomial of degree $m > 0$. In this case the boundary of the Newton polygon (excluding the vertical sides) is the graph of a real valued function, h, on the closed interval $[0, m]$. Likewise if F is not a polynomial then the boundary (excluding the vertical side) is the graph of a real valued function, h, on the positive real line. In either case, h is continuous, piecewise linear with monotonically increasing derivative. Furthermore equation (1.3) shows that the graph of h is asymptotic (if F is not a polynomial) to a line of slope $-b$, where b is the minimal element of the extended real line such that F converges in U_b. If x is not an end point of the interval on which h is defined then $h'(x-0) \leq h'(x+0)$. The points at which the strict inequality holds are called the vertices of the polygon. The abscissa, j, of a vertex is an integer and the vertex is then $(j, \operatorname{ord} A_j)$. Finally, if l is the line obtained by extending in both directions a non-vertical side of the Newton polygon of F then for each $j \in \mathbf{Z}_+$, the point $(j, \operatorname{ord} A_j)$ lies on or above the line l.

Lemma **1.6**. — *Let* $F(t) = \sum_{j=0}^{n} A_j t^j = \prod_{j=1}^{n}(1 - t/\alpha_i)$ *be a polynomial of degree* $n > 0$, *with constant term* 1. *Let* $\lambda_1 < \lambda_2 < \ldots < \lambda_s$ *be the distinct values assumed by* $\operatorname{ord} \alpha_i^{-1}$ *as* i *runs from* 1 *to* n *and for* $j = 1, 2, \ldots, s$, *let* r_j *be the number of zeros,* α, *of* F *(counting multiplicities) such that* $-\operatorname{ord} \alpha = \lambda_j$. *The vertices of the Newton polygon of* F *are the origin* P_0, *and the s points*

$$P_a = \left(\sum_{i=1}^{a} r_i,\ \sum_{=1}^{a} r_i\lambda_i\right) \qquad \textbf{(1.11)}$$

$a = 1, 2, \ldots, s.$

Proof. — Let the zeros of F be so ordered that ord $\alpha_1^{-1} \leq \text{ord}\,\alpha_2^{-1} \leq \ldots \leq \text{ord}\,\alpha_n^{-1}$. The proof may be simplified by letting $r_0 = 0$, λ_0 be any real number, say $\lambda_1 - 1$. Then $P_a = \left(\sum_{i=0}^{a} r_i, \sum_{i=0}^{a} r_i\lambda_i\right)$ for $a = 0, 1, \ldots, s$. Let j_a be the abcissa of P_a, then A_{j_a} is the sum of all products of the α_i^{-1} taken j_a at a time. This sum is dominated by $\prod_{i=1}^{j_a} \alpha_i^{-1}$. Hence ord $A_{j_a} = \text{ord} \prod_{i=1}^{j_a} \alpha_i^{-1} = \sum_{i=0}^{j_a} r_i\lambda_i$. If $a > 0, j_{a-1} < j < j_a$ then

$$\text{ord}\, A_j \geq \text{ord} \prod_{i=1}^{j} \alpha_i^{-1} = \sum_{i=0}^{a-1} r_i\lambda_i + \lambda_a(j - j_{a-1})$$

and hence the point $(j, \text{ord}\, A_j)$ lies on or above the line $P_{a-1}P_a$ since the equation of that line is

$$Y - \sum_{i=0}^{a-1} r_i\lambda_i = \lambda_a(X - j_{a-1}). \tag{1.12}$$

Thus the Newton polygon is the convex closure of the $s+1$ points $P_0, P_1, \ldots, P_s$ and the point $(0, +\infty)$. Equation (1.12) shows that the slope does change at the points $P_1, P_2, \ldots, P_{s-1}$ and this completes the proof.

Corollary. — *The numbers* $\{\text{ord}\,\alpha_i^{-1}\}_{i=1}^{n}$ *are precisely the slopes of the non-vertical sides of the Newton polygon of* F. *If* λ *is such a slope then the number of zeros* α *of* F *such that* $\text{ord}\,\alpha = -\lambda$ *is the length of the projection on the* X*-axis of the side of slope* λ.

We now prove a refined form of a well-known theorem [4, Theorem 10, p. 41] which states roughly that two polynomials of equal degree have approximately the same zeros if the coefficients of the polynomial are approximately equal.

Lemma **1.7.** — *Let* f *and* g *be elements of* $\Omega[t]$ *and let* λ *be an element of the value group* $\mathfrak{G}$ *of* Ω *such that*

a) $f(0) = g(0) = 1$

b) *The number (counting multiplicities) of zeros of* f *on* Γ_λ *is a strictly positive integer*, n.

If N *is a strictly positive real number such that*

$$\min_{x \in \Gamma_\lambda} \text{ord}\,(f(x) - g(x)) > n\text{N}, \tag{1.13}$$

then each (multiplicative) coset of $1 + C_N$ *contains the same number of zeros of* f *in* Γ_λ *as of* g.

Proof. — Let $\alpha_1, \ldots, \alpha_n$ be the zeros of f in Γ_λ, let $\gamma_1, \ldots, \gamma_m$ be the (possibly empty) set of zeros of f in U_λ and let S be the set of zeros of f outside C_λ. Clearly for $\alpha \in S$, ord $\alpha < \lambda$ and hence if $\beta \in \Gamma_\lambda$, ord $(1 - \beta/\alpha) = 0$. Since ord $\gamma_i > \lambda$, we have ord $(1 - \beta/\gamma_i) = \text{ord}\,(\beta/\gamma_i) = \lambda - \text{ord}\,\gamma_i < 0$ for $i = 1, 2, \ldots, m$ if $\beta \in \Gamma_\lambda$. Since

$$f(t) = \prod_{i=1}^{m} (1 - t/\alpha_i) \cdot \prod_{i=1}^{n} (1 - t/\gamma_i) \cdot \prod_{\alpha \in S} (1 - t/\alpha)$$

we may conclude that for $\beta\in\Gamma_\lambda$, $\operatorname{ord} f(\beta) = \sum_{i=1}^{n} \operatorname{ord}(1-\beta/\alpha_i) + \sum_{i=1}^{m} (\lambda - \operatorname{ord} \gamma_i)$. Letting $c = \sum_{i=1}^{m} (-\lambda + \operatorname{ord} \gamma_i)$, we note that c is independent of $\beta\in\Gamma_\lambda$. Letting $\alpha'_1, \alpha'_2, \ldots, \alpha'_n$, be the (possibly empty) set of zeros of g in Γ_λ we conclude by the same argument as above that there exists a constant $c'\geq 0$ such that for $\beta\in\Gamma_\lambda$

$$(1.14)\qquad \begin{cases} \operatorname{ord} f(\beta) = -c + \sum_{i=1}^{n} \operatorname{ord}(1-\beta/\alpha_i) \\ \operatorname{ord} g(\beta) = -c' + \sum_{i=1}^{n'} \operatorname{ord}(1-\beta/\alpha'_i), \end{cases}$$

it being understood that $\operatorname{ord} g(\beta) = -c'$ if $n'=0$. It is easy to see that $n'\neq 0$ for otherwise $\operatorname{ord} g(\alpha_1) = -c' \leq 0 < n\mathrm{N} < \operatorname{ord}(f(\alpha_1)-g(\alpha_1)) = \operatorname{ord} g(\alpha_1)$, a contradiction.

Let $\beta_1, \ldots, \beta_e$ be chosen in Γ_λ such that $\beta_1(1+\mathrm{C_N}), \ldots, \beta_e(1+\mathrm{C_N})$ are disjoint and such that their union contains all zeros of f and g in Γ_λ. If $e>1$, $\operatorname{ord}(1-\beta_j/\beta_1)<\mathrm{N}$ for $j=2, 3, \ldots e$ and hence there exists $\varepsilon>0$ such that

$$(1.15)\qquad 0\leq \operatorname{ord}(1-\beta_j/\beta_1) < \mathrm{N}-\varepsilon \quad \text{for} \quad 2\leq j\leq e.$$

If $e=1$, we interpret this condition to mean simply $0<\varepsilon<\mathrm{N}$. With ε so chosen we shall for the remainder of the proof let β be a variable element of Γ_λ satisfying the condition

$$(1.16)\qquad \mathrm{N}-\varepsilon < \operatorname{ord}(1-\beta_1/\beta) < \mathrm{N}.$$

We now show that if $\alpha\in\beta_i(1+\mathrm{C_N})$ then

$$(1.17)\qquad \mathrm{N} > \operatorname{ord}(1-\beta/\alpha) = \begin{array}{ll} \operatorname{ord}(1-\beta/\beta_1) & \text{if} \quad i=1 \\ \operatorname{ord}(1-\beta_i/\beta_1) & \text{if} \quad i\neq 1. \end{array}$$

For $i=1$ this follows from $\alpha/\beta = (\alpha/\beta_1)(\beta_1/\beta)\in(\beta_1/\beta)(1+\mathrm{C_N})$, while by (1.16) $(\beta_1/\beta)\notin(1+\mathrm{C_N})$. For $i\geq 2$, we have $\alpha/\beta\in(\beta_i/\beta)(1+\mathrm{C_N}) = (\beta_i/\beta_1)(\beta_1/\beta)(1+\mathrm{C_N})$ while by (1.15) and (1.16) $\operatorname{ord}(1-\beta_i/\beta_1)<\mathrm{N}-\varepsilon<\operatorname{ord}(1-\beta_1/\beta)$. This completes the proof of (1.17).

In particular if α is a zero of f in Γ_λ then, by (1.17), $\operatorname{ord}(1-\beta/\alpha)<\mathrm{N}$ and hence by (1.14) since $c\geq 0$, $\operatorname{ord} f(\beta)<n\mathrm{N}$. From (1.13) we now see that $\operatorname{ord} f(\beta) = \operatorname{ord} g(\beta)$ and thus equation (1.14) shows that

$$(1.18)\qquad -c + \sum_{i=1}^{n} \operatorname{ord}(1-\beta/\alpha_i) = -c' + \sum_{i=1}^{n'} \operatorname{ord}(1+\beta/\alpha'_i)$$

For $j=1, 2, \ldots, e$, let n_j (resp. n'_j) be the number of zeros of f (resp. g) in $\beta_j(1+\mathrm{C_N})$. Equations (1.17) and (1.18) now give

$$(1.19)\qquad (n_1-n'_1)\operatorname{ord}(1-\beta/\beta_1) = c-c' + \sum_{j=2}^{e} (n'_j-n_j)\operatorname{ord}(1-\beta_j/\beta_1)$$

the right side being simply $c-c'$ if $e=1$. As β varies under the constraints of (1.16), $\operatorname{ord}(1-\beta_1/\beta)$ varies at least over the rational points in the open interval $(\mathrm{N}-\varepsilon, \mathrm{N})$

while the right side of (1.19) is independent of β. This shows that $n_1 = n'_1$ and by the same argument $n_i = n'_i$ for $i = 2, 3, \ldots, e$. This completes the proof of the lemma.

As an immediate consequence we state the following corollary.

Corollary. — Let f and g be elements of $\Omega[t]$ such that $f(0) = g(0) = 1$. Let b be an element of $\mathfrak{G}$ and let m be the number (counting multiplicities) of zeros of f in C_b.

1. *If* $\operatorname*{Min}_{x \in C_b} \operatorname{ord}(f(x) - g(x)) > 0$ *then the sides of the Newton polygon of f of slope not greater than $-b$ coincide with the corresponding sides of the Newton polygon of g.*

2. *If* N *is a strictly positive real number and*

$$\operatorname*{Min}_{x \in C_b} \operatorname{ord}(f(x) - g(x)) > mN$$

then each coset of $1 + C_N$ *in* C_b *contains the same number of zeros of f as of g.*

We can now demonstrate the main properties of the Newton polygons of power series.

Theorem **1.1.** — *Let $b' < b < \infty$, $b \in \mathfrak{G}$ and let* F *be an element of $\Omega\{t\}$ converging in* $U_{b'}$, $F(0) = 1$. *Let m be the total length of the projection on the* X *axis of all sides of the Newton polygon of* F *of slope not greater than $-b$. There exists a polynomial* G *of degree m,* $(G(0) = 1)$ *and an element* H *of $\Omega\{t\}$ such that the zeros of* G *lie entirely in* C_b *and*

(i) H *converges in* $U_{b'}$, $\operatorname{ord} H(t) = 0$ *everywhere in* C_b.

(ii) $F = GH$.

These conditions uniquely determine G *and* H. *Furthermore* :

(iii) *The Newton polygon of* G *coincides with that of* F *for* $0 \leq X \leq m$ *while the polygon of* H *is obtained from the set*: (Polygon of F) — (Polygon of G) *by the translation which maps the point* $(m, \operatorname{ord} A_m)$ *into the origin.*

(iv) *If* K *is a complete subfield of Ω which contains all the coefficients of* F, *then* $G \in K[t]$.

(v) *If for each partial sum,* F_n, *of* F *we write* $F_n = G_n H_n$, *where* G_n *is the normalized polynomial whose zeros are precisely those of* F_n *(counting multiplicities) in* C_b, *then* G_n *converges to* G *in the weak topology of $\Omega\{t\}$.*

(vi) *If* $n \in \mathbf{Z}_+$ *and* N *is a strictly positive real number such that* $\operatorname{ord}(F(t) - F_n(t)) > mN$ *everywhere on* C_b, *then each coset of* $1 + C_N$ *in* C_b *contains as many zeros of* F *as of* F_n.

Proof. — We follow the procedure of part (v). For $n \geq m$ the Newton polygon of F_n coincides with that of F in the range $0 \leq X \leq m$ and furthermore all sides of the polygon of F_n of slope not greater than $-b$ occur in that range. This shows that for $n \geq m$, F_n has m zeros in C_b. Since the sequence $\{F_n\}$ converges uniformly on C_b to F, we conclude that given $N > 0$, there exists $n_1 \in \mathbf{Z}$, $n_1 \geq m$, such that if n and n' are integers not less than n_1 then $\operatorname{ord}(F_n - F_{n'}) > mN$ everywhere on C_b. We may conclude from the corollary to the previous lemma that each coset of $1 + C_N$ in C_b contains as many zeros of F_n as of $F_{n'}$ and hence the same holds for G_n and $G_{n'}$. This shows that for $n \geq m$ we may write $G_n(t) = \prod_{i=1}^{m} (1 - t/\alpha_{n,i})$ where the zeros $\alpha_{n,1}, \ldots, \alpha_{n,m}$ of G_n are so ordered that $\lim_{n\to\infty}\alpha_{n,i}=\alpha_i$ exists for $i=1, 2, \ldots, m$. This shows that G_n converges to G, a polynomial of degree m whose Newton polygon coincides with that of F_{n_1} and hence with that of F for $0\leq X\leq m$.

For each $n\in\mathbf{Z}_+$, $H_n(t)$ is a product of factors of type $(1-t/\alpha)$ where ord $\alpha<b$. Hence

$$\text{ord}\, H_n(t)=0 \tag{1.20}$$

everywhere on C_b. G_n is a product of factors of type $(1-t/\alpha)$, where $\alpha\in C_b$ and hence if ord $t<b$ then ord $G_n(t)\leq 0$ (equality holds if $G_n(t)=1$). If then $b''\in\mathfrak{G}$, $b>b''>b'$, then ord $G_n(t)\leq 0$ everywhere on $\Gamma_{b''}$ and hence ord $H_n(t)=$ ord $F_n(t)-$ord $G_n(t)\geq$ ord $F_n(t)$ everywhere on $\Gamma_{b''}$. Lemma 1.3 shows that $F_n(t)$ is uniformly bounded on $\Gamma_{b''}$ and hence the same holds for $H_n(t)$. Hence by Lemma 1.2 the sequence $H_1, H_2, \ldots$ is uniformly bounded on $C_{b''}$. We show that the sequence $H_1, H_2, \ldots$ converges in the weak topology of $\Omega\{t\}$. This follows from the fact that $1+t\Omega\{t\}$ is a complete multiplicative group under the weak topology. Certainly $F_n\to F$ and $G_n\to G$ in that topology and hence $H_n=F_n/G_n$ converges weakly to the power series $H=F/G\in 1+t\Omega\{t\}$. It now follows from Lemma 1.3 (part ii) that H converges in $U_{b''}$ (and hence letting $b''\to b'$, in $U_{b'}$) and that for each $\varepsilon>0$, H_n converges uniformly on $C_{b'+\varepsilon}$ to H. Using equation (1.20), it is now clear that $H(t)$ is a unit everywhere on C_b.

This completes the proof of parts (i), (ii), (v). Assertion (iii) has been verified for G, its verification for H follows from Lemma 1.6 and the fact that $H_n\to H$. Assertion (vi) follows from the construction of G, the corollary to Lemma 1.7 and from the fact that the zeros of F in C_b are precisely those of G.

To verify (iv) it is enough to show that $G_n\in K[t]$ for each $n\in\mathbf{Z}_+$ since then $G=\lim G_n\in K[t]$. Since the valuation in a finite field extension of K is invariant under automorphisms which leave K pointwise fixed, we may conclude that the coefficients of G_n are purely inseparable over K. Thus we may suppose K is of characteristic $p\neq 0$. If α is a root of G_n then it is a root of F_n of the same multiplicity and hence the multiplicity must be a multiple, mp^r, of a power of p such that α^{p^r} is separable over K. This shows that the coefficients of G_n are separable over K which now shows that $G_n\in K[t]$. This completes the proof of the theorem.

Part (v) of the above theorem has an important generalization which is the analogue of a theorem of Hurwitz.

Theorem **1.2.** — *Let $b'<b<\infty$, $b\in\mathfrak{G}$ and let $f_1, f_2, \ldots$ be a sequence of elements of $\Omega\{t\}$, each converging in $C_{b'}$ such that $f_j(0)=1$ for each $j\in\mathbf{Z}_+$ and such that the sequence converges uniformly on $C_{b'}$ to $F\in\Omega\{t\}$. By the preceding theorem, $F=GH, f_j=g_jh_j$ where G (resp. g_j) is a polynomial whose zeros are precisely those of F (resp. f_j) in C_b, and $G(0)=g_j(0)=1$. The conclusion is that $G=\lim_{i\to\infty}g_i$ and that for i large enough, g_i and G are polynomials of equal degree.*

Proof. — Let degree $G = m$ and for each $j \in \mathbf{Z}_+$, let F_j be the j^{th} partial sum of F and let $f_{i,j}$ be the j^{th} partial sum of f_i. Let N be a strictly positive real number. Pick $j \in \mathbf{Z}_+$ such that

$$\text{(1.21)} \qquad \operatorname{ord}(F(t) - F_j(t)) > mN$$

everywhere on C_b. Part (vi) of Theorem 1.1 shows that F_j has m zeros in C_b. Pick i_0 such that for each $i > i_0$

$$\text{(1.22)} \qquad \operatorname{ord}(F - f_i) > mN$$

everywhere on C_b. Pick $u \in \mathbf{Z}_+$ such that for given $i > i_0$

$$\text{(1.23)} \qquad \operatorname{ord}(f_i - f_{i,u}) > mN$$

everywhere on C_b. We may conclude from these three relations that

$$\text{(1.24)} \qquad \operatorname{ord}(F_j - f_{i,u}) > mN$$

everywhere on C_b, and the Corollary to Lemma 1.7 now shows that each coset of $1 + C_N$ in C_b has as many zeros of F_j as of $f_{i,u}$ and in particular $f_{i,u}$ has m zeros in C_b. Equation (1.23) together with part (vi) of Theorem 1.1 now shows that f_i has m zeros in C_b. Furthermore equations (1.21) and (1.23) and part (vi) shows that each coset of $1 + C_N$ contains as many zeros of F in C_b as of F_j and as many zeros of f_i as of $f_{i,u}$. We may now conclude that each coset of $1 + C_N$ contains as many zeros of F in C_b as of f_i for each $i > i_0$. It is now clear that $g_i \to G$ and that deg $g_i = m$ for i large enough.

Corollary. — *Under the hypothesis of the theorem, for i large enough, the zeros* $\alpha_{i,1}, \alpha_{i,2}, \ldots \alpha_{i,m}$ *of* f_i *in* C_b *may be so ordered that* $\lim\limits_{i \to \infty} \alpha_{i,j} = \alpha_j, j = 1, 2, \ldots m$ *and* $\alpha_1, \ldots, \alpha_m$ *are the zeros of* F *in* C_b.

We conclude by recalling that in our previous article we left two propositions unverified. Proposition 2 of [1] is contained by Theorem 1.1 above. We now demonstrate Proposition 1.

Proposition. — *If* $b' < b < \infty$ *and* F *converges in* $U_{b'}$ *but is never zero in* U_b, *then the series* $1/F$ *converges in* U_b.

Proof. — As before we may assume $A_0 = 1$. The Newton polygon of F has no side of slope less than $-b$ and hence $\operatorname{ord} A_j \geq -jb$. The conditions $A_0 = 1$, $\operatorname{ord} A_j \geq -jb$ define a subgroup of $1 + t\Omega\{t\}$ and hence are satisfied by the formal power series $1/F$. This shows by Lemma 1.1 that $1/F$ converges in U_b.

§ 2. Spectral Theory.

Let $\mathbf{Q}'$ be the field of rational p-adic numbers, Ω the completion of the algebraic closure of $\mathbf{Q}'$, the valuation of Ω being given by the ordinal function $x \to \operatorname{ord} x$ which is normalized by the condition $\operatorname{ord} p = +1$.

Let q, n, d be integers $q > 1$, $d \geq 1$, $n \geq 0$ which will remain fixed throughout this section. Let $\mathfrak{T}$ be the set of all $u=(u_0, u_1, \ldots, u_n)\in\mathbf{Z}_+^{n+1}$ such that $du_0\geq u_1+\ldots+u_n$. The set, $\mathbf{Z}_+^{n+1}$ may be viewed as imbedded in $n+1$ dimensional Euclidean space, $\mathbf{R}^{n+1}$ and let σ be the projection $(y_0, y_1, \ldots, y_n)\rightarrow y_0$ of $\mathbf{R}^{n+1}$ onto $\mathbf{R}$.

We formalize and reformulate in a manner convenient for our present application the methods appearing in the second half of the proof of Theorem 1 [1].

Lemma 2.1. — *Let c_m be the minimal value of $\left(\sum_{i=1}^{m} u^{(i)}\right)$ as $(u^{(1)}, \ldots, u^{(m)})$ runs through all sets of m distinct elements of $\mathfrak{T}$. Then $c_m/m\rightarrow\infty$ as $m\rightarrow\infty$.*

Let $\mathfrak{M}$ be an infinite matrix with coefficients $\mathfrak{M}_{u,v}$ (in Ω) indexed by $\mathfrak{T}\times\mathfrak{T}$ which have the property $\operatorname{ord}\mathfrak{M}_{u,v}\geq\varkappa\sigma(qu-v)$ where $\varkappa$ is a strictly positive real number. When convenient we write $\mathfrak{M}(u, v)$ instead of $\mathfrak{M}_{u,v}$.

Lemma 2.2. — (i) *If $\mathfrak{M}'$ is any finite submatrix of $\mathfrak{M}$ obtained by restricting the indices (u, v) to $\mathfrak{T}'\times\mathfrak{T}'$ where $\mathfrak{T}'$ is a finite subset of $\mathfrak{T}$, then the coefficient γ_m, of t^m in* $\det(\mathrm{I}-t\mathfrak{M}')$ *satisfies the condition*: $\operatorname{ord}\gamma_m\geq\varkappa(q-1)c_m$. *Hence for $t\in\Omega$,* $\operatorname{ord}\det(\mathrm{I}-t\mathfrak{M}')\geq\operatorname*{Min}_{m=0}(m\operatorname{ord}t+\varkappa(q-1)c_m)$, *an estimate depending only on* ord *t and the constants $\varkappa$, q, d, n, but independent of $\mathfrak{M}'$. In particular for each bounded disk of Ω, $\det(\mathrm{I}-t\mathfrak{M}')$ is uniformly bounded as $\mathfrak{T}'$ varies over all finite subsets of $\mathfrak{T}$.*

(ii) *If $(u, v)\in\mathfrak{T}'\times\mathfrak{T}'$, then the minor of (u, v) in the matrix $(\mathrm{I}-t\mathfrak{M}')$ is a polynomial $\sum_m\gamma_m(u, v)t^m$ and*

$$\operatorname{ord}\gamma_m(u, v)\geq q\varkappa\sigma(v-u)+\varkappa(q-1)c_m.$$

Hence for $t\in\Omega$, ord (*minor of (u, v) in $(\mathrm{I}-t\mathfrak{M}')$*) $\geq q\varkappa\sigma(v-u)+c$, *where c is a constant independent of $\mathfrak{M}'$ and $\mathfrak{T}'$ (if* ord *t is fixed).*

Proof. — (ii) The coefficient, $\gamma_m(u, v)$ is a sum of products $\mathrm{P}=\pm\prod_{i=1}^{m}\mathfrak{M}(u^{(i)}, v^{(i)})$, where $\{u, u^{(1)}, \ldots, u^{(m)}\}$ is a set of $m+1$ distinct elements of $\mathfrak{T}'$ and $\{v, v^{(1)}, \ldots, v^{(m)}\}$ is a permutation of that set. Hence

$$\varkappa^{-1}\operatorname{ord}\mathrm{P}\geq\sum_{i=1}^{m}\sigma(qu^{(i)}-v^{(i)})=\sigma\{q\left(u+\sum_{i=1}^{m}u^{(i)}\right)-\left(v+\sum_{i=1}^{m}v^{(i)}\right)-(qu-v)\}=$$
$$\sigma\{(q-1)\left(v+\sum_{i=1}^{m}v^{(i)}\right)-(qu-v)\}\geq q\sigma(v-u)+(q-1)c_m.$$

Lemma 2.3. — *For $\mathrm{N}\in\mathbf{Z}_+$, let $\mathfrak{M}'_{\mathrm{N}}$ be the submatrix of $\mathfrak{M}$ obtained as in the previous lemma by letting $\mathfrak{T}'=\{u\in\mathfrak{T}\,|\,\sigma(u)\leq\mathrm{N}\}$. Let $\mathfrak{M}_{\mathrm{N}}$ be the matrix obtained from $\mathfrak{M}'_{\mathrm{N}}$ by replacing $\mathfrak{M}_{u,v}$ by zero whenever $\sigma(qu-v)>(q-1)\mathrm{N}$. Then $\lim_{\mathrm{N}\to\infty}\det(\mathrm{I}-t\mathfrak{M}_{\mathrm{N}})=\lim_{\mathrm{N}\to\infty}\det(\mathrm{I}-t\mathfrak{M}'_{\mathrm{N}})$, the limit being in the sense of uniform convergence on each bounded disk of Ω. The limit is an entire function, $\sum_{m=0}^{\infty}\gamma_m t^m$, and* $\operatorname{ord}\gamma_m\geq(q-1)\varkappa c_m$.

The remaining proofs may be omitted since they are consequences of the methods of [1]. Lemma 2.3 follows from Lemma 2.2 and Lemma 1.3 (part (ii)) once it is verified that the two sequences converge weakly to the same limit. However the details concerning weak convergence are very similar to the proof of Lemma 2.2. (We note that the method used in [1, equ. (20.2)] to show weak convergence cannot be used here as that proof made use of the geometrical application.)

Let $\Omega\{X\}$ be the ring of power series and $\Omega[X]$ the ring of polynomials in $n+1$ variables $X_0, X_1, \ldots, X_n$ with coefficients in Ω. If $u=(u_0, u_1, \ldots, u_n)\in\mathbf{Z}_+^{n+1}$, let X^u denote the monomial $\prod_{i=0}^{n} X^{u_i}$. Let ψ be the endomorphism of $\Omega\{X\}$ or $\Omega[X]$ as linear space over Ω defined by $\psi(X^u)=\begin{cases} 0 & \text{if} \quad q\nmid u \\ X^{u/q} & \text{if} \quad q\,|\,u \end{cases}$.

For each ordered pair of real numbers (b, c), let $L(b, c)$ be the additive group of all elements $\Sigma A_u X^u\in\Omega\{X\}$ such that

(i) $$A_u=0 \quad \text{if} \quad u\notin\mathfrak{I}$$
(ii) $$\operatorname{ord} A_u\geq bu_0+c.$$

Let $L(b)=\bigcup_{c\in\mathbf{R}} L(b, c)$, $\mathfrak{L}$ be the subspace of $\Omega[X]$ spanned by $\{X^u\}_{u\in\mathfrak{I}}$. For each integer $N\geq 0$, let $\mathfrak{L}^{(N)}$ be the subspace of $\mathfrak{L}$ consisting of elements of degree not greater than N as polynomials in X_0. Let $\mathfrak{L}(b, c)=\mathfrak{L}\cap L(b, c)$, $\mathfrak{L}^{(N)}(b, c)=\mathfrak{L}^{(N)}\cap L(b, c)$.

If $H\in\Omega\{X\}$, let $\psi\circ H$ denote the linear transformation $\xi\rightarrow\psi(H\xi)$ of $\Omega\{X\}$ into itself.

Lemma **2.4**. — *Let* μ *be any mapping of* $\Omega[X]$ *into the real numbers such that for* ξ_1, ξ_2, $\xi\in\Omega\{X\}$, $c\in\Omega$, $c\neq 0$,

$$(\mathbf{2.1}) \qquad \begin{cases} \mu(\xi_1\xi_2)\leq\mu(\xi_1)+\mu(\xi_2),\ \mu(0)=-\infty \\ \mu(\psi\xi)\leq\mu(\xi)/q,\ \mu(c\xi)=\mu(\xi) \\ \mu(\xi_1+\xi_2)\leq\operatorname{Max}(\mu(\xi_1), \mu(\xi_2)). \end{cases}$$

If s *is an integer,* $s\geq 1$, λ *is a non-zero element of* Ω *and* ξ *is a polynomial such that*

$$(\mathbf{2.2}) \qquad (I-\lambda^{-1}\psi\circ H)^s\xi=0,\ (H\neq 0)$$

then

$$\mu(\xi)\leq\mu(H)/(q-1).$$

The proof may be omitted as it follows trivially from the fact that for $\eta\in\Omega\{X\}$,

$$\mu(\psi(H\eta))\leq(\mu(H)+\mu(\eta))/q.$$

In particular if h is a linear homogeneous function on $\mathbf{R}_+^{n+1}$ and if for each $\eta\in\Omega\{X\}$, $\mu(\eta)$ is the maximum value assumed by $h(u)$ as X^u runs through all monomials occurring in η, then μ satisfies the conditions of Lemma 2.4. In particular if $H\in\mathfrak{L}^{(N(q-1))}$, then letting $h(u)=u_1+\ldots+u_n-du_0$, we may conclude that if ξ satisfies (2.2) then ξ lies in $\mathfrak{L}$ and letting $h(u)=u_0$ we may conclude that ξ lies in $\mathfrak{L}^{(N)}$.

Thus the definition of det $(I-t\psi\circ H)$ appearing in our earlier work is unchanged if $(\psi\circ H)$ is restricted to $\mathfrak{L}^{(m)}$ for any integer $m\geq N$.

Now let $\varkappa$ be a strictly positive rational number. Let $F=\Sigma A_u X^u$ be an element of $L(\varkappa, 0)$ which will remain unchanged in the remainder of this section. We associate with F a power series χ_F, the *characteristic series* of $\psi \circ F$ which generalizes the characteristic polynomial appearing in the case in which F is a polynomial. For each integer $N \geq 0$, let T_N be the linear mapping of $L(-\infty)$ into $\mathfrak{L}^{(N)}$ defined by $T_N(X^u) = \begin{cases} X^u & \text{if } u_0 \leq N \\ 0 & \text{otherwise} \end{cases}$. Let α_N be the mapping $\xi \rightarrow \psi(\xi(T_{N(q-1)}F))$, and let α'_N be the mapping $\xi \rightarrow T_N(\psi(\xi F))$ of (say) $\mathfrak{L}^{(N)}$ into itself. If in the terminology of Lemma 2.3, we set $\mathfrak{M}_{u,v} = A_{qu-v}$ for all $(u, v) \in \mathfrak{I} \times \mathfrak{I}$, then relative to a monomial basis of $\mathfrak{L}^{(N)}$ the matrix form of α_N is $\mathfrak{M}_N$ while that of α'_N is $\mathfrak{M}'_N$. Hence $\lim_{N\rightarrow\infty} \det(I - t\alpha_N)$ and $\lim_{N\rightarrow\infty} \det(I - t\alpha'_N)$ both exist and are equal by Lemma 2.3. The characteristic series, χ_F, is defined to be this common limit. Lemma 2.3 shows that χ_F is entire and lies in $\mathfrak{O}\{t\}$, $\mathfrak{O}$ being the ring of integers of Ω.

The mapping $\alpha : \xi \rightarrow \psi(F\xi)$ of $\Omega\{X\}$ into itself will now be examined. We first show by a general example that a satisfactory theory cannot be obtained if we allow α to operate on the entire space $\Omega\{X\}$. If F has constant term 1 then let $G(X) = \prod_{j=0}^{\infty} F(X^{q^j})$. Clearly, $F(X) = G(X)/G(X^q)$ and hence if $\lambda \neq 0$, $\lambda \in \Omega$ then $\xi = \sum_{j=0}^{\infty} \lambda^j X_0^{q^j}/G(X)$ is a non-zero element of $\Omega\{X\}$, while $\alpha\xi = \lambda\xi$. Thus as an operator on $\Omega\{X\}$ each non-zero element of Ω is an eigenvalue of α. We shall show that χ_F can be explained by restricting α to $L(q\varkappa)$. However to obtain a complete theory it will be necessary to assume that the coefficients of F lie in a finite extension of $\mathbf{Q}'$.

Let $\mathbf{Q}$ be the field of rational numbers. The value group of Ω is the additive group of $\mathbf{Q}$. For $x = (x_0, x_1, \ldots, x_n) \in \Omega^{n+1}$, let $\operatorname{ord} x = (\operatorname{ord} x_0, \operatorname{ord} x_1, \ldots, \operatorname{ord} x_n) \in \mathbf{Q}^{n+1}$ if none of the x_i are zero.

If a and a' are elements of $\mathbf{Q}^{n+1}$, we define the usual inner product

$$\rho(a, a') = \sum_{i=0}^{n} a_i a'_i. \tag{2.3}$$

If $\xi \in \Omega\{X\}$, let S_ξ be the set of all $a \in \mathbf{Q}^{n+1}$ such that ξ converges at x if $\operatorname{ord} x = a$. Writing

$$\xi = \sum_{u \in \mathbf{Z}_+^{n+1}} B_u X^u, \tag{2.4}$$

we have a generalization of Lemma 1.1 : If $a \in \mathbf{Q}^{n+1}$ then $a \in S_\xi$ if and only if $\operatorname{ord} B_u + \rho(u, a) \rightarrow +\infty$ as $u \rightarrow \infty$ in $\mathbf{Z}_+^{n+1}$.

It is convenient to introduce a partial ordering of $\mathbf{Q}^{n+1}$. If a and a' are elements of $\mathbf{Q}^{n+1}$, we write $a' > a$ if $a'_i \geq a_i$ for $i = 0, 1, \ldots, n$. It is clear that if $a' > a$ and $a \in S_\xi$ then $a' \in S_\xi$. We easily check that for $\xi, \eta \in \Omega\{X\}$,

$$\begin{cases} S_{\psi(\xi)} \supset qS_\xi \\ S_{\xi\eta} \supset S_\xi \cap S_\eta \\ S_{\xi+\eta} \supset S_\xi \cap S_\eta \end{cases} \tag{2.5}$$

Let g be a mapping of $\mathbf{Z}_+^{n+1}$ into the set of two elements, $\{0, 1\}$ in Ω. Let γ be the Ω linear mapping of $\Omega\{X\}$ into itself defined by

$$\gamma(X^u) = g(u)X^u. \tag{2.6}$$

For such a mapping we have

$$S_{\gamma(\xi)} \supset S_\xi. \tag{2.7}$$

For each $a \in S_\xi$, let

$$M(\xi, a) = \underset{\operatorname{ord} x = a}{\operatorname{Min}} \operatorname{ord} \xi(x).$$

The generalization of Lemma 1.2 may be stated without proof.

Lemma **2.5**. — *For* $a \in S_\xi$, ξ *as in* (2.4),

$$M(\xi, a) = \underset{u \in \mathbf{Z}_+^{n+1}}{\operatorname{Min}} (\operatorname{ord} B_u + \rho(u, a)).$$

If $a' > a$ *then*

$$M(\xi, a') \geq M(\xi, a).$$

We easily verify for $\xi, \eta \in \Omega\{X\}$, γ as in (2.6) that

$$M(\xi\eta, a) \geq M(\xi, a) + M(\eta, a) \qquad \text{if } a \in S_\xi \cap S_\eta \tag{2.8}$$

$$M(\gamma\xi, a) \geq M(\xi, a) \qquad \text{if } a \in S_\xi \tag{2.9}$$

$$M(\psi\xi, a) \geq M(\xi, a/q) \qquad \text{if } a/q \in S_\xi \tag{2.10}$$

$$M(\xi + \eta, a) \geq \operatorname{Min}\{M(\xi, a), M(\eta, a)\} \qquad \text{if } a \in S_\xi \cap S_\eta \tag{2.11}$$

and equality holds in (2.11) if $M(\xi, a) \neq M(\eta, a)$. Let

$$S = \{a \in \mathbf{Q}^{n+1} \mid a_0 > -q\varkappa, \ da_i + a_0 > -q\varkappa, \ i = 1, 2, \ldots, n\}$$

Elementary computations show that if c is a real number, $\eta \in L(q\varkappa, c)$ then

$$\begin{cases} S_\eta \supset S \\ M(\eta, a) \geq c \text{ for } a \in S, \end{cases} \tag{2.12}$$

and

$$\begin{cases} S_F \supset q^{-1}S \\ M(F, a/q) \geq 0 \ \text{ if } \ a \in S. \end{cases} \tag{2.13}$$

It follows from (2.8), (2.10) and (2.13) that

$$M(\alpha\xi, a) \geq M(\xi, a/q) \qquad \text{if } a \in S \cap qS_\xi. \tag{2.14}$$

This relation remains valid if α is replaced by $\alpha \circ \gamma$ or $\gamma \circ \alpha$, the composition of α with γ on either right or left side.

Let $\mathfrak{O}\{X\}$ be the ring of power series in $X_0, \ldots, X_n$, with coefficients in $\mathfrak{O}$, the ring of integers in Ω. Let L' be the space of all elements of $\Omega\{X\}$ which converge in a polycylinder of radii greater than unity (i.e. an element $\xi \in \Omega\{X\}$ lies in L' if and only if there exists a rational number $b > 0$ such that $(-b, -b, \ldots, -b) \in S_\xi$). We note that $L' \supset L(b)$ for all $b > 0$ but L' is not the union of such subspaces since the monomials, X^u, in L' need not satisfy the condition $u \in \mathfrak{T}$.

Lemma **2.6**. — *Let* $\eta \in L(q\varkappa, -q(q-1)^{-1} \operatorname{ord} \lambda)$, *where* λ *is a non-zero element of* Ω, *and let* ξ *be an element of* $L' \cap \mathfrak{O}\{X\}$ *such that*

$$\alpha\xi = \lambda(\xi + \eta). \tag{2.15}$$

We may then conclude that $\xi \in L(q\varkappa, -q(q-1)^{-1} \operatorname{ord} \lambda)$.

Note. — The same conclusion would hold if α in (2.15) were replaced by $\alpha \circ \gamma$ or by $\gamma \circ \alpha$, with γ as in (2.6). In particular, α may be replaced by α'_N.

Proof. — Writing (2.15) in the form $\xi = -\eta + \lambda^{-1}\alpha\xi$, we see from (2.5) that $S_\xi \supset S_\eta \cap S_{\alpha\xi} \supset S_\eta \cap qS_{F\xi} \supset S_\eta \cap qS_F \cap qS_\xi$, and hence by (2.12) and (2.13) we have

$$S_\xi \supset S \cap qS_\xi. \tag{2.16}$$

By hypothesis, $\xi \in L'$ and hence there exists $b > 0$ such that $a^{(0)} = (-b, -b, \ldots, -b) \in S_\xi$. If $a \in S$ then there exists an integer, $r \geq 0$, so large that $q^{-r}a > a^{(0)}$ and hence

$$q^{-r}a \in S_\xi.$$

Let r be the minimal element of $\mathbf{Z}_+$ such that the displayed relation holds. If $a \in S$ then $q^{-1}a$, $q^{-2}a$, etc., lie in S and hence if $r \geq 1$ then $q^{-(r-1)}a$ lies in S as well as in qS_ξ, so that by (2.16) we have $q^{-(r-1)}a \in S_\xi$, contrary to the minimality of r. This shows that $r = 0$ and hence $S \subset S_\xi$. Since $q^{-1}S \subset S$, we may also conclude that $S \subset qS_\xi$.

Equations (2.14) and (2.15) show that

$$\operatorname{ord} \lambda + M(\xi + \eta, a) \geq M(\xi, a/q) \qquad \text{if} \quad a \in S. \tag{2.17}$$

We write ξ as in (2.4) and we assert that for $a \in S$, $v \in \mathbf{Z}_+^{n+1}$,

$$\operatorname{ord} B_v + \rho(v, a) \geq -q(q-1)^{-1} \operatorname{ord} \lambda. \tag{2.18}$$

To prove this we think of a as fixed and consider two cases.

Case 1. — $M(\xi, a) \geq M(\eta, a)$

In this case Lemma 2.5 and equation (2.12) give a direct verification of (2.18).

Case 2. — $M(\xi, a) < M(\eta, a)$.

Here we may use (2.11) and deduce from (2.17) that

$$\operatorname{ord} \lambda + M(\xi, a) \geq M(\xi, a/q). \tag{2.19}$$

Lemma 2.5 shows that there exists a particular element, $u \in \mathbf{Z}_+^{n+1}$ (depending upon a) such that

$$M(\xi, a/q) = \operatorname{ord} B_u + \rho(u, a/q).$$

On the other hand $M(\xi, a) \leq \operatorname{ord} B_v + \rho(v, a)$, for each $v \in \mathbf{Z}_+^{n+1}$. Thus we have

$$\operatorname{ord} B_v + \rho(v, a) + \operatorname{ord} \lambda \geq \operatorname{ord} B_u + \rho(u, a/q), \tag{2.20}$$

for a particular u and for all $v \in \mathbf{Z}_+^{n+1}$. In particular (2.20) holds for $v=u$ and this gives

$$\textbf{(2.21)} \qquad \operatorname{ord} \lambda \geq (q^{-1}-1)\rho(u, a).$$

We recall that by hypothesis $\xi \in \mathfrak{O}\{X\}$ and hence $\operatorname{ord} B_u \geq 0$. Equation (2.18) now follows from (2.20) and (2.21). This completes our verification of (2.18) for all $a \in S$.

Now let c be a rational number, $c>0$, let $a_0, a_1, \ldots, a_n$ be rational numbers, $a_0 > -q\kappa$, $a_i = c - d^{-1}(q\kappa + a_0)$ for $i = 1, 2, \ldots, n$. Then $a = (a_0, a_1, \ldots, a_n) \in S$ and

$$\rho(v, a) = a_0\left(v_0 - d^{-1}\sum_{i=1}^{n} v_i\right) + (c - d^{-1}q\kappa)\sum_{i=1}^{n} v_i,$$

which shows that if $v_0 < d^{-1}\sum_{i=1}^{n} v_i$ then $\rho(v, a) \to -\infty$ as $a_0 \to +\infty$ if c is kept fixed. Applying this to (2.18) we see that $\operatorname{ord} B_v = +\infty$ if $v_0 < d^{-1}\sum_{i=1}^{n} v_i$, i.e.

$$\textbf{(2.22)} \qquad B_v = 0 \text{ if } v \notin \mathfrak{T}.$$

With $c>0$ as before, let $a_0 = -q\kappa + c$, $a_i = 0$ for $i = 1, 2, \ldots, n$. Once again $a = (a_0, a_1, \ldots, a_n) \in S$ and thus (2.18) shows that

$$\operatorname{ord} B_v \geq v_0(q\kappa - c) - q(q-1)^{-1} \operatorname{ord} \lambda$$

for each $c>0$. Taking limits as $c \to 0$,

$$\textbf{(2.23)} \qquad \operatorname{ord} B_v \geq q\kappa v_0 - q(q-1)^{-1} \operatorname{ord} \lambda.$$

Relations (2.22) and (2.23) show that $\xi \in L(q\kappa, -q(q-1)^{-1} \operatorname{ord} \lambda)$, as asserted.

Note. — If $\eta = 0$ in the statement of the lemma, then equation (2.19) is valid for all $a \in S$. Since $(0, 0, \ldots, 0) \in S$, it follows that $\operatorname{ord} \lambda \geq 0$.

Theorem **2.1.** — *Let* $\lambda_1, \ldots, \lambda_s$ *be a set of non-zero elements of* $\mathfrak{O}$ *and let* $e = \sum_{i=1}^{s} \operatorname{ord} \lambda_i + (q-1)^{-1} \operatorname*{Max}_{1 \leqslant i \leqslant s} \operatorname{ord} \lambda_i$. *Let* ξ *be an element of* $L' \cap \mathfrak{O}\{X\}$ *such that*

$$\textbf{(2.24)} \qquad \prod_{i=1}^{s} (I - \lambda_i^{-1}\alpha)\xi = 0,$$

then $\xi \in L(q\kappa, -e)$.

Proof. — The theorem is a direct consequence of the previous lemma if $s=1$. Hence we may suppose $s>1$ and apply induction on s. Let $\operatorname{ord} \lambda_1 \geq \ldots \geq \operatorname{ord} \lambda_s$ and let $\eta = (\alpha - \lambda_s I)\xi$. Since $\eta \in L' \cap \mathfrak{O}\{X\}$ and $\prod_{i=1}^{s-1} (I - \lambda_i^{-1}\alpha)\eta = 0$, we may conclude that $\eta \in L(q\kappa, -e')$, where $e' = e - \operatorname{ord} \lambda_s$. We may choose $\gamma \in \Omega$ such that $\operatorname{ord} \gamma = e' - (q-1)^{-1} \operatorname{ord} \lambda_s$. Clearly $\gamma\lambda_s^{-1}\eta \in L(q\kappa, -q(q-1)^{-1} \operatorname{ord} \lambda_s)$, while $\alpha(\gamma\xi) = \lambda_s(\gamma\xi + \gamma\lambda_s^{-1}\eta)$. Since $\gamma\lambda_s^{-1} \in \mathfrak{O}$, we may conclude from the previous lemma that $\gamma\xi \in L(q\kappa, -q(q-1)^{-1} \operatorname{ord} \lambda_s)$. The proof is completed by checking that $-\operatorname{ord} \gamma - q(q-1)^{-1} \operatorname{ord} \lambda_s = -e$.

Note. — Although not needed for our applications, we note that we had shown with the aid of Lemma 2.4 that if $F \in \mathfrak{L}^{(N(q-1))}$ and ξ is a polynomial satisfying (2.24), then ξ lies in $\mathfrak{L}^{(N)}$. We can now show that if ξ is known to satisfy (2.24) and is known to lie in L′ then it must be a polynomial (and hence lie in $\mathfrak{L}^{(N)}$). If $F \in \mathfrak{L}^{(N(q-1))}$ then there exists $\gamma \in \mathfrak{O}$ such that $\gamma F \in \mathfrak{O}[X]$ and hence if $r \in \mathbf{Z}_+$, $p^{rN(q-1)}\gamma F \in L(r, 0)$. If $\beta_r = \psi \circ p^{rN(q-1)}\gamma F$, then $\prod_{i=1}^{s} (I - \lambda_{i,r}^{-1}\beta_r)\xi = 0$, where $\lambda_{i,r} = \lambda_i \gamma p^{rN(q-1)}$ and hence the theorem shows that ξ lies in $L(qr, -e-(s+1)(\operatorname{ord}\gamma + rN(q-1)))$. Hence $\xi = \Sigma B_u X^u$, $u \in \mathfrak{T}$ and $\operatorname{ord} B_u \geq qru_0 - e - (s-1) \operatorname{ord}\gamma - rN(q-1)(s+1)$ for each $r \in \mathbf{Z}_+$. Letting $r \to \infty$, it is clear that $B_u = 0$ if $u_0 > N(q-1)(s+1)/q$, which shows that $\xi \in \mathfrak{L}$.

Theorem **2.2.** — *If the coefficients of* F *lie in a field*, K_0, *of finite degree over* $\mathbf{Q}'$ *and if* λ^{-1} *is a zero of order* μ *of* χ_F, *then the dimension of the kernel in* $L(q\varkappa)$ *of* $(I - \lambda^{-1}\alpha)^{\mu}$ *is not less than* μ, *indeed the kernel contains* μ *linearly independent elements which lie in* $L(q\varkappa) \cap K_0(\lambda)\{X\}$.

Proof. — We may suppose that $\mu \geq 1$. Since $\chi_F \in \mathfrak{O}\{t\}$, $\chi_F(0) = 1$, we may conclude from Theorem 1.1 that $\lambda \in \mathfrak{O}$. Let $\chi_N(t) = \det(I - t\alpha_N)$. We recall that Lemma 2.3 shows that $\chi_N \to \chi_F$ uniformly on each bounded disk. There exists a real number, $\rho > 0$ so large that χ_F has no zero distinct from λ^{-1} in $\lambda^{-1}(1 + C_\rho)$. The proof of Theorem 1.2 shows that for N large enough (as will be supposed in the remainder of the proof) there exist (counting multiplicities) precisely μ zeros, $\lambda_{1,N}^{-1}, \ldots, \lambda_{\mu,N}^{-1}$ of χ_N in $\lambda^{-1}(1 + C_\rho)$. Since χ_F, χ_N and the set $\lambda^{-1}(1 + C_\rho)$ are all invariant under automorphisms of Ω which leave $K_0(\lambda)$ pointwise fixed, we conclude that the polynomial $f_N(t) = \prod_{i=1}^{\mu} (1 - \lambda_{i,N} t)$ is also invariant under such automorphisms and hence lies in $K_0(\lambda)[t]$. Let K be the composition of all field extensions in Ω of $K(\lambda)$ of degree not greater than μ. Theorem 1.1 shows that λ is algebraic over K_0, hence $\deg(K_0(\lambda)/\mathbf{Q}') < \infty$. This shows that $\deg(K/K_0) < \infty$ and hence $\deg(K/\mathbf{Q}') < \infty$. The conclusion is that $\lambda_{i,N} \in K$, $\lim_{N \to \infty} \lambda_{i,N} = \lambda$ for $i = 1, 2, \ldots, \mu$ and that K is locally compact. Furthermore f_N is relatively prime to χ_N / f_N.

We now restrict α_N to $K[X] \cap \mathfrak{L}^{(N)}$. This does not change the characteristic equation of α_N and letting W_N be the kernel in that space of $\beta_N = \prod_{i=1}^{\mu} (I - \lambda_{i,N}^{-1}\alpha_N)$, we conclude that the dimension of W_N (as K-space) is μ. An element, ξ, of W_N will be said to be *normalized* if it lies in $\mathfrak{O}\{X\}$ and at least one coefficient is a unit. If ξ is such a normalized element of W_N then by Theorem 2.1, $\xi \in L(q\varkappa, -e)$, where $e = (\mu + (q-1)^{-1}) \operatorname{ord} \lambda$. If we write $\xi = \Sigma B_u X^u$ then $\operatorname{ord} B_u \geq q\varkappa u_0 - e$ and hence B_u must be a unit for at least one element $u \in \mathfrak{T}_1 = \{v \in \mathfrak{T} \mid v_0 \leq e/(q\varkappa)\}$. Conversely if B_u is a unit then $u \in \mathfrak{T}_1$.

It is clear that a subspace W of K[X] of dimension μ has a basis $\xi_1, \ldots, \xi_\mu$ in $\mathfrak{O}[X] \cap K[X]$ for which there exist distinct elements $u_1, \ldots, u_\mu$ of $\mathbf{Z}_+^{n+1}$ such that the coefficient of X^{u_i} in ξ_j is the Kronecker $\delta_{i,j}$ $(i, j = 1, 2, \ldots, \mu)$. Hence for each N there exists a set of μ linearly independent elements $\{\xi_{i,N}\}_{i=1,2,\ldots,\mu}$ in W_N corresponding to which there exist μ distinct elements, $\{u_{i,\mathrm{N}}\}_{i=1,2,\ldots,\mu}$ in $\mathfrak{T}_1$ such that $\xi_{i,\mathrm{N}}\in\mathfrak{O}[\mathrm{X}]$ and the coefficient of $\mathrm{X}^{u_{i,\mathrm{N}}}$ in $\xi_{j,\mathrm{N}}$ is $\delta_{i,j}$ for $i, j=1, 2, \ldots, \mu$. Since $\mathfrak{T}_1$ (and hence $\mathfrak{T}_1^{\mu}$) is a finite set, there exists by the pigeon hole principle, an infinite subset, $\mathfrak{A}$, of $\mathbf{Z}_+$ such that $u_i=u_{i,\mathrm{N}}$ is independent of N for each N in the subset and $i=1, 2, \ldots, \mu$. In the following N will be restricted to this infinite subset.

Now let $\mathfrak{B}=\mathrm{K}\{\mathrm{X}\}\cap\mathrm{L}(q\kappa, -e)$. Generalizing the definition of § 1, we may define the weak topology of $\mathrm{K}\{\mathrm{X}\}$ and by the local compactness of K and the theorem of Tychonoff, $\mathfrak{B}$ is compact under the induced topology. Thus $\mathfrak{B}^{\mu}$, the μ fold cartesian product of $\mathfrak{B}$ is also compact under the product space topology. Clearly the ordered set $\xi^{(\mathrm{N})}=(\xi_{1,\mathrm{N}}, \xi_{2,\mathrm{N}}, \ldots, \xi_{\mu,\mathrm{N}})\in\mathfrak{B}^{\mu}$ and hence an infinite subsequence of the sequence $\{\xi^{(\mathrm{N})}\}_{\mathrm{N}\in\mathfrak{A}}$ must converge. Hence there exists an infinite subset, $\mathfrak{A}'$ of $\mathfrak{A}$ such that $\{\xi^{(\mathrm{N})}\}_{\mathrm{N}\in\mathfrak{A}'}$ converges to an element $(\xi_1, \ldots, \xi_{\mu})\in\mathfrak{B}^{\mu}$. For $j=1, 2, \ldots, \mu$ we have $\{\xi_{j,\mathrm{N}}\}_{\mathrm{N}\in\mathfrak{A}'}\to\xi_j$ and since the coefficient of X^{u_i} in $\xi_{j,\mathrm{N}}$ is $\delta_{i,j}$, the same holds for ξ_j. This shows that $\xi_1, \ldots, \xi_{\mu}$ are elements of $\mathfrak{B}$ which are linearly independent over Ω. Furthermore $\beta_{\mathrm{N}}\xi_{j,\mathrm{N}}=0$ for each $\mathrm{N}\in\mathfrak{A}'$ and hence taking limits as $\mathrm{N}\to\infty$ in $\mathfrak{A}'$, we conclude that $\xi_1, \ldots, \xi_{\mu}$ lie in the kernel of $\beta=(\mathrm{I}-\lambda^{-1}\alpha)^{\mu}$ in $\mathrm{L}(q\kappa)$.

Now let $\omega_1, \ldots, \omega_m$ be a minimal basis of K over $\mathrm{K}_0(\lambda)$. If $\eta\in\mathrm{K}\{\mathrm{X}\}\cap\mathrm{L}(q\kappa, -e)$ then there exist $\eta_1, \ldots, \eta_m\in\mathrm{K}_0(\lambda)\{\mathrm{X}\}$ such that $\eta=\sum_{i=1}^{m}\eta_i\omega_i$ and since the basis is minimal, $\eta_i\in\mathrm{L}(q\kappa, -e-1)$ for $i=1, 2, \ldots, m$. If $0=\beta\eta$ then $0=\sum_{i=1}^{m}\omega_i\beta\eta_i$ and since $\beta\eta_i\in\mathrm{K}_0(\lambda)\{\mathrm{X}\}$ for $i=1, 2, \ldots, m$, we can conclude that η_i lies in the kernel of β. Applying this argument to $\xi_1, \ldots, \xi_{\mu}$ we conclude that the Ω-space spanned by them is spanned by elements of the kernel of β in $\mathrm{L}(q\kappa)\cap\mathrm{K}_0(\lambda)\{\mathrm{X}\}$. This completes the proof of the theorem.

To complete our description of χ_{F} in terms of a spectral theory for α, we must prove a converse of the previous theorem.

Theorem **2.3**. — *Let μ be an integer, $\mu\geq 1$ and λ a non-zero element in Ω. The dimension of the kernel in* L′ *of* $(\mathrm{I}-\lambda^{-1}\alpha)^{\mu}$ *is not greater than the multiplicity of* λ^{-1} *as zero of* χ_{F}.

We defer the proof except to note that we may assume that the kernel of $(\mathrm{I}-\lambda^{-1}\alpha)^{\mu}$ in L′ may be assumed to be of non-zero dimension and to show that $\lambda\in\mathfrak{O}$. If the kernel of $(\mathrm{I}-\lambda^{-1}\alpha)^{\mu}$ is not $\{0\}$ then by an obvious argument, the same holds for the kernel of $(\mathrm{I}-\lambda^{-1}\alpha)$. Hence there exists $\xi\in\mathrm{L}'$ such that $\alpha\xi=\lambda\xi, \xi\neq 0$. Since $\xi\in\mathrm{L}'$ there exists $\gamma\in\Omega, \gamma\neq 0$ such that $\gamma\xi\in\mathfrak{O}\{\mathrm{X}\}$. Hence it may be assumed that $\xi\in\mathfrak{O}\{\mathrm{X}\}$. Thus $\lambda^r\xi=\alpha^r\xi$ for each $r\in\mathbf{Z}_+$ and since α maps $\mathfrak{O}\{\mathrm{X}\}$ into itself, we conclude that $\xi\neq 0, \lambda^r\xi\in\mathfrak{O}\{\mathrm{X}\}$ for all $r\in\mathbf{Z}_+$. This shows that $\lambda\in\mathfrak{O}$. Theorem 2.1 now shows that we can replace L′ in the statement of the theorem by $\mathrm{L}(q\kappa)$.

Before resuming the proof we must recall some formal properties of matrices. Let A be an $m\times m$ matrix with coefficients in some field of characteristic zero. For each subset H of $\{1, 2, \ldots, m\}$, let (A, H) be the square matrix obtained by deleting the j^{th} row and column of A for each $j\in\mathrm{H}$. Let [H] denote the number of elements in H and let t be trancendental over the field, K, generated by the coefficients of A. If $[\mathrm{H}]=m$, we define $\det(\mathrm{A},\mathrm{H})=1$ and for $0\leq[\mathrm{H}]\leq m$, $\mathrm{I}-t(\mathrm{A},\mathrm{H})$ denotes $((\mathrm{I}-t\mathrm{A}),\mathrm{H})$.

Lemma **2.7.** — *For* $1\leq r\leq m$

$$(2.25)\qquad \left(\prod_{i=0}^{r-1}\left(t\frac{d}{dt}-(m-i)\right)\right)\det(\mathrm{I}-t\mathrm{A})=(-1)^r r!\sum_{[\mathrm{H}]=r}\det(\mathrm{I}-t(\mathrm{A},\mathrm{H})),$$

the sum on the right being over all subsets, H, *of* $\{1, 2, \ldots, m\}$ *such that* $[\mathrm{H}]=r$.

Proof. — We recall the classical result that if B is an $m\times m$ matrix whose coefficients are differentiable functions of t then

$$(2.26)\qquad \det \mathrm{B}=\sum_{j=1}^{m}\det \mathrm{B}_j$$

where B_j is the $m\times m$ matrix obtained from B by differentiating each coefficient in the j^{th} row and leaving the other rows unchanged. Thus $\frac{d}{dt}\det(\mathrm{I}t-\mathrm{A})=\sum_{[\mathrm{H}]=1}\det(\mathrm{I}t-(\mathrm{A},\mathrm{H}))$. However $t^{-m}\det(\mathrm{I}-t\mathrm{A})=\det(t^{-1}\mathrm{I}-\mathrm{A})$ and therefore

$$-mt^{-m-1}\det(\mathrm{I}-t\mathrm{A})+t^{-m}\frac{d}{dt}\det(\mathrm{I}-t\mathrm{A})=-t^{-2}\sum_{[\mathrm{H}]=1}\det(t^{-1}\mathrm{I}-(\mathrm{A},\mathrm{H}))=$$
$$-t^{-2}t^{-(m-1)}\sum_{[\mathrm{H}]=1}\det(\mathrm{I}-t(\mathrm{A},\mathrm{H})).$$

The assertion for $r=1$ follows immediately. We may therefore suppose $r>1$ and use induction on r. Hence

$$(2.27)\qquad r!^{-1}\left(\prod_{i=0}^{r-1}\left(t\frac{d}{dt}-(m-i)\right)\right)\det(\mathrm{I}-t\mathrm{A})=$$
$$(-1)^{r-1}r^{-1}\left(t\frac{d}{dt}-(m-(r-1))\right)\sum_{[\mathrm{H}]=r-1}\det(\mathrm{I}-t(\mathrm{A},\mathrm{H})).$$

The lemma is known to be true for $r=1$ and hence for given H such that $[\mathrm{H}]=r-1$, since (A,H) is an $(m-r+1)\times(m-r+1)$ matrix,

$$\left(t\frac{d}{dt}-(m-r+1)\right)\det(\mathrm{I}-t(\mathrm{A},\mathrm{H}))=-\sum_{\mathrm{H}''}\det(\mathrm{I}-t((\mathrm{A},\mathrm{H}),\mathrm{H}'')),$$

the sum being over all $\mathrm{H}''\subset\{1, 2, \ldots, m\}-\mathrm{H}$ such that $[\mathrm{H}'']=1$. However $((\mathrm{A},\mathrm{H}),\mathrm{H}'')=(\mathrm{A},\mathrm{H}')$ where $\mathrm{H}'=\mathrm{H}''\cup\mathrm{H}$ and hence the sum over H″ may be replaced by $\sum_{\mathrm{H}'}\det(\mathrm{I}-t(\mathrm{A},\mathrm{H}'))$, the sum now being over all H′ such that $\mathrm{H}'\supset\mathrm{H}$, $[\mathrm{H}']=r$. Thus the right side of (2.27) is $(-1)^r r^{-1}\sum_{\mathrm{H}}\sum_{\mathrm{H}'}\det(\mathrm{I}-t(\mathrm{A},\mathrm{H}'))$, the sum being over all H such that $[\mathrm{H}]=r-1$ and over all $\mathrm{H}'\supset\mathrm{H}$ such that $[\mathrm{H}']=r$. But given H′ such that $[\mathrm{H}']=r$ there exists exactly r distinct subsets H of H′ such that $[\mathrm{H}]=r-1$. Thus the right side of (2.27) is $(-1)^r\sum_{[\mathrm{H}']=r}\det(\mathrm{I}-t(\mathrm{A},\mathrm{H}'))$, which completes the proof of the lemma.

With the previous conventions, let $S^{(j)}(A), j=0, 1, \ldots, m$ denote the elements of the field K generated by the coefficients of A which satisfy the formal identity

$$(2.28) \qquad \det(I+tA) = \sum_{j=0}^{m} S^{(j)}(A)t^{j}$$

We observe that

$$(2.29) \qquad S^{(m-j)}(A) = \sum_{[H]=j} \det(A, H), j=0, 1, \ldots, m-1$$

the sum being over all subsets H of $\{1, 2, \ldots, m\}$ such that $[H]=j$.

Let $\mu \geq 1$ be a rational integer, let ω be a primitive μ^{th} root of unity in some extension field of K. For $(i_0, i_1, \ldots, i_{\mu-1}) \in \mathbf{Z}_+^{\mu}$, let $g(i_0, i_1, \ldots, i_{\mu-1}) = \omega^{r}$ where $r = \sum_{s=1}^{\mu-1} si_s$. Since $\det(I-t^{\mu}A^{\mu}) = \prod_{s=0}^{\mu-1}\det(I-t\omega^{-s}A)$, we have

$$(2.30) \qquad \sum_{j=0}^{m} t^{j\mu}S^{(j)}(-A^{\mu}) = \prod_{s=0}^{\mu-1}\sum_{j=0}^{m} t^{j}S^{(j)}(-\omega^{-s}A).$$

For $0 \leq i \leq m$, by comparing coefficients of $t^{\mu(m-i)}$ on both sides of (2.30), we conclude that

$$(2.31) \qquad S^{(m-i)}(A^{\mu}) = (S^{(m-i)}A)^{\mu} + (-1)^{i(\mu-1)}\Sigma' g(i_0, \ldots, i_{\mu-1}) \prod_{s=0}^{\mu-1} S^{(m-i_s)}A,$$

the sum, Σ', being over all $(i_0, \ldots, i_{\mu-1}) \in \mathbf{Z}_+^{\mu}, i_j \leq m$ such that $\sum_{s=0}^{\mu-1} i_s = \mu i$, but $i_0 = i_1 = \ldots = i_{\mu-1} = i$ is explicitly excluded.

Proof (*Theorem* **2.3**). — We first outline the proof. Let W be the kernel of $(I-\lambda^{-1}\alpha)^{\mu}$ in L' (and hence by Theorem 2.1) in $L(q\varkappa)$. Suppose $\dim W \geq r > 0$ for some $r \in \mathbf{Z}_+$. We must show that $\chi_F^{(s-1)}(\lambda^{-1}) = 0$ for $s = 1, 2, \ldots, r$. Let $\mathfrak{M}'_N$ (for each $N \in \mathbf{Z}_+$) denote the matrix relative to a monomial basis corresponding to the linear transformation $\alpha'_N = T_N \circ \alpha$ of $\mathfrak{L}^{(N)}$. Explicitly, for each $v \in \mathfrak{I}_N$, $\alpha'_N(X^v) = \Sigma \mathfrak{M}'_N(u, v)X^u$, the sum being over all $u \in \mathfrak{I}_N$.

Let $\chi_N(t) = \det(I - t\mathfrak{M}'_N)$. We know that for all $s \in \mathbf{Z}_+$, $\lim_{N\to\infty} \chi_N^{(s)}(\lambda^{-1}) = \chi_F^{(s)}(\lambda^{-1})$ and thus we must show that $\lim_{N\to\infty} \chi_N^{(s-1)}(\lambda^{-1}) = 0$ for $s = 1, 2, \ldots, r$. Letting N' be the dimension of $\mathfrak{L}^{(N)}$, equations (2.25) and (2.29) show that it is enough to prove that

$$(2.32) \qquad \lim_{N\to\infty} S^{(N'-i)}(I - \lambda^{-1}\mathfrak{M}'_N) = 0 \text{ for } i = 0, 1, \ldots, r-1.$$

We shall prove the existance of a constant c independent of N, such that for $i = 0, 1, \ldots, r-1$

$$(2.33) \qquad \operatorname{ord} S^{(N'-i)}((I - \lambda^{-1}\mathfrak{M}'_N)^{\mu}) \geq c + \varkappa(q-1)N$$

and prove (2.32) by using (2.33) and (2.31) to deduce the existence of a constant c' independent of N such that for $i = 0, 1, \ldots, r-1$

$$(2.34) \qquad \operatorname{ord} S^{(N'-i)}(I - \lambda^{-1}\mathfrak{M}'_N) \geq c' + \varkappa(q-1)N/\mu^{i+1}.$$

Let $\mathfrak{M}''_{\mathrm{N}}=(\mathrm{I}-\lambda^{-1}\mathfrak{M}'_{\mathrm{N}})^{\mu}$. We may view $\mathfrak{M}''_{\mathrm{N}}$ as a matrix whose rows and columns are indexed by the set $\mathfrak{T}_{\mathrm{N}}$ of all $u\in\mathfrak{T}$ such that $u_0\leq\mathrm{N}$. If H is any subset of $\mathfrak{T}_{\mathrm{N}}$, $\mathrm{H}\neq\mathfrak{T}_{\mathrm{N}}$, we may, following our previous convention, denote by $(\mathfrak{M}''_{\mathrm{N}}, \mathrm{H})$ that square matrix obtained from $\mathfrak{M}''_{\mathrm{N}}$ by deleting all rows and columns indexed by elements of H. We shall show that if H is any set of not more than $r-1$ elements of $\mathfrak{T}$ then c may be chosen independent of H and N such that

$$\text{(2.35)} \qquad \operatorname{ord}\det(\mathfrak{M}''_{\mathrm{N}}, \mathrm{H})\geq c+\varkappa(q-1)\mathrm{N}$$

whenever H (if not empty) is contained properly by $\mathfrak{T}_{\mathrm{N}}$. Equation (2.29) shows that equation (2.35) implies (2.33). Our first object is the proof of equation (2.35).

Let H be a set of no more than $r-1$ elements of $\mathfrak{T}$. We know that there exist $\xi_1, \ldots, \xi_r$, a set of r linearly independent elements in W. Let $\xi_j=\Sigma \mathrm{B}_{u,j}\mathrm{X}^u$, $j=1, 2, \ldots, r$, the sum being over all $u\in\mathfrak{T}$. The (possibly empty) set of [H] equations $\sum_{j=1}^{r} a_j\mathrm{B}_{u,j}=0$ for each $u\in\mathrm{H}$, in r unknowns $a_1, a_2, \ldots, a_r$ certainly has a non-trivial solution in Ω (since $r>[\mathrm{H}]$). Since $\xi_1, \ldots, \xi_r$ are linearly independent, we conclude that $\xi=\sum_{j=1}^{r} a_j\xi_j$ is a non-trivial element of W. Since $0\neq\xi\in\mathrm{L}(q\varkappa)$, ξ may be normalized so that $\xi\in\mathfrak{O}\{\mathrm{X}\{$ and at least one coefficient of ξ is a unit. Thus there exists a normalized element $\xi=\Sigma\mathrm{B}_u\mathrm{X}^u$ in W such that $\mathrm{B}_u=0$ for each $u\in\mathrm{H}$. Theorem 2.1 shows that, for all $u\in\mathfrak{T}$,

$$\text{(2.36)} \qquad \operatorname{ord}\mathrm{B}_u\geq q\varkappa u_0-e,$$

where $e=\mu\operatorname{ord}\lambda+(q-1)^{-1}\operatorname{ord}\lambda$. Hence if $\mathrm{N}>\mathrm{N}_0=e/qk$, we may conclude that $\mathrm{T}_{\mathrm{N}}\xi$ is also normalized and the coefficients of $\xi_{\mathrm{N}}=\mathrm{T}_{\mathrm{N}}\xi$ satisfy (2.36).

For typographical reasons we shall when convenient denote the coefficient of X^u in ξ (resp. F) by B(u) (resp. A(u)) instead of B_u (resp. A_u). For given integer $j\geq 1$,

$$(\alpha'_{\mathrm{N}})^j\mathrm{T}_{\mathrm{N}}\xi=(\mathrm{T}_{\mathrm{N}}\circ\alpha)^j\mathrm{T}_{\mathrm{N}}\xi=\Sigma\mathrm{X}^{w^{(j)}}\mathrm{B}(w^{(0)})\mathrm{A}(qw^{(1)}-w^{(0)})\mathrm{A}(qw^{(2)}-w^{(1)})\ldots\mathrm{A}(qw^{(j)}-w^{(j-1)})$$

the sum on the right being over all $(w^{(0)}, w^{(1)}, \ldots, w^{(j)})\in\mathfrak{T}_{\mathrm{N}}^{j+1}$. We may write $\mathrm{T}_{\mathrm{N}}(\alpha^j\xi)$ as a similar sum except in this case the sum is over all $((w^{(0)}, w^{(1)}, \ldots, w^{(j-1)}), w^{(j)})\in\mathfrak{T}^j\times\mathfrak{T}_{\mathrm{N}}$. Since $\operatorname{ord}\mathrm{A}_u\geq\varkappa u_0$ for all $u\in\mathfrak{T}$, we have by (2.36),

$$\varkappa^{-1}\operatorname{ord}\{\mathrm{B}(w^{(0)})\mathrm{A}(qw^{(1)}-w^{(0)})\ldots\mathrm{A}(qw^{(j)}-w^{(j-1)})\}\geq-\varkappa^{-1}e+\sigma\left(qw^{(0)}+\sum_{i=0}^{j-1}(qw^{(i+1)}-w^{(i)})\right)=$$
$$-\varkappa^{-1}e+\sigma\left(qw^{(j)}+(q-1)\sum_{i=0}^{j-1}w^{(i)}\right).$$

If $w^{(0)}, w^{(1)}, \ldots, w^{(j-1)}$ do not all lie in $\mathfrak{T}_{\mathrm{N}}$ then certainly $\sigma\left(\sum_{i=0}^{j-1}w^{(i)}\right)>\mathrm{N}$. Thus we can conclude (using only the fact that $\xi\in\mathrm{L}(q\varkappa, -e)$) that

$$\text{(2.37)} \qquad \mathrm{T}_{\mathrm{N}}(\alpha^j\xi)\equiv(\mathrm{T}_{\mathrm{N}}\circ\alpha)^j\mathrm{T}_{\mathrm{N}}\xi \mod \sum_{u\in\mathfrak{T}_{\mathrm{N}}}\mathrm{X}^u\mathrm{C}(\varkappa qu_0-e+(q-1)\varkappa\mathrm{N}),$$

where for each real number, b, $\mathrm{C}(b)$ is used in the sense of C_b in § 1. Since

$$0=(\mathrm{I}-\lambda^{-1}\alpha)^{\mu}\xi=\sum_{j=0}^{\mu}(-\lambda^{-1})^{j}\tbinom{\mu}{j}\alpha^{j}\xi,$$

we have

$$0=\sum_{j=0}^{\mu}\tbinom{\mu}{j}(-\lambda^{-1})^{j}\mathrm{T}_{\mathrm{N}}(\alpha^{j}\xi)\equiv\sum_{j=0}^{\mu}\tbinom{\mu}{j}(-\lambda^{j})(\mathrm{T}_{\mathrm{N}}\circ\alpha)^{j}\mathrm{T}_{\mathrm{N}}\xi\equiv$$
$$(\mathrm{I}-\lambda^{-1}\alpha'_{\mathrm{N}})^{\mu}\mathrm{T}_{\mathrm{N}}\xi \mod \sum_{u\in\mathfrak{T}_{\mathrm{N}}}\mathrm{X}^{u}\mathrm{C}(\varkappa q u_0-2e+(q-1)\varkappa\mathrm{N}).$$

For each element $(u, v)\in\mathfrak{T}_{\mathrm{N}}\times\mathfrak{T}_{\mathrm{N}}$, let $\mathfrak{M}''_{\mathrm{N}}(u, v)$ denote the coefficient of the matrix $\mathfrak{M}''_{\mathrm{N}}$ in u^{th} row and v^{th} column. We have for each $v\in\mathfrak{T}_{\mathrm{N}}$, $(\mathrm{I}-\lambda^{-1}\alpha'_{\mathrm{N}})^{\mu}\mathrm{X}^{v}=\sum\limits_{u}\mathfrak{M}''_{\mathrm{N}}(u, v)\mathrm{X}^{u}$, the sum being over all $u\in\mathfrak{T}_{\mathrm{N}}$. Thus $(\mathrm{I}-\lambda^{-1}\alpha'_{\mathrm{N}})^{\mu}\mathrm{T}_{\mathrm{N}}\xi=\sum\limits_{v}\mathrm{B}_v\sum\limits_{u}\mathfrak{M}''_{\mathrm{N}}(u, v)\mathrm{X}^{u}$, the sums being over all $u\in\mathfrak{T}_{\mathrm{N}}$ and all $v\in\mathfrak{T}_{\mathrm{N}}$. We conclude that for each $u\in\mathfrak{T}_{\mathrm{N}}$,

$$\sum_{v}\mathfrak{M}''_{\mathrm{N}}(u, v)\mathrm{B}_v\equiv 0 \mod \mathrm{C}(\varkappa q u_0-2e+(q-1)\varkappa\mathrm{N}),$$

the sum being over all $v\in\mathfrak{T}_{\mathrm{N}}$. We recall that $\mathrm{B}_v=0$ for $v\in\mathrm{H}$ and hence if N'' is the number of elements in $\mathfrak{T}_{\mathrm{N}}-\mathrm{H}$, the system of N'' congruences indexed by $u\in\mathfrak{T}_{\mathrm{N}}-\mathrm{H}$,

$$\sum_{v}p^{-\varkappa q u_0}\mathfrak{M}''_{\mathrm{N}}(u, v)\mathrm{B}_v\equiv 0 \mod \mathrm{C}(-2e+(q-1)\varkappa\mathrm{N}), \tag{2.38}$$

(the sum being over all $v\in\mathfrak{T}_{\mathrm{N}}-\mathrm{H}$), has a non-trivial solution if $\mathrm{N}>\mathrm{N}_0$ since B_v is a unit for at least one $v\in\mathfrak{T}_{\mathrm{N}}-\mathrm{H}$. The ring of integers, $\mathfrak{O}$, of Ω is not a principal ideal ring, but finite sums of principal ideals are principal. Hence the theory of elementary divisors may be applied to the matrix E_{N} indexed by $(\mathfrak{T}_{\mathrm{N}}-\mathrm{H})\times(\mathfrak{T}_{\mathrm{N}}-\mathrm{H})$ whose „general" coefficient is $\mathrm{E}_{\mathrm{N}}(u, v)=p^{-\varkappa q u_0}\mathfrak{M}''_{\mathrm{N}}(u, v)$. If $\varepsilon_1|\varepsilon_2|\ldots|\varepsilon_{\mathrm{N}''}$ are the elementary divisors of E_{N} then (2.38) shows that

$$\varepsilon_{\mathrm{N}''}\equiv 0 \mod \mathrm{C}(-2e+(q-1)\varkappa\mathrm{N}). \tag{2.39}$$

Since our object is to prove (2.35), we may assume $\det(\mathfrak{M}''_{\mathrm{N}}, \mathrm{H})\neq 0$. Hence $0\neq\det\mathrm{E}_{\mathrm{N}}$, $0\neq\varepsilon_{\mathrm{N}''}$. If u and v lie in $\mathfrak{T}_{\mathrm{N}}-\mathrm{H}$, let $(\mathrm{E}_{\mathrm{N}}, (u, v))$ denote the matrix obtained from E_{N} by deleting row u and column v. Let $((\mathfrak{M}''_{\mathrm{N}}, \mathrm{H}), (u, v))$ denote the corresponding matrix associated with $(\mathfrak{M}''_{\mathrm{N}}, \mathrm{H})$. It follows from the definitions that

$$\det(\mathrm{E}_{\mathrm{N}}, (u, v))/\det\mathrm{E}_{\mathrm{N}}=p^{\varkappa q u_0}\det((\mathfrak{M}''_{\mathrm{N}}, \mathrm{H}), (u, v))/\det(\mathfrak{M}''_{\mathrm{N}}, \mathrm{H}). \tag{2.40}$$

Ideal theoretically, $(\det\mathrm{E}_{\mathrm{N}})=(\varepsilon_{\mathrm{N}''})\Sigma(\det(\mathrm{E}_{\mathrm{N}}, (u, v)))$, the sum being over all $(u, v)\in(\mathfrak{T}_{\mathrm{N}}-\mathrm{H})^2$. Thus $-\operatorname{ord}\varepsilon_{\mathrm{N}''}=\operatorname{Min}\operatorname{ord}\det(\mathrm{E}_{\mathrm{N}}, (u, v))-\operatorname{ord}\det\mathrm{E}_{\mathrm{N}}$, the minimum being over all $(u, v)\in(\mathfrak{T}_{\mathrm{N}}-\mathrm{H})^2$. This together with (2.40) shows that

$$-\operatorname{ord}\varepsilon_{\mathrm{N}''}=\operatorname{Min}\{\varkappa q u_0+\operatorname{ord}\det((\mathfrak{M}''_{\mathrm{N}}, \mathrm{H}), (u, v))\}-\operatorname{ord}\det(\mathfrak{M}''_{\mathrm{N}}, \mathrm{H}), \tag{2.41}$$

the minimum being over the same set as before. This together with (2.39) would give the proof of (2.35), if it were known that c may be chosen independent of N and H, u and v such that

$$(2.42) \qquad \varkappa q u_0 + \operatorname{ord} \det((\mathfrak{M}''_N, H), (u, v)) \geq c + 2e.$$

Thus the proof of (2.35) has been reduced to that of (2.42).

We observe that $\mathfrak{M}'_N(u, v) = A_{qu-v}$ and hence $\operatorname{ord} \mathfrak{M}'_N(u, v) \geq \varkappa\sigma(qu-v)$. It is easily verified that if two square matrices (each indexed by $\mathfrak{T}_N$) satisfy this estimate then so does their product since $\varkappa\sigma(qu-w) + \varkappa\sigma(qw-v) \geq \varkappa\sigma(qu-v)$. Thus

$$\mathfrak{M}''_N = (I - \lambda^{-1}\mathfrak{M}'_N)^\mu = I + \mathfrak{M}'''_N \lambda^{-\mu},$$

where $\mathfrak{M}'''_N$ is a square matrix indexed by $\mathfrak{T}_N - H$ satisfying the condition

$$(2.43) \qquad \operatorname{ord} \mathfrak{M}'''_N(u, v) \geq \varkappa\sigma(qu-v)$$

for all $(u, v) \in (\mathfrak{T}_N - H)^2$. Equation (2.42) now follows directly from Lemma 2.2 (ii). This completes the proof of (2.42) and hence of (2.35). As we have noted previously, this implies the validity of (2.33). We must now show that (2.33) implies (2.34). This is clearly the case for $r = 1$. Hence we may assume that $r > 1$ and that (2.34) has been verified for $i = 0, 1, \ldots, r-2$. Replacing A by $I - \lambda^{-1}\mathfrak{M}'_N$ in (2.31), we have

$$(S^{(N'-(r-1))}(I - \lambda^{-1}\mathfrak{M}'_N))^\mu = \\ S^{(N'-(r-1))}((I - \lambda^{-1}\mathfrak{M}'_N)^\mu) - \Sigma' g(i_0, \ldots, i_{\mu-1}) \prod_{s=0}^{\mu-1} S^{(N'-i_s)}(I - \lambda^{-1}\mathfrak{M}'_N)$$

the sum, Σ', on the right being over all $i_0, i_1, \ldots, i_{\mu-1}$ in $\{1, 2, \ldots, N'\}$ such that $\sum_{s=0}^{\mu-1} i_s = \mu(r-1)$, but $i_0 = i_1 = \ldots = i_{\mu-1}$ is excluded. In each term in the sum, Σ', at least one factor $S^{(N'-i_s)}(I - \lambda^{-1}\mathfrak{M}'_N)$ occurs such that $i_s < r-1$, while the remaining factors are $\mu - 1$ in number and each of type $S^{(N'-j)}(I - \lambda^{-1}\mathfrak{M}'_N)$, $j \leq \mu(r-1)$. The assertion follows from the induction hypothesis provided we verify the existence of a finite lower bound for $\operatorname{ord} S^{(N'-j)}(I - \lambda^{-1}\mathfrak{M}'_N)$ independent of N and valid for $j \leq \mu(r-1)$. The existence of such a lower bound is an obvious consequence of equation (2.29) and Lemma 2.2 (i). This completes the proof of the theorem.

Note. — No use has been made in Theorem 2.3 of compactness and no hypothesis concerning the field generated by the coefficients of F is needed. On the other hand we do not know if Theorem 2.2 is valid without that hypothesis.

We now summarize some of our information.

Theorem **2.4.** — *For each non-zero element, λ, of Ω, let s_λ be the multiplicity of λ^{-1} as zero of χ_F. If the coefficients of* F *lie in a finite extension,* K_0, *of* $\mathbf{Q}'$, *then for $s \geq s_\lambda$ the space* $W_\lambda =$ *kernel in* L′ *of* $(I - \lambda^{-1}\alpha)^s$ *is independent of s, lies in* $L(\varkappa q)$ *and is of dimension s_λ. Furthermore* W_λ *has a basis consisting of elements of* $K_0(\lambda)\{X\}$.

Proof. — For given $\lambda \in \Omega^*$, let $W^{(s)}$ be the kernel of $(I - \lambda^{-1}\alpha)^s$. Theorem 2.2 shows that for $s \geq s_\lambda$, $\dim W^{(s)} \geq s_\lambda$, while Theorem 2.3 shows that $\dim W^{(s)} \leq s_\lambda$ for all

$s \geq 1$. Since $W^{(s)} \subset W^{(s+1)}$ for all $s \geq 1$ it is clear that $W^{(s)}$ is independent of s and has dimension s_λ for $s \geq s_\lambda$. The remainder of the theorem follows directly from Theorem 2.2.

Corollary. — *If* G *is an element of* $K_0(X)$ *such that for some real number* $b > 0$ *both* G *and* $1/G$ *lie in* $L(b)$ *and if* $H(X) = F(X)G(X)/G(X^q)$ *then* $\chi_F = \chi_H$, *it being understood that* F *and* K_0 *satisfy the conditions of the theorem.*

Proof. — Let $c = \operatorname{Min}(\varkappa, b)$. It is clear that $\xi \to G.\xi$ is a mapping of $L(c)$ onto itself. The corollary now follows from the theorem and the fact that each $\xi \in L(c)$, $\psi(H\xi) = G(X)^{-1}.\alpha(\xi).G(X)$.

We have shown that the zeros of χ_F can be explained in terms of spectral theory if F satisfies the condition of Theorem 2.4. If it were known (as is the case in the geometrical application) that the coefficients of F and the zeros of χ_F all lie in a finite extension, Ω_0, of $\mathbf{Q}'$ then the zeros of χ_F can be explained entirely on the basis of the spectral theory of α as operator on $L'' = \Omega_0\{X\} \cap L(q\varkappa)$. We make no assertion of the type: L″ is a sum of primary subspaces corresponding to α. Our next result serves as a substitute for a statement of this type.

Theorem **2.5.** — *If* λ *is a non-zero element of* Ω *which is algebraic over* $\mathbf{Q}'$, *if* λ^{-1} *is of multiplicity* μ *as a zero of* χ_F, *if the coefficients of* F *lie in a finite extension,* K_0, *of* $\mathbf{Q}'$ *and if* K *is any finite extension of* $K_0(\lambda)$ *then*

$$(2.44) \qquad (I - \lambda^{-1}\alpha)^{\mu+1}(K\{X\} \cap L(q\varkappa)) = (I - \lambda^{-1}\alpha)^{\mu}(K\{X\} \cap L(q\varkappa)).$$

In particular if $\mu = 0$ (i.e., $\chi_F(\lambda^{-1}) \neq 0$) *then* $K\{X\} \cap L(q\varkappa) = (I - \lambda^{-1}\alpha)(K\{X\} \cap L(q\varkappa))$.

Proof. — Let $K' = K_0(\lambda)$. By hypothesis K is a finite extension of K′. For given K we show that (2.44) holds if and only if it is valid when K is replaced by K′. Let $\omega_1, \ldots, \omega_m$ be a minimal basis of K over K′. Suppose (2.44) is valid with K replaced by K′. If $\xi \in K\{X\} \cap L(q\varkappa)$ then $\xi = \sum_{i=1}^{m} \omega_i \xi_i$, where $\xi_i \in K'\{X\} \cap L(q\varkappa)$, $i = 1, 2, \ldots, m$. Hence by hypothesis there exist $\eta_1, \ldots, \eta_m$ each in $K'\{X\} \cap L(q\varkappa)$ such that $(I - \lambda^{-1}\alpha)^{\mu+1}\eta_i = (I - \lambda^{-1}\alpha)^{\mu}\xi_i$, $i = 1, 2, \ldots, m$, and hence $\eta = \sum_{i=1}^{m} \omega_i \eta_i \in K\{X\} \cap L(q\varkappa)$ and furthermore $(I - \lambda^{-1}\alpha)^{\mu+1}\eta = (I - \lambda^{-1}\alpha)^{\mu}\xi$. This shows that

$$(I - \lambda^{-1}\alpha)^{\mu+1}(K\{X\} \cap L(q\varkappa)) \supset (I - \lambda^{-1}\alpha)^{\mu}(K\{X\} \cap L(q\varkappa))$$

and since inclusion in the opposite direction is clear, we may conclude that (2.44) is valid for K if it is valid for K′. Conversely if (2.44) is valid for K, then given $\xi \in K'\{X\} \cap L(q\varkappa)$ there exists $\eta \in K\{X\} \cap L(q\varkappa)$ such that $(I - \lambda^{-1}\alpha)^{\mu+1}\eta = (I - \lambda^{-1}\alpha)^{\mu}\xi$. The relative trace, S, which maps K onto K′ may be extended to a mapping of $K\{X\}$ onto $K'\{X\}$ in an obvious way. The trace, S, commutes with α and hence $(I - \lambda^{-1}\alpha)^{\mu}\xi = (I - \lambda^{-1}\alpha)^{\mu+1}S(\eta/m)$ since $S(\xi) = m\xi$. Since $S(\eta) \in K'\{X\} \cap L(q\varkappa)$ we may conclude that (2.44), if valid for a given K, is also valid for K′.

We have shown that it is enough to prove the theorem for one finite extension, K, of K'. If λ^{-1} is not a zero of χ_F, let $K = K'$. If λ^{-1} is a zero of χ_F then following the procedure of the proof of Theorem 2.2, let $\chi_N(t) = \det(I - t\alpha_N)$, let ρ be real, $\rho > 0$ such that χ_N has no zeros in $\lambda^{-1}(1 + C_\rho)$ distinct from λ^{-1}. For all N large enough, χ_N has precisely μ zeros $\lambda_{1,N}^{-1}, \ldots, \lambda_{\mu,N}^{-1}$ in $\lambda^{-1}(1 + C_\rho)$, these are zeros of a polynomial f_N of degree μ which divides χ_N and is relatively prime to χ_N/f_N. Let K be the composition of all extensions of K' of degree not greater than μ. We know that $\lambda_{1,N}^{-1}, \ldots, \lambda_{\mu,N}^{-1}$ lie in K, approach λ^{-1} as $N \to \infty$ and are distinct from all other zeros of χ_N. In the following α_N will be restricted to $K[X] \cap \mathfrak{L}^{(N)}$.

Let β_N be the endomorphism $\prod_{i=1}^{\mu}(I - \lambda_{i,N}^{-1}\alpha_N)$ of $K[X] \cap \mathfrak{L}^{(N)}$ ($\beta_N = I$ if $\mu = 0$). Since β_N annihilates the primary components of $K[X] \cap \mathfrak{L}^{(N)}$ relative to α_N corresponding to the eigenvalues $\lambda_{1,N}, \ldots, \lambda_{\mu,N}$, it is clear that $\beta_N(K[X] \cap \mathfrak{L}^{(N)})$ is the direct sum of the primary components of $K[X] \cap \mathfrak{L}^{(N)}$ corresponding to the remaining eigenvalues of α_N. Hence if α_N'' denotes the restriction of α_N to $\beta_N(K[X] \cap \mathfrak{L}^{(N)})$, we can conclude that

$$\det(I - t\alpha_N'') = \det(I - t\alpha_N) \Big/ \prod_{i=1}^{\mu}(1 - t\lambda_{i,N}). \tag{2.45}$$

Let ξ be a given element of $K\{X\} \cap L(q\varkappa)$. We must find η in the same space such that $(I - \lambda^{-1}\alpha)^{\mu+1}\eta = (I - \lambda^{-1}\alpha)^{\mu}\xi$. We may suppose that $\xi \in L(q\varkappa, 0)$. Let $\xi_N = T_N\xi$. Since λ is not an eigenvalue of α_N'', there exists $\eta_N \in K[X] \cap \mathfrak{L}^{(N)}$ such that

$$(I - \lambda^{-1}\alpha_N)\beta_N\eta_N = \beta_N\xi_N. \tag{2.46}$$

Eventually we shall complete the proof by taking the limit of this relation as $N \to \infty$. The main problem is the demonstration that η_N may be chosen such that its limit lies in $L(q\varkappa)$. We note that $\beta_N\eta_N$ is uniquely determined by (2.46) and hence η_N is uniquely determined modulo the kernel, W_N, of β_N in $K[X] \cap \mathfrak{L}^{(N)}$, a subspace of dimension μ. We shall show that there exists a real number c' independent of N such that η_N can be chosen so as to satisfy the further requirement

$$\eta_N \in K[X] \cap L(q\varkappa, c') \tag{2.47}$$

for an infinite set of integers, N.

We first construct a basis of $\beta_N(K[X] \cap \mathfrak{L}^{(N)})$. For each $u \in \mathfrak{T}_N$, let $Y_{u,N} = \beta_N X^u$. The set $\{Y_{u,N}\}$ indexed by $u \in \mathfrak{T}_N$, spans $\beta_N(K[X] \cap \mathfrak{L}^{(N)})$ but does not (unless $\mu = 0$) constitute a basis of that space. In the proof of Theorem 2.2, it was shown that there exists an infinite subset, $\mathfrak{A}$, of $\mathbf{Z}_+$ and a set S of μ elements of $\mathfrak{T}$ such that for each $N \in \mathfrak{A}$, the kernel, W_N, of β_N in $(K[X] \cap \mathfrak{L}^{(N)})$ has a basis $\{g_{u,N}\}_{u \in S}$ consisting of elements of $K[X] \cap \mathfrak{L}^{(N)}(q\varkappa, -e) \cap \mathfrak{O}[X]$ indexed by S such that for each $v \in S$ the coefficient of X^v in $g_{u,N}$ is the Kronecker $\delta_{u,v}$. (In the previous remark, $e = (\mu + (q-1)^{-1}) \operatorname{ord} \lambda$, precisely as in the proof of Theorem 2.2.) Thus for each $u \in S$, we have (N being assumed in the remainder of the proof to lie in $\mathfrak{A}$),

$$g_{u,N} = X^u + \Sigma E_N(w, u) X^w, \tag{2.48}$$

the sum being over all $w \in \mathfrak{I}_N - S$, and furthermore $\operatorname{ord} E_N(w, u) \geq q\kappa w_0 - e$. We may now conclude that for each $u \in S$, since $0 = \beta_N(g_{u,N})$, that

$$-Y_{u,N} = \Sigma E_N(w, u) Y_{w,N}, \tag{2.49}$$

the sum being over all $w \in \mathfrak{I}_N - S$. Thus the set $\{Y_{w,N}\}$ indexed by $\mathfrak{I}_N - S$ spans $\beta_N(K[X] \cap \mathfrak{L}^{(N)})$ and hence must be a basis of that space, since it contains the correct number of elements.

We have noted that if η_N is a solution of (2.46), then the sum of η_N and any K-linear combination of the $g_{u,N}$ is also solution of (2.46). Equation (2.48) shows that η_N may be chosen such that the coefficient of X^u in η_N is zero for each $u \in S$. (In fact these additional conditions uniquely determine η_N). Thus we may write $\eta_N = \Sigma B_{v,N} X^v$, the sum being all $v \in \mathfrak{I}_N - S$. By hypothesis $\xi \in L(q\kappa, 0)$ and we write $\xi = \Sigma G_v X^v$, the sum being over all $v \in \mathfrak{I}$. Thus $\xi_N = T_N \xi = \Sigma G_v X^v$, the sum now being over $\mathfrak{I}_N$, and $\operatorname{ord} G_v \geq q\kappa v_0$. Thus $\beta_N \xi_N = \sum_{v \in \mathfrak{I}_N} G_v Y_{v,N} = \sum_{v \in \mathfrak{I}_N - S} G_v Y_{v,N} + \sum_{u \in S} G_u Y_{u,N}$. Applying (2.49) we now obtain $\beta_N \xi_N = \Sigma Y_{v,N}\{G_v - \sum_{u \in S} G_u E_N(v, u)\}$, the sum being over all $v \in \mathfrak{I}_N - S$.

Thus $\beta_N \xi_N = \Sigma G_{v,N} Y_{v,N}$, the sum being again over all $v \in \mathfrak{I}_N - S$. Here

$$G_{v,N} = G_v - \sum_{u \in S} E_N(v, u) G_u$$

and hence $\operatorname{ord} G_{v,N} \geq q\kappa u_0 - c$, c being a real number independent of N.

We now determine the matrix of α''_N relative to the basis $\{Y_{v,N}\}_{v \in \mathfrak{I}_N - S}$ of $\beta_N(K[X] \cap \mathfrak{L}^{(N)})$. Since α_N commutes with β_N, we have

$$\alpha''_N Y_{v,N} = \alpha_N \beta_N X^v = \beta_N \alpha_N X^v = \beta_N \sum_{w \in \mathfrak{I}_N} A_{qw-v} X^w.$$

With the aid of equation (2.49), it is easily seen that for $v \in \mathfrak{I}_N - S$

$$\alpha''_N Y_{v,N} = \Sigma A'_N(w, v) Y_{w,N}, \tag{2.50}$$

the sum being over all $w \in \mathfrak{I}_N - S$, where for $(w, v) \in (\mathfrak{I}_N - S)^2$,

$$A'_N(w, v) = A_{qw-v} - \sum_{u \in S} E_N(w, u) A_{qu-v}.$$

It is easily verified that $\operatorname{ord} A'_N(w, v) \geq \kappa\sigma(qw - v) - e$.

Let A'_N be the square matrix indexed by $\mathfrak{I}_N - S$ whose w, v coefficient is $A'_N(w, v)$. Equation (2.50) shows that $\det(I - \lambda^{-1} A'_N) = \det(I - \lambda^{-1} \alpha''_N)$. Since $\chi_F(t) = \lim_{N \to \infty} \det(I - t\alpha_N)$, we conclude from (2.45) that $\lim_{N \to \infty} \det(I - \lambda^{-1} A'_N)$ is the value assumed at $t = \lambda^{-1}$ by $\chi_F(t)/(1 - \lambda t)^{\mu}$. This value is not zero since μ is the multiplicity of λ^{-1} as zero of χ_F and hence for N large enough, $\det(I - \lambda^{-1} A'_N)$ is bounded away from zero. Explicitly there exists a rational number c'' such that for N large enough,

$$\operatorname{ord} \det(I - \lambda^{-1} A'_N) < c''. \tag{2.51}$$

Equations (2.46) and (2.50) show that the set $\{B_{v,N}\}$ indexed by $v\in\mathfrak{T}_N-S$ is a solution of the system of equations indexed by $w\in\mathfrak{T}_N-S$

$$(2.52) \qquad \Sigma(\delta_{w,v}-\lambda^{-1}A'_N(w, v))B_{v,N}=G_{w,N},$$

the sum being over $v\in\mathfrak{T}_N-S$, it being understood that $\delta_{w,v}$ is the Kronecker δ symbol. To verify equation (2.47), we apply Cramer's rule to equations (2.52) and estimate $\operatorname{ord} B_{v,N}$ for each $v\in\mathfrak{T}_N-S$. For each element (w, v) of $(\mathfrak{T}_N-S)^2$, let$((I-\lambda^{-1}A'_N), (w, v))$ be the square matrix obtained from $I-\lambda^{-1}A'_N$ by deleting row w and column v. Clearly

$$B_{v,N}.\det(I-\lambda^{-1}A'_N)=\Sigma\pm\det((I-\lambda^{-1}A'_N), (w, v))G_{w,N}$$

the sum being over all $w\in\mathfrak{T}_N-S$. In view of (2.51) it is enough to show that there exists c''' independent of N such that

$$(2.53) \qquad \operatorname{ord}\det((I-\lambda^{-1}A'_N), (w, v))+\operatorname{ord} G_{w,N}\geq q\varkappa v_0-c'''$$

for all $(w, v)\in(\mathfrak{T}_N-S)^2$. Equation (2.53) is however a direct consequence of Lemma 2.2 (ii) and our estimates for $\operatorname{ord} G_{w,N}$ and $\operatorname{ord} A'_N(w, v)$. This completes the proof of (2.47).

Since $K\{X\}\cap L(q\varkappa, c')$ is compact, we conclude that the infinite sequence $\{\eta_N\}$ has a limit point η in $L(q\varkappa, c')$. Taking the limit of equation (2.46) as $N\rightarrow\infty$ over a suitable infinite subset of $\mathbf{Z}_+$, we obtain $(I-\lambda^{-1}\alpha)^{\mu+1}\eta=(I-\lambda^{-1}\alpha)^{\mu}\xi$. Thus we have shown that $(I-\lambda^{-1}\alpha)^{\mu+1}(K\{X\}\cap L(q\varkappa))\supset(I-\lambda^{-1}\alpha)^{\mu}(K\{X\}\cap L(q\varkappa))$. This completes the proof of the theorem.

Corollary. — In the notation of the theorem, let $\mathfrak{K}=K\{X\}\cap L(q\varkappa)$ *and let* W *be the kernel of* $(I-\lambda^{-1}\alpha)^{\mu}$ *in* $\mathfrak{K}$. *For each integer* j, $j\geq 1$ *we have*

$$(2.54) \qquad \begin{cases}\mathfrak{K}=W+(I-\lambda^{-1}\alpha)^{j}\mathfrak{K}\\ W\cap(I-\lambda^{-1}\alpha)^{j}\mathfrak{K}=(I-\lambda^{-1}\alpha)^{j}W.\end{cases}$$

If $(I-\lambda^{-1}\alpha)^{\nu}W=\{0\}$ *then*

$$(2.55) \qquad (I-\lambda^{-1}\alpha)^{\nu+1}\mathfrak{K}=(I-\lambda^{-1}\alpha)^{\nu}\mathfrak{K}.$$

Proof. — For simplicity let us use the symbol θ for $(I-\lambda^{-1}\alpha)$. The theorem shows that given $\eta\in\mathfrak{K}$ there exists $\xi\in\mathfrak{K}$ such that $\theta^{\mu}\eta=\theta^{\mu+1}\xi$, which shows that $\theta^{\mu}(\eta-\theta\xi)=0$ and therefore $\eta\in W+\theta\xi$. This shows that $\mathfrak{K}\subset W+\theta\mathfrak{K}$ and hence using the fact that $\theta W\subset W$ we easily see that $\mathfrak{K}\subset W+\theta^{j}\mathfrak{K}$ if $j\geq 1$. This proves the first half of equation (2.54). Writing this with $j=1$ and applying θ^{ν} to both sides we obtain $\theta^{\nu}\mathfrak{K}=\theta^{\nu}W+\theta^{\nu+1}\mathfrak{K}$, which proves (2.55), since $\theta^{\nu}W=0$.

If $\xi\in\mathfrak{K}$ and $\theta^{j}\xi\in W$ then $\theta^{j+\mu}\xi\in\theta^{\mu}W=\{0\}$ and using Theorem 2.4 we see that $\xi\in W$, which shows that $\theta^{j}\xi\in\theta^{j}W$. This completes the proof of (2.54).

§ 3. Differential Operators.

a) *Introduction*

In this section we modify the notation of the previous section so as to facilitate the application of our results to projective varieties. Let $\mathbf{Q}'$ and Ω be as before. Let Ω_0 be a finite extension of $\mathbf{Q}'$ in Ω, whose absolute ramification is divisible by $p-1$. Let $n \geq 0$, $d \geq 1$ be fixed integers as before. Let $\mathfrak{I}$ now be the set of all $u=(u_0, u_1, \ldots, u_{n+1}) \in \mathbf{Z}_+^{n+2}$ such that $du_0 = u_1 + \ldots + u_{n+1}$. The definitions of $\mathrm{L}(b, c)$, $\mathrm{L}(b)$, $\mathfrak{L}$, $\mathfrak{L}^{(\mathrm{N})}$, $\mathfrak{L}^{(\mathrm{N})}(b, c)$ are now precisely as in § 2 except that the set $\mathfrak{I}$ is given a new meaning and furthermore these additive groups now lie in $\Omega_0\{\mathrm{X}_0, \mathrm{X}_1, \ldots, \mathrm{X}_{n+1}\}$ instead of $\Omega\{\mathrm{X}_0, \mathrm{X}_1, \ldots, \mathrm{X}_n\}$. Let S be the set $\{1, 2, \ldots, n+1\}$. For each subset A of S (including the empty subset), let M_A be the monomial $\prod_{i \in \mathrm{A}} \mathrm{X}_i$, $(\mathrm{M}_\emptyset = 1)$ and let $\mathrm{L}^\mathrm{A}(b, c)$, $\mathrm{L}^\mathrm{A}(b)$, $\mathfrak{L}^\mathrm{A}$, $\mathfrak{L}^{\mathrm{A},(\mathrm{N})}$, $\mathfrak{L}^{\mathrm{A},(\mathrm{N})}(b, c)$ be the subsets of the previously defined sets which satisfy the further condition of divisibility by M_A in $\Omega_0\{\mathrm{X}_0, \mathrm{X}_1, \ldots, \mathrm{X}_n\}$.

Let $\mathrm{S}'=\{0, 1, \ldots, n+1\}$, $\mathrm{S}''=\{0, 1, \ldots, n\}$.

Let $\mathfrak{O}_0$ be the ring of integers in Ω_0 and let K be the residue class field of Ω_0. Let E_i be the derivation $\xi \rightarrow \mathrm{X}_i \dfrac{\partial \xi}{\partial \mathrm{X}_i}$ of $\Omega_0\{\mathrm{X}_0, \ldots, \mathrm{X}_{n+1}\}$, $i=0, 1, \ldots, n+1$. A homogeneous form f in $\mathfrak{O}_0\{\mathrm{X}_1, \ldots, \mathrm{X}_{n+1}\}$ will be said to be *regular* (with respect to the variables $\mathrm{X}_1, \ldots, \mathrm{X}_{n+1}$) if the images in $\mathrm{K}[\mathrm{X}_1, \ldots, \mathrm{X}_{n+1}]$ of the polynomials f, $\mathrm{E}_1 f$, $\ldots$, $\mathrm{E}_{n+1} f$ have no common zero in n-dimensional projective space of characteristic p.

Let π be an element of $\mathfrak{O}_0$ such that $\operatorname{ord} \pi = 1/(p-1)$, f a form of degree d in $\mathfrak{O}_0[\mathrm{X}_1, \ldots, \mathrm{X}_{n+1}]$ which is regular with respect to the variables $\mathrm{X}_1, \ldots, \mathrm{X}_{n+1}$ and let H be an element of $\mathrm{L}(1/(p-1))$ such that

$$\mathrm{H} \in \Omega_0\{\mathrm{X}\}$$
$$\mathrm{H} \equiv \pi \mathrm{X}_0 f \bmod \mathrm{X}_0^2$$
$$\mathrm{H}_i = \mathrm{E}_i \mathrm{H} \in \mathrm{L}(p/(p-1), -1),\ i=0, 1, \ldots, n+1.$$

For $i=0, 1, \ldots, n+1$, let D_i be the differential operator $\xi \rightarrow \mathrm{E}_i \xi + \xi . \mathrm{H}_i$, mapping $\mathrm{L}(-\infty)$ into itself. It is easily verified that $d\mathrm{D}_0 = \mathrm{D}_1 + \ldots + \mathrm{D}_{n+1}$, that $\mathrm{D}_i \circ \mathrm{D}_j = \mathrm{D}_j \circ \mathrm{D}_i$ and that D_i maps $\mathrm{L}(b)$ into itself for $b \leq p/p-1$. The object of this section is the study of the factor space $\mathrm{L}(b) / \sum_{i=1}^{n+1} \mathrm{D}_i \mathrm{L}(b)$. To make use of the regularity of f we must recall some well-known facts about polynomial rings.

b) *Polynomial Ideals.*

If A is any set and G is an additive group then a set of elements $\xi_{i,j}$ in G indexed by $\mathrm{A} \times \mathrm{A}$ will be said to be a *skew symmetric set in* G *indexed by* A if $\xi_{i,j} = -\xi_{j,i}$, $\xi_{i,i}=0$ for all $i, j \in \mathrm{A}$.

Let $\mathfrak{K}$ be a field of arbitrary characteristic, and let $\mathfrak{a}$ be a homogeneous ideal in $\mathfrak{K}[\mathrm{X}] = \mathfrak{K}[\mathrm{X}_1, \ldots, \mathrm{X}_{n+1}]$. The ideal $\mathfrak{a}$ has an irredundant decomposition into homogeneous primary ideals, $\mathfrak{a} = \bigcap_{i=1}^{r} \mathfrak{q}_i$. The dimension of $\mathfrak{a}$ is defined to be $\mathrm{Max}\dim \mathfrak{q}_i$ and dimension here is in the projective sense. We recall [5],

I. If $g \in \mathfrak{R}[X]$ then $(\mathfrak{a} : g) = \mathfrak{a}$ if and only if $g \notin \mathfrak{q}_i$, $i = 1, 2, \ldots r$.

II. If g is a non-constant homogeneous form then $\dim(\mathfrak{a} + (g)) = \dim \mathfrak{a} - 1$ if g lies in no primary component $\mathfrak{q}_i$ of maximal dimension, while otherwise $\dim(\mathfrak{a} + (g)) = \dim \mathfrak{a}$.

III. (Unmixedness Theorem): If $\mathfrak{a} = (g_1, g_2, \ldots, g_t)$, $t \leq n+1$ and $\dim \mathfrak{a} = n - t$ then each primary component of $\mathfrak{a}$ is of dimension $n - t$.

Lemma **3.1.** — *If* $g_1, \ldots, g_{n+1}$ *are non-constant homogeneous forms in* $\mathfrak{R}[X_1, \ldots, X_{n+1}]$ *with no common zero in n-dimensional projective space of characteristic equal to that of* $\mathfrak{R}$ *and if* $\{P_i\}_{i \in A}$ *is a set of polynomials indexed by a subset* A *of* $S = \{1, 2, \ldots, n+1\}$ *such that* $\sum_{i \in A} P_i g_i = 0$ *then there exists a skew symmetric set* $\eta_{i,j}$ *in* $\mathfrak{R}[X]$ *indexed by* A *such that* $P_i = \sum_{j \in A} \eta_{ij} g_j$ *for each* $i \in A$. *Furthermore if* $\{P_i\}_{i \in A}$ *consists of homogeneous elements such that* $\deg(P_i g_i) = m$ *is independent of i, then each* $\eta_{i,j}$ *may be chosen homogeneous of degree* $m - \deg(g_i g_j)$.

Proof. — Let $\mathfrak{a}_r = (g_1, \ldots, g_r)$, $1 \leq r \leq n+1$. By hypothesis $\dim \mathfrak{a}_{n+1} = -1$, while by II, $\dim \mathfrak{a}_r - 1 \leq \dim \mathfrak{a}_{r+1} \leq \dim \mathfrak{a}_r$ for $r = 1, 2, \ldots, n$. Also by II, $\dim \mathfrak{a}_1 \geq n - 1$. These inequalities show that $\dim \mathfrak{a}_r = n - r$ for $r = 1, 2, \ldots, n+1$ and that $\dim \mathfrak{a}_{r+1} = \dim \mathfrak{a}_r - 1$ for $r \leq n$. Hence by III, the primary components of $\mathfrak{a}_r$ are all of dimension $n - r$ and by II, g_{r+1} does not lie in any primary component of $\mathfrak{a}_r$ for $r = 1, 2, \ldots, n$. Hence by I, $(\mathfrak{a}_r : g_{r+1}) = \mathfrak{a}_r$. Furthermore since $\dim \mathfrak{a}_1 = n - 1$, we know $g_1 \neq 0$. If $A = \{1\}$ then $P_1 = 0$ and hence we can assume $A = \{1, 2, \ldots, r+1\}$, $r \geq 1$. Since $(\mathfrak{a}_r : g_{r+1}) = \mathfrak{a}_r$, $P_{r+1} \in \mathfrak{a}_r$ and hence there exist polynomials $h_1, h_2, \ldots, h_r$ such that $P_{r+1} = \sum_{i=1}^{r} h_i g_i$. Thus $\sum_{i=1}^{r} (P_i + h_i g_{r+1}) g_i = 0$. Using the obvious induction hypothesis on r, there exists a skew symmetric set $\eta_{i,j}$ in $\mathfrak{R}[X]$ indexed by $\{1, 2, \ldots, r\}$ such that $P_i + h_i g_{r+1} = \sum_{j=1}^{r} \eta_{ij} g_j$ for $i = 1, 2, \ldots, r$. Let $\eta_{r+1,i} = h_i$, $\eta_{i,r+1} = -h_i$, for $i = 1, 2, \ldots r$ and let $\eta_{r+1,\,r+1}, r = 0$. The assertion follows directly.

The valuation of Ω_0 can be extended to a valuation of the polynomial ring $\Omega_0[X]$ in the usual way, if $g(x) = \Sigma a_u x^u$, let $\operatorname{ord} g = \operatorname*{Min}_{u} \operatorname{ord} a_u$.

Lemma **3.2.** — *Let* $g_1, \ldots, g_{n+1}$ *be non-constant homogeneous forms in* $\mathfrak{O}_0[X_1, \ldots, X_{n+1}]$ *whose images in* K[X] *have only the trivial common zero. Let* A *be a non-empty subset of* S *and let g be an element of the ideal* $\sum_{i \in A} (g_i)$ *of* $\Omega_0[X]$. *Then there exist elements* $\{h_i\}_{i \in A}$ *of* $\Omega_0[X]$ *such that* $g = \sum_{i \in A} g_i h_i$ *and such that* $\operatorname{ord} h_i \geq \operatorname{ord} g$ *for each* $i \in A$.

Proof. — We may suppose that $g \neq 0$ and hence that $\operatorname{ord} g = 0$. By hypothesis $g = \sum_{i \in A} g_i h_i$, $h_i \in \Omega_0[X]$. Let e be the absolute ramification of Ω_0 and let $-b = e . \operatorname*{Min}_{j \in A} \operatorname{ord} h_i$. Clearly b is an integer and we complete the proof by showing that if $b > 0$ then there exists a set of elements $\{h'_j\}_{j\in \mathrm{A}}$ indexed by A in $\Omega_0[\mathrm{X}]$ such that $g=\sum_{i\in \mathrm{A}} g_i h'_i$ and such that $e.\operatorname*{Min}_{j\in \mathrm{A}} \operatorname{ord} h'_j \geq -b+1$. Let Π be a prime element of Ω_0. By definition, $\Pi^b h_j \in \mathfrak{O}_0[\mathrm{X}]$ for each $j\in \mathrm{A}$ and if $b>0$ then $\sum_{i\in \mathrm{A}} g_i \Pi^b h_i = \Pi^b g \equiv 0 \bmod(\Pi)$. Let G_i be the image of g_i and let ξ_i be the image of $\Pi^b h_i$ in K[X] for each $i\in \mathrm{A}$. Thus in K[X], $0=\sum_{i\in \mathrm{A}} \mathrm{G}_i\xi_i$ and so by Lemma 3.1, there exists a skew symmetric set, $\{\eta_{i,j}\}$, in K[X] indexed by A such that $\xi_i=\sum_{i\in \mathrm{A}}\eta_{i,j}\mathrm{G}_j$ for each $i\in \mathrm{A}$. We now choose a skew symmetric set $\{\eta'_{i,j}\}$ in $\mathfrak{O}_0[\mathrm{X}]$ indexed by A such that $\eta_{i,j}$ is the image in K[X] of $\eta'_{i,j}$ for each $(i,j)\in \mathrm{A}\times \mathrm{A}$. Hence $\Pi^b h_i \equiv \sum_{j\in \mathrm{A}}\eta'_{i,j}g_j \bmod(\Pi)$ for each $i\in \mathrm{A}$. We now define a set of elements $\{h'_i\}_{i\in \mathrm{A}}$ in $\Omega_0[\mathrm{X}]$ by the equations $\Pi^b h_i = \Pi^b h'_i + \sum_{j\in \mathrm{A}}\eta'_{i,j}g_j$ for each $i\in \mathrm{A}$. Clearly $\Pi^b h'_i \equiv 0 \bmod(\Pi)$ and hence $e.\operatorname*{Min}_{j\in \mathrm{A}}\operatorname{ord} h'_j \geq -b+1$. On the other hand the skew symmetry of the set $\eta'_{i,j}$ shows that $g=\sum_{i\in \mathrm{A}} g_i h'_i$ which completes the proof of the lemma.

Corollary. — If $g_1, \ldots, g_{n+1}$ *satisfy the conditions of the above lemma and* $\{\mathrm{P}_i\}_{i\in \mathrm{A}}$ *is a set of elements of* $\Omega_0[\mathrm{X}]$ *such that* $\sum_{i\in \mathrm{A}} \mathrm{P}_i g_i = 0$, *then the skew symmetric set* $\eta_{i,j}$ *of Lemma* 3.1 *may be chosen such that* $\operatorname*{Min}_{i,j} \operatorname{ord} \eta_{i,j} \geq \operatorname*{Min}_{i} \operatorname{ord} \mathrm{P}_i$.

Let f be the form of degree d in $\mathfrak{O}_0\{\mathrm{X}_1, \ldots, \mathrm{X}_{n+1}\}$ which is regular with respect to the variables $\mathrm{X}_1, \mathrm{X}_2, \ldots, \mathrm{X}_{n+1}$. Let $f_0=f, f_i=\mathrm{E}_i f$ for $i=1, 2, \ldots, n+1$. Since $df_0=f_1+f_2+\ldots+f_{n+1}$, it is clear (letting $\bar{f}_i$ be the image of f_i in $\mathrm{K}[\mathrm{X}_1, \ldots, \mathrm{X}_{n+1}]$) that the regularity of f is equivalent to

(i) $\bar{f}_0, \bar{f}_1, \ldots, \bar{f}_n$ have only the trivial common zero

(ii) $\bar{f}_1, \bar{f}_2, \ldots, \bar{f}_{n+1}$ have only the trivial common zero if $p\nmid d$.

Condition (ii) is simpler for most of our applications but will not be used since it would limit our results to the case in which d is prime to p. However we do note that in any case the regularity of f implies the triviality of the common zeros in Ω of $f_1, f_2, \ldots, f_{n+1}$. Thus Lemma 3.1 shows that $f_1, f_2, \ldots, f_{n+1}$ are linearly independent over Ω_0 (and Ω).

The following lemma refers to ideals in either $\Omega_0[\mathrm{X}]$ or in K[X]. To simplify the statement we use the same symbol for f_i and $\bar{f}_i$.

Lemma **3.3.** — *Let* B *be a non-empty subset of* $\mathrm{S}=\{1, 2, \ldots, n+1\}$.

(i) $$(\mathrm{M_B})\cap \sum_{i\in \mathrm{A}} (f_i) = \sum_{i\in \mathrm{A}-\mathrm{B}} (\mathrm{M_B} f_i) + \sum_{i\in \mathrm{A}\cap \mathrm{B}} (\mathrm{M_B} f_i/\mathrm{X}_i)$$

if A *is any non-empty subset of* S, *provided the characteristic does not divide d* (*i.e. the assertion holds in any case in* $\Omega_0[\mathrm{X}]$ *and if* $p\nmid d$ *in* K[X]).

(ii) *In either characteristic, if* $\mathrm{A} \neq \mathrm{S}$ *then*

$$(\mathrm{M_B})\cap ((f_0) + \sum_{i\in \mathrm{A}} (f_i)) = (\mathrm{M_B} f_0) + \sum_{i\in \mathrm{A}-\mathrm{B}} (\mathrm{M_B} f_i) + \sum_{i\in \mathrm{A}\cap \mathrm{B}} (\mathrm{M_B} f_i/\mathrm{X}_i).$$

unless both $\mathrm{A}\cup \mathrm{B}=\mathrm{S}$ *and* A *contains n elements.*

Proof. — In both statements the ideal on the right side clearly lies in the ideal on the left side. To prove (i) it is clearly enough to show that if $M_B . h \in \sum_{i\in A} (f_i)$ then

$$h \in \sum_{i\in A-B} (f_i) + \sum_{i\in A\cap B} (f_i/X_i) \tag{3.1}$$

Let $B \cap A = C$, $B' = B - C$. Let $h' = M_C h$. We first show that $h' \in \sum_{i\in A} (f_i)$. This is clear if B' is empty, hence we may use induction on the number of elements in B'. If $j \in B'$, then letting $B'' = B' - \{j\}$, $h'' = M_{B''} h'$ then $X_j h'' = M_{B'} h' \in \sum_{i\in A} (f_i)$ and if we can show that $h'' \in \sum_{i\in A} (f_i)$ then by the induction hypothesis we may conclude that the same holds for h'. Thus we consider $j \notin A$, $X_j h'' \in \sum_{i\in A} (f_i)$ and recall that $X_j, \{f_i\}_{i \neq j, i\in S}$ is a set of $n+1$ non-constant polynomials with no non-trivial common zero (since the characteristic does not divide d) and hence Lemma 3.1 shows that $h'' \in \sum_{i\in A} (f_i)$. Hence $M_C h \in \sum_{i\in A} (f_i)$ as asserted. If $d = 1$ then if C is empty, (3.1) is trivial, while if $j \in C$, then f_j/X_j is a non-zero constant which again shows that (3.1) is trivial. Hence it may be supposed that $d > 1$, in which case $f_i' = f_i/X_i$ is a non-constant form for each $i \in S$. We may assume that $C = \{1, 2, \ldots, r\}$, $A = \{1, 2, \ldots, t\}$, $r \leq t \leq n+1$ Thus $X_1 X_2 \ldots X_r h \in \sum_{i=1}^{t} (f_i)$ and hence for some polynomial h_1, $X_1(X_2 \ldots X_r h - f_1' h_1) \in \sum_{i=2}^{t} (f_i)$. We now apply Lemma 3.1 to the $n+1$ polynomials, $X_1, f_2, \ldots, f_t, f_{t+1}, \ldots, f_{n+1}$ and conclude that $X_2 \ldots X_r h \in (f_1') + \sum_{i=2}^{t} (f_i)$ (the left side is h if $r = 1$). Now suppose for some s, $1 \leq s < r$, $X_{s+1} X_{s+2} \ldots X_r h \in \sum_{i=1}^{s} (f_i') + \sum_{j=s+1}^{t} (f_i)$. Then there exists a polynomial, h_{s+1}, such that $X_{s+1}(X_{s+2} \ldots X_r h - h_{s+1} f_{s+1}') \in \sum_{i=1}^{s} (f_i') + \sum_{j=s+2}^{t} (f_j)$. The $n+1$ polynomials $f_1', f_2', \ldots, f_s', X_{s+1}, f_{s+2}, \ldots, f_{n+1}$ are non constant forms satisfying the conditions of Lemma 3.1 and hence $X_{s+2} \ldots X_r h \in \sum_{i=1}^{s+1} (f_i') + \sum_{i=s+2}^{t} (f_i)$. This completes the proof of (3.1), and hence of the first part of the lemma.

(ii) Here it is enough to show that if $M_B h \in (f_0) + \sum_{i\in A} (f_i)$ then

$$h \in (f_0) + \sum_{i\in A-B} (f_i) + \sum_{i\in A\cap B} (f_i'). \tag{3.2}$$

Let C and B′ be defined as before and let $h' = M_C h$. We first show that

$$h' \in (f_0) + \sum_{i\in A} (f_i). \tag{3.3}$$

To show this, it is enough (as before) to show that if $1 \notin A$ and $X_1 h'' \in \sum_{i\in A} (f_i) + (f_0)$ then the same holds for h''. By hypothesis B' is empty if A contains n elements and hence for the proof of (3.3) it may be assumed that A does not contain n elements. Thus

$A \cup \{1\}$ contains at most n elements. Let C' be a subset of S disjoint from $\{1\}$ which contains A and consists of exactly $n-1$ elements. The $n+1$ polynomials, f_0, X_1, $\{f_i\}_{i \in C'}$ satisfy the conditions of Lemma 3.1 and hence $h'' \in \sum_{i \in A} (f_i) + (f_0)$. This proves (3.3). If C is empty then (3.2) is trivially true, hence we may assume C not empty. If $d=1$ the $f_i'=1$ for each $i \in A \cap B \neq \emptyset$ and hence it may again be assumed that $d>1$. We may now let $C=\{1, 2, \ldots, r\}$, $A=\{1, 2, \ldots, t\}$, $r \leq t \leq n$. Since (3.3) now shows that $X_1 \ldots X_r h \in (f_0) + \sum_{i=1}^{t} (f_i)$, we have for some polynomial, h_1,

$$X_1(X_2 \ldots X_r h - h_1 f_1') \in (f_0) + \sum_{i=2}^{t} (f_i).$$

The set of $n+1$ polynomials, $(f_0, X_1, f_2, \ldots, f_n)$ satisfy the conditions of Lemma 3.1 and hence $X_2 \ldots X_r h \in (f_0) + (f_1') + \sum_{i=1}^{t} (f_i)$. We now suppose that for some s, $1 \leq s < r$, we have $X_{s+1} \ldots X_r h \in (f_0) + \sum_{i=1}^{s} (f_i') + \sum_{i=s+1}^{t} (f_i)$. Then for some polynomial h_{s+1}, $X_{s+1}(X_{s+2} \ldots X_r . h - f'_{s+1} h_{s+1}) \in (f_0) + \sum_{i=1}^{s} (f_i') + \sum_{i=s+2}^{t} (f_n)$. The $n+1$ polynomials $f_0, f_1', \ldots, f_s', X_{s+1}, f_{s+2}, \ldots, f_n$ satisfy the conditions of Lemma 3.1 and hence $X_{s+2} \ldots X_r h \in (f_0) + \sum_{i=1}^{s+1} (f_i') + \sum_{i=s+2}^{t} (f_i)$ which completes the proof of (3.2) and hence of the lemma.

c) P-*adic Directness.*

Let W be a vector space of dimension N over Ω_0 which has a « naturally » preassigned basis. For the purpose of the immediate exposition, we may let W be the space all N-tuples, Ω_0^N, with coefficients in Ω_0. However in the applications in the following parts of this section, W will be a subspace of $\Omega_0[X]$ whose « natural » basis is a finite set of monomials.

Let $\mathfrak{W}$ be the $\mathfrak{O}_0$-module, $\mathfrak{O}_0^N$, in W and let φ be the natural map of $\mathfrak{W}$ onto the K-space, $W^* = K^N$. For each subspace W_1 of W there exists a subspace, $W_1^* = \varphi(W_1 \cap \mathfrak{W})$, of W^*. The correspondence $W_1 \rightarrow W_1^*$ maps the set of all subspaces of W onto the set of all subspaces of W^* and preserves dimension. If W_1 and W_2 are subspaces of W then $(W_1 \cap W_2)^* \subset W_1^* \cap W_2^*$, but equality need not hold If however $W_1^* \cap W_2^* = \{0\}$, then equality must hold and hence $W_1 \cap W_2 = \{0\}$. We shall say that $W_1 + W_2$ is a *p-adically direct sum*, written $W_1 [+] W_2$, if $W_1^* \cap W_2^* = \{0\}$. In particular if $W_1 [+] W_2 = W$ then we shall say that W_2 is *p-adically complementary* to W_1 in W. It follows from the above remarks that given a subspace W_1 of Q, there exists a subspace of W which is p-adically complementary to W_1 in W.

The notion of p-adic directness is introduced because of the metric naturally associated with W. If $w = (w_1, \ldots, w_N)$ is an element of W then let $\operatorname{ord} w = \operatorname{Min}_i \operatorname{ord} w_i$.

If $w \in W' + W''$, (W' and W'' being subspaces of W), then $w = w' + w''$ where $w' \in W'$, $w'' \in W''$. Certainly $\operatorname{ord} w \geq \operatorname{Min}(\operatorname{ord} w', \operatorname{ord} w'')$, but if the sum $W' + W''$ is p-adically direct then $\operatorname{ord} w = \operatorname{Min}(\operatorname{ord} w', \operatorname{ord} w'')$ and hence $\operatorname{ord} w' \geq \operatorname{ord} w$.

d) *General Theory.*

Let $\mathfrak{A}$ be the ideal $(f_0, f_1, \ldots, f_n)$ in $\Omega_0[X_1, \ldots, X_{n+1}]$. For each integer $m \geq 0$, let $W^{(m)}$ be the space of forms of degree dm in $\Omega_0[X_1, \ldots, X_{n+1}]$, let $\mathfrak{A}_m = W^{(m)} \cap \mathfrak{A}$ and let $V^{(m)}$ be a subspace of $W^{(m)}$ p-adically complementary to $\mathfrak{A}_m$ in $W^{(m)}$, with respect to the monomial basis of $W^{(m)}$. Since $(f_0, f_1, \ldots, f_n)$ have no common nontrivial zero in Ω, $\mathfrak{A}$ must contain all homogeneous forms of high enough degree and hence there exists an integer, N_0, such that $V^{(m)} = \{0\}$ for $m > N_0$. We shall show eventually that we may take N_0 to be n. We note that $V^{(0)} = \Omega_0$.

We now let $V = \sum_{m=0}^{\infty} X_0^m V^{(m)}$, a subspace of $\mathfrak{L}^{(N_0)}$, and for each pair of real numbers b, c, let $V(b, c) = V \cap L(b, c)$. It follows from Lemma 3.2, the construction of V and the regularity of the polynomial f that if Q is a homogeneous form in $\Omega_0[X_1, \ldots, X_{n+1}]$ of degree dm, then $Q = P + \sum_{i=0}^{n} P_i f_i$, where $P \in V^{(m)}$, $P_0, P_1, \ldots, P_n$ each lie in $W^{(m-1)}$ and $\operatorname{ord} P \geq \operatorname{ord} Q$, $\operatorname{ord} P_i \geq \operatorname{ord} Q$ for $i = 0, 1, \ldots, n$.

We now proceed with the analysis of the differential operators introduced in part a) of this section. We recall that $H_i \in L(p/(p-1), -1)$, and that H_i has no constant term. It follows easily that if $b \leq p/(p-1)$ then $H_i \in L(b, -e)$, where $e = b - (p-1)^{-1}$.

Lemma **3.4**. — $L(b, c) = V(b, c) + \sum_{i=0}^{n} H_i L(b, c+e)$ *if* $b \leq p/(p-1), e = b - 1/(p-1)$.

Proof. — It is clear that the left side contains the right side. If ξ is an element of $L(b, c)$, we show that for each $N \in \mathbf{Z}_+$ there exists $\eta_N \in V(b, c) \cap \mathfrak{L}^{(N)}$ and a set $\xi_{i,N-1}$ of elements in $L(b, c+e)$ indexed by $i \in S'' = \{0, 1, \ldots, n\}$ such that

$$\xi \equiv \eta_N + \sum_{i=0}^{n} H_i \xi_{i,N-1} \bmod (X_0^{N+1}) \tag{3.4}$$

$$\begin{cases} \eta_{N-1} \equiv \eta_N \bmod X_0^N \\ \xi_{i,N-1} \equiv \xi_{i,N-2} \bmod X_0^{N-1} \quad \text{for each } i \in S''. \end{cases} \tag{3.5}$$

Let $P^{(0)}$ be the constant term of ξ, then (3.4) holds for $N = 0$ if we set $\eta_0 = P^{(0)} \in V(b, c)$ and $\xi_{i,-1} = 0$ for each $i \in S''$. We now suppose $N > 0$ and use induction on N. Then $\xi^{(N)} = \xi - \left(\eta_{N-1} + \sum_{i=0}^{n} H_i \xi_{i,N-2}\right)$ lies in $L(b, c)$ and is divisible by X_0^N. Let $P^{(N)}$ be the coefficient of X_0^N in $\xi^{(N)}$. Clearly $\operatorname{ord} P^{(N)} \geq bN + c$ and as noted above there exists $Q^{(N)} \in V^{(N)}$, $P_0^{(N-1)}, \ldots, P_n^{(N-1)}$ each in $W^{(N-1)}$ such that $P^{(N)} = Q^{(N)} + \Sigma \pi P_i^{(N-1)} f_i$, where $\operatorname{ord} Q^{(N)} \geq bN + c$, $\operatorname{ord} P_i^{(N-1)} \geq bN + c - (p-1)^{-1} = (b-1)N + c + e$ for each $i \in S''$. We now let $\eta_N = \eta_{N-1} + X_0^N Q^{(N)} \in V(b, c)$, and for each $i \in S''$ let

$$\xi_{i,N-1} = \xi_{i,N-2} + X_0^{N-1} P_i^{(N-1)} \in L(b, c+e)$$

and compute

$$\xi-\left(\eta_{\mathrm{N}}+\sum_{i=0}^{n}\mathrm{H}_i\xi_{i,\mathrm{N}-1}\right)=$$

$$\xi^{(\mathrm{N})}-\mathrm{X}_0^{\mathrm{N}}\mathrm{Q}^{(\mathrm{N})}-\mathrm{X}_0^{\mathrm{N}-1}\sum_{i=0}^{n}\mathrm{H}_i\mathrm{P}_i^{(\mathrm{N}-1)}\equiv\mathrm{X}_0^{\mathrm{N}}\left(\mathrm{P}^{(\mathrm{N})}-\mathrm{Q}^{(\mathrm{N})}-\pi\sum_{i=0}^{n}f_i\mathrm{P}_i^{(\mathrm{N}-1)}\right)\equiv 0 \bmod \mathrm{X}_0^{\mathrm{N}+1}.$$

This completes the proof of (3.4) and (3.5). The proof is completed by taking weak limits, $\xi_{i,\mathrm{N}}\rightarrow\xi_i\in\mathrm{L}(b, c+e)$ for each $i\in\mathrm{S}''$, $\eta_{\mathrm{N}}\rightarrow\eta\in\mathrm{V}(b, c)$ and hence $\xi=\eta+\sum_{i=0}^{n}\mathrm{H}_i\xi_i$.

Lemma **3.5.** — $\mathrm{V}\cap\sum_{i=0}^{n}\mathrm{H}_i\mathrm{L}(b)=\{0\}$ *if* $b\leq p/(p-1)$.

Proof. — Let ξ lie in the intersection, then $\xi=\sum_{i=0}^{n}\mathrm{H}_i\xi_i$, $\xi_i\in\mathrm{L}(b)$ for each $i\in\mathrm{S}''$. Let m be the minimal integer such that the coefficient $\mathrm{P}_i^{(m)}$ of X_0^m in ξ_i is not zero for at least one $i\in\mathrm{S}''$. For given ξ it may be assumed that $\xi_0, \ldots, \xi_n$ have been chosen in $\mathrm{L}(b)$ such that m is maximal. For $m'<m+1$ it is clear the coefficient of $\mathrm{X}_0^{m'}$ in ξ is zero. Let $\mathrm{Q}^{(m+1)}$ be the coefficient of X_0^{m+1} in ξ. Clearly $\mathrm{Q}^{(m+1)}=\pi\sum_{i=0}^{n}f_i\mathrm{P}_i^{(m)}\in\mathrm{V}^{(m+1)}\cap\mathfrak{A}_{m+1}=\{0\}$. It follows from Lemma 3.1 that there exists a skew symmetric set $\{\mathrm{B}_{i,j}\}$ indexed by S'' in $\mathrm{W}^{(m-1)}$ such that $\mathrm{P}_i^{(m)}=\pi\sum_{j=0}^{n}f_j\mathrm{B}_{i,j}$ for each $i\in\mathrm{S}''$. Let $\eta_{i,j}=\mathrm{B}_{i,j}\mathrm{X}_0^{m-1}$, $\xi_i'=\xi_i-\sum_{j=0}^{n}\mathrm{H}_j\eta_{i,j}\in\mathrm{L}(b)$, then $\xi=\sum_{i=0}^{n}\mathrm{H}_i\xi_i=\sum_{i=0}^{n}\mathrm{H}_i\xi_i'$ and for each $i\in\mathrm{S}''$ the coefficient of $\mathrm{X}_0^{m'}$ in ξ_i' is zero for $m'<m$ and the coefficient of X_0^m is $\mathrm{P}_i^{(m)}-\pi\sum_{j=0}^{n}f_j\mathrm{B}_{i,j}=0$, contradicting the maximality of m.

Lemma **3.6.** —

$$\mathrm{L}(b, c)=\mathrm{V}(b, c)+\sum_{i=0}^{n}\mathrm{D}_i\mathrm{L}(b, c+e)$$

if $(p-1)^{-1}\leq b\leq p/(p-1)$, $e=b-1/(p-1)$.

Proof. — Certainly $\mathrm{L}(b, c)$ contains the space on the right side. We first prove inclusion in the reverse direction if $e>0$ (i.e. $b>1/(p-1)$). Given $\xi\in\mathrm{L}(b, c)$ we construct a sequence of elements indexed by $r\in\mathbf{Z}_+$,

$$(\xi^{(r)}, \eta^{(r)}, \xi_0^{(r)}, \ldots, \xi_n^{(r)})\in\mathrm{L}(b, c+re)\times\mathrm{V}(b, c+re)\times(\mathrm{L}(b, c+(r+1)e))^{n+1}$$

by letting $\xi^{(0)}=\xi$ and the following recursive method. Given $\xi^{(r)}\in\mathrm{L}(b, c+re)$ we choose by Lemma 3.4, $\eta^{(r)}\in\mathrm{V}(b, c+re)$ and $\xi_i^{(r)}\in\mathrm{L}(b, c+(r+1)e)$ for $i=0, 1, \ldots, n$ such that $\xi^{(r)}=\eta^{(r)}+\sum_{i=0}^{n}\mathrm{H}_i\xi_i^{(r)}$. We now define $\xi^{(r+1)}$ by

$$\xi^{(r+1)}=\xi^{(r)}-\eta^{(r)}-\sum_{i=0}^{n}\mathrm{D}_i\xi_i^{(r)}. \tag{3.6}$$

We must show that $\xi^{(r+1)}\in L(b, c+(r+1)e)$. We note that

$$\xi^{(r+1)} = -\sum_{i=0}^{n} E_i \xi_i^{(r)} \in L(b, c+(r+1)e)$$

and this establishes our recursion process. Writing equation (3.6) for $r=0, 1, \ldots, h$ and adding, we obtain

$$(3.7) \qquad \xi^{(h+1)} = \xi^{(0)} - \sum_{r=0}^{h}\eta^{(r)} - \sum_{i=0}^{n} D_i \sum_{r=0}^{h}\xi_i^{(r)}.$$

For $e>0$, $\sum_{r=0}^{\infty} \eta^{(r)}$ converges in $V(b, c)$ and $\sum_{r=0}^{\infty} \xi_i^{(r)}$ converges in $L(b, c+e)$ for each $i\in S''$. Furthermore $\xi^{(h+1)}\to 0$ as $h\to\infty$ and thus taking limits as $h\to\infty$, equation (3.7) shows that ξ lies in the right side of the equation in the statement of the lemma.

We now consider $\xi\in L(b, c)$, $b=1/(p-1)$. If $N\in\mathbf{Z}_+$, $\varepsilon>0$, $s\leq N$ then $s(\varepsilon/N+b)+c-\varepsilon\leq sb+c$ and therefore $T_N\xi\in L^{(N)}(b+\varepsilon/N, c-\varepsilon)$, which shows since $b+\varepsilon/N>1/(p-1)$ that there exists $\eta^{(N)}\in V(\varepsilon/N+b, c-\varepsilon)$, $\xi_i^{(N)}\in L(b+\varepsilon/N, c-\varepsilon+\varepsilon/N)$ for each $i\in S''$ such that

$$(3.8) \qquad T_N\xi = \eta^{(N)} + \sum_{i=0}^{n} D_i\xi_i^{(N)}.$$

The space $V(b, c-\varepsilon)\times(L(b, c-\varepsilon))^{n+1}$ is compact in the weak topology, which shows that the sequence $(\eta^{(N)}, \xi_0^{(N)}, \ldots, \xi_n^{(N)})_{N=0,1,\ldots}$ has an adherent point $(\eta^{(\varepsilon)}, \xi_0^{(\varepsilon)}, \ldots, \xi_n^{(\varepsilon)})$ in that space. Hence taking limits we obtain from equation (3.8),

$$(3.9) \qquad \xi = \eta^{(\varepsilon)} + \sum_{i=0}^{n} D_i\xi_i^{(\varepsilon)}.$$

We now let ε run through a monotonically decreasing sequence of positive real numbers with limit zero. The use of compactness shows that the sequence $(\eta^{(\varepsilon)}, \xi_0^{(\varepsilon)}, \ldots, \xi_n^{(\varepsilon)})$ indexed by ε has an adherent point. Restricting our attention to a converging subsequence we conclude that the adherent point $(\eta, \xi_0, \xi_1, \ldots, \xi_n)$ lies in $V(b, c-\varepsilon)\times(L(b, c-\varepsilon))^{n+1}$ for each ε in an infinite sequence of positive real numbers with limit 0. Thus taking limits in equation (3.9) we obtain $\xi=\eta+\sum_{i=0}^{n} D_i\xi_i$, $\eta\in V(b, c)$, $\xi_i\in L(b, c)$ for each $i\in S''$. This completes the proof of the lemma.

We defer for the moment the discussion of $V\cap\sum_{i=0}^{n} D_i L(b)$.

Lemma **3.7.** — *Let* ρ, c, b *be real numbers,* $b\leq p/(p-1)$, N *an element of* $\mathbf{Z}_+$, $e=b-1/(p-1)$, $\rho+e\geq c$ *and let* A *be a proper subset of* S', $A\neq S$. *Let* $\{\xi_i\}_{i\in A}$ *be a set of elements in* $X_0^N\Omega_0\{X\}\cap L(b, c)$ *indexed by* A *such that* $\sum_{i\in A} H_i\xi_i\in L(b, \rho)$. *Then there exists a set of elements* $\{\eta_i\}_{i\in A}$ *in* $(X_0^N\Omega_0[X_1, \ldots, X_{n+1}])\cap L(b, \rho+e)$ *indexed by* A, *and a* *skew symmetric set* η_{ij} *in* $(X_0^{N-1}\Omega_0[X_1, \dots, X_{n+1}]) \cap L(b, c+e)$ *indexed by* A *such that if we set*

$$\xi_i' = \xi_i - \Big(\eta_i + \sum_{j\in A} H_j\eta_{ij}\Big) \tag{3.10}$$

for each $i \in A$ *then* $\sum_{i\in A} H_i\xi_i' \in L(b, \rho)$ *and* ξ_i' *lies in* $L(b, c)$ *and is divisible by* X_0^{N+1} *for each* $i \in A$.

Proof. — It is quite clear that if the elements η_i are chosen in $L(b, \rho+e)$ and the $\eta_{i,j}$ are chosen in $L(b, c+e)$ then ξ_i' as given by (3.10) certainly lies in $L(b, c)$ and $\sum_{i\in A} H_i\xi_i' = \Sigma H_i\xi_i - \Sigma H_i\eta_i \in L(b, \rho)$. Thus the only important condition to be satisfied by ξ_i' is that of divisibility by X_0^{N+1}.

For each $i \in A$, let $P_i^{(N)}$ be the coefficient of X_0^N in ξ_i and let $Q^{(N+1)}$ be the coefficient of $X_0^{(N+1)}$ in $\sum_{i\in A} H_i\xi_i$. Hence ord $P_i^{(N)} \geq Nb + c$, ord $Q^{(N+1)} \geq (N+1)b + \rho$, $Q^{(N+1)} = \pi \sum_{i\in A} f_i P_i^{(N)}$. Lemma 3.2 now shows that there exists a set of homogeneous forms of degree dN, $\{C_i\}_{i\in A}$ such that $Q^{(N+1)} = \pi \sum_{i\in A} f_i C_i$, ord $C_i \geq Nb + \rho + e$. Thus $0 = \sum_{i\in A} f_i(C_i - P_i^{(N)})$ and hence by the corollary of Lemma 3.2, there exists a skew symmetric set of forms of degree $d(N-1)$, $\{B_{i,j}\}$ indexed by A such that for each $i \in A$.

$$P_i^{(N)} = C_i + \pi \sum_{j\in A} B_{ij} f_j \tag{3.11}$$

and ord $B_{ij} \geq (N-1)b + c + e$ (since by hypothesis, $\rho + e \geq c$). We now let $\eta_{i,j} = B_{i,j}X_0^{N-1}$ for each $(i,j) \in A \times A$ and $\eta_i = C_iX_0^N$ for each $i \in A$. It is clear that X_0^N divides ξ_i' (as given by equation 3.10), while the coefficient of X_0^N in ξ_i' is $P_i^{(N)} - C_i - \pi \sum_{j\in A} B_{i,j} f_j = 0$. This completes the proof of the lemma.

Lemma **3.8**. — *Let* b, c, ρ *be real numbers,* $b \leq p/(p-1)$, $e = b - 1/(p-1)$, $e + \rho \geq c$. *Let* A *be a proper subset of* S', $A \neq S$ *and let* $\{\xi_i\}_{i\in A}$ *be a set of elements of* $L(b, c)$ *indexed by* A *such that* $\sum_{i\in A} H_i\xi_i \in L(b, \rho)$. *Then there exists a set of elements* $\{\eta_i\}$ *in* $L(b, \rho + e)$ *indexed by* A *and a skew symmetric set* $\eta_{i,j}$ *in* $L(b, c+e)$ *indexed by* A *such that*

$$\xi_i = \eta_i + \sum_{j\in A} H_j\eta_{ij}$$

for each $i \in A$.

Proof. — Let $\xi_i^{(0)} = \xi_i$ for each $i \in A$. It is clear that Lemma 3.7 gives a recursive process by which for each $N \in \mathbf{Z}_+$ we may construct a set $\{\eta_i^{(N)}\}$ in $(X_0^N\Omega_0[X_1, \dots, X_{n+1}]) \cap L(b, \rho+e)$ indexed by A and a skew symmetric set $\{\eta_{i,j}^{(N-1)}\}$ in $(X_0^{N-1}\Omega_0[X_1, \dots, X_{n+1}]) \cap L(b, c+e)$ indexed by A such that for each $i \in A$,

$$\xi_i^{(N+1)} = \xi_i^{(N)} - \Big(\eta_i^{(N)} + \sum_{j\in A} H_j\eta_{ij}^{(N)}\Big) \tag{3.12}$$

$\xi_i^{(N)} \in L(b, c)$, X_0^N divides $\xi_i^{(N)}$, $\sum_{i\in A} H_i\xi_i^{(N)}$ is divisible by X_0^N and lies in $L(b, \rho)$. Let $\eta_i = \sum_{N=0}^{\infty} \eta_i^{(N)}$, $\eta_{i,j} = \sum_{N=0}^{\infty} \eta_{i,j}^{(N)}$ for each $i, j \in A$, convergence being obvious in the weak topology.

Clearly $\eta_i \in L(b, \rho + e)$, $\eta_{i,j} \in L(b, c+e)$. For $r \in \mathbf{Z}_+$, we write equation (3.12) for $N = 0, 1, \dots, r$ and add. This gives

$$\xi_i^{(r+1)} = \xi_i^{(0)} - \sum_{N=1}^{r} \eta_i^{(N)} + \sum_{j \in A} H_j \sum_{N=1}^{r} \eta_{ij}^{(N)}$$

The lemma now follows by taking limits as $r \to \infty$ since $\lim\limits_{r \to \infty} \xi_i^{(r+1)} = 0$ in the weak topology.

Lemma **3.9**. — *Let b, c, ρ be real numbers such that $\rho \geq c$, $1/(p-1) < b \leq p/(p-1)$ and let $e = b - 1/(p-1)$. Let A be a proper subset of S′, $A \neq S$ and let ξ_i be a set in $L(b, c)$ indexed by A such that $\sum\limits_{i \in A} D_i \xi_i \in L(b, \rho)$, then there exists a set $\{\eta_i\}$ in $L(b, \rho + e)$ indexed by A and a skew symmetric set $\{\eta_{ij}\}$ in $L(b, c+e)$ indexed by A such that $\xi_i = \eta_i + \sum\limits_{j \in A} D_i \eta_{i,j}$.*

Proof. — There exists a unique element N of $\mathbf{Z}_+$ such that $(N-1)e + c \leq \rho < Ne + c$. For each integer r, $0 \leq r \leq N$ we construct a set $\{\xi_i^{(r)}\}$ in $L(b, c + re)$ indexed by A and, for $0 \leq r < N$ a set $\{\eta_i^{(r)}\}$ in $L(b, c + (r+1)e)$ indexed by A and a skew symmetric set $\{\eta_{i,j}^{(r)}\}$ in $L(b, c + (r+1)e)$ indexed by A such that (letting $\xi = \sum\limits_{i \in A} D_i \xi_i$)

$$\xi = \sum_{i \in A} D_i \xi_i^{(r)} \quad \text{for} \quad 0 \leq r \leq N, \tag{3.13}$$

$$\xi_i^{(r)} = \eta_i^{(r)} + \sum_{j \in A} H_j \eta_{i,j}^{(r)} \quad \text{for} \quad r < N, \tag{3.14}$$

$$\xi_i^{(r+1)} = \xi_i^{(r)} - \sum_{j \in A} D_j \eta_{ij}^{(r)} \quad \text{for} \quad r < N, \tag{3.15}$$

and such that $\xi_i^{(0)} = \xi_i$ for each $i \in A$. Suppose the set $\{\xi_i^{(r)}\}_{i \in A}$ in $L(b, c + re)$ satisfying (3.13) is given for some integer r, $0 \leq r < N$. We then have

$$\sum_{i \in A} H_i \xi_i^{(r)} = \xi - \sum_{i \in A} E_i \xi_i^{(r)} \in L(b, \rho) + L(b, c + re) = L(b, c + re).$$

Hence by Lemma 3.8, elements $\eta_i^{(r)}$ in $L(b, c + e(r+1))$ and $\eta_{i,j}^{(r)}$ in $L(b, c + (r+1)e)$ may be chosen such that equation (3.14) is valid for each $i \in A$. If $\xi_i^{(r+1)}$ is now defined by equation (3.15) then certainly $\xi = \sum\limits_{i \in A} D_i \xi_i^{(r+1)}$ and furthermore, $\xi_i^{(r+1)} = \eta_i^{(r)} - \sum\limits_{j \in A} E_j \eta_{i,j}^{(r)} \in L(b, c + (r+1)e)$. This completes the construction of $\xi_i^{(r)}$ for $r = 0, 1, \dots, N$, since $\xi_i^{(0)}$ is specified, and also of $\eta_i^{(r)}$ and $\eta_{i,j}^{(r)}$ for $r = 0, 1, \dots, N-1$. In particular, $\xi_i^{(N)} \in L(b, c + Ne) \subset L(b, \rho)$ and therefore $\sum\limits_{i \in A} H_i \xi_i^{(N)} = \xi - \sum\limits_{i \in A} E_i \xi_i^{(N)} \in L(b, \rho)$. Since $\rho + e \geq c + Ne$, we may conclude from Lemma 3.8 that there exists a set $\{\eta_i^{(N)}\}$ in $L(b, \rho + e)$ indexed by A and a skew symmetric set $\{\eta_{i,j}^{(N)}\}$ in $L(b, c + (N+1)e)$ indexed by A such that equation (3.14) is valid for $r = N$. If now we define for each $i \in A$, $\xi_i^{(N+1)}$ by setting $r = N$ in equation (3.15) we have

$$\xi_i^{(N+1)} = \eta_i^{(N)} - \sum_{j \in A} E_j \eta_{ij}^{(N)} \in L(b, \rho + e) + L(b, c + (N+1)e) = L(b, \rho + e).$$

If now we write equation (3.15) for $r = 0, 1, \ldots, N$ and add, we obtain after obvious cancellation, $\xi_i^{(N+1)} = \xi_i - \sum_{j\in A} D_j\left(\sum_{r=0}^{N} \eta_{i,j}^{(N)}\right)$. The lemma follows directly by setting $\eta_i = \xi_i^{(N+1)} \in L(b, \rho + e)$ and $\eta_{i,j} = \sum_{r=0}^{N} \eta_{i,j}^{(r)} \in L(b, c+e)$.

Lemma **3.10.** — *If* A *is a proper subset of* S', $A \neq S$; b, c *are real numbers,* $1/(p-1) < b \leq p/(p-1)$ *and if* $\{\xi_i\}$ *is a set in* $L(b, c)$ *indexed by* A *such that* $\sum_{i\in A} D_i\xi_i = 0$ *then there exists a skew symmetric set* $\{\eta_{i,j}\}$ *in* $L(b, c+e)$ *indexed by* A *such that* $\xi_i = \sum_{j\in A} D_i\eta_{i,j}$ *for each* $i \in A$.

Proof. — Let ρ be any real number, then $\sum_{i\in A} D_i\xi_i \in L(b, \rho)$ and hence if $\rho > c$ there exists a set $\{\eta_i^{(\rho)}\}$ in $L(b, \rho+e)$ indexed by A and a skew symmetric set $\{\eta_{i,j}^{(\rho)}\}$ in $L(b, c+e)$ indexed by A such that

$$\xi_i = \eta_i^{(\rho)} + \sum_{j\in A} D_i\eta_{i,j}^{(\rho)} \tag{3.16}$$

for each $i \in A$. Let ρ run through an infinite sequence of real numbers towards $+\infty$, then by the compactness of the cartesian product of copies of $L(b, 0)$ indexed by A and of copies of $L(b, c+e)$ indexed by $A \times A$ there exists an infinite subsequence such that if ρ is restricted to the subsequence, then, as $\rho \to \infty$, $\eta_i^{(\rho)}$ converges (necessarily to 0) and $\eta_{i,j}^{(\rho)}$ converges to $\eta_{ij} \in L(b, c+e)$. Clearly the set $\{\eta_{ij}\}$ is skew symmetric and taking limits in equation (3.16) as $\rho \to \infty$, the assertion is proved.

Lemma **3.11.** — *For* $b > 1/(p-1)$, $V \cap \sum_{i=0}^{n} D_i L(b) = \{0\}$.

Proof. — Let ξ be an element of the intersection. It may be assumed that $1/(p-1) < b \leq p/(p-1)$. With b fixed in this range, let ρ be chosen such that $\xi \in L(b, \rho)$, $\xi \notin L(b, \rho+e)$. If $\xi \neq 0$ then ρ certainly exists. Since $\xi \in \sum_{i=0}^{n} D_i L(b)$, Lemma 3.9 shows that there exist $\eta_0, \eta_1, \ldots, \eta_n$ in $L(b, \rho+e)$ such that $\xi = \sum_{i=0}^{n} D_i\eta_i$. Thus $\xi - \sum_{i=0}^{n} H_i\eta_i = \sum_{i=0}^{n} E_i\eta_i \in L(b, \rho+e)$. Lemma 3.4 shows that there exists $\xi' \in V(b, \rho+e)$, $\eta_0', \ldots, \eta_n'$ in $L(b, \rho+2e)$ such that $\xi - \sum_{i=0}^{n} H_i\eta_i = \xi' + \sum_{i=0}^{n} H_i\eta_0'$. This shows that $\xi - \xi'$ lies in $V \cap \sum_{i=0}^{n} H_i L(b)$, and hence by Lemma 3.5, $\xi - \xi' = 0$. Thus $\xi = \xi' \in L(b, \rho+e)$, which contradicts the choice of ρ. Hence $\xi = 0$.

This completes the « general » theory of the differential operators. We note that if $b \leq p/(p-1)$ then for each subset A of S, the subspace $L^A(b)$ of $L(b)$ is invariant under each D_i. The action of the differential operators on these subspaces must now be discussed in greater detail.

e) *Special Theory.*

In this section we cannot avoid distinctions [1] depending upon whether or not p divides d. Furthermore some of our results will be valid only if H and the $H_i = E_i H$ are subject to further restrictions. To avoid confusion, for each $i \in S$, let $H'_i = \pi' X_0 f_i$, and let $\mathfrak{D}_i$ be the mapping $\xi \rightarrow E_i \xi + \xi H'_i$, where $\pi' \in \Omega_0$, $\operatorname{ord} \pi' = 1/(p-1)$.

For each subset, A, of $S = \{1, 2, \ldots, n+1\}$, let X_A be the set of variables $\{X_i\}_{i \in A}$. The ring, $\Omega_0[X_A]$, of polynomials in the variables X_A with coefficients in Ω_0, is viewed as a subring of $\Omega_0[X_S] = \Omega_0[X_1, \ldots, X_{n+1}]$ and in particular if A is empty then $\Omega_0[X_A]$ is the field Ω_0. Let $\mathfrak{J}_A$ be the homomorphism of $\Omega_0[X_S]$ onto $\Omega_0[X_A]$ defined by

$$\mathfrak{J}_A(X_i) = \begin{cases} X_i & \text{if} \quad i \in A \\ 0 & \text{if} \quad i \notin A \end{cases}$$

As before $W^{(m)}$ denotes, for each $m \in \mathbf{Z}_+$, the space of forms of degree dm in $\Omega_0[X_S]$. For each subset A of S let $W_A^{(m)} = \mathfrak{J}_A(W^{(m)})$ and for each subset B of A, let $W_A^{B,(m)} = W_A^{(m)} \cap (M_B)$, where (M_B) denotes the principal ideal in $\Omega_0[X_S]$ generated by the monomial $M_B = \prod_{i \in B} X_i$. (Unfortunately, our notation permits the same space to be designated by several symbols. Thus if $\emptyset$ is the empty subset of S, then $W_A^{(m)} = W_A^{\emptyset,(m)}$ and $W^{(m)} = W_S^{(m)} = W_A^{\emptyset,(m)}$).

For each subset A of S let $\mathfrak{B}_A^{A,(m)}$ be a subspace of $W_A^{A,(m)}$ which is p-adically complementary (with respect to the monomial basis of $W_A^{A,(m)}$) in $W_A^{A,(m)}$ to $W_A^{A,(m)} \cap \mathfrak{J}_A(\mathfrak{A})$.

Thus we have

$$W_A^{A,(m)} = \mathfrak{B}_A^{A,(m)} [+] (W_A^{A,(m)} \cap \mathfrak{J}_A(\mathfrak{A})) \tag{3.17}$$

For each subset, A, of S, let

$$\mathfrak{B}_S^{A,(m)} = \sum_B \mathfrak{B}_B^{B,(m)}, \tag{3.18}$$

the sum being over all subsets, B, of S which contain A.

Lemma **3.12**. — *Let* A *be a subset of* S.

(i) $W_S^{A,(m)} = \sum_{A \supset B} W_B^{B,(m)}$, *the sum being over subsets*, B, *of* S *which contain* A.

(ii) $W_S^{A,(m)} \cap (\text{Kernel of } \mathfrak{J}_A) = \sum_{\substack{B \supset A \\ \neq}} W_S^{B,(m)}$, *the sum being over all subsets*, B, *of* S *which contain but are not equal to* A.

(iii) $W_A^{A,(m)} \cap (\mathfrak{J}_A \mathfrak{A}) = \mathfrak{J}_A(\mathfrak{A} \cap W_S^{A,(m)})$.

Proof. — The first assertion is trivial. For (ii) we observe that a polynomial, ξ, lies in the kernel of $\mathfrak{J}_A$ if and only if each monomial, X^u, appearing in ξ is divisible by at

[1] The theory in the case $p|d$ is hampered by the fact that Lemma 3.3 fails to give an explicit basis for the ideal $(M_S) \cap (\bar{f}_0, \bar{f}_1, \ldots, \bar{f}_n)$ in K[X]. This ideal contains but is not necessarily equal to $(M_S \bar{f}_0) + \sum_{i=1}^{n+1} (M_S \bar{f}_i / X_i)$, a counter-example being given (for $n = 3$) by $\xi = \bar{f}_2 \delta_1 (\delta_4 - I) \bar{f}_3 + \bar{f}_3 \delta_1 (I - \delta_4) \bar{f}_2 - \bar{f}_4 \delta_1 \delta_4 \bar{f}_2$, where for $i = 1, 2, 3, 4$, δ_i is the specialization of $K[X_1, X_2, X_3, X_4]$ defined by $\delta_i X_i = 0$, $\delta_i X_j = X_j$ for $j \neq i$.

least one variable X_i such that $i \in S - A$. If X^u is also divisible by M_A then certainly there exists a subset, B, of S containing A properly such that M_B divides X^u. For the proof of (iii), we use Lemma 3.3 (i) which shows that as an ideal in $\Omega_0[X_A]$, $(M_A) \cap \mathfrak{J}_A(\mathfrak{A}) = \sum_{i \in A} (M_A(\mathfrak{J}_A f_i)/X_i)$. Intersecting both sides of this last relation with $W_A^{(m)}$, we see $W_A^{A,(m)} \cap \mathfrak{J}_A \mathfrak{A}$ is the set of all homogeneous polynomials of degree *dm* of the form $\sum_{i \in A} g_i M_A(\mathfrak{J}_A f_i)/X_i$, the g_i being elements of $\Omega_0[X_A]$. By homogeneity it may be assumed that $g_i M_A f_i/X_i$ is a form of degree *dm* in $\Omega_0[X_s]$ and hence lies in $\mathfrak{A} \cap W_s^{A,(m)}$. This shows that the left side of (iii) lies in the right side, which completes the proof since inclusion in the reverse direction is trivial.

Lemma **3.13**. — *Let* A *be a subset of* S

(i) $\mathfrak{B}_S^{A,(m)} = \Sigma[+]\mathfrak{B}_B^{B,(m)}$, *the sum being over all subsets* B *of* S *which contain* A.

(ii) $W_S^{A,(m)} = \mathfrak{B}_S^{A,(m)} \oplus (\mathfrak{A} \cap W_S^{A,(m)})$ *and the sum is p-adically direct if* $p \nmid d$.

Proof. — (i) The definition of $\mathfrak{B}_S^{A,(m)}$ shows that it is enough to prove the p-adic directness. For each set B containing A, let ξ_B be an element of $V_B^{B,(m)}$ such that $\text{ord}\,(\Sigma\xi_B) > 0$. Let C be a minimal subset of S which contains A such that $\text{ord}\,\xi_C \leq 0$. Clearly $\text{ord}\,\xi_C = \text{ord}(\mathfrak{J}_C \Sigma\xi_B) \geq \text{ord}(\Sigma\xi_B) > 0$, which shows that $\text{ord}\,\xi_B > 0$ for each B.

(ii) We first prove this assertion without any claim concerning directness. The assertion is equivalent to equation (3.17) if $A = S$. Thus we may assume that $A \neq S$ and use induction on the number of elements in A. By Lemma 3.12 (i), $W_S^{A,(m)} = W_A^{A,(m)} + \sum_{\substack{B \supset A \\ \neq}} W_B^{B,(m)}$. Equation (3.17) and Lemma 3.12 (iii) show that $W_A^{A,(m)} = \mathfrak{B}_A^{A,(m)} + \mathfrak{J}_A(\mathfrak{A} \cap W_S^{A,(m)})$ and since $\mathfrak{J}_A$ acts like the identity on $W_A^{A,(m)}$, we may conclude that $W_A^{A,(m)} \subset \mathfrak{B}_A^{A,(m)} + \mathfrak{A} \cap W_S^{A,(m)} + (\text{Kernel}\,\mathfrak{J}_A) \cap W_S^{A,(m)}$. Lemma 3.12 (ii) now shows that $W_A^{A,(m)} \subset \mathfrak{B}_A^{A,(m)} + \mathfrak{A} \cap W_S^{A,(m)} + \sum_{\substack{B \supset A \\ \neq}} W_S^{B,(m)}$ and it is clear from the previous relations that $W_S^{A,(m)}$ also lies in this space. The induction hypothesis now shows that $W_S^{A,(m)} \subset \mathfrak{B}_A^{A,(m)} + \mathfrak{A} \cap W_S^{A,(m)} + \sum_{\substack{B \supset A \\ \neq}} (\mathfrak{B}_S^{B,(m)} + \mathfrak{A} \cap W_S^{B,(m)})$. Equation (3.18) now shows that $W_S^{A,(m)} \subset \mathfrak{B}_S^{A,(m)} + \mathfrak{A} \cap W_S^{A,(m)}$ and equality is clear.

To show directness (in the ordinary sense) of the sum, let ξ be an element in $\mathfrak{B}_S^{A,(m)} \cap (\mathfrak{A} \cap W_S^{A,(m)})$. Equation (3.18) shows that for each set B containing A, there exists $\xi_B \in \mathfrak{B}_B^{B,(m)}$ such that $\xi = \Sigma\xi_B$. Let C be a minimal set containing A such that $\xi_C \neq 0$. Clearly $\xi_C = \mathfrak{J}_C \xi \in \mathfrak{J}_C \mathfrak{A}$ and hence $\xi_C \in (W_C^{C,(m)} \cap \mathfrak{J}_C \mathfrak{A}) \cap \mathfrak{B}_C^{C,(m)}$, which shows by equation (3.17) that $\xi_C = 0$. This contradiction shows that $\xi_B = 0$ for all B and hence $\xi = 0$.

Let ξ be an element of $\mathfrak{B}_S^{A,(m)}$ and η an element of $W_S^{A,(m)} \cap \mathfrak{A}$, both in $\mathfrak{O}_0[X_S]$ such that $\text{ord}\,(\xi - \eta) > 0$. To complete the proof of the lemma, we must show (if $p \nmid d$) that $\text{ord}\,\xi > 0$. By definition, for each set B containing A there exists $\xi_B \in \mathfrak{B}_B^{B,(m)}$ such that $\xi = \Sigma\xi_B$. We show that $\text{ord}\,\xi_B > 0$ for each B. Suppose otherwise, then there exists a minimal set C containing A such that $\text{ord}\,\xi_C = 0$. Then $\text{ord}(\xi_C - \mathfrak{J}_C \xi) > 0$, while $\operatorname{ord}(\mathfrak{I}_C\xi - \mathfrak{I}_C\eta) \geq \operatorname{ord}(\xi-\eta) > 0$. Thus $\xi_C \in \mathfrak{O}_0[X_C]$ and $\operatorname{ord}(\xi_C - \mathfrak{I}_C\eta) > 0$. Let ξ_C^* be the image of ξ_C in $K[X_C]$ under the residue class map. Clearly ξ_C^* is divisible by M_C and lies in the image in $K[X_C]$ of $\mathfrak{O}_0[X_C] \cap \mathfrak{I}_C\mathfrak{A}$. Using the asterisk to denote images in $K[X_C]$ under the residue class map, we may conclude from Lemma 3.3 (i) (since $p \nmid d$) that there exist a set of forms of degree dm, $\{g_i\}$ in $K[X_C]$ indexed by C such that $\xi_C^* = \sum_{i \in C} g_i M_C(\mathfrak{I}_C f_i)^*/X_i$. Choosing forms G_i of degree dm in $\mathfrak{O}_0[X_C]$ which represent the g_i and setting $\xi'_C = \sum_{i\in C} G_i M_C(\mathfrak{I}_C f_i)/X_i \in W_C^{C,(m)} \cap \mathfrak{I}_C\mathfrak{A}$, we have $\operatorname{ord}(\xi_C - \xi'_C) > 0$. Since $\xi_C \in \mathfrak{B}_C^{C,(m)}$, this contradicts equation (3.17) and so the proof of the lemma is completed.

For each subset A of S, let $V_S^{A,(m)} = \mathfrak{B}_S^{A,(m)}$ if $p \nmid d$, while otherwise let $V_S^{A,(m)}$ be chosen in $W_S^{A,(m)}$ p-adically complementary to $(\mathfrak{A} \cap W_S^{A,(m)})$. (Clearly we may let $V_S^{S,(m)} = \mathfrak{B}_S^{S,(m)}$ in any case.) It follows from the definitions and Lemmas 3.2, 3.3 and 3.13 that if $A \subset S'' \cap S$ and $P \in W_S^{A,(m)}$ then there exists $Q^{(m)} \in V_S^{A,(m)}$ and a set of homogeneous elements $\{P_i\}$ indexed by S'' in $\Omega_0[X_S]$ such that

$$\textbf{(3.19)} \qquad P = Q^{(m)} + \sum_{i\in A} P_i f_i M_A/X_i + \sum_{i \in S''-A} P_i f_i M_A,$$

$\operatorname{ord} Q^{(m)} \geq \operatorname{ord} P$, $\operatorname{ord} P_i \geq \operatorname{ord} P$. If A is any subset of S, there exists $Q^{(m)} \in \mathfrak{B}_S^{A,(m)}$ and a set of homogeneous elements $\{P_i\}$ indexed by S in $\Omega_0[X_S]$ such that

$$\textbf{(3.20)} \qquad P = Q^{(m)} + \sum_{i\in A} P_i f_i M_A/X_i + \sum_{i \in S-A} P_i f_i M_A,$$

but in this situation the previous estimates for $\operatorname{ord} Q^{(m)}$ and $\operatorname{ord} P_i$ do not hold unless p does not divide d.

Finally let $V_S^A = \sum_{m=0}^{\infty} V_S^{A,(m)} X_0^m$, $\mathfrak{B}_S^A = \sum_{m=0}^{\infty} \mathfrak{B}_S^{A,(m)} X_0^m$, $V_S^A(b, c) = V_S^A \cap L(b, c)$. In particular the space, V, defined previously, may in our present notation be written $V_S^{\emptyset}$. We shall write V^A (resp. $\mathfrak{B}^A$) instead of V_S^A (resp. $\mathfrak{B}_S^A$) and likewise V (resp. $\mathfrak{B}$) instead of $V^{\emptyset}$ (resp: $\mathfrak{B}^{\emptyset}$) whenever there is no danger of confusion. In particular $\mathfrak{B} = \Sigma \mathfrak{B}_A^A$, the sum being over all subsets A, of S. We note that for each subset A of S, V^A and $\mathfrak{B}^A$ lie in $\mathfrak{L}^{(N_0)}$ and have equal dimension.

Lemma **3.14**. — *If* $b \leq p/(p-1)$, *and* A *is any subset of* $S \cap S''$ *then*

$$L^A(b, c) = V^A(b, c) + \sum_{i\in A} H_i L^{A-\{i\}}(b, c+e) + \sum_{i\in S''-A} H_i L^A(b, c+e).$$

If $p \nmid d$ *then*

$$L^S(b, c) = V^S(b, c) + \sum_{i \in S} H_i L^{S-\{i\}}(b, c+e).$$

The proof is a step by step repetition of that of Lemma 3.4 and therefore may be omitted. We note that the statement of Lemma 3.4 is obtained from this lemma by setting $A = \emptyset$.

Lemma **3.15**. — *If* $(p-1)^{-1} \leq b \leq p/(p-1)$ *and if* A *is any subset of* $S \cap S''$ *then*

$$L^A(b, c) = V^A(b, c) + \sum_{i\in A} D_i L^{A-\{i\}}(b, c+e) + \sum_{i\in S''-A} D_i L^A(b, c+e)$$

If $p \nmid d$ then

$$\mathrm{L}^{\mathrm{S}}(b, c) = \mathrm{V}^{\mathrm{S}}(b, c) + \sum_{i \in \mathrm{S}} \mathrm{D}_i \mathrm{L}^{\mathrm{S}-\{i\}}(b, c+e).$$

This generalization of Lemma 3.6 follows from Lemma 3.14 in precisely the same way that Lemma 3.6 follows from Lemma 3.4.

We must now overcome some of the difficulties caused by the incompleteness of Lemma 3.15.

Lemma **3.16.** — *For each subset* A *of* S *and each* $\mathrm{N} \in \mathbf{Z}_+$, $\mathrm{N} \geq \mathrm{N}_0$,

$$\mathfrak{L}^{\mathrm{A},(\mathrm{N})} = \mathfrak{V}^{\mathrm{A}} + \sum_{i \in \mathrm{A}} \mathfrak{D}_i \mathfrak{L}^{\mathrm{A}-\{i\},(\mathrm{N}-1)} + \sum_{i \in \mathrm{S}-\mathrm{A}} \mathfrak{D}_i \mathfrak{L}^{\mathrm{A},(\mathrm{N}-1)}$$

and the equality remains valid if $\mathfrak{V}^{\mathrm{A}}$ *is replaced by* V^{A}.

Proof. — Since the left side of our assertion clearly contains the right side it is enough to shows that the right side contains the left side. We show this inclusion for each $\mathrm{N} \in \mathbf{Z}_+$. This is trivial for $\mathrm{N} = 0$ since $\mathfrak{L}^{\mathrm{A},(0)} = \mathfrak{V}^{\mathrm{A},(0)} = \mathrm{V}^{\mathrm{A},(0)} = \{0\}$ (resp. Ω_0) if $\mathrm{A} \neq \emptyset$ (resp. $\mathrm{A} = \emptyset$). We now suppose that $\mathrm{N} > 0$ and use induction on N. Let $\xi \in \mathfrak{L}^{\mathrm{A},(\mathrm{N})}$ and let P be the coefficient of $\mathrm{X}_0^{\mathrm{N}}$ in ξ. Let homogeneous forms $\mathrm{Q}^{(\mathrm{N})}$, $\{\mathrm{P}_i\}_{i \in \mathrm{S}}$ be chosen as indicated by equation (3.20) (with m replaced by N). Let $\xi_i = \mathrm{X}_0^{\mathrm{N}-1}\mathrm{M}_{\mathrm{A}}\mathrm{P}_i/\mathrm{X}_i$ for $i \in \mathrm{A}$ and $\xi_i = \mathrm{X}_0^{\mathrm{N}-1}\mathrm{M}_{\mathrm{A}}\mathrm{P}_i$ for $i \in \mathrm{S}-\mathrm{A}$. Let $\eta = \mathrm{Q}^{(m)}\mathrm{X}_0^{\mathrm{N}} \in \mathfrak{V}^{\mathrm{A}}$ and then $\xi - (\eta + \sum_{i=1}^{n+1} \mathfrak{D}_i \xi_i) \in \mathfrak{L}^{\mathrm{A},(\mathrm{N}-1)}$. This shows that

$$\mathfrak{L}^{\mathrm{A},(\mathrm{N})} \subset \mathfrak{V}^{(\mathrm{A})} + \sum_{i \in \mathrm{A}} \mathfrak{D}_i \mathfrak{L}^{\mathrm{A}-\{i\},(\mathrm{N}-1)} + \sum_{i \in \mathrm{S}-\mathrm{A}} \mathfrak{D}_i \mathfrak{L}^{\mathrm{A},(\mathrm{N}-1)} + \mathfrak{L}^{\mathrm{A},(\mathrm{N}-1)}$$

and the assertion now follows from the induction hypothesis. The above argument can be used for $\mathfrak{V}^{\mathrm{A}}$ replaced by $\mathrm{V}^{(\mathrm{A})}$, since $\mathrm{Q}^{(\mathrm{N})}$ may be chosen in $\mathrm{V}^{\mathrm{A},(\mathrm{N})}$ instead of $\mathfrak{V}^{\mathrm{A},(\mathrm{N})}$.

The following lemma is a special case of Lemma 3.11 unless p divides d.

Lemma **3.17.** — *For* $b > 1/(p-1)$, $\mathfrak{V} \cap \sum_{i=1}^{n+1} \mathfrak{D}_i \mathrm{L}(b) = 0$.

Proof. — The previous lemma shows that $\mathrm{V} \subset \mathfrak{L}^{(\mathrm{N}_0)} = \mathfrak{V} + \sum_{i=1}^{n+1} \mathfrak{D}_i \mathfrak{L}^{(\mathrm{N}_0-1)} \subset \mathfrak{V} + \sum_{i=1}^{n+1} \mathfrak{D}_i \mathrm{L}(b)$. We may assume that $b \leq p/(p-1)$ and use Lemma 3.6 which shows that $\mathrm{L}(b) = \mathrm{V} + \sum_{i=1}^{n+1} \mathfrak{D}_i \mathrm{L}(b)$ and thus conclude that $\mathrm{L}(b) = \mathfrak{V} + \sum_{i=1}^{n+1} \mathfrak{D}_i \mathrm{L}(b)$. Lemma 3.11 shows that $\mathrm{V} \approx \mathrm{L}(b)/\sum_{i=1}^{n+1} \mathfrak{D}_i \mathrm{L}(b) \approx \mathfrak{V}/(\mathfrak{V} \cap \sum_{i=1}^{n+1} \mathfrak{D}_i \mathrm{L}(b))$. Since V and $\mathfrak{V}$ are vector spaces of the same (finite) dimension, this completes the proof of the lemma.

Our next lemma is a weak form of Lemma 3.15 of interest only if p divides d.

Lemma **3.18.** — *If* A *is any subset of* S *and if* $1/(p-1) < b \leq p/(p-1)$ *then*

$$\mathrm{L}^{\mathrm{A}}(b) \subset \mathfrak{V}^{\mathrm{A}} + \sum_{i=1}^{n+1} \mathfrak{D}_i \mathrm{L}(b).$$

Proof. — By Lemmas 3.15 and 3.16 we have if $A \neq S$,

$$L^{A}(b) \subset V^{A} + \sum_{i=1}^{n+1} \mathfrak{D}_i L(b) \subset \mathfrak{L}^{A,(N_0)} + \sum_{i=1}^{n+1} \mathfrak{D}_i L(b) \subset \mathfrak{B}^{A} + \sum_{i=1}^{n+1} \mathfrak{D}_i L(b).$$

To prove the lemma for $A = S$, let $B = \{1, 2, \ldots, n\}$ and let $\mathfrak{J}_{n+1}$ denote the mapping of $\Omega_0\{X_S\}$ onto $\Omega_0\{X_B\}$ obtained by replacing X_{n+1} by 0. For each $i \in B$ let $\mathfrak{D}_i'$ be the mapping $\xi \rightarrow E_i \xi + \xi \mathfrak{J}_{n+1} H_i'$ of $\Omega_0\{X_B\}$ into itself. For $\xi \in \Omega_0\{X_S\}$, $i \in B$ we have $\mathfrak{J}_{n+1} \mathfrak{D}_i \xi = \mathfrak{D}_i' \mathfrak{J}_{n+1} \xi$, while $\mathfrak{J}_{n+1} \mathfrak{D}_{n+1} \xi = 0$. If $\xi \in L^{S}(b)$ then from the part of the lemma already proven, there exists $\eta \in \mathfrak{B}_S^B$ such that $\xi \in \eta + \sum_{i=1}^{n+1} \mathfrak{D}_i L_S(b)$. Applying $\mathfrak{J}_{n+1}$ to this relation we have $0 = \mathfrak{J}_{n+1} \eta + \sum_{i=1}^{n} \mathfrak{D}_i' L_B(b)$. However equation (3.18) shows that $\mathfrak{B}_S^B = \mathfrak{B}_B^B + \mathfrak{B}_S^S$ and hence $\mathfrak{J}_{n+1} \eta \in \mathfrak{B}_B^B$ and hence lies in $\mathfrak{B}_B \cap \sum_{i=1}^{n} \mathfrak{D}_i' L_B(b)$, which according to Lemma 3.17 (with S replaced by B) is $\{0\}$ since $b > 1/(p-1)$. Thus $\mathfrak{J}_{n+1} \eta = 0$, which shows that $\eta \in \mathfrak{B}_S^S$. This completes the proof of the lemma.

f) *Exact Sequences.*

The object of this section is the computation of the dimension of the space V_S^S defined in the previous section. For this purpose we shall need a theorem concerning exact sequences which will be used again in the geometric application of our theory.

Let $\mathfrak{K}$ be a field of arbitrary characteristic and let W be a vector space over $\mathfrak{K}$ with an infinite family of subspaces indexed by both $\mathbf{Z}$ and by the subsets of $S = \{1, 2, \ldots, n+1\}$. That is, for each $t \in \mathbf{Z}$ and each subset, A, of S, let $W(A, t)$ be a subspace of W. Let $\varphi_1, \ldots, \varphi_{n+1}$ be a commutative set of endomorphisms of W with the property

$$\varphi_i W(A, t) \subset W(A \cup \{i\}, t+1) \tag{3.21}$$

for each $i \in S$, $t \in \mathbf{Z}$, and each (not necessarily proper) subset, A, of S.

For each $r \in \mathbf{Z}_+$ and each pair of subsets A, B of S such that $\emptyset \neq A \subseteq B$, let $\mathfrak{F}(t, r; A, B)$ be the space of all antisymmetric functions g on A^r such that $g(a_1, \ldots, a_r) \in W(-t-r, B - \{a_1, a_2, \ldots, a_r\})$, it being understood that $\mathfrak{F}(t, 0; A, B)$ is to be identified with $W(-t, B)$. For $r \geq 1$, let $\delta(t, r; A, B)$ be the mapping of $\mathfrak{F}(t, r; A, B)$ into $\mathfrak{F}(t, r-1; A, B)$ defined by

$$(\delta(t, r; A, B)g)(a_1, \ldots, a_{r-1}) = \sum_{j \in A} \varphi_j g(a_1, \ldots, a_{r-1}, j) \tag{3.22}$$

for each $g \in \mathfrak{F}(t, r; A, B)$. This mapping shall be denoted by δ when no confusion can arise.

Theorem **3.1**. — *If the sequence*

$$\mathfrak{F}(t, r+2; A, B) \xrightarrow{\delta} \mathfrak{F}(t, r+1; A, B) \xrightarrow{\delta} \mathfrak{F}(t, r; A, B)$$

is exact when $r = 0$ *for all pairs of subsets* A, B *of* S *such that* $\emptyset \neq A \subset B$ *then the sequence is exact for all* $r \in \mathbf{Z}_+$.

Proof. — We must show that Kernel $\delta(t, r+1; \mathrm{A}, \mathrm{B}) = $ Image $\delta(t, r+2; \mathrm{A}, \mathrm{B})$. We show that the right side is contained by the left side by showing that $\delta(t, r+1; \mathrm{A}, \mathrm{B})\delta(t, r+2; \mathrm{A}, \mathrm{B}) = 0$. Let $g \in \mathfrak{F}(t, r+2, \mathrm{A}, \mathrm{B})$, then

$$(\delta(t, r+1; \mathrm{A}, \mathrm{B})\delta(t, r+2; \mathrm{A}, \mathrm{B})g)(a_1, a_2, \ldots, a_r) = \\ \sum_{j\in\mathrm{A}} \varphi_j(\delta(r+2; \mathrm{A}, \mathrm{B})g)(a_1, \ldots, a_r, j) = \sum_{i,j\in\mathrm{A}} \varphi_i\varphi_j g(a_1, \ldots, a_r, j, i) = 0$$

by the commutativity of the endomorphism φ_i and the skew symmetry of g.

To complete the proof we must show:

$$\text{Kernel } \delta(t, r+1; \mathrm{A}, \mathrm{B}) \subset \text{Image } \delta(t, r+2; \mathrm{A}, \mathrm{B}).$$

This is true by hypothesis for $r=0$ and hence we may assume that $r \geq 1$. Antisymmetry shows that if A contains just one element then $\mathfrak{F}(t, r+1; \mathrm{A}, \mathrm{B}) = \mathfrak{F}(t, r+2; \mathrm{A}, \mathrm{B}) = 0$ for $r \geq 1$. The assertion is thus trivial if A contains only one element. We now may assume that A contains at least two elements, that $r \geq 1$ and we use induction on r for all t. Let $g \in$ Kernel $\delta(t, r+1; \mathrm{A}, \mathrm{B})$. Renumbering the elements of S if necessary we may suppose that $\mathrm{A} = \{1, 2, \ldots, s\}$, $s \geq 2$ and hence $0 = \sum_{j=1}^{s} \varphi_j g(a_1, \ldots, a_r, j)$ for all $(a_1, \ldots, a_r) \in \mathrm{A}^r$. With a_1 fixed, say $a_1 = 1$, we consider the mapping $(a_2, \ldots, a_{r+1}) \to g(1, a_2, \ldots, a_{r+1})$ as a function on $(\mathrm{A}-\{1\})^r$, indeed as an element of $\mathfrak{F}(t+1, r; \mathrm{A}-\{1\}, \mathrm{B}-\{1\})$ since it is skew symmetric in the « variables » $a_2, \ldots, a_{r+1}$ and $g(1, a_2, \ldots, a_{r+1}) \in \mathrm{W}(-t-r-1, \mathrm{B}-\{1\}-\{a_2, \ldots, a_{r+1}\})$. In this sense the mapping lies in Kernel $\delta(t+1, r; \mathrm{A}-\{1\}, \mathrm{B}-\{1\})$ and hence by induction on r there exists $h' \in \mathfrak{F}(t+1, r+1; \mathrm{A}-\{1\}, \mathrm{B}-\{1\})$ such that

$$(\delta(t+1, r+1 \ ; \mathrm{A}-\{1\}, \mathrm{B}-\{1\})h')(a_2, \ldots, a_{r+1}) = g(1, a_2, \ldots, a_{r+1})$$

for all $(a_2, \ldots, a_{r+1}) \in (\mathrm{A}-\{1\})^r$. Let h be the function on $\{1\} \times (\mathrm{A}-\{1\})^{r+1}$ defined by $h(1, a_2, \ldots, a_{r+2}) = h'(a_2, \ldots, a_{r+2})$ for all $(a_2, \ldots, a_{r+1}) \in (\mathrm{A}-\{1\})^{r+1}$. Let A_1 be the set of all $(b_1, b_2, \ldots, b_{r+2}) \in \mathrm{A}^{r+2}$ such that at least one « coordinate » is 1. By anticommutativity, h may be extended uniquely to a mapping (again denoted by h) of A_1 into W. Furthermore it is easily verified that if $(b_1, \ldots, b_{r+1}) \times \mathrm{A} \subset \mathrm{A}_1$ (i.e. at least one $b_i = 1$) then $g(b_1, b_2, \ldots, b_{r+1}) = \sum_{j\in\mathrm{A}} \varphi_j h(b_1, b_2, \ldots, b_{r+1}, j)$. If $(b_1, \ldots, b_{r+2}) \in \mathrm{A}$ then $h(b_1, b_2, \ldots, b_{r+2}) \in \mathrm{W}(-t-(r+2), \mathrm{B}-\{b_1, \ldots, b_{r+2}\})$ as follows directly from the corresponding property of h'.

For each integer m, $1 \leq m \leq s$, let $\mathrm{A}_m = \{(a_1, \ldots, a_{r+2}) \in \mathrm{A}^{r+2} \mid a_i \in \{1, 2, \ldots, m\}$ for at least one $i \in \{1, 2, \ldots, r+2\}\}$. Let $\mathrm{A}'_m = \{(a_1, \ldots, a_{r+1}) \in \mathrm{A}^{r+1} \mid a_i \in \{1, 2, \ldots, m\}$ for at least one $i \in \{1, 2, \ldots, r+1\}\}$. Suppose (second induction hypothesis) that h has been extended to a skew symmetric function on A_m such that for all $(a_1, \ldots, a_{r+2}) \in \mathrm{A}_m$ and $(b_1, \ldots, b_{r+1}) \in \mathrm{A}'_m$ we have $h(a_1, \ldots, a_{r+2}) \in \mathrm{W}(-t-(r+2), \mathrm{B}-\{a_1, a_2, \ldots, a_{r+2}\})$ and $g(b_1, \ldots, b_{r+1}) = \sum_{j\in\mathrm{A}} \varphi_j h(b_1, \ldots, b_{r+1}, j)$. If $m = s$, we are done, i.e. $h \in \mathfrak{F}(t, r+2; \mathrm{A}, \mathrm{B})$ and $\delta(t, r+2; \mathrm{A}, \mathrm{B})h = g$. Hence we may assume that $1 \leq m < s$. If $m = s-1$ then since $r+1 \geq 2$, $g(m+1, a_2, \dots, a_{r+1}) = 0$ unless $(m+1, a_2, \dots, a_{r+1}) \in A'_m$. Likewise h is defined on A_m and can be extended to an anticommutative mapping of A^{r+2} into W by letting h map elements of A^{r+2} not in A_m into 0. Thus for $(m+1, a_2, \dots, a_{r+1}) \in A^{r+1}$, $g(m+1, a_2, \dots, a_{r+1}) = \sum_{i \in A} \varphi_i h(m+1, a_2, \dots, a_{r+1}, i)$ since this is certainly true if $(m+1, a_2, \dots, a_{r+1}) \in A'_m$, while otherwise $m+1 = a_2 = \dots = a_{r+1}$ and hence both sides are zero.

Thus our second induction hypothesis may be applied to the case in which $1 \leq m < s-1$. We know that $\sum_{j \in A} \varphi_j g(m+1, a_2, \dots, a_r, j) = 0$ for all $(a_2, \dots, a_r) \in A^{r-1}$. We restrict $(a_2, \dots, a_r)$ to $(A - \{1, 2, \dots, m\})^{r-1}$. For $j \leq m$, the second induction hypothesis gives $g(m+1, a_2, \dots, a_r, j) = \sum_{i=1}^{s} \varphi_i h(m+1, a_2, \dots, a_r, j, i)$ and hence

$$0 = \sum_{j=1}^{m} \sum_{i=1}^{s} \varphi_j \varphi_i h(m+1, a_2, \dots, a_r, j, i) + \sum_{j=m+1}^{s} \varphi_j g(m+1, a_2, \dots, a_r, j).$$

The anticommutativity of h on A_m shows that

$$0 = \sum_{j=1}^{m} \sum_{i=1}^{m} \varphi_j \varphi_i h(m+1, a_2, \dots, a_r, j, i)$$

and hence

$$0 = \sum_{j=m+1}^{s} \varphi_j \left\{ g(m+1, a_2, \dots, a_r, j) + \sum_{i=1}^{m} \varphi_i h(m+1, a_2, \dots, a_r, i, j) \right\}.$$

Since $m+1 < s$, this last relation may be written $0 = \sum_{j=m+2}^{s} g'(m+1, a_2, \dots, a_r, j)$ where g' is the mapping $(a_2, \dots, a_{r+1}) \rightarrow g(m+1, a_2, \dots, a_{r+1}) - \sum_{i=1}^{m} \varphi_i h(m+1, a_2, \dots, a_{r+1}, i)$, of $(A - \{1, 2, \dots, m+1\})^r$ into W. It is easily verified that

$$g' \in \mathfrak{F}(t+1, r; A - \{1, 2, \dots, m+1\}, B - \{m+1\})$$

and we have just shown that g' lies in the kernel of

$$\delta(t+1, r; A - \{1, 2, \dots, m+1\}, B - \{m+1\})$$

and hence by induction on r, there exists

$$h'' \in \mathfrak{F}(t+1, r+1; A - \{1, 2, \dots, m+1\}, B - \{m+1\})$$

such that $\delta(t+1, r+1; A - \{1, 2, \dots, m+1\}, B - \{m+1\}) h'' = g'$. Thus

$$g(m+1, a_2, \dots, a_{r+1}) = \sum_{i=1}^{m} \varphi_i h(m+1, a_2, \dots, a_{r+1}, i) + \sum_{k=m+2}^{s} \varphi_k h''(a_2, \dots, a_{r+1}, k)$$

for all $(a_2, \dots, a_{r+1}) \in (A - \{1, 2, \dots, 1+m\})^r$. We now define for all $(a_2, \dots, a_{r+1}) \in (A - \{1, 2, \dots, m+1\})^{r+1}$, $h(m+1, a_2, \dots, a_{r+1}, a_{r+2}) = h''(a_2, \dots, a_{r+2})$ and extend h by antisymmetry to $\Gamma = \{(a_1, \dots, a_{r+2}) \in (A - \{1, 2, \dots, m\})^{r+2}$ such that at least one $a_i = m+1\}$. (We note that $\Gamma \cap A_m = \emptyset$ while $\Gamma \cup A_m = A_{m+1}$). Thus h is now well defined and antisymmetric on A_{m+1}. If now $(a_1, \ldots, a_{r+1}) \in A'_{m+1}$ then $g(a_1, \ldots, a_{r+1}) = \sum_{i \in A} \varphi_i h(a_1, \ldots, a_{r+1}, i)$ since this is known by the induction hypothesis to be true if $(a_1, \ldots, a_{r+1}) \in A'_m$ and hence we may assume that

$$(a_1, \ldots, a_{r+1}) \in (A - \{1, 2, \ldots, m\})^{r+1}$$

and that at least one of the a_i is $m+1$ in which case we may use our relation involving h'', our extension of h and the antisymmetry of both h and g. Finally we note that for $(a_1, \ldots, a_{r+2}) \in A_{m+1}$, $h(a_1, \ldots, a_{r+2}) \in W(-t-r-2, B-\{a_1, \ldots, a_{r+2}\})$ since this holds by the induction hypothesis if $(a_1, \ldots, a_{r+2}) \in A_m$, while otherwise we may suppose $a_1 = m+1$, $(a_2, \ldots, a_{r+1}) \in (A - \{1, 2, \ldots, m+1\})^r$ so that

$$h(a_1, \ldots, a_{r+2}) = h''(a_2, \ldots, a_{r+2})$$

which lies in the asserted space since $h'' \in \mathfrak{F}(t+1, r+1; A-\{1, 2, \ldots, m+1\}, B-\{m+1\})$. This completes the proof of the theorem.

For subsequent applications it is convenient to make available a weaker form of the theorem. Let W now be a vector space over K with an infinite family of subspaces, $W(t)$, indexed by $t \in \mathbf{Z}$. Let $\varphi_1, \ldots, \varphi_{n+1}$ be a commutative set of endomorphisms of V with the property $\varphi_i W(t) \subset W(t+1)$ for each $i \in S$, $t \in \mathbf{Z}$. For each $r \in \mathbf{Z}_+$ and each non-empty subset, A, of S, let $\mathfrak{F}(t, r; A)$ be the space of all antisymmetric functions, g, on A^r such that $g(a_1, \ldots, a_r) \in W(-t-r)$, it being again understood that $\mathfrak{F}(t, 0; A)$ is to be identified with $W(-t)$. For $r \geq 1$, let $\delta(t, r; A)$ be the mapping of $\mathfrak{F}(t, r; A)$ into $\mathfrak{F}(t, r-1; A)$ defined as in equation (3.22). The second corollary follows directly from the theorem.

Corollary. — If the sequence

$$\mathfrak{F}(t, r+2; A) \xrightarrow{\delta} \mathfrak{F}(t, r+1; A) \xrightarrow{\delta} \mathfrak{F}(t, r; A)$$

is exact when $r=0$ *for each non-empty subset* A *of* S *then it is exact for all* $r \in \mathbf{Z}_+$.

For our final result of this section we use the notation of § 3 *e*.

Lemma **3.19.** — *Consider the polynomial,* $Y^{n+1}(1-Y^{d-1})^{n+1}/(1-Y)^{n+1} = \Sigma \gamma_j Y^j$ *in one variable,* Y. *Then for each* $m \in \mathbf{Z}$,

$$\dim \mathfrak{V}_S^{S,(m)} = \gamma_{md}, \textit{ and hence}$$
$$\dim \mathfrak{V}_S^S = d^{-1}\{(d-1)^{n+1} + (-1)^{n+1}.(d-1)\}.$$

Proof. — In the notation of Theorem 3.1, let $W = \mathfrak{L}$ and let $W(t, A) = W_S^{A,(t)}$ for each $t \in \mathbf{Z}$ and for each subset A of S. For each $i \in S$, let φ_i be the mapping $\xi \to f_i \xi$ of W into itself. It is clear that condition (3.21) is satisfied. To apply the theorem we must verify that if $\emptyset \neq A \subset B \subset S$ then Kernel $\delta(t, 1; A, B) =$ Image $\delta(t, 2; A, B)$. This is equivalent to the assertion that if h_i is a set of elements of $W_S^{(-t-1)}$ indexed by A such that $h_i \in (M_B/X_i)$ and such that $\sum_{i \in A} h_i f_i = 0$ then there exists a skew symmetric set $\{\eta_{ij}\}$

in $W_S^{(-t-2)}$ indexed by A such that $h_i = \sum_{j \in A} \eta_{ij} f_j$ for each $i \in A$ and such that $\eta_{i,j} \in (M_B/X_i X_j)$. This assertion may be proven without difficulty by means of Lemma 3.3 (i), using the fact that the proof of Lemma 3.1 shows that $(f_1, \ldots, f_r) : f_{r+1} = (f_1, \ldots, f_r)$ for $r = 1, 2, \ldots, n$. Thus, Theorem 3.1 may be applied and denoting $\delta(-m, r; S, S)$ by δ_r for fixed m, and $\mathfrak{F}(-m, r; S, S)$ by $\mathfrak{F}_r$, we may conclude that

$$\text{Kernel } \delta_r = \text{Image } \delta_{r+1} \tag{3.33}$$

for $r = 1, 2, \ldots$ Furthermore $\mathfrak{F}_r$ being the space of all skewsymmetric functions, g, on S^r taking values in $W^{(m-r)}$ such that $g(a_1, \ldots, a_r) \in W^{S-\{a_1, \ldots, a_r\}, (m-r)}$, we easily compute

$$\dim \mathfrak{F}_r = \binom{n+1}{r} \binom{d(m-r)+r-1}{n} \tag{3.34}$$

Since Image $\delta_r \simeq \mathfrak{F}_r/\text{Kernel } \delta_r$ and since $\mathfrak{F}_r$ is of finite dimension, we have

$$\dim \mathfrak{F}_r = \dim \text{Kernel } \delta_r + \dim \text{Image } \delta_r. \tag{3.35}$$

Writing $[\text{Im } \delta_r]$ (resp: $[\text{Ker } \delta_r]$) for dim Image δ_r (resp: dim Kernel δ_r), we now have as power series in Y, $\sum_{r=1}^{\infty} Y^r \dim \mathfrak{F}^r = \sum_{r=1}^{\infty} Y^r [\text{Im } \delta_r] + \sum_{r=1}^{\infty} Y^r [\text{Ker } \delta_r]$. Equation (3.33) now gives

$$\sum_{r=1}^{\infty} Y^r \dim \mathfrak{F}_r = Y[\text{Im } \delta_1] + (1 + Y^{-1}) \sum_{r=2}^{\infty} Y^r [\text{Im } \delta_r]. \tag{3.36}$$

Since $\dim \mathfrak{F}_r = 0$ for $r > n+1$, this equation is a relation between polynomials and hence setting $Y = -1$ in (3.36), we have $-[\text{Im } \delta_1] = \sum_{r=1}^{\infty} (-1)^r \dim \mathfrak{F}_r$. By definition $\mathfrak{V}_S^{S,(m)}$ is isomorphic to the factor space $W_S^{S,(m)}/(\mathfrak{A} \cap W_S^{S,(m)})$ and Lemma 3.3 (i) shows that $\mathfrak{A} \cap W_S^{S,(m)} = \text{Image } \delta_1$. Furthermore $\mathfrak{F}_0 = W_S^{S,(m)}$ and hence

$$\dim \mathfrak{V}_S^{S,(m)} = \dim \mathfrak{F}_0 - [\text{Im } \delta_1].$$

We may conclude that

$$\dim \mathfrak{V}_S^{S,(m)} = \sum_{r=0}^{\infty} (-1)^r \dim \mathfrak{F}_r. \tag{3.37}$$

It is easy to verify with the aid of (3.34) that the right side of (3.37) is the coefficient γ_{md} of Y^{md} in the polynomial

$$h(Y) = Y^{n+1}(1 - Y^{d-1})^{n+1}/(1 - Y)^{n+1} = Y^{n+1}(1 + Y + \ldots + Y^{d-2})^{n+1}.$$

Clearly $\dim \mathfrak{V}_S^S = \sum_{m=0}^{\infty} \gamma_{md} = d^{-1} \sum_{\omega} h(\omega)$, the sum being over the d^{th} roots of unity. Clearly $1 - \omega^{d-1} = -\omega^{-1}(1 - \omega)$ and hence $h(\omega) = (-1)^{n+1}$ if $\omega \neq 1$, while $h(1) = (d-1)^{n+1}$. This completes the proof of the lemma.

We now observe that $\dim V = \dim \mathfrak{V} = \sum_{A} \dim \mathfrak{V}_A^A$, the sum being over all subsets A of S. In particular for $A = \emptyset$, $\dim \mathfrak{V}_\emptyset^\emptyset = 1$ and this coincides with the formula of the previous lemma if we replace $n+1$ by 0. It is easily verified that $\dim V = d^n$, a result that could have been obtained directly by an argument similar to that of the lemma in which the corollary of Theorem 3.1 is used instead of Theorem 3.1.

Since the polynomial h in the proof of the previous lemma has the property

$$h(Y) = Y^{d(n+1)}h(Y^{-1})$$

it is clear that $\gamma_{d(n+1)-j} = \gamma_j$ for all $j \in \mathbf{Z}$. In particular

$$\gamma_{md} = \gamma_{(n+1-m)d}$$

for all m, a result which may be related to the conjectured functional equation of the zeta function. We also note that $\gamma_d = 0$ if and only if $d < n+1$, a fact related to the results of Warning.

§ 4. Geometrical Theory.

The notation of § 3 shall be used whenever possible. In this section $q = p^a$, $a \in \mathbf{Z}_+$, $a \geq 1$. The first subsection involves power series in one variable, t, with coefficients in Ω. Such a power series, $\Sigma \gamma_m t^m$, will be said to lie in $L(b, c)$ if $\operatorname{ord} \gamma_m \geq mb + c$ for all $m \in \mathbf{Z}_+$.

a) *Splitting functions.*

In [1] we gave two examples of a power series, θ, in one variable satisfying the conditions

(i) $\theta \in L(\varkappa, 0), \varkappa > 0$.

(ii) $\theta(1)$ is a primitive p^{th} root of unity.

(iii) If $\gamma^{p^s} = \gamma$ for some integer s, $s > 0$ then

$$\prod_{j=0}^{s-1} \theta(\gamma^{p^j}) = \theta(1)^{\sum_{j=0}^{s-1} \gamma^{p^j}}.$$

(iv) The coefficients of θ lie in a finite extension of $\mathbf{Q}'$.

A power series in one variable satisfying these four conditions will be called a *splitting function.* We shall construct an infinite family of such functions indexed by $\mathbf{Z}^* = \{+\infty\} \cup \{s \in \mathbf{Z} \mid s \geq 1\}$. Indeed the theory of Newton polygons shows that for each $s \in \mathbf{Z}^*$, the polynomial (or power series),

$$\sum_{j=0}^{s} Y^{p^j}/p^j$$

has a zero, γ_s, such that $\operatorname{ord} \gamma_s = 1/(p-1)$. While there are $p-1$ such zeros, we shall suppose one has been chosen for each $s \in \mathbf{Z}^*$. For each $s \in \mathbf{Z}^*$ we now set

$$\theta_s(t) = \exp\left\{\sum_{j=0}^{s} (t\gamma_s)^{p^j}/p^j\right\}. \tag{4.1}$$

Lemma **4.1.** — *For each* $s\in\mathbf{Z}^*$, θ_s *is a splitting function.*

Proof. — In the following the symbol y shall denote a parameter to be chosen in Ω subject to the condition $\operatorname{ord} y=1/(p-1)$. For each $s\in\mathbf{Z}_+$, let $g_s(t,y)=\exp\{-(ty)^{p^s}/p^s\}$. It is easily verified that $g_s\in \mathrm{L}(a_s, 0)$, where

$$a_s=(p-1)^{-1}-p^{-s}(s+(p-1)^{-1}) \tag{4.2}$$

for $s\in\mathbf{Z}_+$, while a_∞ is taken to be $(p-1)^{-1}$ for later use.

For $s\in\mathbf{Z}_+$ let $\mathrm{G}_s(t,y)=\prod_{j=s+1}^{\infty} g_j(t,y)$. Since $a_{j+1}\geq a_j$ for each $j\in\mathbf{Z}_+$, we conclude that $\mathrm{G}_s(t,y)\in\mathrm{L}(a_{s+1}, 0)$. Let $\mathrm{E}(t)$ denote the Artin-Hasse exponential series

$$\mathrm{E}(t)=\exp\left\{\sum_{j=0}^{\infty} t^{p^j}/p^j\right\}. \tag{4.3}$$

It is well known that

$$\begin{gathered}\mathrm{E}(t)\in\mathrm{L}(0,0)\\ \mathrm{E}(t)\equiv 1+t \bmod t^2\mathbf{Q}'\{t\}.\end{gathered} \tag{4.4}$$

Let $h_\infty(t,y)=\mathrm{E}(ty)$ and for $s\in\mathbf{Z}_+$, $s\geq 1$ let

$$h_s(t,y)=h_\infty(t,y)\mathrm{G}_s(t,y) \tag{4.5}$$

and so for $s\in\mathbf{Z}^*$

$$h_s(t,y)=\exp\left\{\sum_{j=0}^{s}(ty)^{p^j}/p^j\right\}. \tag{4.6}$$

Clearly $h_\infty(t,y)\in\mathrm{L}(a_\infty, 0)$ and for $s\in\mathbf{Z}$, $s\geq 1$, equation (4.5) shows that

$$h_s(t,y)\in\mathrm{L}(a_{s+1}, 0). \tag{4.7}$$

Since $a_2=(p-1)/p^2>0$, we may conclude that $h_s(t,y)$ converges for $\operatorname{ord} t\geq 0$. Furthermore equation (4.5) shows that

$$h_s(t,y)+(ty)^{p^{s+1}}/p^{s+1}\equiv h_\infty(t,y) \bmod t^{1+p^{s+1}}\Omega\{t\}. \tag{4.8}$$

Combining this relation with (4.7) we conclude that for $s\geq 1$

$$\operatorname{ord}(h_s(1,y)+y^{p^{s+1}}/p^{s+1}-h_\infty(1,y))\geq a_{s+1}(1+p^{s+1}). \tag{4.9}$$

Since $h_\infty(1,y)=\mathrm{E}(y)$, we conclude with the aid of (4.4), and (4.2) that for $s\in\mathbf{Z}_+$, $s\geq 1$

$$\operatorname{ord}(h_s(1,y)-1)=1/p-1 \tag{4.10}$$

and (4.4) shows that (4.10) is valid for all $s\in\mathbf{Z}^*$. Furthermore equation (4.6) shows that for $s\in\mathbf{Z}^*$

$$\log h_s(1,y)=\sum_{i=0}^{s} y^{p^j}/p^j \tag{4.11}$$

and hence $\log h_s(1,\gamma_s)=0$.

Since $\theta_s(t)=h_s(t,\gamma_s)$, we conclude from (4.7) that $\theta_s\in L(a_{s+1},0)$, and from (4.11) that $\theta_s(1)$ is a p^r—th root of unity for some r while (4.10) shows that $\theta_s(1)$ is a primitive p^{th} root of unity.

If $\gamma^{p^r}=\gamma$ where $r\in\mathbf{Z}$, $r\geq 1$ then as a power series in y, $\prod_{j=0}^{r-1} h_s(\gamma^{p^j},y)=h_s(1,y)^{\sum_{j=0}^{r-1}\gamma^{p^j}}$ as may be seen from equation (4.6). Replacing y by γ_s we conclude that θ_s satisfies condition (iii) in the definition of a splitting function. We have already verified conditions (i) and (ii). Finally we note that $\mathbf{Q}'(\gamma_s)$ is a purely ramified extension of $\mathbf{Q}'$ of degree $p-1$, while condition (ii) shows that $\mathbf{Q}'(\gamma_s)$ contains a primitive p^{th} root of unity. We conclude that for each $s\in\mathbf{Z}^*$ the coefficients of θ_s lie in the field of p^{th} roots of unity. This completes the proof of the lemma.

If $g\in 1+t\Omega\{t\}$, let $\hat{g}(t)=\prod_{j=0}^{\infty} g(t^{p^j})$, an infinite product which converges in the formal topology of $\Omega\{t\}$. Clearly $g(t)=\hat{g}(t)/\hat{g}(t^p)$ and if $q=p^a$, $a\geq 1$ then

$$\textbf{(4.12)}\qquad \prod_{j=1}^{a-1} g(t^{p^j})=\hat{g}(t)/\hat{g}(t^q).$$

It follows from the definitions that for each $s\in\mathbf{Z}^*$

$$\textbf{(4.13)}\qquad \hat{\theta}_s(t)=\exp\left\{\sum_{j=0}^{s-1}\gamma_{s,j}t^{p^j}\right\},$$

where

$$\textbf{(4.14)}\qquad \gamma_{s,j}=\sum_{i=0}^{j}\gamma_s^{p^i}/p^i$$

It is worth observing that

$$\textbf{(4.15)}\qquad \operatorname{ord}\gamma_{s,j}=(p-1)^{-1}p^{j+1}-(j+1)$$

In particular $\hat{\theta}_1=\exp(\gamma_1 t)$. In the application use will be made only of θ_∞ and θ_1.

b) Let $f(X)$ be a homogeneous polynomial of degree d in $n+1$, $(n\geq 0)$ variables, $X_1, X_2, \ldots, X_{n+1}$ whose coefficients are either zero or $(q-1)$—th roots of unity in Ω. We may write

$$\textbf{(4.16)}\qquad f(X)=\sum_{i=1}^{\rho} A_iM_i,$$

where $A_i^q=A_i$ and M_i is a monomial in $X_1, \ldots, X_{n+1}$ for $i=1, 2, \ldots, \rho$. Let $\mathfrak{S}_n$ denote n dimensional projective space of characteristic p and let $\mathfrak{H}$ be the variety in $\mathfrak{S}_n$ defined over the field k of q elements by the equation $f(X)\equiv 0 \bmod p$. For $n=0$, extending in the obvious way the usual identifications associated with projective coordinates, $\mathfrak{S}_0$ consists of just one point which is of course rational over the prime field. In any case $\mathfrak{H}=\mathfrak{S}_n$ if f is trivially congruent to zero $\bmod p$. If f is not trivial $\bmod p$ then $\mathfrak{H}$ is a *hypersurface* in $\mathfrak{S}_n$, to which we attatch the conventional meaning if $n\geq 2$, while if $n=0$ then $\mathfrak{H}$ is empty and if $n=1$ then $\mathfrak{H}$ is a set of at most d points on the projective line which are algebraic over k and closed under field automorphisms which leave the elements of k fixed.

For $n \geq 0$ we say that $\mathfrak{H}$ is a non-singular hypersurface of degree d in $\mathfrak{S}_n$ if the polynomials $f, \frac{\partial f}{\partial X_1}, \ldots, \frac{\partial f}{\partial X_{n+1}} \pmod p$ have no common zero in $\mathfrak{S}_n$. For $n \geq 2$ this coincides with the usual definition, while for $n = 1$ it means that $\mathfrak{H}$ is a set of d distinct points and for $n = 0$, it means that $\mathfrak{H}$ is empty (i.e. f is not trivial mod p).

Let $\zeta(\mathfrak{H}, t)$ be the zeta function of $\mathfrak{H}$ as variety defined over k and let $P(\mathfrak{H}, t)$ be the rational function defined by

$$P(\mathfrak{H}, t)^{(-1)^n} = \zeta(\mathfrak{H}, t)(1 - q^n t)^{-1} \prod_{i=0}^{n} (1 - q^i t) \tag{4.17}$$

According to the Weil hypothesis, if $n \geq 2$ and $\mathfrak{H}$ is a nonsingular hypersurface of degree d in $\mathfrak{S}_n$ then $P(\mathfrak{H}, t)$ is a polynomial of degree $d^{-1}\{(d-1)^{n+1} + (-1)^{n+1}(d-1)\}$. Using the above conventions this hypothesis is easily verified for $n = 0$, 1 as for $n = 0$,

$$\zeta(\mathfrak{H}, t) = \begin{cases} 1 & \text{if } \mathfrak{H} \text{ is empty} \\ (1-t)^{-1} & \text{if } \mathfrak{H} = \mathfrak{S} \end{cases} \tag{4.18}$$

while if $n = 1$ and $\mathfrak{H}$ consists of d distinct points, then $\mathfrak{H}$ is a union of e disjoint sets of points, the i^{th} subset consisting of b_i points conjugate over k and each point generating an extension of k of degree b_i. In this case $d = \sum_{i=1}^{e} b_i$ and

$$\zeta(\mathfrak{H}, t) = \prod_{i=1}^{e} (1 - t^{b_i})^{-1} \tag{4.19}$$

Thus if $\mathfrak{H}$ is a non-singular hypersurface of degree d in $\mathfrak{S}_n$ then

$$P(\mathfrak{H}, t) = \begin{cases} 1 & \text{if } n = 0 \\ \left(\prod_{i=1}^{e} (1 - t^{b_i})\right) / (1-t) & \text{if } n = 1, \end{cases} \tag{4.20}$$

which is precisely the Weil hypothesis in these trivial cases.

We know from [1] that the zeta function of $\mathfrak{H}$ is related to the linear transformation $\psi \circ F$, where

$$F(X) = \prod_{i=1}^{\rho} \prod_{j=0}^{a-1} \theta((X_0 A_i M_i)^{p^j}), \tag{4.21}$$

θ being any splitting function. If $\hat{\theta}$ is defined as before then since $A_i^q = A_i$,

$$F(X) = \hat{F}(X)/\hat{F}(X^q) \tag{4.22}$$

where

$$\hat{F}(X) = \prod_{i=1}^{\rho} \hat{\theta}(A_i X_0 M_i)$$

If we take the splitting function to be θ_s, $s=1, 2, \dots, +\infty$, then $\hat{F}$ takes the form

$$(4.23) \qquad \hat{F}_s(X)=\exp\left\{\sum_{j=0}^{s-1}\gamma_{s,j}X_0^{pj}f^{\tau^j}(X^{p^j})\right\},$$

where τ is the Frobenius automorphism over $\mathbf{Q}'$ of a sufficiently large, unramified extension field.

Since $\theta_s\in L(a_{s+1}, 0)$, equation (4.12) shows that $\hat{\theta}_s(t)/\hat{\theta}_s(t^q)\in L(pa_{s+1}/q, 0)$. It follows without difficulty that $F_s(X)=\hat{F}_s(X)/\hat{F}_s(X^q)\in L(pa_{s+1}/q, 0)$ in the sense of § 3, a_{s+1} being given by (4.2).

We now recall and clarify the geometrical significance of the characteristic series, χ_F, where F is given by (4.21) If $g\in\Omega\{t\}$, let g^φ be the power series $g(qt)$ and if $g\in 1+t\Omega\{t\}$, let g^δ be the power series $g^{1-\varphi}=g(t)/g(qt)$.

If $\mathfrak{H}'$ is the « hypersurface », $\prod_{i=1}^{n+1}X_i=0$ in $\mathfrak{S}_n$, then by [1, equation (21)] (recalling that although F now involves a total of $n+2$ variables, we are now counting points in projective rather than affine space)

$$(4.24) \qquad \zeta(\mathfrak{H}-\mathfrak{H}', qt)=\chi_F^{-(-\delta)^{n+1}}(1-t)^{-(-\delta)^n}$$

For each non-empty subset A of $S=\{1, 2, \dots, n+1\}$, let $1+m(A)$ be the number of elements in A and let $\mathfrak{H}_A$ be the variety in $\mathfrak{S}_{m(A)}$ defined by the equation in X_A,

$$\mathfrak{I}_A f\equiv 0 \bmod p$$

and let $\mathfrak{H}'_A$ be the hypersurface $\prod_{i\in A}X_i=0$ in $\mathfrak{S}_{m(A)}$. Let Δ_A be the power series in one variable defined by

$$(4.25) \qquad \zeta(\mathfrak{H}_A-\mathfrak{H}'_A, qt)=\Delta_A^{-(-\delta)^{1+m(A)}}(1-t)^{-(-\delta)^{m(A)}}.$$

The precise formulation of Δ_A as a characteristic series in the sense of § 2 does not concern us here, except that we observe that $\mathfrak{H}_S=\mathfrak{H}$, $\Delta_S=\chi_F$. To simplify notation let $P_A(t)$ denote $P(\mathfrak{H}_A, t)$ as defined by (4.17), so that

$$(4.26) \qquad P_A(t)^{(-1)^{m(A)}}=\zeta(\mathfrak{H}_A, t)(1-q^{m(A)}t)^{-1}\prod_{i=0}^{m(A)}(1-q^it).$$

If B is a non-empty subset of S then

$$(4.27) \qquad \mathfrak{H}_B=\bigcup_{A\subset B}(\mathfrak{H}_A-\mathfrak{H}'_A),$$

a *disjoint* union indexed by all non-empty subsets, A, of B. We may conclude with the aid of (4.25) that

$$\zeta(\mathfrak{H}_B, qt)=\prod_{A\subset B}\zeta(\mathfrak{H}_A-\mathfrak{H}'_A, qt)=\prod_{A\subset B}\{\Delta_A^{-(-\delta)^{1+m(A)}}(1-t)^{-(-\delta)^{m(A)}}\}.$$

But an elementary computation gives $\sum_{B\subset A}-(-\delta)^{m(A)}=\delta^{-1}(\varphi^{1+m(B)}-1)$ and hence

$$(4.28) \qquad \zeta(\mathfrak{H}_B, qt)=(1-t)^{\delta^{-1}(\varphi^{1+m(B)}-1)}\prod_{A\subset B}\Delta_A^{-(-\delta)^{1+m(A)}},$$

while equation (4.26) shows that

$$(4.29) \qquad \zeta(\mathfrak{H}_B, qt) = \{P_B(t)^{(-1)^{m(B)}}(1-t)^{-\delta^{-1}(1-\varphi^{m(B)})}\}^{\varphi}$$

comparing (4.28) and (4.29) we obtain

$$(4.30) \qquad (1-t)P_B(t)^{\varphi(-1)^{m(B)}} = \prod_{A \subset B} \Delta_A^{-(-\delta)^{1+m(A)}}$$

Relations such as (4.30) can easily be inverted by an analogue of the Möbius inversion formula. Explictly if $A \to \mathfrak{G}_A$ is a mapping of subsets of S into a multiplicative abelian group and if for each subset B of S

$$(4.31) \qquad G_B = \prod_{A \subset B} \mathfrak{G}_A,$$

the product being over all subsets, A, of B, then

$$(4.32) \qquad \mathfrak{G}_B = \prod_{A \subset B} G_A^{(-1)^{m(B)-m(A)}}$$

The inductive proof of (4.32) may be omitted since it depends entirely on the well known fact that $\sum_{i=1}^{r} \binom{r}{i}(-1)^i = -1$ for each integer $r \geq 1$. Applying this to (4.30) and letting $B = S$, we obtain $\Delta_S^{-(-\delta)^{1+n}} = \prod_A \{P_A(t)^{\varphi(-1)^{m(A)}}(1-t)\}^{(-1)^{n-m(A)}}$. Since $\chi_F = \Delta_S$ we obtain

$$(4.33) \qquad \chi_F^{\delta^{1+n}} = (1-t)\prod_A P_A(qt),$$

the product being over the non-empty subsets, A of S. (A similar formula appeared in an earlier work [6, equation 21].) We believe this equation is quite significant since χ_F is entire even if $\mathfrak{H}$ is singular.

Since $\zeta(\mathfrak{H}_A, t)$ is rational, P_A is also rational and hence (4.33) shows that the zeros of χ_F and the $(q-1)p$ roots of unity generate a finite extension, Ω_0, of $\mathbf{Q}'$. With this choice of Ω_0, the results of § 2 show that the zeros of χ_F are explained by the action of $\psi \circ F$ as linear transformation of $L(q\varkappa)$ if $F \in L(\varkappa, 0)$.

We now fix $s \in \mathbf{Z}^*$, let $F = F_s$ so that $\varkappa = pa_{s+1}/q$, $\hat{F} = \exp H$, where $H = \sum_{j=0}^{s-1} \gamma_{s,j} X_0^{p^j} f^{\tau^j}(X^{p^j})$. We shall assume unless otherwise indicated that f is a regular polynomial ([1]). Equation (4.15) shows that H satisfies the conditions of § 3. It follows from (4.22) that $\alpha = \psi \circ F$, may be written

$$(4.34) \qquad \alpha = \hat{F}^{-1} \circ \psi \circ \hat{F},$$

while with this choice of H, the mappings D_i of § 3 are simply $\xi \to \hat{F}^{-1}E_i(\xi\hat{F})$. Since $qE_i \circ \psi = \psi \circ E_i$, we conclude for $i = 0, 1, \ldots, n+1$ that

$$(4.35) \qquad \alpha \circ D_i = qD_i \circ \alpha.$$

[1] This condition on f is equivalent to the condition that $\mathfrak{H}_A$ is non-singular for each non-empty subset, A, of S. It will be shown that this condition involves no essential loss in generality.

If λ is any non-zero element of Ω_0, let W_λ be defined as in Theorem 2.4, i.e. $W_\lambda=\{0\}$ if λ^{-1} is not a zero of χ_F, while $W_\lambda=\text{Kernel of } (I-\lambda^{-1}\alpha)^\mu$ in $L(q\varkappa)$ if λ^{-1} is a zero of multiplicity μ. We note that $\varkappa$, α, F, the D_i, H, and the spaces W_λ depend upon our choice of s. The maximum value of $q\varkappa$ is $p/(p-1)$ and corresponds to $s=\infty$. The minimum value of $q\varkappa$ is $(p-1)/p$ and this exceeds $1/(p-1)$ unless $p=2$. This minimum value of $q\varkappa$ corresponds to $s=1$. It is assumed in the following that $q\varkappa>1/(p-1)$.

Lemma **4.2**. — *If* A *is any subset of* S *and* $0<b\leq q\varkappa$ *then*

$$W_\lambda\cap\sum_{i\in A}D_iL(b)=\sum_{i\in A}D_iW_{\lambda/q}$$

Proof. — Let $\{\xi_i\}$ be a set of elements in $L(b)$ indexed by A such that $\sum\limits_{i\in A}D_i\xi_i=\xi\in W_\lambda$. Let $\mu=\max\{\dim W_\lambda, \dim W_{\lambda/q}\}$. It follows from the corollary to Theorem 2.5 that for each $i\in A$ there exists $\eta_i\in W_{\lambda/q}$ and $\eta_i'\in L(b)$ such that

$$\xi_i=\eta_i+(I-(\lambda/q)^{-1}\alpha)^\mu\eta_i'.$$

Thus $(I-\lambda^{-1}\alpha)^\mu\sum\limits_{i\in A}D_i\eta_i'=\sum\limits_{i\in A}D_i\xi_i-\sum\limits_{i\in A}D_i\eta_i=\xi-\sum\limits_{i\in A}D_i\eta_i$, which lies in W_λ by hypothesis, choice of the η_i and equation (4.35). We may now conclude from equation (2.54) that $(I-\lambda^{-1}\alpha)^\mu\sum\limits_{i\in A}D_i\eta_i'\in W_\lambda\cap(I-\lambda^{-1}\alpha)^\mu L(b)=(I-\lambda^{-1}\alpha)^\mu W_\lambda=\{0\}$. This shows that $\xi-\sum\limits_{i\in A}D_i\eta_i=0$ and hence $\xi\in\sum\limits_{i\in A}D_iW_{\lambda/q}$. Thus $W_\lambda\cap\sum\limits_{i\in A}D_iL(b)\subset\sum\limits_{i\in A}D_iW_{\lambda/q}$ and equality follows without difficulty.

Lemma **4.3**. — *If* A *is a non-empty subset of* S *and* $\{\xi_i\}_{i\in A}$ *is a set of elements in* W_λ *such that* $\sum\limits_{i\in A}D_i\xi_i=0$ *then there exists a skew symmetric set* $\{\eta_{ij}\}$ *in* $W_{\lambda/q}$ *indexed by* A *such that* $\xi_i=\sum\limits_{j\in A}D_j\eta_{ij}$ *for each* $i\in A$.

Proof. — Let $A=\{1, 2, \dots, r\}$, $1\leq r\leq n+1$. If $r=1$ then $D_1\xi_1=0$, $\xi_1\in L(q\varkappa)$ and hence Lemma 3.10 shows that $\xi_1=0$. We may therefore assume $r>1$ and use induction on r. Lemma 3.10 shows that there exist $\xi_1', \dots, \xi_{r-1}'$ in $L(q\varkappa)$ such that

$$\xi_r=\sum_{i=1}^{r-1}D_i\xi_i'.$$

Since $\xi_r\in W_\lambda$, the previous lemma shows that the ξ_i' may be chosen in $W_{\lambda/q}$. Hence $0=\sum\limits_{i=1}^{r-1}D_i(\xi_i+D_r\xi_i')$ and since $\xi_i+D_r\xi_i'\in W_\lambda$ for $i=1, 2, \dots, r-1$, the induction hypothesis shows the existence of a skew symmetric set $\{\eta_{i,j}\}$ in $W_{\lambda/q}$ indexed by $\{1, 2, \dots, r-1\}$ such that for $i=1, 2, \dots, r-1$

$$\xi_i+D_r\xi_i'=\sum_{j=1}^{r-1}D_j\eta_{ij}$$

We now extend the skew symmetric set by defining $\eta_{i,r} = -\xi_i' = -\eta_{r,i}$ for $i = 1, 2, \ldots, r-1$ and $\eta_{r,r} = 0$. It is readily seen that the $\eta_{i,j}$ satisfy the conditions of the lemma.

Let λ be an eigenvalue of α. We now compute the dimension (as vector space over Ω_0) of the factor space $W_\lambda \Big/ \sum\limits_{i=1}^{n+1} D_i W_{\lambda/q}$.

Lemma 4.4. — $\mathrm{Dim}\left(W_\lambda \Big/ \sum\limits_{i=1}^{n+1} D_i W_{\lambda/q}\right) = \sum\limits_{r=0}^{n+1} \binom{n+1}{r}(-1)^r \dim W_{\lambda/q^r}$.

Proof. — In the statement of the Corollary of Theorem 3.1, let $W = L(q\varkappa)$ and for each $t \in \mathbf{Z}$, let $W(t) = W_{\lambda q^t}$, $\varphi_i = D_i$ for $i = 1, 2, \ldots, n+1$. The previous lemma shows that the sequence of the Corollary is exact when $r = 0$ and hence the Corollary may be applied. In this application $\mathfrak{F}(0, r; S)$ is the space of all skew symmetric maps of S^r into $W(-r) = W_{\lambda/q^r}$ and hence $\dim \mathfrak{F}(0, r; S) = \binom{n+1}{r} \dim W_{\lambda/q^r}$.

The corollary may be used to obtain an identity similar to equation (3.36), where $\mathfrak{F}_r = \mathfrak{F}(0, r; S)$, $\delta_r = \delta(0, r; S)$ and the assertion follows without difficulty since

$$\dim\left(W_\lambda \Big/ \sum_{i=1}^{n+1} D_i W_{\lambda/q}\right) = \dim \mathfrak{F}_0 - [\mathrm{Im}\, \delta_1] = \sum_{r=0}^{\infty} (-1)^r \dim \mathfrak{F}_r.$$

We can now show that $\chi_F^{\delta^{1+n}}$ is a polynomial.

Theorem 4.1. — *For each* $\lambda \in \Omega_0^*$, *let* $b_\lambda = \dim W_\lambda \Big/ \sum\limits_{i=1}^{n+1} D_i W_{\lambda/q}$, *then*

$$\chi_F^{\delta^{1+n}} = \Pi(1-\lambda t)^{b_\lambda}$$

the product being over all eigenvalues λ *of* α.

Proof. — Let λ be an eigenvalue of α with the property that λ/q^r is not an eigenvalue for any $r \geq 1$. For each eigenvalue, λ', of α, there exists an eigenvalue λ with this property such that $\lambda' = q^i\lambda$ for some $i \in \mathbf{Z}_+$. Let $a_j = \dim W_{\lambda q^j}$ for each $j \in \mathbf{Z}_+$. The factors of χ_F corresponding to terms of type $(1 - \lambda q^r t)$, $r \in \mathbf{Z}_+$ may be written

$$H_\lambda(t) = \prod_{i=0}^{\infty} (1 - t\lambda q^i)^{a_i} = (1 - t\lambda)^{\sum\limits_{i=0}^{\infty} a_i \varphi^i}.$$

The previous lemma shows that

$$b_{\lambda q^i} = \sum_{j=0}^{n+1} (-1)^j \binom{n+1}{j} a_{i-j}$$

and hence

$$(1-\varphi)^{n+1} \sum_{i=0}^{\infty} a_i \varphi^i = \sum_{i=0}^{\infty} b_{\lambda q^i} \varphi^i.$$

It follows that

$$H_\lambda(t)^{\delta^{n+1}} = (1 - \lambda t)^{\sum\limits_{i=0}^{\infty} b_{\lambda q^i} \varphi^i}.$$

This completes the proof of the theorem.

Equations (4.26) and (4.33), together with the known rationality of zeta functions, show that $\chi_F^{\delta 1+n}$ is a rational function. The theorem shows that the function is entire in the p-adic sense and hence it must be a polynomial.

Let $\mathfrak{W}$ be the factor space $L(q\varkappa)/\sum_{i=1}^{n+1} D_i L(q\varkappa)$. For $q\varkappa > 1/(p-1)$, we have shown in § 3 that $\dim \mathfrak{W} = d^n$. Since $\sum_{i=1}^{n+1} D_i L(q\varkappa)$ is a subspace of $L(q\varkappa)$ which is invariant under α, there exists an endomorphism $\bar{\alpha}$ of $\mathfrak{W}$ deduced from α by passage to quotients.

Theorem **4.2.**

$$\chi_F^{\delta 1+n} = \det(I - t\bar{\alpha}),$$

provided $q\varkappa > 1/(p-1)$.

Proof. — It is quite clear that the characteristic equation of $\bar{\alpha}$ is independent of Ω_0 and hence it may be assumed that Ω_0 contains the zeros of $\det(I - t\bar{\alpha})$. For each non-zero element λ of Ω_0, let $\mathfrak{W}_\lambda$ be the primary component of λ in $\mathfrak{W}$ with respect to $\bar{\alpha}$. To prove the theorem it is enough in view of Theorem 4.1 to show that

$$\dim \mathfrak{W}_\lambda = \dim \left(W_\lambda / \sum_{i=1}^{n+1} D_i W_{\lambda/q} \right) \tag{4.36}$$

Under the natural mapping, J, of $L(q\varkappa)$ onto $\mathfrak{W}$, W_λ is mapped into $\mathfrak{W}_\lambda$ with kernel $W_\lambda \cap \sum_{i=1}^{n+1} D_i L(q\varkappa)$, which by Lemma 4.2 is $\sum_{i=1}^{n+1} D_i W_{\lambda/q}$. This shows that $\dim \mathfrak{W}_\lambda$ is at least as large as the right side of (4.36). To complete the proof it is enough to show that $\mathfrak{W}_\lambda$ is the image of W_λ under J. To prove this let $\xi' \in \mathfrak{W}_\lambda$, hence there exists $r \geq 1$ such that $(I - \lambda^{-1}\bar{\alpha})^r \xi' = 0$. Let ξ be a representative of ξ' in $L(q\varkappa)$, then $(I - \lambda^{-1}\alpha)^r \xi \in \sum_{i=1}^{n+1} D_i L(q\varkappa)$. Hence there exists elements $\eta_1, \ldots, \eta_{n+1}$ in $L(q\varkappa)$ such that

$$(I - \lambda^{-1}\alpha)^r \xi = \sum_{i=1}^{n+1} D_i \eta_i.$$

Let μ be the multiplicity of $(\lambda/q)^{-1}$ as zero of χ_F, then

$$(I - \lambda^{-1}\alpha)^{r+\mu} \xi = \sum_{i=1}^{n+1} D_i (I - q\lambda^{-1}\alpha)^{\mu} \eta_i.$$

Theorem 2.5 shows that there exist $\eta_1', \ldots, \eta_{n+1}'$ in $L(q\varkappa)$ such that for $i = 1, 2, \ldots, n+1$

$$(I - q\lambda^{-1}\alpha)^{\mu+r} \eta_i' = (I - q\lambda^{-1}\alpha)^{\mu} \eta_i.$$

The last two displayed formulas show that

$$(I - \lambda^{-1}\alpha)^{\mu+r} \left(\xi - \sum_{i=1}^{n+1} D_i \eta_i' \right) = 0.$$

This shows that $\xi \in W_\lambda + \sum_{i=1}^{n+1} D_i L(q\varkappa)$ and hence $\xi' = J(\xi) \in J(W_\lambda)$, which completes the proof of (4.36) and hence of the theorem.

Theorem 4.3. — *The mapping*, $\bar{\alpha}$, *is a non-singular endomorphism of* $\mathfrak{W}$ (*and hence* $\chi_{\mathrm{F}}^{\delta 1+n}$ *is a polynomial of degree* d^n).

Proof. — It is enough to show that $\bar{\alpha}(\mathfrak{W})=\mathfrak{W}$, which by Lemma 3.6 is equivalent to the assertion that

$$\alpha \mathrm{V}+\sum_{i=1}^{n+1} \mathrm{D}_i \mathrm{L}(q\varkappa) \supset \mathrm{V}. \tag{4.37}$$

We recall that α depends upon the choice of $s \in \mathbf{Z}^*$ in our construction of $\mathrm{F}=\mathrm{F}_s$, but the degree of $\chi_{\mathrm{F}}^{\delta 1+n}$ is clearly independent of s and Theorem 4.2 therefore shows that $\dim \bar{\alpha}(\mathfrak{W})$ is independent of s provided $q\varkappa > 1/(p-1)$. Since $\dim \mathfrak{W}$ is also independent of s (subject to the same condition) we conclude that if equation (4.37) holds when $s=\infty$ then it holds for all s such that $q\varkappa > 1/(p-1)$. We may suppose in the remainder of the proof that $s=\infty$. Let τ be an extension, which leaves fixed a primitive p^{th} root of unity, to Ω_0 of the Frobenius automorphism over $\mathbf{Q}'$ of the maximal unramified subfield of Ω_0. Our proof is based on the fact that while $\hat{\mathrm{F}}(\mathrm{X})/\hat{\mathrm{F}}(\mathrm{X}^q)$ lies in $\mathrm{L}(p/q(p-1), 0)$, $\hat{\mathrm{F}}(\mathrm{X})/\hat{\mathrm{F}}^{\tau}(\mathrm{X}^p)$ lies in $\mathrm{L}(1/(p-1), 0)$.

Let ψ_p denote the mapping ψ with q replaced by p, (i.e. $\psi=\psi_p^a$). Let Φ_p be the mapping $\mathrm{X}^u \rightarrow \mathrm{X}^{pu}$ of $\Omega_0\{\mathrm{X}\}$ onto itself. Let α_0, β_0 be the $\mathbf{Q}'$-linear mappings of $\Omega_0\{\mathrm{X}\}$ into itself defined by

$$\alpha_0=\hat{\mathrm{F}}^{-1} \circ \tau^{-1} \circ \psi_p \circ \hat{\mathrm{F}}$$
$$\beta_0=\hat{\mathrm{F}}^{-1} \circ \tau \circ \Phi_p \circ \hat{\mathrm{F}}$$

We note that α_0 and β_0 are endomorphisms of $\Omega_0\{\mathrm{X}\}$ as $\mathbf{Q}'$-space, not (necessarily) as Ω_0-space. In view of our previous remarks we easily verify since $\hat{\mathrm{F}}(\mathrm{X})/\hat{\mathrm{F}}^{\tau}(\mathrm{X}^p) \in \mathrm{L}(1/(p-1))$ that

$$\begin{cases} \beta_0 \mathrm{L}(p/(p-1)) \subset \mathrm{L}(1/(p-1)) \\ \alpha_0 \mathrm{L}(1/(p-1)) \subset \mathrm{L}(p/(p-1)) \end{cases} \tag{4.38}$$

and since $\psi_p \circ \Phi_p=1$, we conclude trivially that

$$\alpha_0 \circ \beta_0=1. \tag{4.39}$$

Since τ^a leaves $\hat{\mathrm{F}}$ invariant, the definitions show that

$$\alpha=\alpha_0^a \circ \tau^a=\tau^a \circ \alpha_0^a. \tag{4.40}$$

Equations (4.38) and (4.39) give

$$\mathrm{L}(p/(p-1))=\alpha_0 \beta_0 \mathrm{L}(p/(p-1)) \subset \alpha_0 \mathrm{L}(1/(p-1)) \subset \mathrm{L}(p/(p-1))$$

which shows that

$$\alpha_0 \mathrm{L}(1/(p-1))=\mathrm{L}(p/(p-1)). \tag{4.41}$$

Furthermore, the definitions show that for $i=0, 1, \ldots, n+1$

$$\alpha_0 \circ \mathrm{D}_i=p \mathrm{D}_i \circ \alpha_0. \tag{4.42}$$

Lemma 3.6 shows that $L(1/(p-1)) = V + \sum_{i=1}^{n+1} D_i L(1/(p-1))$; applying α_0 to both sides of this relation and applying (4.41) and (4.42) we find

$$L(p/(p-1)) \subset \alpha_0 V + \sum_{i=1}^{n+1} D_i L(p/(p-1)). \tag{4.43}$$

$$\alpha_0\left(\sum_{i=1}^{n+1} D_i L(p/(p-1))\right) \subset \sum_{i=1}^{n+1} D_i L(p/(p-1)). \tag{4.44}$$

Since $V \subset L(p/(p-1))$, we may conclude that for $j = 0, 1, \ldots, a-1$

$$\alpha_0^j V \subset \alpha_0^{j+1} V + \sum_{i=1}^{n+1} D_i L(p/(p-1))$$

an elementary consequence of which is

$$V \subset \alpha_0^a V + \sum_{i=1}^{n+1} D_i L(p/(p-1)).$$

Since $L(p/(p-1))$ is stable under τ and V may be assumed to have been constructed so as to be stable under τ^a, equation (4.40) and this last relation give

$$V \subset \alpha V + \sum_{i=1}^{n+1} D_i L(p/(p-1)),$$

which is the form taken by (4.37) when $s = \infty$. This completes the proof of the theorem.

We have thus shown that if f is a regular polynomial then $(1-t)\Pi P_A(qt)$ (the product being over all non-empty subsets, A, of S) is a polynomial of degree d^n; and if s is chosen such that $q\varkappa > (p-1)^{-1}$ then this polynomial is simply the characteristic equation of $\bar{\alpha}$. Since $\varkappa = pa_{s+1}/q$, equation (4.2) shows that $q\varkappa$ certainly exceeds $(p-1)^{-1}$ if $s \geq 1$ (resp. $s \geq 3$) when $p > 2$ (resp. $p = 2$).

We now propose to investigate the factor $P_S(qt)$ under the restriction that the hypersurface is of odd degree if the characteristic is 2. To do this we now specialize s. If p divides d let $s = 1$. If p does not divide d let s be so large that $q\varkappa > 1/(p-1)$ (say $s = \infty$).

For each subset A of S, a ring homomorphism, $\mathfrak{J}_A$ of $\Omega_0[X_S]$ onto $\Omega_0[X_A]$ was defined in § 3. We now use the same symbol to denote the extension of this homomorphism to one of $\Omega_0\{X_0, X_S\}$ onto $\Omega_0\{X_0, X_A\}$ which is defined by $\mathfrak{J}_A(X_0) = X_0$.

For each subset, A (including the empty subset) of S and for each subset B of A and each real number b, let

$$L_A(b) = \mathfrak{J}_A L(b)$$
$$L_A^B(b) = \{\xi \in L_A(b) \text{ such that } M_B \text{ divides } \xi\}.$$

For $i \in A \cup \{0\}$, let $D_{i,A}$ be the mapping $\xi \to \mathfrak{J}_A D_i \xi$ of $\Omega_0\{X_0, X_A\}$ into itself. Let α_A be the mapping $\xi \to \mathfrak{J}_A(\alpha\xi)$ of $L_A(q\varkappa)$ into itself. Using an obvious analogue of equation 4.35, the subgroup $\sum_{i \in A} D_{i,A} L_A(q\varkappa)$ of $L_A(q\varkappa)$ is mapped into itself by α_A and hence by passage to quotients we define an endomorphism $\bar{\alpha}_A$ of the factor space $\mathfrak{W}_A = L_A(q\varkappa) \big/ \sum_{i \in A} D_{i,A} L_A(q\varkappa)$. (Thus in the notation of Theorem 4.2, $\mathfrak{W} = \mathfrak{W}_S$, $\bar{\alpha} = \bar{\alpha}_S$).

Now let $\mathfrak{W}_A^A$ be the image in $\mathfrak{W}_A$ of $L_A^A(q\varkappa)$. We note that $L_A^A(q\varkappa)$ is mapped into itself by α_A and hence $\bar{\alpha}_A$ maps $\mathfrak{W}_A^A$ into itself. Let $\bar{\alpha}_A^A$ be the restriction of $\bar{\alpha}_A$ to $\mathfrak{W}_A^A$.

For the empty subset, $\emptyset$, of S, we have $\mathfrak{J}_A F = 1$, $L_\emptyset(q\varkappa) = \Omega_0$, $D_{0,\emptyset} L_\emptyset(q\varkappa) = \{0\}$, $\mathfrak{W}_\emptyset = \mathfrak{W}_\emptyset^\emptyset \approx \Omega_0$, $\alpha_\emptyset$ is the mapping $\xi \to \psi\xi$ of Ω_0 into itself. Clearly $\alpha_\emptyset$ operates as the identity mapping on Ω_0 and hence

$$\det(I - t\bar{\alpha}_\emptyset^\emptyset) = 1 - t.$$

Theorem **4.4.**

$$\det(I - t\bar{\alpha}) = \prod_A \det(I - t\bar{\alpha}_A^A),$$

the product being over all subsets, A, *of* S.

Proof. — Lemmas 3.11, 3.15, 3.17, 3.18 show that under the natural mapping of $L_A(q\varkappa)$ onto $\mathfrak{W}_A$, $\mathfrak{V}_A^A$ is mapped isomorphically onto $\mathfrak{W}_A^A$. The proof of lemma 3.17 shows that $\mathfrak{W} \approx \mathfrak{V} = \Sigma \mathfrak{V}_A^A$ and here the isomorphism is given by the natural map of $L_S(q\varkappa) = L(q\varkappa)$ onto $\mathfrak{W}$. For each subset A of S, let $\mathfrak{P}_A$ be a basis of $\mathfrak{V}_A^A$ and let $\mathfrak{P} = \cup \mathfrak{P}_A$. Lemma 3.13 shows that $\mathfrak{P}$ is a basis of $\mathfrak{V}$. We use this basis to construct a matrix corresponding to $\bar{\alpha}$. For each $\omega \in \mathfrak{P}$ we may write (by virtue of Lemmas 3.15 and 3.18)

$$\alpha(\omega) \in \sum_{\omega' \in \mathfrak{P}} \mathfrak{M}(\omega, \omega')\omega' + \sum_{i \in S} D_i L(q\varkappa), \tag{4.45}$$

where $\mathfrak{M}(\omega, \omega') \in \Omega_0$. It follows from Lemmas 3.11 and 3.17 that this relation uniquely determines $\mathfrak{M}(\omega, \omega')$. If M_A divides ω then $\alpha(\omega) \in L_S^A(q\varkappa)$ and hence by Lemmas 3.15, 3.18, $\sum_{\omega' \in \mathfrak{P}} \mathfrak{M}(\omega, \omega')\omega' \in \mathfrak{V}^A$, which shows that $\mathfrak{M}(\omega, \omega') = 0$ unless M_A divides ω'. We now order the elements of $\mathfrak{P}$ so that the elements of $\mathfrak{P}_A$ preceed those of $\mathfrak{P}_B$ if the number of elements in B exceed the number in A and such that for $A \neq B$ no element of $\mathfrak{P}_B$ lies between two elements of $\mathfrak{P}_A$. Let $\mathfrak{M}$ be the matrix indexed by $\mathfrak{P} \times \mathfrak{P}$ with general coefficient $\mathfrak{M}(\omega, \omega')$ and with the elements of $\mathfrak{P}$ ordered as indicated. Let $\mathfrak{M}_A$ be the submatrix obtained from $\mathfrak{M}$ by restricting (ω, ω') to $\mathfrak{P}_A \times \mathfrak{P}_A$. It is clear that $\mathfrak{M}_A$ is a square matrix, its diagonal lies along the diagonal of $\mathfrak{M}$ and the coefficients of $\mathfrak{M}$ lying below $\mathfrak{M}_A$ are zero since these coefficients are of type $\mathfrak{M}(\omega, \omega')$, where $\omega' \in \mathfrak{P}_A$ and ω is divisible by M_B for some B not contained by A. It now follows that

$$\det(I - t\mathfrak{M}) = \prod \det(I - t\mathfrak{M}_A), \tag{4.46}$$

the product being over all subsets, A, of S. It follows from (4.45) that $\det(I - t\bar{\alpha}) = \det(I - t\mathfrak{M})$. For $\omega \in \mathfrak{P}_A$ if we apply $\mathfrak{J}_A$ to both sides of equation (4.45) we obtain

$$\alpha_A^A(\omega) = \alpha_A(\omega) = \mathfrak{J}_A(\alpha\omega) \in \sum_{\omega' \in \mathfrak{P}_A} \mathfrak{M}(\omega, \omega')\omega' + \sum_{i \in A} D_{i,A} L_A(q\varkappa).$$

Since $\mathfrak{P}_A$ is a set of representatives of a basis of $\mathfrak{W}_A^A$, this shows that for each subset A

$$\det(I - t\mathfrak{M}_A) = \det(I - t\bar{\alpha}_A^A).$$

The theorem now follows from (4.46).

Corollary.

$$\mathrm{P}_{\mathrm{S}}(qt) = \det(\mathrm{I}-t\bar{\alpha}^{\mathrm{S}}_{\mathrm{S}})$$
$$\deg \mathrm{P}_{\mathrm{S}} = d^{-1}\{(d-1)^{n+1}+(d-1)(-1)^{n+1}\}$$

Proof. — Theorem 4.2, equation 4.33 and Theorem 4.4 show that for each non-empty subset B, of S,

$$\prod_{\mathrm{A}} \det(\mathrm{I}-t\bar{\alpha}^{\mathrm{A}}_{\mathrm{A}}) = \prod_{\mathrm{A}} \mathrm{P}_{\mathrm{A}}(qt)$$

the products being over all non-empty subsets A of B. This system of relations can be solved for $\mathrm{P}_{\mathrm{A}}(qt)/\det(\mathrm{I}-t\bar{\alpha}^{\mathrm{A}}_{\mathrm{A}})$ by means of equation (4.32). This gives the first assertion of the corollary. The assertion concerning the degree follows from the computation of $\dim \mathfrak{B}^{\mathrm{S}}_{\mathrm{S}}$ (Lemma 3.19) and the proof (Theorem 4.4) that $\bar{\alpha}$ (and hence $\bar{\alpha}^{\mathrm{S}}_{\mathrm{S}}$) is non-singular.

c) Let k (as previously) be the field of q elements and let us extend the notion of regularity (in the obvious way) to polynomials in $k[\mathrm{X}_1, \ldots, \mathrm{X}_{n+1}]$. We have verified a part of the Weil hypothesis for a non-singular hypersurface, $\mathfrak{H}$, in $\mathfrak{S}_n$ defined over k provided d is odd if $p=2$ and provided the defining polynomial $\bar{f} \in k[\mathrm{X}]$ of $\mathfrak{H}$ is regular. ($\bar{f}$ = image for f under the residue class map). We now consider the situation in which $\bar{f}$ is not necessarily regular. Let $\mathrm{A}=(a_{ij})$ be an $(n+1)\times(n+1)$ matrix whose coefficients are algebraically independent over $k[\mathrm{X}_1, \ldots, \mathrm{X}_{n+1}]$. We consider the coordinate transformation

$$\mathrm{X}_i = \sum_{j=1}^{n+1} a_{ij}\mathrm{Y}_j, \quad j=1, 2, \ldots, n+1$$

and consider $\bar{f}$ as a polynomial in $\mathrm{Y}_1, \ldots, \mathrm{Y}_{n+1}$ with coefficients in $k(a_{11}, \ldots, a_{n+1,n+1})$. We easily compute

$$\bar{f}'_i = \mathrm{Y}_i\frac{\partial\bar{f}}{\partial\mathrm{Y}_i} = \sum_{j,l=1}^{n+1} \mathrm{X}_l\frac{\partial\bar{f}}{\partial\mathrm{X}_j}a_{ji}\mathrm{A}_{li}/\det\mathrm{A}$$

where A_{ij} is the cofactor of a_{ij} in A. Our problem is to specialize the matrix A subject to the conditions

(1) $$\det\mathrm{A} \neq 0$$

(2) $\bar{f}$, $(\det\mathrm{A})\bar{f}'_i$, $\ldots$, $(\det\mathrm{A})\bar{f}'_{n+1}$ have no common zero in $\mathfrak{S}_n$.

Let U be the set of all A with coefficients in the algebraic closure of k which fail to satisfy these conditions, i.e. U is the set of all A such that either $\det\mathrm{A}=0$ or $\bar{f}$, $(\det\mathrm{A})\bar{f}'_1$, $\ldots$, $(\det\mathrm{A})\bar{f}'_{n+1}$ have a common zero in $\mathfrak{S}_n$. It follows from elimination theory that U is an algebraic variety in $\mathfrak{S}_m$, where $m=(n+1)^2-1$. On the other hand it is known ([7], Chap. VIII, prop. 13) that the generic hyperplane section of a non-singular variety is non-singular and therefore $\mathrm{U}\neq\mathfrak{S}_m$. Hence the dimension of U is at most $m-1$ (and hence must in fact be $m-1$). Thus if k_r is the field of q^r elements, the number of points of U rational over k_r is no greater than $b(q^{r(m-1)}-1)/(q^r-1)$ for some fixed real number b. On the other hand there are $(q^{rm}-1)/(q^r-1)$ points in $\mathfrak{S}_m$ rational over k_r. Thus there exists an integer r_0 such that for each $r>r_0$, there exists a point of $\mathfrak{S}_m$ rational over k_r but not in U. This means that for each $r>r_0$ there exists a coordinate transformation rational over k_r such that $\mathfrak{H}$ is defined by a regular polynomial with respect to the new coordinates. For each integer r, let ζ_r be the zeta function of $\mathfrak{H}$ as hypersurface over k_r and let P_r be the rational function defined by

$$\mathrm{P}_r(t)^{(-1)^n} = \zeta_r(t) \prod_{i=0}^{n-1} (1-q^{ri}t).$$

It follows that for each $r\in\mathbf{Z}$, $r\geq 1$

$$\mathrm{P}_r(t) = \prod_{\nu^r=1} \mathrm{P}_1(\nu t^{1/r}),$$

the product being over all r^{th} roots of unity, ν. Furthermore if $r>r_0$ then P_r is a polynomial of a certain predicted degree m'. If P_1 is a polynomial then clearly it must also be of degree m', and hence to complete our treatment of P_1 it is enough to show that P_1 is a polynomial. Since P_1 is a power series with constant term 1, we may write

$$\mathrm{P}_1(t) = \prod_{i=1}^{c} (1-b_i t) \Big/ \prod_{i=1}^{c'} (1-b'_i t)$$

where the b'_i are distinct from the b_i. Consider b'_1. If P_r is a polynomial then there must be an r^{th} root of unity, ν, such that $b'_1\nu = b_i$ for some integer i, $1\leq i\leq c$. Let r run through $c+1$ distinct primes each greater than r_0. By the pigeon hole principle there exists one integer i such that $b'_1\nu' = b_i = b'_1\nu''$, where ν' (resp. ν'') is a p'-th (resp. p''-th) root of unity, p', p'' being distinct prime numbers. It is clear that $\nu'=\nu''=1$ and $b'_1=b_i$, contrary to hypothesis.

It is now clear that for the treatment of a non-singular hypersurface, the hypothesis that the defining polynomial is regular is no essential restriction.

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Reçu le 15 août 1961.

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