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MULTI-DIMENSIONAL RIEMANN PROBLEMS FOR LINEAR HYPERBOLIC SYSTEMS (*)

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Abstract. — *In this paper we prove that each homogeneous linear first order hyperbolic system of two unknowns in N space dimensions with constant coefficients can be reduced to one of three canonical systems. Then we give the explicit solution of the multi-dimensional Riemann problem associated with the most interesting canonical system on a structured or unstructured mesh.*

Résumé. — *Nous montrons d'abord que tout système hyperbolique linéaire 2 × 2 à coefficients constants en N dimensions d'espace peut être réduit à un parmi trois systèmes canoniques. Nous donnons ensuite la solution explicite du problème de Riemann multi-dimensionnel associé au système canonique le plus intéressant, pour un maillage structuré ou non structuré.*

1. INTRODUCTION

Let $(A_i)_{1 \leqslant i \leqslant N}$ be 2×2 matrices with constant coefficients; we are interested in solving the Riemann Problem for a linear system of partial differential equations

$$\forall (x, t) = (x_1, \dots, x_N, t) \in \mathbb{R}^N \times \mathbb{R}^+ , \quad \partial_t W(x, t) + \sum_{i=1}^{N} A_i \, \partial_{x_i} W(x, t) = 0 , \tag{1.1}$$

$$W(x, t) = \begin{pmatrix} u \\ v \end{pmatrix} (x, t) \in \mathbb{R}^2 , \quad A_i \in \mathbb{R}^{2 \times 2} , \quad 1 \leqslant i \leqslant N \tag{1.2}$$

$$\forall x = (x_1, \dots, x_N) \in \mathbb{R}^N , \quad W(x, t = 0) = W_0(x) \tag{1.3}$$

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where $W_0(x)$ is a simple piecewise constant vector valued function.

The system is supposed to be hyperbolic according to [1] in the time direction for any N-uple $\alpha = (\alpha_1, ..., \alpha_N) \in S^N$ (the unit sphere in $(\mathbb{R}^N)$, which means that for all α, the matrix $\sum_{i=1}^{N} \alpha_i A_i$ is diagonalizable with real eigenvalues.

2. THREE CANONICAL 2 × 2 LINEAR HYPERBOLIC SYSTEMS

In [3], a particular case of the result pointed out by Lax shows that system (1.1) cannot be strictly hyperbolic in the time direction for all $\alpha = (\alpha_1, ..., \alpha_N) \in S^N$, as soon as $N \geqslant 3$, more precisely we have :

THEOREM 2.1 : *If system (1.1) is hyperbolic in the time direction for all α, then :*

— *either system (1.1) is not strictly hyperbolic in the time direction for any α and it can be reduced to :*

$$\forall (x, t) \in \mathbb{R}^N \times \mathbb{R}^+ , \quad \partial_t W(t, x) = 0 \tag{2.1}$$

— *either system (1.1) is strictly hyperbolic in the time direction for some α ' s generating a two-dimensional subspace of $\mathbb{R}^N$ and it can be reduced to the canonical system*

$$\forall (x, t) = (x_1, x_2, t) \in \mathbb{R}^2 \times \mathbb{R}^+ ,$$

$$\partial_t W(x, t) + A \partial_{x_1} W(x, t) + B \partial_{x_2} W(x, t) = 0$$

$$\text{with } A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad x = (x_1, x_2) \in \mathbb{R}^2 \tag{2.2}$$

— *or system (1.1) can be reduced to*

$$\forall (x, t) \in \mathbb{R} \times \mathbb{R}^+ , \quad \partial_t W(x, t) + A \partial_x W(x, t) = 0 \text{ with } A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.3}$$

In order to prove the theorem, we define three types of reversible actions on the class of hyperbolic systems (1.1) which preserve hyperbolicity in the time direction for all $\alpha \in S^N$.

• **A : Galilean translations.** To each $(x_i)_{1 \leqslant i \leqslant N}$ corresponds $(\tilde{x}_i)_{1 \leqslant i \leqslant N} := (x_i - t\lambda_i)$, this galilean translation moves W into $\tilde{W}$ defined by :

$$\forall((x_i), t) \in \mathbb{R}^N \times \mathbb{R}^+ , \quad \tilde{W}((\tilde{x}_i), t) = W((\tilde{x}_i + t\lambda_i), t) .$$

If $\tilde{A}_i := A_i - \lambda_i Id$, the system (1.1) becomes (2.4) and remains hyperbolic in the time direction since the matrix $\sum_{i=1}^{N} \alpha_i \tilde{A}_i = \sum_{i=1}^{N} \alpha_i A_i + \beta Id$ (with $\beta \in \mathbb{R}$) is diagonalizable over the real field whenever $\sum_{i=1}^{N} \alpha_i A_i$ is.

$$\partial_t \tilde{W} + \sum_{i=1}^{N} \tilde{A}_i \, \partial_{\tilde{x}_i} \tilde{W} = 0 \quad \text{with} \quad \operatorname{Tr}(\tilde{A}_i) = \operatorname{Tr}(A_i) - \lambda_i \tag{2.4}$$

• **B : Linear one-to-one change of space variables.** Let P be a $N \times N$ invertible matrix with constant coefficients. Defining $\tilde{x} := Px$ and $\tilde{W}(\tilde{x}, t) := W(P^{-1}\tilde{x}, t)$ leads to :

$$\partial_t \tilde{W} + \sum_{j=1}^{N} \tilde{A}_j \, \partial_{\tilde{x}_j} \tilde{W} = 0 \quad \text{with} \quad \tilde{A}_j = \sum_{i=1}^{N} P_{ji} A_i . \tag{2.5}$$

If system (1.1) is hyperbolic in the time direction, the same holds for system (2.5) because of the following :

$$\tilde{A}_j = \sum_{i=1}^{N} P_{ji} A_i \Leftrightarrow A_i = \sum_{j=1}^{N} (P^{-1})_{ij} \tilde{A}_j$$

$$\sum_{i=1}^{N} \alpha_i A_i = \sum_{i=1}^{N} \alpha_i \sum_{j=1}^{N} (P^{-1})_{ij} \tilde{A}_j = \sum_{j=1}^{N} \left(\sum_{i=1}^{N} \alpha_i (P^{-1})_{ij} \right) \tilde{A}_j = \sum_{j=1}^{N} \beta_j \tilde{A}_j .$$

• **C : Linear one-to-one change of unknowns.** Let Q be a $N \times N$ invertible matrix with constant coefficients. The new unknowns $\tilde{W} := QW$ satisfy :

$$Q^{-1} \, \partial_t \tilde{W} + \sum_{i=1}^{N} A_i Q^{-1} \, \partial_{x_i} \tilde{W} = 0$$

multiplying by Q on the left, one obtains :

$$\partial_t \tilde{W} + \sum_{i=1}^{N} \tilde{A}_i \, \partial_{x_i} \tilde{W} = 0 \quad \text{with} \quad \tilde{A}_i = Q A_i Q^{-1} . \tag{2.6}$$

System (2.6) still remains hyperbolic in the time direction because two conjugate matrices are diagonalizable over $\mathbb{R}$ at the same time and

$$\sum_{i=1}^{N} \alpha_i A_i = \sum_{i=1}^{N} \alpha_i Q^{-1} \tilde{A}_i Q = Q^{-1} \left(\sum_{i=1}^{N} \alpha_i \tilde{A}_i \right) Q .$$

The proof of Theorem 2.1 is carried out in eight steps.

Step 1. One performs a galilean translation (action of type A) in order to set the trace of all the coefficient matrices to 0.

$$\partial_t W + \sum_{i=1}^{N} A_i \, \partial_{x_i} W = 0 \quad \text{with} \quad \text{Tr}\,(A_i) = 0 . \tag{2.7}$$

Step 2. Then, either all matrices A_i are null and the reduction is achieved (we have reached (2.1)), or at least one of them, A_{i_0}, is not null. As system (2.7) is strictly hyperbolic in the coordinate $\alpha = (0, \dots, \alpha_{i_0} = 1, \dots, 0) \in S^N$, A_{i_0} must have two real eigenvalues with opposit signs (because $(\text{Tr}\,(A_{i_0}) = 0)$. Re-ordering coordinates with an action of type B exchanges x_{i_0} and x_1, so one can suppose that A_1 is diagonizable with real eigenvalues and $\text{Tr}\,(A_1) = 0$.

Step 3. An action of type C moves A_1 into a diagonalized form : there exists a 2×2 invertible matrix Q such that $QAQ^{-1} = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}$. Then an action of type B introduces a new space coordinate $\tilde{x}_1 := x_1 / \lambda$ which leads to an equivalent hyperbolic system with :

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \quad \forall 2 \leqslant i \leqslant N , \quad A_i = \begin{pmatrix} \gamma_i & \mu_i \\ \beta_i & -\gamma_i \end{pmatrix} . \tag{2.8}$$

The aim of the next steps is to reduce the other matrices A_2 to A_N.

Step 4. A new action of type B changes x_i into $x_i - \gamma_i x_1$ for all $2 \leqslant i \leqslant N$, the first coordinate x_1 remaining unchanged, the new equivalent system satisfies

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \forall 2 \leqslant i \leqslant N, \quad A_i = \begin{pmatrix} 0 & \mu_i \\ \beta_i & 0 \end{pmatrix}. \tag{2.9}$$

Step 5. The spatial coordinates are ordered (action de type B) to obtain :

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \forall 1 < i \leqslant k \leqslant N, A_i = \begin{pmatrix} 0 & \mu_i \\ \beta_i & 0 \end{pmatrix},$$

$$\forall k < j \leqslant N, A_j = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

If $k = 1$ the reduction of (1.1) is achieved (we have reached (2.3)).

Step 6. If $k > 1$, one performs the change of space coordinates (action of type B) defined by :

$$\forall 1 < i \leqslant k \leqslant N, \quad x_i := \frac{x_i}{\sqrt{\mu_i \beta_i}}$$

which is well defined since $\mu_i \beta_i > 0$ as the system is hyperbolic in the time direction for $\alpha = (0, \ldots, \alpha_i = 1, \ldots, 0)$; one gets

$$\forall 1 < i \leqslant k \leqslant N, \quad A_i = \begin{pmatrix} 0 & \delta_i \\ \frac{1}{\delta_i} & 0 \end{pmatrix}, \quad \delta_i > 0 \tag{2.10}$$

a system with coefficients such that (2.10) holds remains hyperbolic in the time direction, thus, for each $2 < i \leqslant k \leqslant N$, the matrices $\delta_2 A_2 - \delta_i A_i$ need to be diagonalizable which cannot be realized unless $\delta_2 = \delta_i$ or equivalently $A_2 = A_i$ as we can see :

$$\forall 2 < i \leqslant k \leqslant N, \quad \delta_2 A_2 - \delta_i A_i = \begin{pmatrix} 0 & \delta_2^2 - \delta_i^2 \\ 0 & 0 \end{pmatrix}.$$

Step 7. The change of unknowns defined by $Q = \begin{pmatrix} 1 & 0 \\ 0 & \delta_2 \end{pmatrix}$ gives

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \forall 1 < i \leqslant k, A_i = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\forall k < j \leqslant N, A_j = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{2.11}$$

Step 8. A last change of coordinates moves $x_i - x_2$ for all $2 < i \leqslant k$ and completes the proof.

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad A_2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \forall 3 \leqslant i \leqslant N, \quad A_i = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{2.12}$$

3. ANALYTIC SOLUTION OF THE MULTI-DIMENSIONAL RIEMANN PROBLEM FOR THE CANONICAL LINEAR SYSTEM

We look for the solution of the multi-dimensional Riemann problem for the three canonical systems pointed out by the previous theorem. For systems (2.1) and (2.3), one can easily exhibit the exact solution of any Cauchy problem as linear combinations of plane waves with various propagation speed, so we focus on the interesting case (3.1).

$$\forall (x, y, t) \in \mathbb{R}^2 \times \mathbb{R}^+ \quad \partial_t W(x, y, t) + A\, \partial_x W(x, y, t) + B\, \partial_y W(x, y, t) = 0$$

$$\text{with} \quad A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad W = \begin{pmatrix} u \\ v \end{pmatrix}. \tag{3.1}$$

The two-dimensional elementary Riemann data are :

$$W_0(x, y) = W^0_{++}\, H(x)\, H(y) + W^0_{-+}\, H(-x)\, H(y) +$$

$$+ W^0_{--}\, H(-x)\, H(-y) + W^0_{+-}\, H(x)\, H(-y) \tag{3.2}$$

where $W^0_{\pm\pm}$ refer to four constant vectors of $\mathbb{R}^2$ and $H(\,.\,)$ refers to the Heaviside function.

THEOREM 3.1 : *When $\mathbb{R}^2 \times \mathbb{R}^+$ is decomposed into the six domains $(D^i)_{1 \leqslant i \leqslant 6}$ which sections at time $t > 0$ are given on figure 1, the solution of the multi-dimensional Riemann Problem $(3.1) + (3.2)$ is constant on every $(D^i)_{2 \leqslant i \leqslant 6}$, regular in D^1 (inside a light cone), its values are given in table 1.*

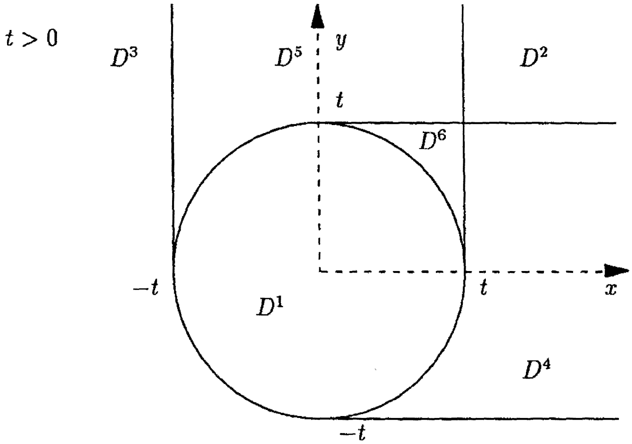

Figure 1. — Plane decomposition associated to the solution of the Riemann problem (3.1) + (3.3).

Table 1. — Explicit formulae for u and v on a structured mesh.

Domain	$u(x,y,t) =$	$v(x,y,t) =$
D^1	$\frac{u^0}{2\pi}\cos^{-1}\frac{x\,y}{\sqrt{t^2-x^2}\sqrt{t^2-y^2}} - \frac{v^0}{2\pi}\cos^{-1}\frac{-x}{\sqrt{t^2-y^2}} - \frac{u^0}{2\pi}\cos^{-1}\frac{-y}{\sqrt{t^2-x^2}}$	$\frac{v^0}{2\pi}\cos^{-1}\frac{x\,y}{\sqrt{t^2-x^2}\sqrt{t^2-y^2}} - \frac{u^0}{2\pi}\cos^{-1}\frac{-x}{\sqrt{t^2-y^2}} + \frac{v^0}{2\pi}\cos^{-1}\frac{-y}{\sqrt{t^2-x^2}}$
D^2	u^0	v^0
D^3	0	0
D^4	$\frac{u^0-v^0}{2}$	$\frac{v^0-u^0}{2}$
D^5	0	v^0
D^6	$-\frac{u^0+v^0}{2}$	$\frac{v^0-u^0}{2}$

According to [4] and [5], system (3.1) is well posed in all $H^s_{\mathrm{loc}}(\mathbb{R}^2)$, ($s \in \mathbb{R}$) because it is a linear symmetric system with constant coefficients. Given initial data W_0 in $H^s_{\mathrm{loc}}(\mathbb{R}^2)$, the solution $W(x, y, .)$ belongs to $\mathscr{C}^k(\mathbb{R}, H^{s-k}_{\mathrm{loc}}(\mathbb{R}^2))$ for each nonnegative integer k. As the initial data W_0 is in $H^0_{\mathrm{loc}}(\mathbb{R}^2)$, there exists a unique solution $W(x, y, .)$ of (3.1) + (3.2) in $\mathscr{C}^0(\mathbb{R}, H^0_{\mathrm{loc}}(\mathbb{R}^2))$.

Moreover it is well known that the solution of (3.1) + (3.2) being self-similar is a function of $\frac{x}{t}, \frac{y}{t}$ and that it depends linearly on the constants $W^0_{\pm\pm}$. Considering linear combinations of plane waves solutions, one can reduce the general Riemann problem to the simpler case where $W^0_{+-} = W^0_{--} = W^0_{-+} = 0$, in other words

$$W_0(x, y) = W^0 H(x) H(y), \quad W^0 \in \mathbb{R}^2. \tag{3.3}$$

It is easy to verify that each component of a smooth solution $W = {}^T(u, v)$ of (3.1) + (3.2) belongs to $\mathscr{C}^\infty(\mathbb{R}^+ \times \mathbb{R}^2)$ and is a solution of the d'Alembert equation

$$\partial_{tt} w - \Delta w = 0. \tag{3.4}$$

We take two sets of initial data for these d'Alembert equations

$$\begin{cases} u_0(x, y) & = u(x, y, t = 0) \\ u_{0t}(x, y) & := \partial_t u(x, y, t = 0) = -\partial_x u_0(x, y) - \partial_y v_0(x, y) \\ & = -u^0 \delta_{|x=0} H(y) - v^0 H(x) \delta_{|y=0} \end{cases} \tag{3.5}$$

the previous one for u and the next one for v :

$$\begin{cases} v_0(x, y) & = v(x, y, t = 0) \\ v_{0t}(x, y) & := \partial_t v(x, y, t = 0) = -\partial_y u_0(x, y) + \partial_x v_0(x, y) \\ & = v^0 \delta_{|x=0} H(y) - u^0 H(x) \delta_{|y=0}. \end{cases} \tag{3.6}$$

Thus the Kirchhof formula (3.9) is used to get explicit formulae that will be proven to define the unique solution of (3.1) + (3.2). They rely on two auxiliary functions $F_w(x, y, t)$ and $G_w(x, y, t)$ ($w \in \{u, v\}$) defined in the sense of distribution by (3.7) and (3.8) :

$$F_w(x, y, t) := \frac{1}{2\pi} \frac{H(t^2 - x^2 - y^2)}{\sqrt{t^2 - x^2 - y^2}} \underset{(x,y)}{*} w_{0t}(x, y) \tag{3.7}$$

$$G_w(x, y, t) := \frac{1}{2\pi} \frac{H(t^2 - x^2 - y^2)}{\sqrt{t^2 - x^2 - y^2}} \underset{(x,y)}{*} w_0(x, y) \tag{3.8}$$

$$\forall (x, y, t) \in \mathbb{R}^2 \times \mathbb{R}^+, \quad w(x, y, t) = F_w(x, y, t) + \partial_t G_w(x, y, t). \tag{3.9}$$

We now evaluate F_w and G_w in the domains denoted $(D^i)_{1 \leqslant i \leqslant 6}$ (*fig.* 1) suggested by elementary considerations of finite propagation speed. For each domain, the results are given for both $u(x, y, t)$ and $v(x, y, t)$ but we only carry out the calculations for the first one.

- $D^1 := \{(x, y, t) \in \mathbb{R}^2 \times \mathbb{R}^+ \quad \text{such} \quad \text{that} \quad x^2 + y^2 \leqslant t^2\}$.

Inside D^1, we have :

$$F_u(x, y, t) = \frac{-u^0}{2\pi} \int_{\substack{\eta \leqslant y \\ x^2 + \eta^2 \leqslant t^2}} \frac{d\eta}{\sqrt{t^2 - x^2 - \eta^2}} + \frac{-v^0}{2\pi} \int_{\substack{\xi \leqslant x \\ \xi^2 + y^2 \leqslant t^2}} \frac{d\xi}{\sqrt{t^2 - \xi^2 - y^2}}$$

$$G_u(x, y, t) = \frac{u^0}{2\pi} \iint_{\begin{cases} \xi \leqslant x, \eta \leqslant y \\ \xi^2 + \eta^2 \leqslant t^2 \end{cases}} \frac{d\xi \, d\eta}{\sqrt{t^2 - \xi^2 - \eta^2}}$$

and similarly for $v(x, y, t)$. Explicit formulae in terms of elementary functions follow from a simple calculations for F_u and the use of Gauss-Bonnet theorem (3.11) after a change in space variables for G_u. We can express the solution in terms of Φ and Ψ defined below :

$$\forall a, b \text{ such that } a^2 + b^2 \leqslant 1 , \quad \begin{cases} \Phi(a, b) := \cos^{-1} \dfrac{-b}{\sqrt{1 - a^2}} \\ \Psi(a, b) := \cos^{-1} \dfrac{ab}{\sqrt{1 - b^2} \sqrt{1 - a^2}} \end{cases} \tag{3.10}$$

$$\iint_{\begin{cases} u \geqslant a, v \geqslant b \\ u^2 + v^2 \leqslant 1 \end{cases}} \frac{du \, dv}{\sqrt{1 - u^2 - v^2}} = \Psi(a, b) + a\Phi(a, b) + b\Phi(b, a) - \pi(a + b)$$

$$\forall (a, b) \in (-1, 1)^2 \tag{3.11}$$

one gets finally

$$F_u(x, y, t) = -\frac{u^0}{2\pi} \Phi\left(\frac{x}{t}, \frac{y}{t}\right) - \frac{v^0}{2\pi} \Phi\left(\frac{y}{t}, \frac{x}{t}\right), \quad \partial_t G_u(x, y, t) = \frac{u^0}{2\pi} \Psi\left(\frac{x}{t}, \frac{y}{t}\right)$$

and the solutions are :

$$\forall (x, y, t) \in D^1, \begin{cases} u(x, y, t) = \dfrac{u^0}{2\pi} \Psi\left(\dfrac{x}{t}, \dfrac{y}{t}\right) - \dfrac{u^0}{2\pi} \Phi\left(\dfrac{x}{t}, \dfrac{y}{t}\right) - \dfrac{v^0}{2\pi} \Phi\left(\dfrac{y}{t}, \dfrac{x}{t}\right) \\ v(x, y, t) = \dfrac{v^0}{2\pi} \Psi\left(\dfrac{x}{t}, \dfrac{y}{t}\right) + \dfrac{v^0}{2\pi} \Phi\left(\dfrac{x}{t}, \dfrac{y}{t}\right) - \dfrac{u^0}{2\pi} \Phi\left(\dfrac{y}{t}, \dfrac{x}{t}\right). \end{cases} \tag{3.12}$$

One can easily verify that $u(x, y, t)$ and $v(x, y, t)$ belong to $\mathscr{C}^{\infty}(D^1)$ and are classical smooth solutions of (3.1) inside this light cone.

- $D^2 := \{(x, y, t) \in \mathbb{R}^2 \times \mathbb{R}^+$ such that $x \geqslant t$ and $y \geqslant t\}$.
Inside D^2, both functions F_u and F_v are identically 0 and

$$G_u(x, y, t) = \frac{u^0}{2\pi} \iint_{\xi^2 + \eta^2 \leqslant t^2} \frac{d\xi \, d\eta}{\sqrt{t^2 - \xi^2 - \eta^2}} = u^0 t .$$

The same computation holds for $G_v(x, y, t) = v^0 t$ thus :

$$\forall (x, y, t) \in D^2 , \quad u(x, y, t) \equiv u^0 \quad \text{and} \quad v(x, y, t) \equiv v^0 . \tag{3.13}$$

- $D^3 := \{x^2 + y^2 \geqslant t^2, x \leqslant 0, y \leqslant 0\} \cup \{x \leqslant -t\} \cup \{y \leqslant -t\}$.

Inside D^3, all of the functions F_u, F_v, G_u and G_v are identically 0 and

$$\forall (x, y, t) \in D^3 , \quad u(x, y, t) = v(x, y, t) \equiv 0 . \tag{3.14}$$

- $D^4 := \{x^2 + y^2 \geqslant t^2, 0 \leqslant x \leqslant t, -t \leqslant y \leqslant 0\} \cup \{x \geqslant t, |y| \leqslant t\}$.

Inside D^4 we have :

$$F_u(x, y, t) = \frac{-u^0}{2\pi} \int_{\substack{\eta \leqslant y \\ x^2 + \eta^2 \leqslant t^2}} \frac{d\eta}{\sqrt{t^2 - x^2 - \eta^2}}$$

$$- \frac{v^0}{2\pi} \int_{\substack{\xi \leqslant x \\ \xi^2 + y^2 \leqslant t^2}} \frac{d\xi}{\sqrt{t^2 - \xi^2 - y^2}} = \frac{-v^0}{2}$$

$$G_u(x, y, t) = \frac{u^0}{2\pi} \int_{\eta = -t}^{\eta = y} \left(\int_{\xi = -\sqrt{t^2 - \eta^2}}^{\xi = \sqrt{t^2 - \eta^2}} \frac{d\xi}{\sqrt{t^2 - \xi^2 - \eta^2}} \right) d\eta = \frac{u^0}{2} (y + t) .$$

Similar calculations give $F_v(x, y, t) = \frac{-u^0}{2}$ and $G_v(x, y, t) = \frac{v^0}{2}(y + t)$, so

$$\forall (x, y, t) \in D^4 , \quad u(x, y, t) \equiv \frac{u^0 - v^0}{2} \quad \text{and} \quad v(x, y, t) \equiv \frac{v^0 - u^0}{2} . \tag{3.15}$$

- $D^5 := \{x^2 + y^2 \geq t^2, -t \leq x \leq 0, 0 \leq y \leq t\} \cup \{|x| \leq t, y \geq t\}$.

This Domain D^5 is treated in the same way as D^4, one has :

$$F_u(x, y, t) = \frac{-u^0}{2}, \quad G_u(x, y, t) = \frac{u^0}{2}(x + t),$$

$$F_v(x, y, t) = \frac{v^0}{2}, \quad G_v(x, y, t) = \frac{v^0}{2}(x + t)$$

$$\forall (x, y, t) \in D^5, \quad u(x, y, t) \equiv 0 \quad \text{and} \quad v(x, y, t) \equiv v^0. \tag{3.16}$$

- $D^6 := \{(x, y, t) \in \mathbb{R}^2 \times \mathbb{R}^+, \; x^2 + y^2 \geq t^2, \; 0 \leq x \leq t \text{ and } 0 \leq y \leq t\}$.

Inside D^6, we obtain :

$$F_u(x, y, t) = \frac{-u^0}{2\pi} \int_{\substack{\eta \leq y \\ x^2 + \eta^2 \leq t^2}} \frac{d\eta}{\sqrt{t^2 - x^2 - \eta^2}}$$

$$- \frac{v^0}{2\pi} \int_{\substack{\xi \leq x \\ \xi^2 + y^2 \leq t^2}} \frac{d\xi}{\sqrt{t^2 - \xi^2 - y^2}} = \frac{-(u^0 + v^0)}{2}$$

$$G_u(x, y, t) = \frac{u^0}{2\pi} \int_{\xi = -t}^{\xi = x} \left(\int_{\eta = -\sqrt{t^2 - \xi^2}}^{\eta = \sqrt{t^2 - \xi^2}} \frac{d\eta}{\sqrt{t^2 - \xi^2 - \eta^2}} \right) d\xi$$

$$- \frac{u^0}{2\pi} \int_{\eta = y}^{\eta = t} \left(\int_{\xi = -\sqrt{t^2 - \eta^2}}^{\xi = \sqrt{t^2 - \eta^2}} \frac{d\xi}{\sqrt{t^2 - \xi^2 - \eta^2}} \right) d\eta = \frac{u^0}{2}(x + y).$$

For v, we get $F_v(x, y, t) = \frac{v^0 - u^0}{2}$ and $G_v(x, y, t) = \frac{v^0(x + y)}{2}$, which finally gives

$$\forall (x, y, t) \in D^6, \quad u(x, y, t) \equiv -\frac{u^0 + v^0}{2} \quad \text{and} \quad v(x, y, t) \equiv \frac{v^0 - u^0}{2}. \tag{3.17}$$

We have to verify that $W = {}^T(u, v)$, defined in $\mathbb{R}^2 \times \mathbb{R}^+$ with initial data (3.2), is a weak solution of (3.1). $W(x, y, t)$ must satisfy :

$$\begin{gathered} \forall \Phi \in \mathscr{C}_0^1(\mathbb{R}^2 \times \mathbb{R}^+), \\ \langle W, \partial_t \Phi + A\, \partial_x \Phi + B\, \partial_y \Phi \rangle = \langle W(x, y, 0), \Phi(x, y, 0) \rangle. \end{gathered} \tag{3.18}$$

Define $Q = (-R, R)^2 \times [0, T)$ which contains the support of Φ. Let $D^i := D^i \cap Q$ and let $\Gamma^{i,j} = \bar{D}^i \cap \bar{D}^j$ be the boundary separating D^i and D^j. We consider $\mathscr{I}$ the set of all (i, j) such that $\Gamma^{i,j}$ has codim 1 : for $(i, j) \in \mathscr{I}$, $\overrightarrow{n}_i^j$ is the normal to $\Gamma^{i,j}$ from D^i to D^j ($\overrightarrow{n}_i^j = -\overrightarrow{n}_j^i$). Let

$T^i W$ be the trace of $W_{|D^i}$ on the boundary ∂D^i and $[W]_i^j := T_j W_{|\Gamma^{i,j}} - T_i W_{|\Gamma^{i,j}}$ be the jump of W across $\Gamma^{i,j}$. We finally set $\Gamma^i := \bar{D}^i \cap \partial Q$ when it is not empty and denote by $\vec{n}_i$ the exterior normal to Γ^i.

For each test function Φ we integrate by parts the left hand side of (3.18) separately on each D^i :

$$\iiint_Q {}^T W(\partial_t \Phi + A\,\partial_x \Phi + B\,\partial_y \Phi)\,dx\,dy\,dt =$$

$$\sum_i \iiint_{D^i} {}^T\Phi(\partial_t W + A\,\partial_x W + B\,\partial_y W)\,dx\,dy\,dt$$

$$- \sum_{(i,j)\in \mathcal{J}} \iint_{\Gamma^{i,j}} {}^T[W]_i^j\,(\vec{n}_i^j \,.\, \vec{t}\,Id + \vec{n}_i^j \,.\, \vec{x}A + \vec{n}_i^j \,.\, \vec{y}B)\,\Phi\,d\gamma$$

$$- \iint_{\partial Q} {}^T W(\vec{n} \,.\, \vec{t}\,Id + \vec{n} \,.\, \vec{x}A + \vec{n} \,.\, \vec{y}B)\,\Phi\,d\gamma$$

where $(\vec{x}, \vec{y}, \vec{t})$ is the canonical basis of $\mathbb{R}^2 \times \mathbb{R}$. One has

$$\forall i, \quad \iiint_{D_i} {}^T\Phi(\partial_t W + A\,\partial_x W + B\,\partial_y W)\,dx\,dy\,dt = 0$$

since for all i the exact solution W belongs to $(\mathscr{C}^1(D^i))^2$ and verifies

$$\partial_t W + A\,\partial_x W + B\,\partial_y W = 0$$

in the classical sense inside D^i.

Now, we indicate how to calculate the integrals over $\Gamma^{i,j}$ taking $(i,j) = (1,6)$ as an example. On $\Gamma^{1,6}$, with $x^2 + y^2 = t^2 - 0$, x and y nonnegative, the trace of W, $T^1 W$, is given by :

$$u(x,y,t) = -\frac{u^0}{2\pi}\cos^{-1}\frac{-y}{|y|} - \frac{v^0}{2\pi}\cos^{-1}\frac{-x}{|x|} + \frac{u^0}{2\pi}\cos^{-1}\frac{xy}{|x|\,|y|} \equiv -\frac{u^0}{2}$$

$$v(x,y,t) = \frac{v^0}{2\pi}\cos^{-1}\frac{-y}{|y|} - \frac{u^0}{2\pi}\cos^{-1}\frac{-x}{|x|} + \frac{v^0}{2\pi}\cos^{-1}\frac{xy}{|x|\,|y|} \equiv \frac{v^0 - u^0}{2}$$

on $\Gamma^{1,6}$, with $x^2 + y^2 = t^2 + 0$, x and y being nonnegative, the trace of W, $T^6\, W$, is given by :

$$u(x, y, t) = -\frac{u^0 + v^0}{2}, \quad v(x, y, t) = \frac{v^0 - u^0}{2}$$

thus, we obtain that $[W]_6^1$ is identically 0. Similarly, one can prove that :

$$\forall (i, j) \in \mathcal{J}, \quad \iint_{\Gamma^{i,j}} {}^T[W]_i^j (\vec{n}_j^i \cdot \vec{t}\, Id + \vec{n}_j^i \cdot \vec{x} A + \vec{n}_j^i \cdot \vec{y} B)\, \Phi\, d\gamma \equiv 0$$

The last identity is quite obvious and completes the proof :

$$\iint_{\partial Q} {}^T W(\vec{n} \cdot \vec{t}\, Id + \vec{n} \cdot \vec{x} A + \vec{n} \cdot \vec{y} B)\, \Phi\, d\gamma =$$

$$\iint_{(-R, R)^2} W(x, y, 0)\, \Phi(x, y, 0)\, dx\, dy$$

We next present as an example in figure 2 and figure 3 isovalue curves at time $t = 1$ for the two components u, v of the solution of problem (3.1) with initial data (3.3) when the parameters are set to

$$u^0 = 1, \quad v^0 = 0.$$

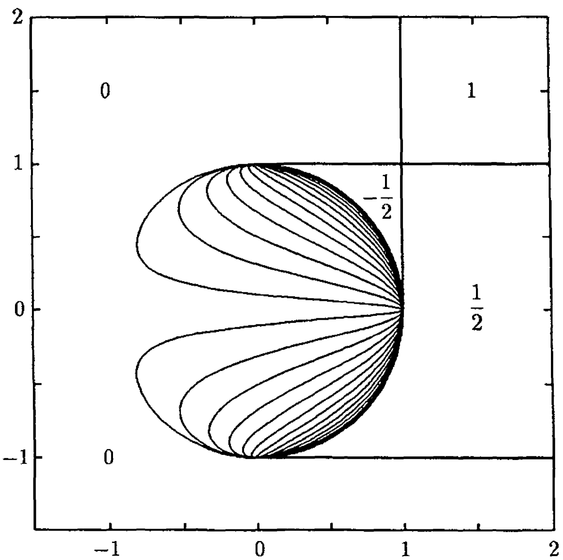

Figure 2. — Isovalue curves for u solution of (3.1) + (3.3) + (3.19).

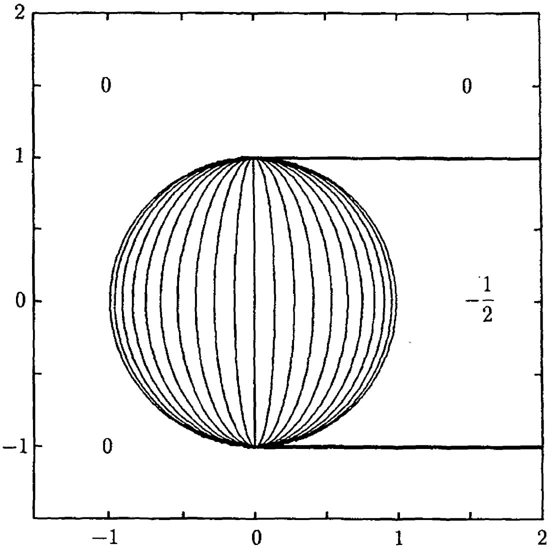

Figure 3. — Isovalue curves for v solution of (3.1) + (3.3) + (3.19).

4. ANALYTIC EXPRESSION OF THE SOLUTION OF THE MULTI-DIMENSIONAL RIEMANN PROBLEM FOR A GENERAL LINEAR SYSTEM

We now consider the hyperbolic system (4.1) of two equations in three space dimensions, where $(A_i)_{i=1,2,3}$ are 2×2 matrices with constant coefficients, W_σ^0 is a constant vector of $\mathbb{R}^2$ and σ is in $\{-1, 1\}^{\{1,2,3\}}$.

$$\begin{cases} \partial_t W(x,t) + \sum_{i=1}^{3} A_i \, \partial_{x_i} W(x,t) = 0, \quad x \in \mathbb{R}^3, t \in \mathbb{R}^+ \\ W(x_1, x_2, x_3) = W_\sigma^0 \, H(\sigma(1)\, x_1)\, H(\sigma(2)\, x_2)\, H(\sigma(3)\, x_3)\,. \end{cases} \tag{4.1}$$

According to Theorem 2.1, system (4.1) can be reduced to one of the three forms (2.1), (2.2) and (2.3). The solution of the Cauchy problem for (2.1) and (2.3) is easy to obtain ; thus the following system is the most interesting.

$$\partial_t W' + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \partial_{x_1'} W' + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \partial_{x_2'} W' + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \partial_{x_3'} W' = 0\,.$$

The new coordinates (x_1', x_2', x_3') are linear functions of (t, x_1, x_2, x_3) and W' is a linear function of W, they are defined by the transformations given in the proof of theorem 2.1. In order to determine the modified initial data, we have to intersect the plane $\{x_3' = 0\}$ with the initial data of system (4.1) ; then $W'(t = 0)$ is locally constant except along the lines Δ_1 , Δ_2, and/or Δ_3 drawn in figure 4.

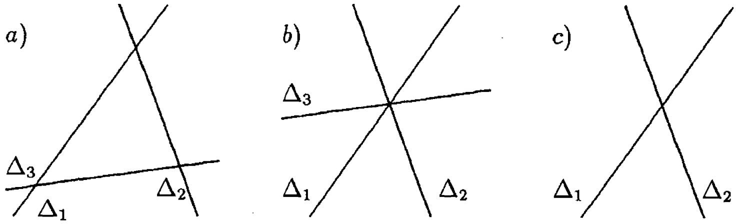

Figure 4. — The Riemann data in new coordinates.

Since all the systems are linear, cases *a*) and *b*) of figure 4 directly stem from the last one. So we will only give the explicit solution of the following Riemann problem with scalar unknowns u and v :

$$\begin{cases} \partial_t u + \partial_x u + \partial_y v = 0 \\ \partial_t v - \partial_x v + \partial_y u = 0 \end{cases} \tag{4.2}$$

with the simplest interesting initial data described in figure 5 where u^0 and v^0 are scalar constants and θ lies between 0 and $\frac{\pi}{2}$:

$$\forall x, y \in \mathbb{R}, \quad \begin{pmatrix} u \\ v \end{pmatrix} (x, y, t = 0) = \begin{pmatrix} u^0 \\ v^0 \end{pmatrix} H(y)\, H(x \sin\theta - y \cos\theta) . \tag{4.3}$$

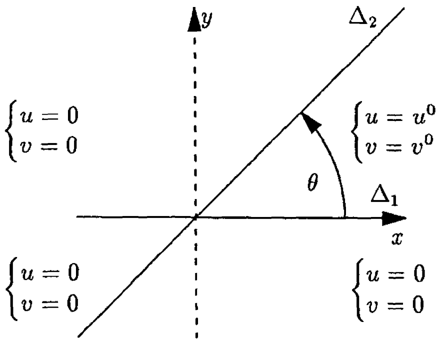

Figure 5. — Simplest Riemann data (4.3).

Let us use the rotated pair of coordinates

$$\begin{cases} X_\theta(x, y) = x \cos\theta + y \sin\theta \\ Y_\theta(x, y) = x \sin\theta - y \cos\theta , \end{cases} \quad X_\theta^2 + Y_\theta^2 = x^2 + y^2 . \tag{4.4}$$

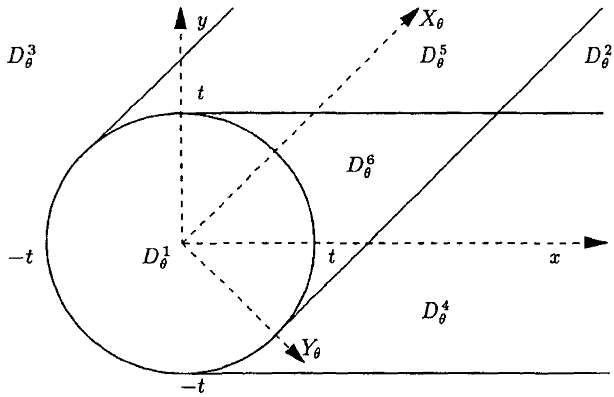

Figure 6. — Plane decomposition associated to the Riemann data (4.3).

As previously, the exact solution of the Riemann problem (4.2) + (4.3) presents a simple structure and is given in figure 6 for each domain $(D_\theta^i)_{1 \leqslant i \leqslant 6}$ of $\mathbb{R}^2 \times \mathbb{R}^+$. The interesting behaviour of the solution is confined to the interior of the cone $\{x^2 + y^2 \leqslant t^2\}$ and is discussed here in more details. Inside this cone, the solution is defined through two auxiliary functions defined in the sense of distributions :

$$w \in \{u, v\} , \quad w(x, y, t) = F_w(x, y, t) + \partial_t G_w(x, y, t) .$$

The expression for $F_w(w \in \{u, v\})$ is a straightforward extension of (3.7) with new initial data :

$$u_{0t}(x, y) = (-u^0 \sin\theta + v^0 \cos\theta) H(y) \delta_{|Y_\theta = 0} - v^0 \delta_{|y=0} H(Y_\theta)$$

$$v_{0t}(x, y) = (u^0 \cos\theta + v^0 \sin\theta) H(y) \delta_{|Y_\theta = 0} - u^0 \delta_{|y=0} H(Y_\theta)$$

$$F_u(x, y, t) = \frac{-u^0 \sin\theta + v^0 \cos\theta}{2\pi} \cos^{-1} \frac{-X_\theta}{\sqrt{t^2 - Y_\theta^2}} - \frac{v^0}{2\pi} \cos^{-1} \frac{-x}{\sqrt{t^2 - y^2}}$$

$$F_v(x, y, t) = \frac{u^0 \cos\theta + v^0 \sin\theta}{2\pi} \cos^{-1} \frac{-X_\theta}{\sqrt{t^2 - Y_\theta^2}} - \frac{u^0}{2\pi} \cos^{-1} \frac{-x}{\sqrt{t^2 - y^2}} .$$

The expression for $G_w(w \in \{u, v\})$ requires more specific calculations.

$$G_w(x, y, t) = \frac{w^0}{2\pi} \iint_{\left\{\begin{array}{l}(x-\xi)^2+(y-\eta)^2 \leqslant t^2 \\ 0 \leqslant \eta \cos\theta \leqslant \xi \sin\theta\end{array}\right.} \frac{d\xi\, d\eta}{\sqrt{t^2 - (x-\xi)^2 + (y-\eta)^2}} .$$

For all θ in $\left[0, \frac{\pi}{2}\right]$, we define (See *fig.* 8)

$$D_\theta(a, b) := \{\xi^2 + \eta^2 \leqslant 1 \,,\, 0 \leqslant (\xi - a)\cos\theta \leqslant (\eta - b)\sin\theta\} \quad (4.5)$$

$$I_\theta(a, b) = \iint_{D_\theta(a, b)} \frac{d\xi\, d\eta}{\sqrt{1 - \xi^2 - \eta^2}} . \quad (4.6)$$

This allows to simplify G_w by an homogeneous change of variables :

$$G_w(x, y, t) = \frac{w^0}{2\pi} t I_\theta\left(\frac{x}{t}, \frac{y}{t}\right) .$$

In order to estimate the integral I_θ, a change of variables in (4.6) leads to

$$2\, I_\theta(a, b) = \text{area}\, [\Omega_\theta(a, b)] \,,$$
$$\Omega_\theta(a, b) := \{(x, y, z) \in S^2 / (x, y) \in D_\theta(a, b)\} .$$

In figure 7, the domain of the sphere $\Omega_\theta(a, b)$ is enclosed between the curves Γ^1 and Γ^2 which are both part of circles drawn on S^2 and have constant geodesic curvatures. Let (ρ, φ) be the polar coordinates of the point $M(a, b)$ of the (ξ, η)-plane in figure 8. We give the coordinates of center C and the radius R of the circle supporting Γ^1 (point C and distance CP in figure 8) and the length L of the arc.

$$C(\Gamma^1) = \rho \sin(\varphi - \theta) \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix} ,$$

$$\left\{\begin{array}{l} R(\Gamma^1) = \sqrt{1 - \rho^2 \sin^2(\varphi - \theta)}, \\ L(\Gamma^1) = 2\, R(\Gamma^1) \cos^{-1} \dfrac{\rho\cos(\varphi - \theta)}{R(\Gamma^1)} . \end{array}\right.$$

If the orientation is specified by figure 7, the geodesic curvature $\kappa(\Gamma^1)$ is constant along the curves and given by

$$\kappa(\Gamma^1) = \left(\frac{PT}{CP^2} = \frac{OC}{OP\, CP}\right) = \frac{\rho \sin(\varphi - \theta)}{R(\Gamma^1)} .$$

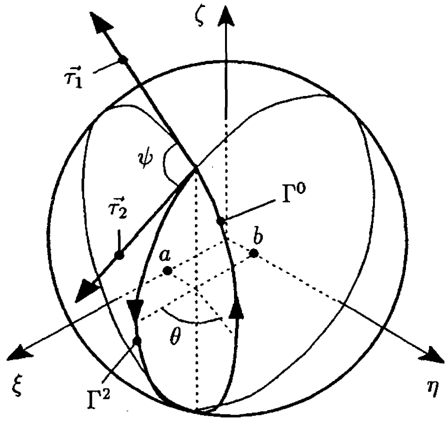

Figure 7. — Circles drawn on the sphere S^2.

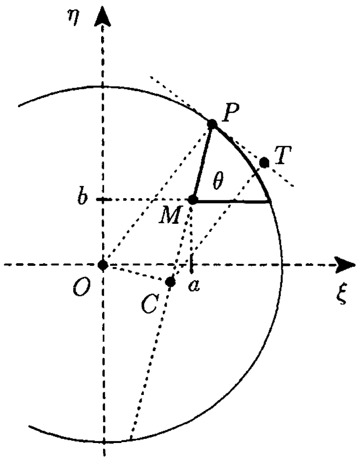

Figure 8. — Geometrical construction for the Gauss-Bonnet theorem.

We apply similar results for Γ^2 taking into account that these results correspond to $\theta = 0$ and to a change in the orientation of the curve. We evaluate $I_0(a, b)$ through the application of the Gauss-Bonnet theorem from which we obtain

$$2\, I_0(a, b) = 2(\pi - \text{meas}\, (\widehat{\vec{\tau}_1, \vec{\tau}_2})) + \kappa(\Gamma^1)\, L(\Gamma^1) + \kappa(\Gamma^2)\, L(\Gamma^2)\,. \quad (4.7)$$

In order to complete the calculation of the integral $I_\theta(a, b)$, it remains to estimate the measure of the angle $(\widehat{\vec{\tau}_1, \vec{\tau}_2})$ We introduce $(\vec{X}_i)_{i=1,2}$ two vectors tangent to $(\Gamma^i)_{i=1,2}$ at the edge oriented in the convenient way.

$$\vec{X}_1 = \begin{pmatrix} -\cos\theta \\ -\sin\theta \\ \dfrac{\rho\cos(\varphi-\theta)}{\sqrt{1-\rho^2}} \end{pmatrix}, \quad \vec{X}_2 = \begin{pmatrix} 1 \\ 0 \\ -\dfrac{\rho\cos\varphi}{\sqrt{1-\rho^2}} \end{pmatrix}$$

$$\operatorname{meas}(\vec{\tau}_1, \vec{\tau}_2) = \cos^{-1}\frac{\vec{X}_1 \cdot \vec{X}_2}{\|\vec{X}_1\|\,\|\vec{X}_2\|} =$$

$$\cos^{-1}\frac{-(1-\rho^2)\cos\theta - \rho^2\cos(\varphi-\theta)\cos\varphi}{\sqrt{1-\rho^2\sin^2\varphi}\sqrt{1-\rho^2\sin^2(\varphi-\theta)}}.$$

Finally, we can evaluate the left hand side of (4.7) in terms of a and b given two straight-forward formulae :

$$\forall a = \rho\cos\varphi,\ b = \rho\sin\varphi, \quad \text{with} \quad \rho \leqslant 1$$

$$\begin{cases} \dfrac{\rho\cos(\varphi-\theta)}{\sqrt{1-\rho^2\sin^2(\varphi-\theta)}} = \dfrac{a\cos\theta + b\sin\theta}{\sqrt{1-(a\sin\theta - b\cos\theta)^2}} \\ \dfrac{(1-\rho^2)\cos\theta + \rho^2\cos(\varphi-\theta)\cos\varphi}{\sqrt{1-\rho^2\sin^2\varphi}\sqrt{1-\rho^2\sin^2(\varphi-\theta)}} = \dfrac{(1-b^2)\cos\theta + ab\sin\theta}{\sqrt{1-b^2}\sqrt{1-(a\sin\theta-b\cos\theta)^2}} \end{cases}$$

and we obtain

$$I_\theta(a, b) = \cos^{-1}\frac{(1-b^2)\cos\theta + ab\sin\theta}{\sqrt{1-b^2}\sqrt{1-(a\sin\theta - b\cos\theta)^2}}$$

$$+ (b\cos\theta - a\sin\theta)\cos^{-1}\frac{a\cos\theta + b\sin\theta}{\sqrt{1-(a\sin\theta-b\cos\theta)^2}} - b\cos^{-1}\frac{a}{\sqrt{1-b^2}}. \tag{4.8}$$

We describe the exact solution of the Riemann problem (4.2) + (4.3) as we did in Section 3, giving the analytical expression associated to each significative domain $(D^i_\theta)_{1 \leqslant i \leqslant 6}$ of $\mathbb{R}^2 \times \mathbb{R}^+$.

• $D_\theta^1 := \{(x, y, t) \in \mathbb{R}^2 \times \mathbb{R}^+, \quad x^2 + y^2 \leqslant t^2\}$.

$\forall (x, y, t) \in D_\theta^1$,

$$u(x, y, t) = \frac{-u^0 \sin(\theta) + v^0 \cos(\theta)}{2\pi} \cos^{-1} \frac{-X_\theta(x, y)}{\sqrt{t^2 - Y_\theta^2(x, y)}} \tag{4.9}$$

$$-\frac{v^0}{2\pi} \cos^{-1} \frac{-x}{\sqrt{t^2 - y^2}} + \frac{u^0}{2\pi} \cos^{-1} \frac{t^2 \cos(\theta) + y Y_\theta(x, y)}{\sqrt{t^2 - y^2}\sqrt{t^2 - Y_\theta^2(x, y)}}$$

$$v(x, y, t) = \frac{u^0 \cos(\theta) + v^0 \sin(\theta)}{2\pi} \cos^{-1} \frac{-X_\theta(x, y)}{\sqrt{t^2 - Y_\theta^2(x, y)}} \tag{4.10}$$

$$-\frac{u^0}{2\pi} \cos^{-1} \frac{-x}{\sqrt{t^2 - y^2}} + \frac{v^0}{2\pi} \cos^{-1} \frac{t^2 \cos(\theta) + y Y_\theta(x, y)}{\sqrt{t^2 - y^2}\sqrt{t^2 - Y_\theta^2(x, y)}}.$$

• Inside $(D_\theta^i)_{2 \leqslant i \leqslant 6}$ the solution of table 2 is obtained as in the previous section.

Table 2. — The explicit formulae for u and v on an unstructured mesh.

Domain	$u(x, y, t) =$	$v(x, y, t) =$
D_θ^2	u^0	v^0
D_θ^3	0	0
D_θ^4	$\frac{u^0 - v^0}{2}$	$\frac{v^0 - u^0}{2}$
D_θ^5	$\frac{1 - \sin\theta}{2} u^0 + \frac{\cos\theta}{2} v^0$	$\frac{\cos\theta}{2} u^0 + \frac{1 + \sin\theta}{2} v^0$
D_θ^6	$\frac{-\sin\theta}{2} u^0 + \frac{\cos\theta - 1}{2} v^0$	$\frac{\cos\theta - 1}{2} u^0 + \frac{\sin\theta}{2} v^0$

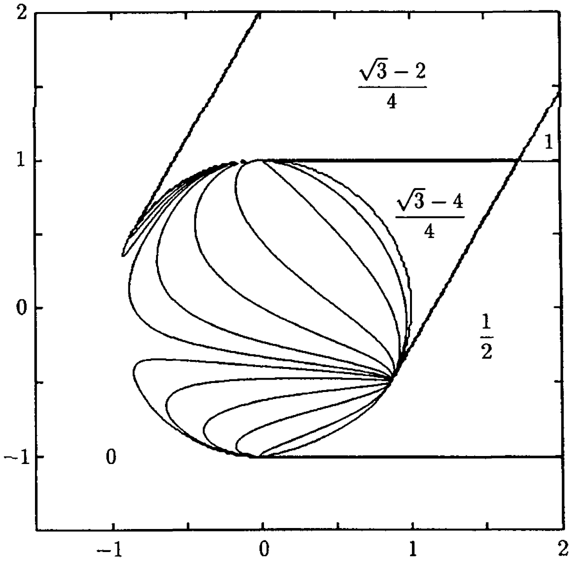

Figure 9. — Isovalue curves for u solution of (4.2) + (4.3) + (4.11).

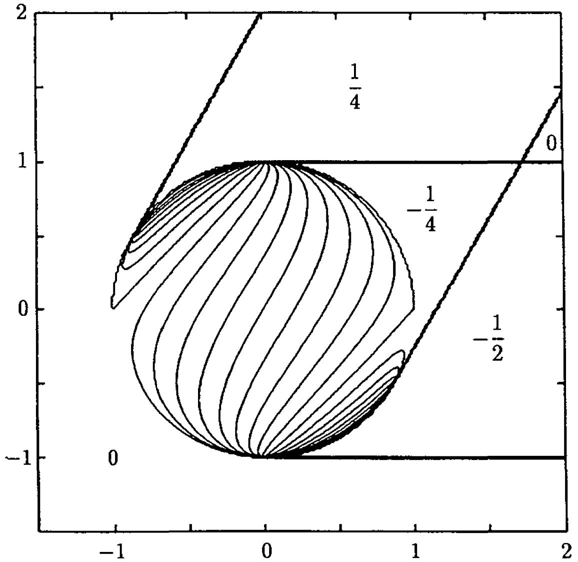

Figure 10. — Isovalue curves for v solution of (4.2) + (4.3) + (4.11).

We next present as an example in figure 9 and figure 10 isovalue curves at time $t = 1$ for the two components u, v of the solution of problem (4.2) with the initial data (4.3) when the parameters are set to

$$u^0 = 1\,, \quad v^0 = 0\,, \quad \theta = \frac{\pi}{3}\,.$$

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