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EQUICONVERGENCE OF SOME SIMULTANEOUS HERMITE-PADÉ INTERPOLANTS (*)

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Abstract. — *In several papers a result by J. L. Walsh on equiconvergence of polynomial interpolation in the roots of unity to analytic functions, has been extended using methods from complex analysis into the direction of rational interpolation to meromorphic functions having a given number of poles (E. B. Saff, A. Sharma and R. S. Varga ; followed by M. P. Stojanova who introduced an extra integer parameter* $\ell \geq 1$ *that governed the degree of the roots of unity in the first stage of the interpolation process, for* $\ell = \infty$ *both stages use the same roots of unity).*

The aim of this paper is to extend the results indicated to the situation of simultaneous (or vector) rational interpolation to d-tuples of meromorphic functions, each analytic at the origin, having disjoint sets of poles of given (finite) cardinality ; the main result exhibits so-called overconvergence : the difference between the rational interpolant for a fixed ℓ, $1 \leq \ell < \infty$ *and that for* $\ell = \infty$ *converges to zero (geometrically) on a larger disk centered at the origin, than the disk of analyticity of the function that is interpolated.*

AMS subject classification : 41A05 (primary), 41A28 (secondary)

Keywords : simultaneous interpolation, simultaneous Padé approximation, uniform and geometric convergence.

Résumé. — *Dans plusieurs articles un résultat de J. L. Walsh sur equiconvergence des polynômes d'interpolation sur les racines d'unité des fonctions analytiques est généralisé par des méthodes d'analyse complexe au cas d'interpolation rationnelle des fonctions méromorphe avec un nombre de pôles donnés (E. B. Saff, A. Sharma et R. S. Varga ; suivit par M. P. Stojanova, qui a introduit un paramètre entier* $\ell \geq 1$ *gouvernant le degré des racines d'unité dans la première étape de l'interpolation, pour* $\ell = \infty$ *la première et la seconde étape ont les mêmes racines d'unité).*

Le but de l'article est maintenant une extension des résultats indiqués dans la direction d'interpolation rationnelle simultanée de vecteurs de d fonctions méromorphe, chaque régulier a $z = 0$, *avec ensembles de pôles différentes de cardinalité finie donné ; le résultat le plus important est ici la situation appelée overconvergence : la différence entre l'interpolation rationnelle avec une valeur fixée pour* ℓ, $1 \leq \ell < \infty$ *et avec* $\ell = \infty$ *converge vers zéro*

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(de vitesse géométrique) dans un disque plus grand — centrée à zero — que le disque d'analyticité de la fonction interpolée.

AMS subject classification : 41A05 (primaire), 41A28 (secondaire)

Mots clefs : interpolation simultanée, approximation de Padé simultanée, convergence uniforme et géométrique.

1. INTRODUCTION

Our object here is to extend some recent results of E. B. Saff, A. Sharma and R. S. Varga [1]. They used complex analysis methods to extend a result of J. L. Walsh on equiconvergence of polynomial interpolants for functions in class A_ρ to Hermite-Padé interpolants for functions belonging to $M_\rho(\nu)$, $\rho > 1$. Here A_ρ denotes the set of functions analytic in the disc D_ρ of radius ρ but not analytic in its closure $\bar{D}_\rho$, while $M_\rho(\nu)$ refers to functions meromorphic in D_ρ with precisely ν given poles inside. Later M. P. Stojanova [4] extended the result of E. B. Saff *et al.* For any integer $\ell \geqslant 1$ and any function in $M_\rho(\nu)$, she defined a rational function which satisfies properties analogous to Padé approximation. For $\ell = 1$, her results give the result of Saff *et al.* but for $\ell > 1$, the sequence of rationals do not reduce to Padé approximants.

Our main concern is to extend the results of Saff *et al.* and of Stojanova to simultaneous Hermite-Padé approximation for d given functions $F_i(z)$ which belong to $M_{\rho_i}(\nu_i)$ $(i = 1, \dots, d)$. Here $\nu_1, \dots, \nu_d$ are d positive integers and $\{\rho_i\}_1^d$ are positive real numbers which satisfy $1 < \rho_1 \leqslant \rho_2 \leqslant \cdots \leqslant \rho_d$. When $d = 1$ and $\ell = 1$, our results will give those of Saff *et al.* [1], but for $\ell > 1$, $d = 1$, our results are analogous to those of Stojanova, but are different from hers. In fact, the rational interpolants which we define for any $\ell \geqslant 1$ and $d \geqslant 1$ can be stated in terms of Padé interpolants. We shall show how our rationals differ from those of Stojanova.

In Section 2, we give the preliminaries on simultaneous Hermite-Padé interpolation for d given functions in $M_{\rho i}(\nu_i)$, $(i = 1, \dots, d)$. We shall suppose that the pole sets of the given functions are disjoint. In an extension of the theorem of de Montessus de Ballore to vector valued interpolation using rational functions with a common denominator, P. R. Graves-Morris and E. B. Saff introduced an analogous condition (termed « polewise independence ») in [2]. We shall first consider the case when the nodes of interpolation are simple and consist of the zeros of $z^{\sigma+1} - \alpha^{\sigma+1}$ (or of $z^{\sigma+1} - \beta^{\sigma+1}$) where $\alpha, \beta \in \mathbb{C}$ and $\max(|\alpha|, |\beta|) < \rho_1$. In Section 3 we prove the existence and uniqueness of the operators defined in Section 2. In Section 4, we prove some lemmas which are used in Section 5 to prove the main result. Section 6 will deal with the formulation of the problem for the case of multiple nodes ; i.e., the nodes of interpolation are the zeros of $(z^{\sigma+1} - 1)^r$, $r \geqslant 1$.

2. PRELIMINARIES AND MAIN RESULT

Let d be a natural number and let $F_1, \dots, F_d$ be d given functions, each F_i belonging to the class $M_{\rho_i}(\nu_i)$, where $\nu_1, \dots, \nu_d$ are natural numbers and

$1 < \rho_1 \leqslant \rho_2 \leqslant \cdots \leqslant \rho_d$ are real numbers. Since $M_{\rho_i}(\nu_i)$ is the class of meromorphic functions in the disc D_{ρ_i} with precisely ν_i poles inside, we may set

$$F_i(z) := \frac{f_i(z)}{B_i(z)}, \quad f_i(z) := \sum_{k=0}^{\infty} a_{i,k} z^k, \quad \overline{\lim_{k \to \infty}} |a_{i,k}|^{\frac{1}{k}} \leqslant \frac{1}{\rho_i} \, (i = 1, \dots, d) \tag{2.1}$$

where

$$B_i(z) := \sum_{k=0}^{\nu_i} \alpha_{i,k} z^k, \quad \alpha_{i,\nu_i} = 1$$

$$= \prod_{j=1}^{\mu_i} (z - z_{i,j})^{\lambda_{ij}}, \quad \sum_{j=1}^{\mu_i} \lambda_{i,j} = \nu_i . \tag{2.2}$$

For the sake of simplicity, we shall only consider the case when the pole sets of the d functions F_i are distinct, so that

$$z_{i,j} \neq z_{i',j'} \quad \text{for} \quad i \neq i' \quad (j, j' \; arbitrary) .$$

Let ν_0 be any positive integer (when $d = 1$, we generally take $\nu_0 = n$) and let $\sigma := \nu_0 + \nu_1 + \cdots + \nu_d$.

The problem of simultaneous Hermite-Padé interpolation on the zeros of $z^{\sigma+1} - \alpha^{\sigma+1}$ for some $\alpha \in \mathbb{C}$ with $|\alpha| < \rho_1$ is to find $d + 1$ polynomials

$$P_0(z), P_1(z), \dots, P_d(z)$$

satisfying the following three conditions :

(i) $P_0(z)$ is monic and of degree $\sigma - \nu_0$

(ii) $P_j(z)$ is of degree $\sigma - \nu_j \; (j = 1, \dots, d)$

and

(iii) $f_j(z)\, P_0(z) - P_j(z)\, B_j(z) = O(z^{\sigma+1} - \alpha^{\sigma+1}), \qquad j = 1, \dots, d.$

The last condition means that $P_j(z)/P_0(z)$ interpolates $F_j(z)$ on the zeros of $z^{\sigma+1} - \alpha^{\alpha+1}$ where $P_j(z)$ and $P_0(z)$ satisfy (i) and (ii). It is easy to observe that the number of unknowns is $\sigma - \nu_0 + \sum_{j=1}^{d} (\sigma - \nu_j + 1)$ which on simplifying becomes $d(\sigma + 1)$ which agrees with the number of interpolatory conditions for the d functions. For a detailed historical survey on the subject for $\alpha = 0$, we refer to [5].

For any $\beta \in \mathbb{C}$, with $|\beta| < \rho_1$, $\beta \neq \alpha$, and for any integer $\ell \geqslant 1$, we shall denote the Lagrange interpolant to f_i on the zeros of $z^{\ell(\sigma+1)} - \beta^{\ell(\sigma+1)}$ by $L_{\ell(\sigma+1)}(f_i\,;\beta, z)$. If $f_i(z)$ is given by (2.1), then it is easy to see that

$$\begin{cases} L_{\ell(\sigma+1)}(f_i\,;\beta, z) := \sum_{k=0}^{\ell(\sigma+1)-1} B_{i,k}^{\ell} z^k \\ B_{i,k}^{\ell} := \sum_{m=0}^{\infty} a_{i,m(\sigma+1)\ell+k}\, \beta^{m\ell(\sigma+1)} . \end{cases} \tag{2.3}$$

We shall require the Lagrange interpolant to the polynomial $L_{\ell(\sigma+1)}(f_i\,;\beta, z)$ in the zeros of $z^{\sigma+1} - \alpha^{\sigma+1}$. Denoting this by $L_{\sigma+1}(L_{\ell(\sigma+1)}^{\beta}\,;\alpha, z)$, we see that

$$L_{\sigma+1}(L_{\ell(\sigma+1)}^{\beta}\,;\alpha, z) = \sum_{j=0}^{\sigma} A_{i,j}^{\ell} z^j , \tag{2.4}$$

where

$$\begin{aligned} A_{i,j}^{\ell} &= \sum_{r=0}^{\ell-1} B_{i,r(\sigma+1)+j}\, \alpha^{r(\sigma+1)} \\ &= \sum_{r=0}^{\ell-1} \sum_{m=0}^{\infty} a_{i,(m\ell+r)(\sigma+1)+j}\, \beta^{m\ell(\sigma+1)}\, \alpha^{r(\sigma+1)} . \end{aligned} \tag{2.5}$$

For $i = 1, \ldots, d$, we set

$$F_{i,\ell}^{\beta,\alpha}(z) := \frac{\sum_{j=0}^{\sigma} A_{i,j}^{\ell} z^j}{B_i(z)} = \frac{L_{\sigma+1}(L_{\ell(\sigma+1)}^{\beta}\,;\alpha, z)}{B_i(z)} . \tag{2.6}$$

We are interested in finding the simultaneous Hermite-Padé interpolant of the d functions $F_{i,\ell}^{\beta,\alpha}$ $(i = 1, \ldots, d)$ on the zeros of $z^{\sigma+1} - \alpha^{\sigma+1}$. We denote the Hermite-Padé interpolant by $P_{i,\sigma}^{\ell}(z)$ which is a rational function whose numerator is of degree $\sigma - \nu_i$ and whose denominator is monic of degree $\sigma - \nu_0$. We set

$$P_{i,\sigma}^{\ell}(z) := \frac{U_i^{\ell}(z)}{B_{\ell}(z)} := \frac{\sum_{s=0}^{\sigma-\nu_i} p_{i,s}^{\ell} z^s}{\sum_{s=0}^{\sigma-\nu_0} \gamma_s^{\ell} z^s} , \quad \gamma_{\sigma-\nu_0}^{\ell} = 1 . \tag{2.7}$$

Then $P^{\ell}_{i,\sigma}(z)$ interpolates $F^{\beta,\alpha}_{i,\ell}$ on the zeros of $z^{\sigma+1} - \alpha^{\sigma+1}$ so that

$$B_i(z)\, U_i^{\ell}(z) = B_{\ell}(z) \sum_{j=0}^{\sigma} A^{\ell}_{i,j} z^j , \quad z = \alpha\omega , \quad \omega^{\sigma+1} = 1 . \tag{2.8}$$

The left side is a polynomial of degree $\nu_i + (\sigma - \nu_i) = \sigma$ while the right side is of degree $\sigma + (\sigma - \nu_0)$. But on putting $z = \alpha\omega$ on both sides, since $\omega^{\sigma+1} = 1$, the right side becomes a polynomial of degree σ. Then comparing coefficients of z^s on both sides in (2.6) we get the relations

$$\sum_{k=0}^{s} \alpha_{i,k}\, p^{\ell}_{i,s-k} = \sum_{k=0}^{s} \gamma^{\ell}_k A^{\ell}_{i,s-k} + \alpha^{\sigma+1} \sum_{k=s+1}^{\sigma-\nu_0} \gamma^{\ell}_k A^{\ell}_{i,\sigma+1+s-k} \tag{2.9}$$

$$(i = 1, \dots, d\,;\, 0 \leqslant s \leqslant \sigma) .$$

If we make the convention that

$$A^{\ell}_{i,s} := \alpha^{\sigma+1} A^{\ell}_{i,\sigma+1+s} , \quad s < 0$$

then the system of equations (2.9) can be written

$$\sum_{k=0}^{\sigma-\nu_0} \alpha_{i,k}\, p^{\ell}_{i,s-k} = \sum_{k=0}^{\sigma-\nu_0} \gamma^{\ell}_k A_{i,s-k} \quad (i = 1, \dots, d\,;\, 0 \leqslant s \leqslant \sigma) \tag{2.10}$$

where $\alpha_{i,k} = 0$, $k > \nu_i$, $p^{\ell}_{i,n} = 0$ for $n > \sigma - \nu_i$. This system of equations determines the Hermite-Padé interpolants on the zeros of $z^{\sigma+1} - \alpha^{\sigma+1}$ of the d functions $F^{\beta,\alpha}_{i,\ell}$ $(i = 1, \dots, d)$.

Remark 1: If $\beta = 0$, then from (2.5), we have $A^{\ell}_{i,j} = \sum_{r=0}^{\ell-1} a_{i,r(\sigma+1)+j}\, \alpha^{r(\sigma+1)}$ so that

$$F^{0,\alpha}_{i,\ell}(z) = \sum_{j=0}^{\sigma} z^j \sum_{r=0}^{\ell-1} a_{i,r(\sigma+1)+j}\, \alpha^{r(\sigma+1)} / B_i(z) . \tag{2.11}$$

When $\ell = \infty$, then for any β,

$$A^{\infty}_{i,j} = \begin{cases} \displaystyle\sum_{r=0}^{\infty} a_{i,r(\sigma+1)+j}\, \alpha^{r(\sigma+1)} , & j \geqslant 0 \\ \displaystyle\sum_{r=1}^{\infty} a_{i,r(\sigma+1)+j}\, \alpha^{r(\sigma+1)} , & j < 0 . \end{cases} \tag{2.12}$$

Remark 2 : Comparing the coefficients $A^{\ell}_{i,j}$ with those of Stojanova [4], we see that there is only a difference for $j < 0$. Denote the A's from [4] by $\tilde{A}$, then :

$$\tilde{A}^{\ell}_{j;\,n,\,\nu} = A^{\ell}_{i,j} - a_{i,\,(\ell-1)(\sigma+1)+\sigma+1+j} \quad (j < 0\,;\, \sigma = n + \nu)\,,$$

where i on the right hand side can be omitted because in [4] there is only one function.

We shall show that for each i $(1 \leqslant i \leqslant d)$ and any integer $\ell \geqslant 1$, the difference between the rational interpolants $P^{\infty}_{i,\sigma}(z)$ and $P^{\ell}_{i,\sigma}(z)$ converges to zero as $\sigma \to \infty$ in a region larger than the region of meromorphy of $F_i(z)$ with the poles of $F_i(z)$ deleted. More precisely, we prove

THEOREM 1 : *Let $\vec{F} = (F_1, \dots, F_d)$ be d functions meromorphic in the discs D_{ρ_i} $(1 < \rho_1 \leqslant \rho_2 \leqslant \cdots \leqslant \rho_d)$ respectively as given by (2.1).*

Let α, $\beta \in \mathbb{C}$, $\alpha \neq \beta$, $\max(|\alpha|, |\beta|) < \rho_1$ and suppose $\alpha\omega^j$ and $\beta\omega^j$ $(0 \leqslant j \leqslant \sigma)$ with $\omega^{\sigma+1} = 1$ are distinct from the pole sets of F_i's. Suppose for some integer $\ell \geqslant 1$, $P^{\ell}_{i,\sigma}(z) = U^{\ell}_i(z)/B_{\ell}(z)$ denotes the simultaneous Hermite-Padé interpolant to $F^{\beta,\alpha}_{i,\ell}$ on the zeros of $z^{\sigma+1} - \alpha^{\sigma+1}$. Suppose $P^{\infty}_{i,\sigma}(z) = U^{\infty}_i(z)/B^{\infty}(z)$ is the Hermite-Padé interpolant to F_i on the zeros of $z^{\sigma+1} - \alpha^{\sigma+1}$. Then as $\sigma \to \infty$, the following holds :

$$\lim_{\sigma \to \infty} P^{\infty}_{i,\sigma}(z) - P^{\ell}_{i,\sigma}(z) = 0 \quad \text{for} \quad |z| < \frac{\rho_i}{R^{\ell}}, \tag{2.13}$$

$(i = 1, \dots, d)$ where $R = \max\big(|\alpha| / \rho_1, |\beta| / \rho_1\big) < 1$. The convergence is uniform and geometric on compact subsets of the region.

Sharpness of Theorem : We shall show that there are two functions $\in M_{\rho}(1)$ for which the result of Theorem 1 is sharp. Let

$$F_j(z) = \frac{\rho}{(z - \alpha_j)(\rho - z)}, \quad j = 1, 2, \quad |\alpha_j| < \rho, \quad |\alpha_1| \neq |\alpha_2|, \quad (\rho < 1).$$

Then as in (2.5), we have

$$A^{\ell}_{ij} = \begin{cases} C_{\ell}\dfrac{1}{\rho^j}, & j \geqslant 0 \\ C_{\ell}\dfrac{\alpha^{\sigma+1}}{\rho^{\sigma+1+j}}, & j < 0, \end{cases} \quad \text{where} \quad C_{\ell} = \frac{1 - \left(\frac{\alpha}{\rho}\right)^{\ell(n+3)}}{1 - \left(\frac{\alpha}{\rho}\right)^{n+3}} \cdot \frac{1}{1 - \left(\frac{\beta}{\rho}\right)^{\ell(n+3)}}.$$

Following the method in Section 3, we see that γ_1^ℓ and γ_0^1 in the denominator $z^2+\gamma_1^\ell z+\gamma_0^\ell$ of the simultaneous interpolants to $F_j(z)$ on the zeros of $z^{n+1}-\alpha^{n+1}$, are given by the system of equations given by

$$\begin{pmatrix} 1-\left(\frac{\alpha_1}{\rho}\right)^{n+3} & \left(\frac{\alpha}{\rho}\right)^{n+3}(\rho-\alpha_1)+\alpha_1\left[1-\left(\frac{\alpha_1}{\rho}\right)^{n+2}\right] \\ 1-\left(\frac{\alpha_2}{\rho}\right)^{n+3} & \left(\frac{\alpha}{\rho}\right)^{n+3}(\rho-\alpha_2)+\alpha_2\left[1-\left(\frac{\alpha_2}{\rho}\right)^{n+2}\right] \end{pmatrix}\begin{pmatrix}\gamma_0^\ell \\ \gamma_1^\ell\end{pmatrix}$$

$$=-\begin{pmatrix} \left(\frac{\alpha}{\rho}\right)^{n+3}(\rho^2-\alpha_1^2)+\alpha_1^2\left(1-\left(\frac{\alpha_1}{\rho}\right)^{n+1}\right) \\ \left(\frac{\alpha}{\rho}\right)^{n+3}(\rho^2-\alpha_2^2)-\alpha_2^2\left(1-\left(\frac{\alpha_2}{\rho}\right)^{n+1}\right)\end{pmatrix}$$

It is interesting to observe from the above that γ_i^ℓ $(i=0,1)$ do not depend on ℓ. Note that as $n\to\infty$, we get $\gamma_1^\ell=-(\alpha_2+\alpha_1)$, $\gamma_0^\ell=\alpha_1\alpha_2$ so that the denominator of each of the simultaneous interpolants tends to $(z-\alpha_1)(z-\alpha_2)$.

Since γ_k^ℓ $(k=0,1,2)$ are independent of ℓ, it follows easily that $B^\infty(z)=B^\ell(z)$. Also,

$$p_{i,m-1}^\infty-p_{i,m-1}^\ell=\sum_{k=0}^{2}\left\{\sum_{s=m}^{n+2}(A_{i,s-k}^\infty-A_{i,s-k}^\ell)\,\alpha_i^{s-m}\right\}\gamma_k^\ell,\quad i=1,2$$

where

$$A_{i,j}^\infty-A_{i,j}^\ell=\tilde{C}_\ell A_{i,j}^\infty,\quad \tilde{C}_\ell=1-\frac{1-\left(\frac{\alpha}{\rho}\right)^{\ell(\sigma+1)}}{1-\left(\frac{\beta}{\rho}\right)^{\ell(\sigma+1)}}.$$

Therefore

$$\frac{U_i^\infty(z)}{B^\infty(z)}-\frac{U_i^\ell(z)}{B^\ell(z)}=(1-\tilde{C}_\ell)\frac{U_i^\infty(z)}{B^\infty(z)}.$$

Since for $z\neq\alpha_1,\alpha_2$, we have

$$\lim_{n\to\infty}\left|\frac{1-\tilde{C}_\ell}{B^\infty(z)}\right|^{\frac{1}{n+3}}=R^\ell,\quad R:=\max\left(\frac{|\alpha|,|\beta|}{\rho}\right)$$

it remains to find $\lim_{n\to\infty}|U_i^\infty(z)|^{1/n+3}$.

After some simplification we get

$$U_i^\infty(z) = \sum_{m=0}^{n+1} p_{i,m}^\infty z^m$$

$$= C_\infty \frac{B^\infty(\rho)}{\rho - \alpha_i} h_{i,n+2}(z) + \rho C_\infty\left(-1 + \left(\frac{\alpha}{\rho}\right)^{n+3}\right), \quad i = 1, 2,$$

where

$$h_{i,n+2}(z) = \rho\left(\frac{z}{\rho}\right)^{n+3}\left(\frac{(\rho - \alpha_i)}{(z-\rho)(z-\alpha_i)}\right) - \left\{\frac{\rho}{z-\rho} - \left(\frac{\alpha_i}{\rho}\right)^{n+3}\frac{\alpha_i}{z-\alpha_i}\right\}.$$

Because $\lim_{n\to\infty} C_\infty = 1$, we get for $|z| = \rho R^{-\ell}$, the following :

$$\lim_{n\to\infty} |U_i^\infty(z)|^{\frac{1}{n+3}} = R^{-\ell}.$$

Hence for $|z| = \rho R^{-\ell}$, we get

$$\lim_{n\to\infty}\left|\frac{U_i^\infty(z)}{B^\infty(z)} - \frac{U_i^\ell(z)}{B^\ell(z)}\right|^{\frac{1}{n+3}} = 1$$

which proves the sharpness.

3. EXISTENCE AND UNIQUENESS OF HERMITE-PADÉ INTERPOLANTS

We shall prove the following

THEOREM 2 : *Let $\overrightarrow{F} = (F_1, \dots, F_d)$ be d functions, $F_i \in M_{\rho_i}(\nu_i)$, where ν_i's are integers and $1 \leqslant \rho_1 \leqslant \rho_2 \leqslant \cdots \leqslant \rho_d$ are real numbers. If $\sigma := \nu_0 + \nu_1 + \cdots + \nu_d$ is sufficiently large, then for any integer $\ell \geqslant 1$, there exists for each i $(1 \leqslant i \leqslant d)$, a rational function $P_{i,\sigma}^\ell(z)$, (and $P_{i,\sigma}^\infty(z)$) which satisfies* (2.8).

Proof : The rational function $P_{i,\sigma}^\ell(\overrightarrow{F}; z)$ is determined by the system of equations (2.10). We shall show that the values of γ_k^ℓ $(k = 0, 1, \dots, \sigma - \nu_0)$ can be uniquely determined. For a fixed $i \in \{1, \dots, d\}$, we multiply the equations (2.10) by

$$z_{i,j}^{s-t}\binom{s}{t}, \quad 0 \leqslant t \leqslant \lambda_{i,j} - 1, \quad 1 \leqslant j \leqslant \mu_i$$

and sum both sides from $s = 0$ to σ. We then get a total of $\sum_{j=1}^{\mu_i} \lambda_{i,j} = \nu_i$ equations for each i. Then the left side becomes

$$\sum_{s=0}^{\sigma} \sum_{k=0}^{\sigma-\nu_0} \alpha_{i,k} p^{\ell}_{i,s-k} \binom{s}{t} z_{ij}^{s-t} = \sum_{n=0}^{\sigma-\nu_i} \left\{ \sum_{k=0}^{\nu_i} \alpha_{i,k} z_{i,j}^{k+n-t} \binom{n+k}{t} \right\} p^{\ell}_{i,n}$$

where we put $s - k = n$ and observe that $\alpha_{i,k} = 0$ for $k > \nu_i$ and $p^{\ell}_{i,n} = 0$ for $n > \sigma - \nu_i$. The coefficient of $p^{\ell}_{i,n}$ in the above

$$\sum_{k=0}^{\nu_i} \alpha_{i,k} \binom{n+k}{t} z_{i,j}^{n+k-t} = \frac{1}{t!} \sum_{k=0}^{\nu_i} (n+k)\dots(n+k-t+1)\, \alpha_{i,k} z_{i,j}^{n+k-t}$$

$$= \frac{1}{t!} D^t [z^n B_i(z)]_{z = z_{i,j}}, \quad D \equiv \frac{d}{dz}$$

$$= 0 \quad \text{for} \quad 0 \leqslant t \leqslant \lambda_{i,j} - 1, \quad 1 \leqslant j \leqslant \mu_i .$$

Thus we get from the right side of (2.10) for each i

$$\sum_{k=0}^{\sigma-\nu_0} \gamma^{\ell}_k \left\{ \sum_{s=0}^{\sigma} A^{\ell}_{i,s-k} z_{i,j}^{s-t} \binom{s}{t} \right\} = 0, \quad 0 \leqslant t \leqslant \lambda_{i,j} - 1, \quad 1 \leqslant j \leqslant \mu_i . \tag{3.1}$$

This is a system of $\sigma - \nu_0$ linear equations in the unknowns γ^{ℓ}_k $(k = 0, \dots, \sigma - \nu_0 - 1)$ with $\gamma^{\ell}_{\sigma-\nu_0} = 1$. We shall show that

$$\sum_{s=0}^{\sigma} A^{\ell}_{i,s-k} z_{i,j}^{s-t} \binom{s}{t} = \frac{1}{t!} D^t [z^k f_i(z)]_{z_{i,j}} + R^{(i)}_{\sigma} \tag{3.2}$$

and will prove that

$$|R^{(i)}_{\sigma}| \leqslant c_0 \sigma^{\sigma-\nu_0} \left\{ \max \left(\frac{|\alpha|}{\rho_i - \varepsilon}, \left(\frac{|\beta|}{\rho_i - \varepsilon} \right)^{\ell}, q \right) \right\}^{\sigma+1} \tag{3.3}$$

where

$$q = \max_{1 \leqslant i \leqslant d} \frac{1}{\rho_i - \varepsilon} \left[\max_{1 \leqslant j \leqslant \mu_i} \{1, |z_{i,j}|\} \right].$$

To prove this, we set the left side of (3.2) equal to $I_1 - I_2 + I_3$, where

$$\begin{cases} I_1 = \sum_{s=0}^{\infty} a_{i,s-k}\, z_{i,j}^{s-t} \binom{s}{t}, \quad I_2 = \sum_{s=\sigma+1}^{\infty} a_{i,s-k}\, z_{i,j}^{s-t} \binom{s}{t} \\ I_3 = \sum_{s=0}^{\sigma} (A_{i,s-k}^{\ell} - a_{i,s-k})\, z_{i,j}^{s-t} \binom{s}{t}. \end{cases} \tag{3.4}$$

We shall now obtain an estimate for I_2 and prove

LEMMA 1 : *If I_2 is given by* (3.4), *then*

$$|I_2| \leqslant c_1\, \sigma^{\sigma - \nu_0}\, q^{\sigma+1}, \quad \text{where} \quad q = \max_{1 \leqslant i \leqslant d} \frac{1}{\rho_i - \varepsilon} \left[\max_{1 \leqslant j \leqslant \mu_i} \{1, |z_{i,j}|\} \right] \tag{3.5}$$

and c_1 is an absolute constant not depending on n.

Proof: Observe that if $0 \leqslant x < 1$, then for any positive integer t, we have

$$D^t\left(\frac{x^{\sigma+1}}{1-x}\right) \leqslant 2^t(\sigma+1)\,\sigma\ldots(\sigma-t+2)\,\frac{x^{\sigma-t+1}}{1-x} \tag{3.6}$$

for $\sigma \geqslant t + \frac{x}{1+x}$, $t \geqslant 0$. This follows from the fact that

$$\frac{d}{dx}\left(\frac{x^{\sigma+1}}{1-x}\right) = \frac{(\sigma+1)\,x^{\sigma}}{(1-x)} + \frac{x^{\sigma+1}}{(1-x)^2}$$

$$= \frac{(\sigma+1)\,x^{\sigma}}{(1-x)}\left(1 + \frac{x}{1-x} \cdot \frac{1}{\sigma+1}\right) \leqslant 2(\sigma+1)\,\frac{x^{\sigma}}{1-x}.$$

From this we get the result by iteration. Putting q as indicated above, and $x = (|z_{i,j}|/(\rho_i - \varepsilon))$, we get

$$|I_2| = \left| \sum_{s=\sigma+1}^{\infty} a_{i,s-k} z_{i,j}^{s-t} \binom{s}{t} \right|$$

$$\leqslant \frac{1}{t!} \sum_{s=\sigma+1}^{\infty} s(s-1)\dots(s-t+1) \frac{|z_{i,j}|^{s-t}}{(\rho_i - \varepsilon)^{s-k}}$$

$$\leqslant (\rho_i - \varepsilon)^{k-t} \frac{1}{t!} \sum_{s=\sigma+1}^{\infty} s(s-1)\dots(s-t+1)\, x^{s-t}$$

$$\leqslant \frac{(\rho_i - \varepsilon)^{k-t}}{t!} \left| D^t \sum_{s=\sigma+1}^{\infty} x^s \right|$$

$$= \frac{(\rho_i - \varepsilon)^{k-t}}{t!} D^t \left(\frac{x^{\sigma+1}}{1-x} \right)$$

$$\leqslant \frac{(\rho_i - \varepsilon)^{k-t}}{t!} 2^t (\sigma+1)\dots(\sigma-t+2) \frac{x^{\sigma+1-t}}{1-x}, \quad \text{using } (3.6)$$

$$\leqslant \frac{(\rho_i - \varepsilon)^{k}}{t!} 2^t q^t (2\sigma)^{\sigma-\nu_0} \frac{q^{\sigma+1-t}}{1-q}$$

from which (3.5) follows immediately. □

It remains to obtain an estimate for I_3. We shall prove.

LEMMA 2 : *If I_3 is given by* (3.4), *then*

$$|I_3| \leqslant c_5\, \sigma^{\sigma-\nu_0} \left(\frac{|\alpha|}{\rho_i - \varepsilon} \right)^{\sigma+1} + c_6\, \sigma^{\sigma-\nu_0} \left(\frac{|\beta|}{\rho_i - \varepsilon} \right)^{\ell(\sigma+1)} . \qquad (3.7)$$

Proof: We shall first consider the case when $s \leqslant k - 1$. Then

$$|A^{\ell}_{i,s-k} - a_{i,s-k}| =$$

$$= |\alpha^{\sigma+1} A^{\ell}_{i,\sigma+1+s-k}|$$

$$= \left| \alpha^{\sigma+1} \sum_{r=0}^{\ell-1} \sum_{m=0}^{\infty} \left(\frac{1}{\rho_i - \varepsilon} \right)^{m\ell(\sigma+1)+(\sigma+1)+s-k} |\beta|^{m\ell(\sigma+1)} |\alpha|^{r(\sigma+1)} \right|$$

$$= \left(\frac{|\alpha|}{\rho_i - \varepsilon} \right)^{\sigma+1} \left(\frac{1}{\rho_i - \varepsilon} \right)^{s-k} \frac{1}{1 - \left(\frac{|\beta|}{\rho_i - \varepsilon} \right)^{\ell(\sigma+1)}}$$

$$\times \frac{1 - \left(\frac{|\alpha|}{\rho_i - \varepsilon} \right)^{\ell}}{1 - \frac{|\alpha|}{\rho_i - \varepsilon}}$$

$$\leqslant c_2 \left(\frac{|\alpha|}{\rho_i - \varepsilon} \right)^{\sigma+1} \frac{1}{(\rho_i - \varepsilon)^{s-k}},$$

$$\text{since} \quad 1 - \left(\frac{|\beta|}{\rho_i - \varepsilon} \right)^{\ell(\sigma+1)} > 1 - \frac{|\beta|}{\rho_i - \varepsilon} \tag{3.8}$$

for all $k \in \{0, 1, ..., \sigma - \nu_0\}$. For $\ell = \infty$ a similar bound is derived (then there is no contribution from β).

For $s \geqslant k$, we have

$$|A^{l}_{i,s-k} - a_{i,s-k}| = \left| \sum_{r=0}^{\ell} \sum_{m=0}^{\infty} a_{i,m\ell(\sigma+1)+r(\sigma+1)+s-k} \beta^{m\ell(\sigma+1)} \alpha^{r(\sigma+1)} - a_{i,s-k} \right|$$

$$= \left| \sum_{r=0}^{\ell} \sum_{m=0}^{\infty} a_{i,m\ell(\sigma+1)+r(\sigma+1)+s-k} \beta^{m\ell(\sigma+1)} \alpha r^{r(\sigma+1)} \right|$$

$$+ \left| \sum_{m=1}^{\infty} a_{i,m\ell(\sigma+1)+s-k} \beta^{m\ell(\sigma+1)} \right|$$

$$\leqslant \sum_{r=1}^{\ell-1} \sum_{m=0}^{\infty} \frac{|\beta|^{m\ell(\sigma+1)} |\alpha|^{r(\sigma+1)}}{(\rho_i - \varepsilon)^{(m\ell+r)(\sigma+1)+s-k}}$$

$$+ \sum_{m=1}^{\infty} \frac{|\beta|^{m\ell(\sigma+1)}}{(\rho_i - \varepsilon)^{m\ell(\sigma+1)+s-k}} = \frac{1}{(\rho_i - \varepsilon)^{s-k}}$$

$$\times \left\{ \left(\frac{|\alpha|}{\rho_i - \varepsilon} \right)^{\sigma+1} \frac{1 - \left(\frac{|\alpha|}{\rho_i - \varepsilon} \right)^{(\ell-1)(\sigma+1)}}{1 - \frac{|\alpha|}{(\rho_i - \varepsilon)}} \right.$$

$$\times \frac{1}{1 - \left(\frac{|\beta|}{(\rho_i - \varepsilon)} \right)^{\ell(\sigma+1)}}$$

$$\left. + \left(\frac{|\beta|}{\rho_i - \varepsilon} \right)^{\ell(\sigma+1)} \frac{1}{1 - \left(\frac{|\beta|}{\rho_i - \varepsilon} \right)^{\ell(\sigma+1)}} \right\}$$

$$\leqslant \frac{1}{(\rho_i - \varepsilon)^{s-k}} \left\{ c_3 \left(\frac{|\alpha|}{\rho_i - \varepsilon} \right)^{\sigma+1} \right.$$

$$\left. + c_4 \left(\frac{|\beta|}{\rho_i - \varepsilon} \right)^{\ell(\sigma+1)} \right\}, \quad k \leqslant s \leqslant \sigma . \tag{3.9}$$

For $\ell = \infty$ we get a bound like (3.9) with the β-part missing

From (3.8) and (3.9), we see that

$$\left|\sum_{s=0}^{\sigma} (A^{\ell}_{i,s-k} - a_{i,s-k})\, z_{ij}^{s-t}\binom{s}{t}\right| \leqslant$$

$$\leqslant c_2 \sum_{s=0}^{k-1} \frac{|z_{i,j}|^{s-t}}{(\rho_i-\varepsilon)^{s-k}} \left(\frac{|\alpha|}{\rho_i-\varepsilon}\right)^{\sigma+1} \binom{s}{t}$$

$$+ \sum_{s=k}^{\sigma} \left\{ c_3\left(\frac{|\alpha|}{\rho_i-\varepsilon}\right)^{\sigma+1} + c_4\left(\frac{|\beta|}{\rho_i-\varepsilon}\right)^{\ell(\sigma+1)} \right\} \frac{|z_{i,j}|^{s-t}}{(\rho_i-\varepsilon)^{s-t}} \binom{s}{t}$$

$$\leqslant \left(\frac{|\alpha|}{\rho_i-\varepsilon}\right)^{\sigma+1} \left\{ \sum_{s=0}^{k-1} c_2 \binom{s}{t} \left(\frac{|z_{i,j}|}{\rho_i-\varepsilon}\right)^{s-t} \cdot \frac{1}{(\rho_i-\varepsilon)^{t-k}} \right.$$

$$\left. + \sum_{s=k}^{\sigma} c_3 \binom{s}{t} \left(\frac{|z_{i,j}|}{\rho_i-\varepsilon}\right)^{s-t} \frac{1}{(\rho_i-\varepsilon)^{t-k}} \right\}$$

$$+ c_4\left(\frac{|\beta|}{\rho_i-\varepsilon}\right)^{\ell(\sigma+1)} \sum_{s=k}^{\sigma} \left(\frac{|z_{i,j}|}{\rho_i-\varepsilon}\right)^{s-t} \binom{s}{t} \frac{1}{(\rho_i-\varepsilon)^{t-k}}$$

$$\leqslant \left(\frac{|\alpha|}{\rho_i-\varepsilon}\right)^{\sigma+1} \left\{ c_2 \sum_{s=0}^{k-1} \binom{s}{t} q^{s-t} q^{t-k} + c_3 \sum_{s=k}^{\sigma} \binom{s}{t} q^{s-t} q^{t-k} \right\}$$

$$+ \left(\frac{|\beta|}{\rho_i-\varepsilon}\right)^{\ell(\sigma+1)} c_4 \cdot \sum_{s=k}^{\sigma} \binom{s}{t} q^{s-t} q^{t-k} .$$

Using the fact that $\binom{s}{t} \leqslant \sigma^{\sigma - \nu_0}$, $q < 1$ and the boundedness of k ($k \leqslant \sigma - \nu_0 = \nu_1 + \cdots + \nu_d$), we find (3.7). □

Combining the results of Lemma 1 and Lemma 2, we find (3.3). Furthermore

$$I_1 = \sum_{s=k}^{\infty} a_{i, s-k} \binom{s}{t} z_{i,j}^{s-t} = \sum_{s=0}^{\infty} a_{i,s} \binom{s+k}{t} z_{i,j}^{s+k-t}$$

$$= \frac{1}{t!} D^t [z^k f_i(z)]_{z_{i,j}},$$

showing (3.2).

Proof of Theorem 2 : We are now ready to prove Theorem 2. Denote the matrix of the system of equations (3.1) for the γ_k^ℓ by Δ_σ^ℓ. Then (3.2) and (3.3) imply that

$$\operatorname{Det} \Delta_\sigma^\ell = \operatorname{Det} \frac{1}{t!} D^t [z^k f_i(z)]_{z_{i,j}} + O\left(c_0 \, \sigma^{\sigma - \nu_0} \max_{1 \leqslant i \leqslant d} \left(q, \frac{|\alpha|}{\rho_i - \varepsilon}, \left(\frac{|\beta|}{\rho_i - \varepsilon} \right)^\ell \right)^{\sigma + 1} \right)$$

where $q = \max_{1 \leqslant i \leqslant d} \frac{1}{\rho_i - \varepsilon} \left[\max_{1 \leqslant j \leqslant \mu_i} \{1, |z_{ij}|\} \right]$. Remark that k ($0 \leqslant k \leqslant \sigma - \nu_0 - 1$) is used for the columns and t, j, i with $0 \leqslant t \leqslant \lambda_{i,j} - 1$, $1 \leqslant j \leqslant \mu_i$, $1 \leqslant i < d$ for the rows.

For the matrix Δ with elements $\frac{1}{t!} D^t [z^k f_i(z)] \, z_{i,j}$, observe that

$$\frac{1}{t!} D^t [z^k f_i(z)]_{z_{i,j}} = \sum_{\tau=0}^{t} \binom{k}{\tau} z_{i,j}^{k-\tau} \frac{f_i^{(t-\tau)}(z_{i,j})}{(t-\tau)!}.$$

Introduce the confluent Vandermonde matrix (known to be different from zero) by

$$V\binom{z_{i,j}}{\lambda_{i,j}} = \left(\binom{k}{\tau} z_{i,j}^{k-r}, \quad 0 \leqslant k \leqslant \sigma - \nu_0 - 1, \quad 0 \leqslant \tau \leqslant \lambda_{i,j} - 1, \quad 1 \leqslant j \leqslant \mu_i, \quad 1 \leqslant i \leqslant d \right).$$

Then

$$\Delta = \mathbb{D} V\binom{z_{i,j}}{\lambda_{i,j}}$$

where $\mathbb{D} = \text{Diag}\,(D_{1,1}, \dots, D_{1,\mu_1}, \dots, D_{d,1}, \dots, D_{d,\mu_d})$ is a block diagonal matrix with building blocks

$$D_{i,j} = \begin{pmatrix} f_i(z_{i,j}) & 0 & \dots & 0 \\ f_i'(z_{i,j}) & f_i(z_{i,j}) & \dots & 0 \\ \dots\dots\dots & & & \\ f_i^{(\mu_i-1)} & f_i^{(\mu_i-2)}(z_{i,j}) & \dots & f_i(z_{i,j}) \end{pmatrix}.$$

We conclude that $\Delta_\sigma^\ell \neq 0$ for σ sufficiently large.

The equations (3.1) uniquely define the numbers $\gamma_k^\ell\ (0 \leqslant k \leqslant \sigma - \gamma_0 - 1)$. Moroever the rank of the system (3.1) is $\sigma - \nu_0$.

We now return to the system of equations (2.10) with $d(\sigma+1)$ equations and unknowns $\gamma_k^\ell\ (0 \leqslant k \leqslant \sigma - \nu_0 - 1)$; $p_{i,j}^\ell\ (0 \leqslant j \leqslant \sigma - \nu_i, 1 \leqslant i \leqslant d)$. We shall show that this system of equations has full rank. This will prove Theorem 2.

To show this, we rewrite the equations (2.10) putting the γ_k^ℓ's first and look at the following three systems of equations :

(1) M_1 : Equations (2.10).
(2) M_2 : We replace in (2.10), the equations with $0 \leqslant s \leqslant \nu_i - 1$, $1 \leqslant i \leqslant d$ by (3.1).
(3) M_3 : Equations (3.1) and the system (2.10) together.

The determinant of the system M_2 is det Δ_σ^ℓ det $\tilde{\Delta}$, where $\tilde{\Delta}$ is the algebraic complement of Δ_σ^ℓ, where $\tilde{\Delta}$ is a block diagonal matrix with upper triangular matrices having $\alpha_{i,\nu_0}(=1)$ on their main diagonals. Thus det $\tilde{\Delta} = 1$ and det $\Delta_\sigma^\ell \neq 0$ for $\sigma \geqslant \sigma_0$, so that rank $M_2 = d(\sigma+1)$.

Since the equations (3.1) have been derived from (2.10), rank M_1 = rank M_3. From the trivial observation that rank $M_1 \leqslant d(\sigma+1)$, and that

$$d(\sigma+1) \geqslant \text{rank}\, M_1 = \text{rank}\, M_3 \geqslant \text{rank}\, M_2 = d(\sigma+1)$$

for σ sufficiently large, we see rank $M_1 = d(\sigma+1)$, which proves the (unique) existence of the interpolants. □

Remark 1 : The asymptotics for det Δ_σ^ℓ ($\sigma \to \infty$) imply the following result

$$\lim_{\sigma\to\infty} \sum_{k=0}^{\sigma-\nu_0} \gamma_k^\ell z^k = \prod_{i=1}^{d} B_i(z) \quad (1 \leqslant \ell \leqslant \infty).$$

Indeed the set of equations for $\gamma_k^\ell\ (0 \leqslant k \leqslant \sigma - \nu_0 - 1)$ has an asymptotic right hand side given by

$$-\frac{D^t[z^{\sigma-\nu_0} f_i(z)]\, z_{i,j}}{t!} = -\sum_{\tau=1}^{t} \binom{\sigma-\nu_0}{\tau} z_{i,j}^{\sigma-\nu_0-\tau} \frac{f_i^{(t-\tau)}(z_{i,j})}{(t-\tau)},$$

which shows that the solution γ_k^ℓ of the system is asymptotically equal to

$$\gamma_k^\ell \sim (-1)^{\sigma-\nu_0-k} \frac{\det V_k\binom{z_{i,j}}{\lambda_{i,j}}}{\det V\binom{z_{i,j}}{\lambda_{i,j}}}, \quad (0 \leqslant k \leqslant \sigma - \nu_0 - 1).$$

Here $V_k\binom{z_{i,j}}{\lambda_{i,j}}$ arises from $V\binom{z_{i,j}}{\lambda_{i,j}}$ by replacing the column $\binom{k}{j} z_{i,r}^{k-j}$ (row number j running like: $0 \leqslant j \leqslant \lambda_{i,r} - 1$, $1 \leqslant r \leqslant \mu_i$, $1 \leqslant i \leqslant d$) by the column $\binom{\sigma-\nu_0}{j} z_{i,r}^{\sigma-\nu_0-j}$ and $\mathbb{D}$ is not used. The quotient of these two Vandermonde determinants is just the elementary symmetric polynomial $\xi_{\sigma-\nu_0-k}$ of degree $\sigma - \nu_0 - k$ on the zeros of $\prod_{i=1}^{d} B_i(z) := \sum_{k=0}^{\sigma-\nu_0} (-1)^{\sigma-\nu_0-k} \xi_{\sigma-\nu_0-k} z^k$.

4. SOME TECHNICAL LEMMAS

In order to prove (2.13), it is necessary to have bounds on the differences of the denominators $B^\ell(z)$ and $B^\infty(z)$ for $1 \leqslant \ell < \infty$ and bounds on $B^\ell(z)$ for $1 \leqslant \ell \leqslant \infty$. We have to do the same for $U_i^\ell(z)$ and $U_i^\infty(z)$. This will be done in the following two lemmas:

LEMMA 3: *Let $\mathcal{K}$ be a compact subset of $|z|\langle\tau, \tau\rangle 0$ omitting the singularities of $F_1, ..., F_d$. Then*

$$\limsup_{\sigma\to\infty} \left\{\max_{z\in\mathcal{K}} |B^\infty(z) - B^\ell(z)|\right\}^{\frac{1}{\sigma}} \leqslant \max\left\{\frac{|\alpha|}{\rho_i}, \frac{|\beta|}{\rho_i}\right\}^\ell, \quad 1 \leqslant \ell < \infty \tag{4.1}$$

$$\lim_{\sigma\to\infty} \left\{\max_{z\in\mathcal{K}} |B_\ell(z)|\right\}^{\frac{1}{\sigma}} = \lim_{\sigma\to\infty} \left\{\min_{z\in\mathcal{K}} |B_\ell(z)|\right\}^{\frac{1}{\sigma}} = 1, \quad 1 \leqslant \ell \leqslant \infty. \tag{4.2}$$

Proof: We shall need an estimate on the differences of the coefficients of $B^\infty(z)$ and $B^\ell(z)$, which are determined by the system of equations (3.4), with $\gamma_{\sigma-\nu_0}^\ell = 1 \quad (1 \leqslant \ell \leqslant \infty)$. Observe that

$$\left|\sum_{s=0}^{\sigma} A_{i,s-k}^\infty z_{i,j}^{s-t}\binom{s}{t} - \sum_{s=0}^{\sigma} A_{i,s-k}^\ell z_{i,j}^{s-t}\binom{s}{t}\right| \leqslant I_4 + I_5, \tag{4.3}$$

where

$$I_4 = \sum_{s=0}^{k-1} |A^{\infty}_{i,s-k} - A^{\ell}_{i,s-k}| \binom{s}{k} |z_{i,j}|^{s-t}$$

$$I_5 = \sum_{s=k}^{\sigma} |A^{\infty}_{i,s-k} - A^{\ell}_{i,s-k}| \binom{s}{t} |z_{i,j}|^{s-t}.$$

For the case $k \leqslant s$, we have

$$|A^{\infty}_{i,s-k} - A^{\ell}_{i,s-k}| = \left| \sum_{r=0}^{\infty} \alpha^{r(\sigma+1)} a_{i,r(\sigma+1)+s-k} - \sum_{r=0}^{\ell-1} \sum_{m=0}^{\infty} \alpha^{r(\sigma+1)} \beta^{m\ell(\sigma+1)} a_{i,m\ell(\sigma+1)+r(\sigma+1)+s-k} \right|$$

$$= \left| \sum_{r=\ell}^{\infty} a_{i,r(\sigma+1)+s-k} \alpha^{r(\sigma+1)} - \sum_{r=0}^{\ell-1} \sum_{m=0}^{\infty} a_{i,m\ell(\sigma+1)+r(\sigma+1)+s-k} \alpha^{r(\sigma+1)} \beta^{m\ell(\sigma+1)} \right|$$

$$\leqslant \frac{1}{\left(\rho_i - \frac{\varepsilon}{2}\right)^{s-k}} \left\{ c_5 \left(\frac{|\alpha|}{\rho_i - \frac{\varepsilon}{2}} \right)^{\ell(\sigma+1)} + c_6 \left(\frac{|\beta|}{\rho_i - \frac{\varepsilon}{2}} \right)^{\ell(\sigma+1)} \right\}. \tag{4.4}$$

For $k > s$, we have

$$|A^{\infty}_{i,s-k} - A^{\ell}_{i,s-k}| = |\alpha|^{\sigma+1} |A^{\infty}_{i,\sigma+1+s-k} - A^{\ell}_{i,\sigma+1+s-k}|$$

and now with $\sigma + 1 + s$ playing the role of s in the above, we have

$$|A^{\infty}_{i,s-k} - A^{\ell}_{i,s-k}| \leqslant \left(\frac{|\alpha|}{\rho_i - \frac{\varepsilon}{2}} \right)^{\sigma+1} \frac{1}{\left(\rho_i - \frac{\varepsilon}{2}\right)^{s-k}} \times \left\{ c_5 \left(\frac{|\alpha|}{\rho_i - \frac{\varepsilon}{2}} \right)^{\ell(\sigma+1)} + c_6 \left(\frac{|\beta|}{\rho_i - \frac{\varepsilon}{2}} \right)^{\ell(\sigma+1)} \right\}. \tag{4.5}$$

For all ε sufficiently small, $0 < \varepsilon < \min\left\{\rho_i - \max_j\{1, |z_{i,j}|\}\right\}$ (since $\frac{|\alpha|}{\rho_i - \varepsilon/2} < 1$ and $\binom{s}{t} \leqslant s^t < \sigma^t$ and $\binom{s}{t} = 0$ for $s < t$), we get from (4.4) and (4.5) that

$$I_4 + I_5 \leqslant \sum_{\substack{s=0\\ s \geqslant t}}^{k-1} \left(\frac{|\alpha|}{\rho_i - \frac{\varepsilon}{2}}\right)^{\sigma+1} \frac{1}{\left(\rho_i - \frac{\varepsilon}{2}\right)^{s-k}} \times$$

$$\times \left\{ c_5 \left(\frac{|\alpha|}{\rho_i - \frac{\varepsilon}{2}}\right)^{\ell(\sigma+1)} + c_6 \left(\frac{|\beta|}{\rho_i - \frac{\varepsilon}{2}}\right)^{\ell(\sigma+1)} \right\} \sigma^t (\rho_i - \varepsilon)^{s-t}$$

$$+ \sum_{\substack{s=k\\ s \geqslant t}}^{\sigma} \frac{1}{\left(\rho_i - \frac{\varepsilon}{2}\right)^{s-k}} \left\{ c_5 \left(\frac{|\alpha|}{\rho_i - \frac{\varepsilon}{2}}\right)^{\ell(\sigma+1)} \right.$$

$$\left. + c_6 \left(\frac{|\beta|}{\rho_i - \frac{\varepsilon}{2}}\right)^{\ell(\sigma+1)} \right\} \sigma^t (\rho_i - \varepsilon)^{s-t}$$

$$\leqslant \sum_{s=t}^{\sigma} \frac{(\rho_i - \varepsilon)^{s-t}}{\left(\rho_i - \frac{\varepsilon}{2}\right)^{s-k}} \left\{ c_7 \left(\frac{|\alpha|}{\rho_i - \frac{\varepsilon}{2}}\right)^{\ell(\sigma+1)} + c_8 \left(\frac{|\beta|}{\rho_i - \frac{\varepsilon}{2}}\right)^{\ell(\sigma+1)} \right\} \sigma^t .$$

Since $\left(\frac{1}{\rho_i - \frac{\varepsilon}{2}}\right)^{t-k}$ is bounded (the range of t, k being $[0, \nu_1 + \cdots + \nu_d]$) and $\left(\frac{\rho_i - \varepsilon}{\rho_i - \frac{\varepsilon}{2}}\right)^{\ell(\sigma+1)} \sigma^t$ is also bounded, it follows that

$$I_4 + I_5 \leqslant c_9 \left(\frac{|\alpha|}{\rho_i - \varepsilon}\right)^{\ell(\sigma+1)} + c_{10} \left(\frac{|\beta|}{\rho_i - \varepsilon}\right)^{\ell(\sigma+1)} \tag{4.6}$$

$$(0 \leqslant t \leqslant \lambda_{i,j} - 1, \quad 1 \leqslant j \leqslant \mu_i, \quad 1 \leqslant i \leqslant d, \quad 0 \leqslant k \leqslant \sigma - \nu_0) .$$

From (4.3) and (4.4), we see that the determinants $\det \Delta_\sigma^\infty$ and $\det \Delta_\sigma^\ell$ of the system (2.4) are related by

$$\det \Delta_\sigma^\infty = \det \Delta_\sigma^\ell + c_{11} \max \left\{ \frac{|\alpha|}{\rho_i - \varepsilon}, \frac{|\beta|}{\rho_i - \varepsilon} \right\}^{\ell(\sigma+1)} . \tag{4.7}$$

Applying Cramer's rule, this implies that

$$\gamma_k^\infty = \gamma_k^\ell + c_{12} \max \left\{ \frac{|\alpha|}{\rho_i - \varepsilon}, \frac{|\beta|}{\rho_i - \varepsilon} \right\}^{\ell(\sigma+1)} \tag{4.8}$$

since $\varepsilon > 0$ was arbitrarily small, this yields (4.1).

Finally, $\lim_{\sigma \to \infty} B^{\infty}(z) = \lim_{\sigma \to \infty} B^{\ell}(z) = \prod_{i=1}^{d} B_i(z)$ (from the remark at the end of Section 3) and this limit polynomial is different from zero in $\mathcal{H}$ which yields (4.2).□

LEMMA 4: *Let* $\tau > 0$ *and* $\overline{D}_\tau = \{z \in \mathbb{C} \,|z| \leqslant \tau\}$. *Then for* $1 \leqslant \ell < \infty$,

$$\limsup_{\sigma \to \infty} \left\{ \max_{z \in \overline{D}_\tau} |U_i^{\infty}(z) - U_i^{\ell}(z)| \right\}^{\frac{1}{\sigma}} \leqslant \begin{cases} R^{\ell}, & \tau < \rho_i \\ \dfrac{R^{\ell}\tau}{\rho_i}, & \tau \geqslant \rho_i \end{cases} \tag{4.9}$$

$$\limsup_{\sigma \to \infty} \left\{ \max_{z \in \overline{D}_\tau} |U_i^{\infty}(z)| \right\}^{\frac{1}{\sigma}} \leqslant \begin{cases} 1, & \tau < \rho_i \\ \tau/\rho_i, & \tau \geqslant \rho_i . \end{cases} \tag{4.10}$$

Proof: Putting $\delta_{i,k} := \delta_{i,\ell,k} := p_{i,k}^{\infty} - p_{i,k}^{\ell}$ $(0 \leqslant \ell < \infty)$ with $p_{i,k}^{0} := 0$, we derive bounds on $\delta_{i,k}$ which lead to (4.7) and to (4.8) for $\ell = 0$.

We again start with the system of equations (2.10) for all unknowns $\gamma_k^{\ell}, p_{i,k}^{\ell}$. We multiply the equations (2.10) by $\binom{s-m}{t} z_{i,j}^{s-m-t}$ and sum over s from m to σ. The range for t, j, i is as before: $0 \leqslant t \leqslant \lambda_{i,j} - 1$, $1 \leqslant j \leqslant \mu_i$, $1 \leqslant i \leqslant d$. This leads to the following:

$$\sum_{s=m}^{\sigma} \sum_{k=0}^{\sigma - \nu_0} \alpha_{i,k} p_{i,s-k}^{\ell} \binom{s-m}{t} z_{i,j}^{s-m-t} = \sum_{k=0}^{\sigma-\nu_0} \left\{ \sum_{s=m}^{\sigma} A_{i,s-k}^{\ell} \binom{s-m}{t} z_{i,j}^{s-m-t} \right\} \gamma_k^{\ell} . \tag{4.11}$$

As $\binom{s-m}{t} = 0$ for $s - m < t$, there are no negative powers of $z_{i,j}$. The γ_k^{ℓ}'s are known, so we leave the right side and rearrange the summation on the left taking $s - k$ fixed say r. We keep in mind that $\alpha_{i,k} = 0$ for $k > \nu_i$ and $p_{i,r}^{\ell} = 0$ for $r > \sigma - \nu_i$. Then the left side of (4.11) becomes

$$\sum_{r=m-1}^{m-\nu_i} \left\{ \sum_{k=m-r}^{\nu_i} \alpha_{i,k} \binom{r+k-m}{t} z_{i,j}^{r+k-m-t} \right\} p_{i,r}^{\ell} +$$

$$+ \sum_{r=m}^{\sigma-\nu_i} \left\{ \sum_{k=0}^{\nu_i} \alpha_{i,k} \binom{r+k-m}{t} z_{i,j}^{r+k-m-t} \right\} p_{i,r}^{\ell} .$$

Now the coefficient of $p^{\ell}_{i,r}$ in the second sum can be written as

$$D^t[z^{r-m}B_i(z)]_{z_{i,j}}$$

which is zero, $0 \leqslant t \leqslant \lambda_{i,j} - 1$, $1 \leqslant j \leqslant \mu_i$, $1 \leqslant i \leqslant d$.

In the first sum, we replace r by $m - r$ and find that (4.11) reduces to

$$\sum_{r=1}^{\nu_i} \left\{ \sum_{k=r}^{\nu_i} \alpha_{i,k} \binom{k-r}{t} z_{i,j}^{k-r-t} \right\} p^{\ell}_{i,m-r}$$

$$= \sum_{k=0}^{\sigma-\nu_0} \left\{ \sum_{s=m}^{\sigma} A^{\ell}_{i,s-k} \binom{s-m}{t} z_{i,j}^{s-m-t} \right\} \gamma^{\ell}_k \quad (4.12)$$

$$(0 \leqslant t \leqslant \lambda_{i,j} - 1, \quad 1 \leqslant j \leqslant \mu_i).$$

We do not write these equations for $1 \leqslant i \leqslant d$ but we fix a value of $i \in \{1, ..., d\}$, i.e., we treat one polynomial in the numerator at a time.

(4.12) is a system of $\sum_{j=1}^{\mu_i} \lambda_{ij} = \nu_i$ equations of the ν_i unknowns $p^{\ell}_{i,m-1}, ..., p^{\ell}_{i,m-\nu_i}$. Letting m run over the values ν_i, $\nu_i + 1, ..., \sigma - \nu_i + 1$, we can give estimates for all the coefficients.

As the coefficients of $p^{\ell}_{i,m-r}$ $(1 \leqslant \ell \leqslant \infty)$ do not depend upon ℓ, we can write down the equations for $\delta_{i,m-r}(:= p^{\infty}_{i,m-r} - p^{\ell}_{i,m-r})$:

$$\sum_{r=1}^{\nu_i} \left\{ \sum_{k=r}^{\nu_i} \alpha_{i,k} \binom{k-r}{t} z_{i,j}^{k-r-t} \right\} \delta_{i,m-r} = \sum_{k=0}^{\sigma-\nu_0} \gamma^{\infty}_k \sum_{s=m}^{\sigma} A^{\infty}_{i,s-k} \binom{s-m}{t} z_{ij}^{s-m-t}$$

$$- \sum_{k=0}^{\sigma-\nu_0} \gamma^{\ell}_k \sum_{s=m}^{\sigma} A^{\ell}_{i,s-k} \binom{s-m}{t} z_{ij}^{s-m-t} \quad (4.13)$$

$$(0 \leqslant t \leqslant \lambda_{i,j} - 1, \quad 1 \leqslant j \leqslant \mu_i).$$

The coefficient matrix of (4.13) can be written as a product of two matrices. A typical row of the matrix is given by

$$\sum_{k=1}^{\nu_i} \alpha_{i,k} \binom{k-1}{t} z_{i,j}^{k-1-t}, \sum_{k=2}^{\nu_i} \alpha_{i,k} \binom{k-2}{t} z_{i,j}^{k-2-t}, ..., \sum_{k=\nu_i}^{\nu_i} \alpha_{i,k} \binom{k-\nu_i}{t} z_{i,j}^{k-\nu_i-t}$$

with t, j fixed.

This row helps us to write the matrix as a product of two matrices :

$$\left(\binom{0}{t} z_{i,j}^{-t} \binom{1}{t} z_{i,j}^{1-t} \cdots \binom{\nu_i - 2}{t} z_{i,j}^{\nu_i - 2 - t} \binom{\nu_i - 1}{t} z_{i,j}^{\nu_i - 2 - t} \right)$$

$$\times \begin{pmatrix} \alpha_{i,1} & \alpha_{i,2} & \cdots & \alpha_{i,\nu_i - 1} & \alpha_{i,\nu_i} \\ \alpha_{i,2} & \alpha_{i,3} & \cdots & \alpha_{i,\nu_i} & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ \alpha_{i,\nu_i - 1} & \alpha_{i,\nu_i} & \cdots & 0 & 0 \\ \alpha_{i,\nu_i} & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Thus the coefficient matrix is the confluent Vandermonde $V\binom{z_{i,j}}{\lambda_{i,j}}$ multiplied by a matrix whose determinant is 1, that is the determinant of the system is different from zero and we can apply Cramer's rule to find

$$\delta_{i,m-r} = \sum_{j=1}^{\mu_i} \sum_{t=0}^{\lambda_{i,j}-1} d_{r,j,t} \left\{ \sum_{k=0}^{\sigma-\nu_0} \gamma_k^{\infty} \sum_{s=m}^{\sigma} A_{i,s-k}^{\infty} \binom{s-m}{t} z_{i,j}^{s-m-t} \right.$$

$$\left. - \sum_{k=0}^{\sigma-\nu_0} \gamma_k^{\ell} \sum_{s=m}^{\sigma} A_{i,s-k}^{\ell} \binom{s-m}{t} z_{i,j}^{s-m-t} \right\} \tag{4.14}$$

where the $d_{r,j,t}$ are the minors when developing the determinant with the column of γ's on the r^{th} column place with respect to this column. These $d_{r,j,t}$ depend only on the poles of the function F_i, and on ν_i and t and so are uniformly bounded :

$$|d_{r,j,t}| \leqslant c_{16} \quad (\text{all} \quad r, j, t \quad \text{and} \quad m). \tag{4.15}$$

To estimate the difference between the curly brackets on the right side of (4.14), we denote it by Φ and we see that

$$\Phi = \sum_{k=0}^{\sigma-\nu_0} (\gamma_k^{\infty} - \gamma_k^{\ell}) \sum_{s=m}^{\sigma} A_{i,s-k}^{\infty} \binom{s-m}{t} z_{i,j}^{s-m-t}$$

$$+ \sum_{k=0}^{\sigma-\nu_0} \gamma_k^{\ell} \sum_{s=m}^{\infty} (A_{i,s-k}^{\infty} - A_{i,s-k}^{\ell}) \binom{s-m}{t} z_{i,j}^{s-m-t}. \tag{4.16}$$

If we set $R_\varepsilon := \max\left(\dfrac{|\alpha|}{\rho_1 - \frac{\varepsilon}{2}}, \dfrac{|\beta|}{\rho_1 - \frac{\varepsilon}{2}}\right)$, then we know that

(1) $|\gamma_k^{\infty} - \gamma_k^{\ell}| \leqslant c R_\varepsilon^{\ell(\sigma+1)}$ from (4.6)

(2) $|\gamma_k^\ell| \leqslant c$ from the remark at the end of Section 3

(3) the bounds for $|A^\infty_{i,s-k} - A^\ell_{i,s-k}|$ for $0 \leqslant s \leqslant k-1$ and $k \leqslant s \leqslant \sigma$, in (4.4) and (4.5)

(4) $|A^\infty_{i,s-k}| = |\alpha^{\sigma+1} A^\infty_{i,\sigma+1+s-k}| \leqslant c_{10}\left(\dfrac{|\alpha|}{\rho_i - \varepsilon}\right)^{\sigma+1} \dfrac{1}{\left(\rho_i - \frac{\varepsilon}{2}\right)^{s-k}}$,

$0 \leqslant s \leqslant k-1$, method of (4.4)

(5) $|A^\infty_{i,s-k}| = \left|\sum\limits_{r=0}^{\infty} \alpha^{r(\sigma+1)} a_{i,r(\sigma+1)+s-k}\right| \leqslant c_{20} \dfrac{1}{(\rho_i - \varepsilon)^{s-k}}$.

Using (1)-(5), it is easy to see from (4.16) that

$$|\Phi| \leqslant c_{21} \cdot \frac{R_\varepsilon^{\ell(\sigma+1)}}{(\rho_i - \varepsilon)^m}. \tag{4.17}$$

Then (4.14) and (4.17) gives

$$|\delta_{i,m-r}| \leqslant c_{22} \cdot \frac{R_\varepsilon^{\ell(\sigma+1)}}{(\rho_i - \varepsilon)^{m-r}}, \quad (r = 1, \ldots, \nu_i),$$

where c_{22} does not depend on m.

Repeating the argument with different values of m, we get

$$|\delta_{i,k}| \leqslant c_{23} \frac{R_\varepsilon^{\ell(\sigma+1)}}{(\rho_i - \varepsilon)^k}, \quad (0 \leqslant k \leqslant \sigma - \nu_i, \quad 1 \leqslant i \leqslant d, \quad \ell \geqslant 0). \tag{4.18}$$

Finally, we have

$$\begin{aligned}\max_{z \in D_r} |U_i^\infty(z) - U_i^\ell(z)| &= \max_{z \in D_r} \left|\sum_{k=0}^{\sigma-\nu_i} \delta_{i,k} z^k\right| \\ &\leqslant \sum_{k=0}^{\sigma-\nu_i} c_{23} \cdot \frac{R_\varepsilon^{\ell(\sigma+1)}}{(\rho_i - \varepsilon)^k} \tau^k \\ &= c_{23} R_\varepsilon^{\ell(\sigma+1)} \sum_{k=0}^{\sigma-\nu_i} \left(\frac{\tau}{\rho_i - \varepsilon}\right)^k \leqslant M \cdot R_\varepsilon^{\ell(\sigma+1)},\end{aligned}$$

if $\tau < \rho_i - \varepsilon$.

For $\tau \geqslant \rho_i$, we have

$$(\sigma - \nu_i + 1)\left(\frac{\tau}{\rho_i - \varepsilon}\right)^{\sigma - \nu_i} \geqslant \sum_{k=0}^{\sigma - \nu_i} \left(\frac{\tau}{\rho_i - \varepsilon}\right)^k$$

$$= \frac{\left(\frac{\tau}{\rho_i - \varepsilon}\right)^{\sigma - \nu_i + 1} - 1}{\frac{\tau}{\rho_i - \varepsilon} - 1} > \frac{\left(\frac{\tau}{\rho_i - \varepsilon}\right)^{\sigma - \nu_i + 1}}{\frac{\tau}{\rho_i - \varepsilon} - 1}$$

whence we obtain

$$\lim_{\sigma \to \infty} \left\{ \sum_{k=0}^{\sigma - \nu_i} \left(\frac{\tau}{\rho_i - \varepsilon}\right)^k \right\}^{\frac{1}{\sigma}} = \frac{\tau}{\rho_i - \varepsilon}.$$

The first part of the lemma, that is (4.9), follows immediately as $\varepsilon > 0$ may be taken arbitrarily small. As stated before, (4.10) follows on taking $\ell = 0$. □

5. PROOF OF THEOREM 1

Let $\mathcal{H}$ be a compact subset of $|z| < \tau, \tau > 0$. Then

$$\limsup_{\sigma \to \infty} \left\{ \max_{z \in \mathcal{H}} |P^{\infty}_{i,\sigma}(z) - P^{\ell}_{i,\sigma}(z)| \right\}^{\frac{1}{\sigma}} =$$

$$= \limsup_{\sigma \to \infty} \left\{ \max_{z \in \mathcal{H}} |U^{\infty}_i(z)\, B^{\ell}(z) - U^{\ell}_i(z)\, B^{\infty}(z)| \right\}^{\frac{1}{\sigma}} \text{ by } (4.2)$$

$$\leqslant \limsup_{\sigma \to \infty} \left\{ \max_{z \in \mathcal{H}} |U^{\infty}_i(z)|\ |B^{\ell}(z) - B^{\infty}(z)| + \max_{z \in \mathcal{H}} |B^{\infty}(z)|\ |U^{\infty}_i(z) - U^{\ell}_i(z)| \right\}^{\frac{1}{\sigma}}$$

$$\leqslant \max\left\{ \frac{\tau}{\rho_i} R^{\ell}, 1 \cdot \frac{\tau}{\rho_i} R^{\ell} \right\} \text{ by } (4.10), (4.1) \text{ and } (4.2), (4.9)$$

$$= \frac{\tau}{\rho_i} R^{\ell} \quad \textit{for} \quad \tau \geqslant \rho_i.$$

Now fix a value of $i \in \{1, ..., d\}$, choosing $\tau \geq \rho_i$ in such a manner that

$$\frac{\tau R^{\ell}}{\rho_i} < 1 .$$

Then the above implies that

$$\lim_{\sigma \to \infty} P^{\infty}_{i,\sigma}(z) - P^{\ell}_{i,\sigma}(z) = 0$$

and the convergence is uniform on compact sets omitting the poles of $F_1, ..., F_d$.□

6. INTERPOLATION WITH MULTIPLE NODES

Let $\vec{F} = (F_1, ..., F_d)$ be as in (2.1), (2.2) and let $r \geq 1$ be a natural number ; ℓ is an integer, $\ell \geq 1$. Finally put $n = \sigma + 1$. If the Taylor-sections of the f_i are given by

$$\sum_{k=0}^{n\ell r - 1} a_{i,k} z^k , \tag{6.1}$$

then the Hermite-interpolant to (6.1) on the zeros of $(z^n - 1)^r$ will be denoted by

$$\tilde{f}_{i,\ell}(z) = \sum_{t=0}^{nr-1} A^{\ell}_{i,t} z^t \tag{6.2}$$

with coefficients

$$A^{\ell}_{i,t} = a_{i,t} + \sum_{j=0}^{(\ell-1)r-1} b_{j,s}\, a_{i,(r+j)n+k} \quad \text{for} \quad t = sn + k \begin{cases} 0 \leq s \leq r-1 \\ 0 \leq k \leq n-1 \end{cases}, \tag{6.3}$$

where

$$b_{j,s} = (-1)^{r+1-s} \binom{r+j-s-1}{r-s-1} \binom{r+j}{s}, \quad (0 \leq s \leq r-1, \quad j \geq 0) . \tag{6.4}$$

For $\ell = \infty$ we write

$$\tilde{f}_{i,\infty}(z) = \sum_{t=0}^{nr-1} A^{\infty}_{i,t} z^t \tag{6.5}$$

with

$$A_{i,t}^{\infty} = a_{i,t} + \sum_{j=0}^{\infty} b_{j,s}\, a_{i,(r+j)n+k} \quad \text{for} \quad t = sn + k \begin{cases} 0 \leqslant s \leqslant r-1 \\ 0 \leqslant k \leqslant n-1 \end{cases} \tag{6.6}$$

and $b_{j,s}$ as in (6.4).

The interpolation problem with multiple nodes is now formulated for $1 \leqslant \ell \leqslant \infty$ as :

$$\begin{cases} \text{find d rational functions } \dfrac{U_i^{\ell}(z)}{B^{\ell(z)}},\ \deg U_i^{\ell}(z) \leqslant nr - v_i - 1 \\ (1 \leqslant i \leqslant d), \\ \deg B^{\ell} \leqslant n - v_0 - 1, \text{ that interpolate the } d \text{ rationals} \\ \dfrac{\sum_{t=0}^{nr-1} A_{i,t}^{\ell} z^t}{B_i(z)}\ (i = 1, \dots, d) \text{ on the zeros of } (z^n - 1)^r \text{ ; moreover the} \\ \text{coefficient of } z^{n - v_0 - 1} \text{ in } B^{\ell} \text{ will be 1 ;} \\ B^{\ell}(z) = \sum_{k=0}^{n - v_0 = 1} \gamma_k^{\ell} z^k,\ \gamma_{n - v_0 - 1}^{\ell} = 1. \end{cases} \tag{6.7}$$

Using the same methods as in the previous sections, it is possible to derive the following result.

THEOREM 3 :

A. *For n sufficiently large, the interpolation problem* (6.7) *has a unique solution that moreover satisfies*

$$\lim_{n \to \infty} \gamma_k^{\ell} = \zeta_k, \quad \textit{with} \quad \sum_{k=0}^{n - v_0 - 1} \zeta_k z^k = \prod_{i=1}^{d} B_i(z) \text{ ;} \quad 1 \leqslant \ell \leqslant \infty .$$

B. *Let $\mathcal{H}$ be a compact subset of $|z| < \tau$, $\tau > 0$, that omits the singularities of the functions $F_i(1 \leqslant i \leqslant d)$, then*

$$\limsup_{n \to \infty} \left\{ \max_{z \in \mathcal{H}} \left| \frac{U_i^{\infty}(z)}{B^{\infty}(z)} - \frac{U_i^{\ell}(z)}{B^{\ell}(z)} \right| \right\}^{\frac{1}{n}} \leqslant \begin{cases} R^{(\ell-1)r+1} \left(\frac{\tau}{\rho_i}\right)^r & (\tau \geqslant \rho_i) \\ R^{(\ell-1)r+1} & (\tau < \rho_i) \end{cases},$$

with $R = \max_{1 \leqslant i \leqslant d} \dfrac{1}{\rho_i}$.

C. *Specifically we have for* $|z| < \rho_i R^{-\left(\ell - 1 + \frac{1}{r}\right)}$:

$$\lim_{n \to \infty} \frac{U_i^{\infty}(z)}{B^{\infty}(z)} - \frac{U_i^{\ell}(z)}{B^{\ell}(z)} = 0 ,$$

uniformly and geometrically in compact subsets of $\mathcal{H}$ *omitting the singularities.*

The proofs of these results will be given elsewhere.

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