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VIBRATIONS OF A FOLDED PLATE (*)

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Abstract. — *In this article, it is shown that the eigenvalues and eigenvectors of the three-dimensional linearized elasticity operator in a thin folded plate converge toward the eigenvalues and eigenvectors of a limit 2d-2d model as the thickness of the plates tends to 0. The convergence of the associated stresses is also established.*

Résumé. — *On montre dans cet article que les valeurs propres et vecteurs propres de l'opérateur de l'élasticité linéarisée tridimensionnelle dans un domaine en forme de plaque pliée convergent vers les valeurs propres et vecteurs propres d'un modèle limite 2d-2d quand l'épaisseur des plaques tend vers 0. On établit également que les contraintes qui leur sont associées convergent.*

0. INTRODUCTION

The purpose of this article is to derive two-dimensional eigenvalue problems that describe the limit behavior of the three-dimensional eigenvalue problem of linearized elasticity in thin folded plates when. the thickness of the plates tends to 0. This is a question of interest since the result provides a model for the free vibrations of folded plate structures. This purpose is achieved by combining the techniques of Le Dret [10]-[11]-[12], which deal with the modeling of folded plates in the static case, with the techniques of Ciarlet and Kesavan [4], who consider the limit eigenvalue problem for a single plate. Both works contain a good part of the ingredients we need here and we have thus felt free to refer to them rather extensively, in order to keep the size of the article within reasonable bounds.

Let us recall that the problem at hand is a special case of the more general problem of modeling and controlling elastic "multi-structures", i.e., structures combining $3d$-bodies with plates and rods that are held together by appropriate junctions. Significant progress in this area from the mathematical viewpoint has recently been achieved for $3d$-$2d$ junctions in the works of Ciarlet, Le Dret and Nzengwa [6]-[7] and Ciarlet and Le Dret [5] in the static case and of Bourquin and Ciarlet [3] for the eigenvalue problem. The resulting models give good numerical results, see Aufranc [2]. For $2d$-$2d$ junctions, i.e., folded plates, we have already mentioned Le Dret [10]-[11]-[12] who treat the static case. The case of $1d$-$1d$ junctions (junctions between rods) is also analyzed in the static case in Le Dret [13].

The central idea behind all these works is always the same. It consists in scaling each part of the elastic structure independently of the others, in the same way as is usually done for single plates or rods. These scalings must be performed in such a way that the junction region between two parts is taken into account in each of the scaled parts. The scaled displacements are then defined on separate domains, but satisfy some compatibility relations in each of the scaled images of the junction region. Passing to the limit in these relations yield the limit junction conditions.

To be more specific, we consider here the same standard folded plate as in [11]-[12], i.e., an homogeneous isotropic linearly elastic body consisting of two plates of thickness ε perpendicular to each other (see [12] for more general geometries). The Lamé moduli of the materials are supposed to be of the form $\varepsilon^{-2}(\mu, \lambda)$ and the structures are assumed to be clamped on one plate only. It is shown that the eigenvalues η_p^ε of the three-dimensional problem converge as $\varepsilon \to 0$ toward the eigenvalues η_p^0 of a well-posed $2d$-$2d$ eigenvalue problem. Accordingly, the scaled eigenfunctions converge toward eigenfunctions of the limit $2d$-$2d$ model. This model is as follows. The limit eigenfunctions are of Kirchhoff-Love type in each plate with no in-plane components. They are thus determined by pairs $(\zeta_2^{p\prime}, \zeta_1^{p\prime\prime})$ of H^2-functions of the in-plane variables of each plate (i.e., with the coordinate convention assumed throughout, x_1, x_3 and x_2, x_3 respectively) that correspond to the flexural displacements of the plates. These displacements are such that (assuming that $\zeta_2^{p\prime}$ is the displacement of the clamped plate and denoting the fold by γ)

(i) $\zeta_1^{p\prime\prime} = 0$ on γ and $\zeta_2^{p\prime} = -a(x_3 - 1/2) + b$ on γ with $(a, b) \in \mathbb{R}^2$, which indicates a stiffening effect of the fold,

(ii) $\partial_1 \zeta_2^{p\prime} = -\partial_2 \zeta_1^{p\prime\prime}$ on γ, i.e., the angle of the plates stays equal to $\pi/2$ during the vibrations of the structure.

These two relations, which we call limit junction conditions, determine a Hilbert space $\mathcal{V}$ in which the limit eigenvalue problem is set. The limit

eigenvalues are characterized as min-max of Rayleigh quotients over the usual sets. The numerator of the Rayleigh quotient is the sum of the usual elastic energies of each plate and the denominator is the sum of the L^2-norms of the test-functions $(\xi', \xi'') \in \mathcal{V}$ plus an extra term of the form $\frac{5}{12} a(\xi')^2 + b(\xi')^2$, where $a(\xi')$, $b(\xi')$ denote the constants a and b of (i) for an arbitrary element (ξ', ξ'') of $\mathcal{V}$. This extra term represents the contribution of the overall rigid motion of the free plate, which follows the rigid motion of the fold, to the limit eigenvalue problem. The particular factors 5/12 and 1 are due to the specific square shape of the plates we consider here.

In addition, a very simple and general proof of convergence of the scaled stresses is given. This proof extends that of Destuynder [9]. It is shown that the scaled stresses $\sigma_{\alpha\beta}(\varepsilon)$, $\sigma_{\alpha3}(\varepsilon)$ and $\sigma_{33}(\varepsilon)$ (with the standard index convention, not used in this article) converge respectively in the spaces $L^2(\Omega)$, $H^1(0, 1 ; H^{-1}(\omega))$ and $H^1(0, 1 ; H^{-2}(\omega))$ (again with standard notation) in the strong sense, toward the limits that are usually found by asymptotic expansions. Note that we do not use such asymptotic expansions here and that the proofs are direct.

NOTATION

Let Ω be an open subset of $\mathbb{R}^n$ and let m be a positive integer. We denote by

$\mathcal{D}(\Omega)$ the space of C^∞-functions with compact support in Ω,

$\mathcal{D}'(\Omega)$ the space of distributions on Ω,

$L^2(\Omega)$ the space of (classes) of measurable square-integrable real functions on Ω,

$H^m(\Omega)$ the space of functions of $L^2(\Omega)$ whose distributional derivatives up to the order m belong to $L^2(\Omega)$,

$H_0^m(\Omega)$ the closure of $\mathcal{D}(\Omega)$ in $H^m(\Omega)$,

$H^{-m}(\Omega)$ the dual space of $H_0^m(\Omega)$.

More generally $H_\Gamma^m(\Omega)$ is the space of H^m-functions whose traces vanish on a part Γ of the boundary of Ω.

Finally, if X is a Hilbert space, $L^2(0, 1 ; X)$ is the space of measurable functions from $]0, 1[$ into X such that $\int_0^1 \|u(t)\|_X^2 \, dt < + \infty$, and $H^m(0, 1 ; X)$ is the space of functions of $L^2(0, 1 ; X)$ such that all their distributional derivatives with respect to t up to the order m belong to $L^2(0, 1 ; X)$.

We refer to [1], [14] and [15] for the general properties of these spaces.

1. THE THREE-DIMENSIONAL PROBLEM

We consider the standard eigenvalue problem of three-dimensional linearized elasticity for domains that model thin folded plates in the sense of [11], [12]. Let us briefly review the notation which is the same as in those two papers. The reference configurations of the elastic bodies under consideration are the sets Ω_ε, defined for $\varepsilon > 0$ as

$$\Omega_\varepsilon \overset{\text{def}}{=} \Omega'_\varepsilon \cup \Omega''_\varepsilon \,,$$

where :

$$\Omega'_\varepsilon \overset{\text{def}}{=} \left\{ x \in \mathbb{R}^3 \,;\, 0 < x_1, x_3 < 1, 0 < x_2 < \varepsilon \right\} \,,$$
$$\Omega''_\varepsilon \overset{\text{def}}{=} \left\{ x \in \mathbb{R}^3 \,;\, 0 < x_2, x_3 < 1, 0 < x_1 < \varepsilon \right\} \,.$$

The bodies are made of homogeneous elastic materials that depend on ε in the following way. We assume that the Lamé moduli $(\mu^\varepsilon, \lambda^\varepsilon)$ of the bodies are of the form

$$(\mu^\varepsilon, \lambda^\varepsilon) = \varepsilon^{-2}(\mu, \lambda) \qquad (1.1)$$

with $\mu, \lambda > 0$ and independent of ε. The specific choice (1.1) does not restrict the generality of our results. Indeed, the results corresponding to any other choice (including $(\mu^\varepsilon, \lambda^\varepsilon)$ independent of ε as in [4]) may be deduced from the results given here via an appropriate rescaling. The reason for assuming (1.1) lies in the fact that, in this case, the eigenvalues of the $3d$-problem turn out to converge to the eigenvalues of the $2d$-$2d$ model as $\varepsilon \to 0$ *without rescaling* (compare with the eigenvalues of the order ε^2 in [4]). From the viewpoint of mechanics, (1.1) is an assumption on the *rigidity* of the materials, i.e., the materials are assumed to become more and more rigid as the thickness of the plates goes to 0, with the specific order ε^{-2} indicated in (1.1). This assumption is thus the only one that yields such a limit behavior for the eigenvalues and we find it more agreeable to work with, instead of with any other (mathematically) equivalent assumption.

We also assume that the boundary conditions are as in [12], i.e., clamping $u = 0$ on Γ'_ε and the rest of the boundary $\partial\Omega_\varepsilon \backslash \Gamma'_\varepsilon$ is traction-free, where

$$\Gamma'_\varepsilon \overset{\text{def}}{=} \bar{\Omega}'_\varepsilon \cap \{x_1 = 1\} \,, \quad \Gamma''_\varepsilon \overset{\text{def}}{=} \bar{\Omega}''_\varepsilon \cap \{x_2 = 1\}$$

(the simpler case when clamping holds on parts of both plates, e.g. also, on Γ''_ε is left to the reader).

The eigenvalue problem for the folded plates under consideration consists in finding pairs of scalars η^ε and nonzero displacement fields u^ε satisfying the following set of partial differential equations:

$$
\begin{cases}
- \operatorname{div} \sigma^\varepsilon = \eta^\varepsilon u^\varepsilon \text{ in } \Omega_\varepsilon , \\[4pt]
\quad \sigma^\varepsilon = 2 \, \mu^\varepsilon \, e(u^\varepsilon) + \lambda^\varepsilon \operatorname{tr} e(u^\varepsilon) \operatorname{Id} \text{ in } \Omega_\varepsilon , \\[4pt]
e_{ij}(u^\varepsilon) = \dfrac{1}{2} \, (\partial_i u_j^\varepsilon + \partial_j u_i^\varepsilon) \text{ in } \Omega_\varepsilon , \\[4pt]
\quad u^\varepsilon = 0 \text{ on } \Gamma_\varepsilon' , \\[4pt]
\quad \sigma^\varepsilon n^\varepsilon = 0 \text{ on } \partial\Omega_\varepsilon \backslash \Gamma_\varepsilon' ,
\end{cases}
\tag{1.2}
$$

where σ^ε is the stress tensor, $e(u^\varepsilon)$ is the linearized strain tensor corresponding to the displacement u^ε, $(\operatorname{div} \sigma)_i = \partial_j \sigma_{ij}$ for any tensor field σ and n^ε is the outer unit normal vector to $\partial\Omega_\varepsilon$ which is defined almost everywhere on $\partial\Omega_\varepsilon$.

The mechanical motivation for problem (1.2) comes from linearized elastodynamics. In elastodynamics, the response $u^\varepsilon(x, t)$ of the structures Ω_ε under the action of body force densities $f^\varepsilon(x, t)$ and surface tractions $g^\varepsilon(x, t)$ is governed by the equations

$$
\begin{cases}
\rho^\varepsilon \dfrac{\partial^2 u^\varepsilon(x, t)}{\partial t^2} = \operatorname{div} \sigma^\varepsilon(x, t) + \rho^\varepsilon \, f^\varepsilon(x, t) \text{ in } \Omega_\varepsilon , \\[6pt]
\quad u^\varepsilon(x, t) = 0 \text{ on } \Gamma_\varepsilon' , \\[6pt]
\sigma^\varepsilon(x, t) \, n^\varepsilon(x) = g^\varepsilon(x, t) \text{ on } \partial\Omega_\varepsilon \backslash \Gamma_\varepsilon' , \\[6pt]
\text{plus appropriate initial data for } u^\varepsilon \text{ and } \dfrac{\partial u^\varepsilon}{\partial t} \text{ at } t = 0 ,
\end{cases}
\tag{1.3}
$$

where ρ^ε is the density of the folded plates. If we look for *stationary solutions* to (1.3) under zero loading, i.e., solutions of the special form $u^\varepsilon(x, t) = u^\varepsilon(x) \, e^{i \omega^\varepsilon t}$, then, clearly, the pair $(\rho^\varepsilon (\omega^\varepsilon)^2, u^\varepsilon(x))$ is a solution of the eigenvalue problem (1.2) and vice versa. Thus, solving problem (1.2) is equivalent to finding the *free vibration modes* u^ε of the structures and the corresponding *proper frequencies* $\dfrac{1}{2 \pi} \sqrt{\dfrac{\eta^\varepsilon}{\rho^\varepsilon}}$.

Problem (1.2) falls within the classical framework of abstract spectral theory for self-adjoint compact operators. Let us introduce the relevant spaces

$$
\mathbf{H}^\varepsilon = L^2(\Omega^\varepsilon)^3 , \quad \mathbf{V}^\varepsilon = H_{\Gamma_\varepsilon'}^1(\Omega^\varepsilon)^3 .
\tag{1.4}
$$

Then it is well-known, see e.g. [16], that, for all $\varepsilon > 0$ fixed, the eigenvalues

are arranged in an increasing sequence

$$0 < \eta_1^\varepsilon \leqslant \eta_2^\varepsilon \leqslant \ldots \leqslant \eta_p^\varepsilon \leqslant \ldots \tag{1.5}$$

with $\eta_p^\varepsilon \to +\infty$ as $p \to +\infty$, and that the corresponding eigenvectors $u^{\varepsilon,p}$ (normalized in $\mathbf{H}^\varepsilon$) form an hilbertian orthonormal basis of $\mathbf{H}^\varepsilon$ and also an hilbertian orthogonal basis of $\mathbf{V}^\varepsilon$ equipped with the inner product $\int_{\Omega^\varepsilon} \sigma^\varepsilon(e(u^\varepsilon)) : e(v^\varepsilon) \, dx$.

As in [4], we will make an essential use of the following variational characterization of the eigenvalues. First of all, for any $\varepsilon > 0$ we define the *Rayleigh quotient*

$$R^\varepsilon(v^\varepsilon) \overset{\text{def}}{=} \frac{\displaystyle\int_{\Omega_\varepsilon} \sigma^\varepsilon(e(v^\varepsilon)) : e(v^\varepsilon) \, dx}{\displaystyle\int_{\Omega_\varepsilon} v^\varepsilon \cdot v^\varepsilon \, dx} \quad \text{for all } v^\varepsilon \in \mathbf{V}^\varepsilon \setminus \{0\} \; . \tag{1.6}$$

Then we have

$$\eta_p^\varepsilon = \min_{E_p \in \mathscr{W}_p^\varepsilon} \; \max_{\substack{v^\varepsilon \in E_p \\ v^\varepsilon \neq 0}} R^\varepsilon(v^\varepsilon) \; , \tag{1.7}$$

where $\mathscr{W}_p^\varepsilon$ is the set of all vector subspaces of $\mathbf{V}^\varepsilon$ of dimension p, see e.g. [8] or [16]. Moreover, we will use the following variational characterization of the eigenvectors:

$$\int_{\Omega_\varepsilon} \sigma^\varepsilon(e(u^{\varepsilon,p})) : e(v^\varepsilon) \, dx = \eta_p^\varepsilon \int_{\Omega_\varepsilon} u^{\varepsilon,p} \cdot v^\varepsilon \, dx \tag{1.8}$$

for all $v^\varepsilon \in \mathbf{V}^\varepsilon$. For our purposes here, it will be convenient to normalize the eigenvectors according to

$$\int_{\Omega_\varepsilon} u^{\varepsilon,n} \cdot u^{\varepsilon,p} \, dx = \varepsilon \delta_{np} \; . \tag{1.9}$$

2. RESCALING THE STRUCTURE

The rescaling we use here is exactly the same as in [11], [12]. We will thus only briefly describe it. First of all, we introduce two copies of $\mathbb{R}^3$, which we call $(\mathbb{R}^3)'$ and $(\mathbb{R}^3)''$ (to be shortly identified to one another), and two rescaled plates independent of ε

$$\begin{cases} \Omega' \overset{\text{def}}{=} \left\{ x \in (\mathbb{R}^3)' \; ; \; 0 < x_i < 1 \right\}, \\ \Omega'' \overset{\text{def}}{=} \left\{ x \in (\mathbb{R}^3)'' \; ; \; 0 < x_i < 1 \right\}. \end{cases} \quad (2.1)$$

We only need to define the primed objects, since their double-primed counterparts are defined by similar expressions, up to a switching of the indices 1 and 2. Define thus

$$\Gamma' = \bar{\Omega}' \cap \{x_1 = 1\}, \quad \bar{\omega}' = \bar{\Omega}' \cap \{x_2 = 0\}, \quad J'_\varepsilon = \Omega' \cap \{0 < x_1 < \varepsilon\}. \quad (2.2)$$

We introduce a scaling mapping ϕ^ε

$$\phi^\varepsilon : \Omega' \cup \Omega'' \to \Omega_\varepsilon$$

$$x \mapsto \begin{cases} (x_1, \varepsilon x_2, x_3) & \text{if } x \in \Omega', \\ (\varepsilon x_1, x_2, x_3) & \text{if } x \in \Omega''. \end{cases} \quad (2.3)$$

Note this crucial property of ϕ^ε which is that ϕ^ε takes the junction region into account twice, once in J'_ε and another time in J''_ε. Let

$$\mathbf{V} \overset{\text{def}}{=} H^1_{\Gamma'}(\Omega')^3 \times H^1(\Omega'')^3, \quad (2.4)$$

and

$$\Theta^\varepsilon : \mathbf{V}^\varepsilon \to \mathbf{V}$$
$$v \mapsto \left((\varepsilon^{-1} v_1, v_2, \varepsilon^{-1} v_3) \circ \phi^\varepsilon, (v_1, \varepsilon^{-1} v_2, \varepsilon^{-1} v_3) \circ \phi^\varepsilon \right). \quad (2.5)$$

The scaling operator Θ^ε is not onto. Its range, $\Theta^\varepsilon \mathbf{V}^\varepsilon$ consists exactly of those pairs (v', v'') in $\mathbf{V}$ that satisfy the fundamental relations

$$\begin{cases} \varepsilon v'_1(\varepsilon x_1, x_2, x_3) = v''_1(x_1, \varepsilon x_2, x_3), \\ v'_2(\varepsilon x_1, x_2, x_3) = \varepsilon v''_2(x_1, \varepsilon x_2, x_3), \\ v'_3(\varepsilon x_1, x_2, x_3) = v''_3(x_1, \varepsilon x_2, x_3), \end{cases} \quad (2.6)$$

for almost all (x_1, x_2, x_3) in $]0, 1[^3$. Then with any eigenfunction $u^{\varepsilon, p}$, we associate a rescaled eigenfunction $u^p(\varepsilon) \in \Theta^\varepsilon \mathbf{V}^\varepsilon$ by

$$u^p(\varepsilon) = \Theta^\varepsilon u^{\varepsilon, p}. \quad (2.7)$$

For brevity, we also set $\mathbf{V}(\varepsilon) = \Theta^\varepsilon \mathbf{V}^\varepsilon$. As in [11]-[12], we then introduce two quadratic forms

$$B'_\varepsilon(u, v) \overset{\text{def}}{=} 2\,\mu e_{\alpha'\beta'}(u')\,e_{\alpha'\beta'}(v') + \lambda e_{\alpha'\alpha'}(u')\,e_{\beta'\beta'}(v') +$$

$$+ \varepsilon^{-2}[4\,\mu e_{\alpha'2}(u')\,e_{\alpha'2}(v') + \lambda\,(e_{\alpha'\alpha'}(u')\,e_{22}(v') + e_{22}(u')\,e_{\alpha'\alpha'}(v'))] +$$

$$+ \varepsilon^{-4}(2\,\mu + \lambda)\,e_{22}(u')\,e_{22}(v')\,, \tag{2.8}$$

and $B''_\varepsilon(u, v)$ as explained above. Then, if we perform the change of variables (2.3), (2.5) in the integrals, formulas (1.6), (1.7), (1.8) and (1.9) become respectively

$$R^\varepsilon(v(\varepsilon)) = \int_{\Omega'} B'_\varepsilon(v'(\varepsilon), v'(\varepsilon))\, dx + \int_{\Omega''\backslash J''_\varepsilon} B''_\varepsilon(v''(\varepsilon), v''(\varepsilon))\, dx \Bigg/$$

$$\tag{2.9}$$

$$\int_{\Omega'} (v'_2(\varepsilon)\, v'_2(\varepsilon) + \varepsilon^2\, v'_{\alpha'}(\varepsilon)\, v'_{\alpha'}(\varepsilon))\, dx + \int_{\Omega''\backslash J''_\varepsilon} (v''_1(\varepsilon)\, v''_1(\varepsilon) + \varepsilon^2\, v''_{\alpha''}(\varepsilon)\, v''_{\alpha''}(\varepsilon))\, dx$$

$$\eta^\varepsilon_p = \min_{E_p \in \mathscr{W}_p(\varepsilon)}\ \max_{\substack{v(\varepsilon) \in E_p \\ v(\varepsilon) \neq 0}} R^\varepsilon(v(\varepsilon))\,, \tag{2.10}$$

where $\mathscr{W}_p(\varepsilon)$ is the set of all vector subspaces of $V(\varepsilon)$ of dimension p,

$$\int_{\Omega'} B'_\varepsilon(u^{p\prime}(\varepsilon), v'(\varepsilon))\, dx + \int_{\Omega''\backslash J''_\varepsilon} B''_\varepsilon(u^{p\prime\prime}(\varepsilon), v''(\varepsilon))\, dx =$$

$$= \eta^\varepsilon_p \Bigg[\int_{\Omega'} (u^p_2{}'(\varepsilon)\, v'_2(\varepsilon) + \varepsilon^2\, u^{p\,\prime}_{\alpha'}(\varepsilon)\, v'_{\alpha'}(\varepsilon))\, dx +$$

$$+ \int_{\Omega''\backslash J''_\varepsilon} (u^{p\prime\prime}_1(\varepsilon)\, v''_1(\varepsilon) + \varepsilon^2\, u^{p\prime\prime}_{\alpha''}(\varepsilon)\, v''_{\alpha''}(\varepsilon))\, dx \Bigg] \tag{2.11}$$

for all $v(\varepsilon)$ in $V(\varepsilon)$, and

$$\int_{\Omega'} (u^{n\prime}_2(\varepsilon)\, u^{p\prime}_2(\varepsilon) + \varepsilon^2\, u^{n\prime}_{\alpha'}(\varepsilon)\, u^{p\prime}_{\alpha'}(\varepsilon))\, dx +$$

$$+ \int_{\Omega''\backslash J''_\varepsilon} (u^{n\prime\prime}_1(\varepsilon)\, u^{p\prime\prime}_1(\varepsilon) + \varepsilon^2\, u^{n\prime}_{\alpha''}(\varepsilon)\, u^{p\prime\prime}_{\alpha''}(\varepsilon))\, dx = \delta_{np}\,. \tag{2.12}$$

Finally, we recall the following version of Korn's inequality whose proof may be found in [12]. Let $y^\varepsilon = (1/2, \varepsilon/2, 1/2)$. For all v in $H^1(\Omega'')^3$, we

denote by $W(v) = (\nabla v - \nabla v^T)/2$ the skew-symmetric part of the gradient of v. Let $a(v)$ be the vector of $\mathbb{R}^3$ associated with the skew-symmetric matrix

$$\int_{\Omega''} W(v) \, dx \text{ and } b(v) = \int_{\Omega''} v \, dx + \left(\int_{\Omega''} W(v) \, dx \right) \left(\int_{\Omega''} (x-y^\varepsilon) \, dx \right).$$

Then we have

LEMMA 2.1 : *There exists a constant $C > 0$ independent of ε such that, if we set*

$$\bar{v}(x) = v(x) - a(v) \wedge (x - y^\varepsilon) - b(v) \, , \tag{2.13}$$

then

$$\| e(v) \|_{L^2(\Omega'')^9} = \| e(\bar{v}) \|_{L^2(\Omega'')^9} \geq C \| \bar{v} \|_{H^1(\Omega'')^3} \, . \tag{2.14}$$

3. THE LIMIT PROBLEM

We follow the scheme of the proof of Theorem 1 in [4], namely, we first prove that the various unknowns involved $(\eta_p^\varepsilon, u^p(\varepsilon))$ satisfy appropriate bounds, which, upon extraction of a subsequence, will allow us to consider limits for these unknowns as $\varepsilon \to 0$. Then, we will identify limit *2d-2d* problems for the limit unknowns by using the techniques of [11]-[12]. As these limit problems will turn out to be well-posed eigenvalue problems, we will then be able to determine precisely the limit unknowns as being the eigenvalues and eigenvectors of the limit problems. This will show that the whole family $(\eta_p^\varepsilon, u^p(\varepsilon))$ converge.

To begin with, let us consider the eigenvalues η_p^ε.

LEMMA 3.1. *For any integer $p \geq 1$, there exists a constant η_p^* independent of ε such that*

$$\eta_p^\varepsilon \leq \eta_p^* \, . \tag{3.1}$$

Proof: Let us rewrite the Rayleigh quotient (2.9) as

$$R^\varepsilon(v(\varepsilon)) = \frac{N(\varepsilon)(v(\varepsilon))}{D(\varepsilon)(v(\varepsilon))} \, .$$

For $\varepsilon < 1/2$, let $\mathbf{W}$ be the subspace of $\mathbf{V}(\varepsilon)$ of functions (v', v'') such that $v'' = 0$, $v'(x) = 0$ for $x_1 \leq 1/2$ and

$$v'(x) = (-(x_2 - 1/2) \, \partial_1 \xi(x_1, x_3), \xi(x_1, x_3), -(x_2 - 1/2) \, \partial_3 \xi(x_1, x_3))$$

for $x_1 > 1/2$, with $\xi \in H^2(\omega)$. Let $\mathscr{W}_p$ be the set of all vector subspaces of $\mathbf{W}$ of dimension p. Then, as $\mathscr{W}_p \subset \mathscr{W}_p(\varepsilon)$ for $\varepsilon < 1/2$, we see that

$$\eta_p^\varepsilon \leqslant \min_{\substack{E_p \in \mathcal{W}_p}} \max_{\substack{v \in E_p \\ v \neq 0}} R^\varepsilon(v). \tag{3.2}$$

Now, for v in **W**, we have

$$N(\varepsilon)(v) = \int_{\Omega'} (2\,\mu e_{\alpha'\beta'}(v)\,e_{\alpha'\beta'}(v) + \lambda e_{\alpha'\alpha'}(v)\,e_{\beta'\beta'}(v))\,dx\,, \tag{3.3}$$

so that, $N(\varepsilon)(v) = N(\xi)$ does not depend on ε. Moreover,

$$D(\varepsilon)(v) \geqslant \int_{\omega'} \xi^2\,dx_1\,dx_3 \overset{\text{def}}{=} D(\xi). \tag{3.4}$$

It follows immediately from (3.2)-(3.4) that:

$$\eta_p^\varepsilon \leqslant \min_{\substack{E_p \in \mathcal{W}_p}} \max_{\substack{v \in E_p \\ v \neq 0}} \frac{N(\xi)}{D(\xi)} = \eta_p^*\,, \tag{3.5}$$

and the proof is complete. $\square$

Let us now consider the eigenfunctions.

LEMMA 3.2 : *For any integer* $p \geqslant 1$, *let* $\bar{u}^{p''}(\varepsilon)$, $a(u^{p''}(\varepsilon))$ *and* $b(u^{p''}(\varepsilon))$ *be associated with* $u^{p''}(\varepsilon)$ *by Lemma 2.1. There exists a constant* $C_p > 0$ *independent of* ε *such that*

$$\|u^{p'}(\varepsilon)\|_{H^1(\Omega')^3} \leqslant C_p\,, \quad \|\bar{u}^{p''}(\varepsilon)\|_{H^1(\Omega'')^3} \leqslant C_p\,, \tag{3.6}$$

$$\begin{cases} |a_{\alpha''}(u^{p''}(\varepsilon))| \leqslant C_p\,, & |b_{\alpha'}(u^{p''}(\varepsilon))| \leqslant C_p\,, \\ \varepsilon\,|a_1(u^{p''}(\varepsilon))| \leqslant C_p\,, & \varepsilon\,|b_2(u^{p''}(\varepsilon))| \leqslant C_p\,. \end{cases} \tag{3.7}$$

Proof : Let $v(\varepsilon) = u^p(\varepsilon)$ in equation (2.11). Then, by formula (2.12), we obtain

$$\int_{\Omega'} B_\varepsilon'(u^{p'}(\varepsilon), u^{p'}(\varepsilon))\,dx + \int_{\Omega'' \setminus J_\varepsilon''} B_\varepsilon''(u^{p''}(\varepsilon), u^{p''}(\varepsilon))\,dx = \eta_p^\varepsilon\,. \tag{3.8}$$

By Lemma 3.1, the right-hand side of equation (3.8) is bounded from above by η_p^*. As in [11], we may rewrite the left-hand side as

$$\int_{\Omega' \setminus J_\varepsilon'} B_\varepsilon'(u^{p'}(\varepsilon), u^{p'}(\varepsilon))\,dx + \frac{1}{2} \int_{J_\varepsilon'} B_\varepsilon'(u^{p'}(\varepsilon), u^{p'}(\varepsilon))\,dx +$$

$$+ \int_{\Omega'' \setminus J_\varepsilon''} B_\varepsilon''(\bar{u}^{p''}(\varepsilon), \bar{u}^{p''}(\varepsilon))\,dx + \frac{1}{2} \int_{J_\varepsilon''} B_\varepsilon''(\bar{u}^{p''}(\varepsilon), \bar{u}^{p''}(\varepsilon))\,dx \tag{3.9}$$

and as $\varepsilon < 1$, it follows that

$$C\left(\|e(u^{p'}(\varepsilon))\|_{L^2(\Omega')^9} + \|e(\bar{u}^{p''}(\varepsilon))\|_{L^2(\Omega'')^9}\right) \le \eta_p^* \qquad (3.10)$$

from which the bounds (3.6) follow, by Lemma 2.1. Then we apply estimates (1.3.10), (1.3.12) and (1.3.19) of [12], proof of Proposition 1.3.1, to conclude the proof of Lemma 3.2. Let us simply mention here that these estimates rely crucially upon the compatibility relations (2.6). □

We may therefore extract a subsequence ε_n (which can be chosen to be the same for all p, by use of the diagonal procedure) such that

LEMMA 3.3 : *For all $p \ge 1$, we have*

$$\eta_p^{\varepsilon_n} \to \eta_p^0, \qquad (3.11)$$

$$(u^{p'}(\varepsilon_n), \bar{u}^{p''}(\varepsilon_n)) \to (u^{p'}(0), \bar{u}^{p''}(0)) \quad weakly\ in \quad \mathbf{V}, \qquad (3.12)$$

$$a_{\alpha''}(u^{p''}(\varepsilon_n)) \to a_{\alpha''}^{p0}, \quad \varepsilon_n\, a_1(u^{p''}(\varepsilon_n)) \to \bar{a}_1^{p0}, \qquad (3.13)$$

$$b_{\alpha'}(u^{p''}(\varepsilon_n)) \to b_{\alpha'}^{p0}, \quad \varepsilon_n\, b_2(u^{p''}(\varepsilon_n)) \to \bar{b}_2^{p0}. \qquad (3.14)$$

Remarks : In the sequel, we will omit the subscript n for the sake of brevity. As in [12], we may as well incorporate the components of the rigid displacement that converge into $\bar{u}^{p''}(\varepsilon)$, thus defining

$$\bar{\bar{u}}^{p''}(\varepsilon)(x) = \bar{u}^{p''}(\varepsilon)(x) +$$

$$+ \begin{pmatrix} a_2(u^{p''}(\varepsilon))(x_3 - 1/2) - a_3(u^{p''}(\varepsilon))(x_2 - \varepsilon/2) + b_1(u^{p''}(\varepsilon)) \\ a_3(u^{p''}(\varepsilon))(x_1 - 1/2) \\ -\varepsilon a_1(u^{p''}(\varepsilon))/2 - a_2(u^{p''}(\varepsilon))(x_1 - 1/2) + b_3(u^{p''}(\varepsilon)) \end{pmatrix}, \qquad (3.15)$$

so that

$$u^{p''}(\varepsilon)(x) = \bar{\bar{u}}^{p''}(\varepsilon)(x) + \begin{pmatrix} 0 \\ -a_1(u^{p''}(\varepsilon))(x_3 - 1/2) + b_2(u^{p''}(\varepsilon)) \\ a_1(u^{p''}(\varepsilon))\, x_2 \end{pmatrix}. \qquad (3.16)$$

The interpretation of formula (3.16) is the same as in the static case of [12]. Namely, if we « descale » equation (3.16) and restrict it to the mid-plane of the free plate, we see that $u^{\varepsilon, p}$ in this plate consists of a flexural displacement that converges toward $u_1^{p''}(0)$ and of in-plane displacements that converge toward a rigid displacement $(0, -\bar{a}_1^{p0}(x_3 - 1/2) + \bar{b}_2^{p0}, \bar{a}_1^{p0} x_2)$. As the eigenvectors correspond to vibration modes for the whole structure, it follows that these modes comprise a vertical rigid motion of the free plate which, as we will show in the next lemma, is equal to the motion of the edge of the clamped plate at the fold, *cf.* formula (3.20) below.

We next state without proof the following properties of $u^p(0)$. The proofs for the various formulas in Lemma 3.4 can be found in [11] and [12]. Although they were written for the static case, it is clear that they apply equally well here.

LEMMA 3.4 : *For each $p \geqslant 1$, the limit displacements $u^{p\prime}(0)$ and $\bar{u}^{p\prime\prime}(0)$ are of Kirchhoff-Love type, that is, there exist six functions $\zeta_{\alpha'}^{p\prime} \in H^1(\omega')$, $\zeta_{\alpha''}^{p\prime\prime} \in H^1(\omega'')$, $\zeta_2^{p\prime} \in H^2(\omega')$ and $\zeta_1^{p\prime\prime} \in H^2(\omega'')$ such that*

$$u^{p\prime}(0)(x) = (\zeta_1^{p\prime}(x_1, x_3) - (x_2 - 1/2)\partial_1\zeta_2^{p\prime}(x_1, x_3), \zeta_2^{p\prime}(x_1, x_3), \\ \zeta_3^{p\prime}(x_1, x_3) - (x_2 - 1/2)\,\partial_3\zeta_2^{p\prime}(x_1, x_3)), \tag{3.17}$$

and

$$\bar{u}^{p\prime\prime}(0)(x) = (\zeta_1^{p\prime\prime}(x_2, x_3), \zeta_2^{p\prime\prime}(x_2, x_3) - (x_1 - 1/2)\,\partial_2\zeta_1^{p\prime\prime}(x_2, x_3), \\ \zeta_3^{p\prime\prime}(x_2, x_3) - (x_1 - 1/2)\,\partial_3\zeta_1^{p\prime\prime}(x_2, x_3)). \tag{3.18}$$

These functions satisfy the boundary condition

$$\zeta_i^{p\prime}(1, x_3) = \partial_1\zeta_2^{p\prime}(1, x_3) = 0, \tag{3.19}$$

and the junction conditions

$$\begin{cases} \zeta_1^{p\prime\prime}(0, x_3) = 0, \\ \zeta_2^{p\prime}(0, x_3) = -\bar{a}_1^{p0}(x_3 - 1/2) + \bar{b}_2^{p0}, \\ \partial_1\zeta_2^{p\prime}(0, x_3) = -\partial_2\zeta_1^{p\prime\prime}(0, x_3), \\ \zeta_3^{p\prime}(0, x_3) = \zeta_3^{p\prime\prime}(0, x_3). \end{cases} \tag{3.20}$$

Moreover,

$$\begin{cases} \varepsilon^{-1}\, e_{\alpha'2}(u^{p\prime}(\varepsilon)) \rightharpoonup 0 \\ \varepsilon^{-2}\, e_{22}(u^{p\prime}(\varepsilon)) \rightharpoonup -\dfrac{\lambda}{2\,\mu + \lambda}\, e_{\alpha'\alpha'}(u^{p\prime}(0)) \end{cases} \qquad weakly\ in\ L^2(\Omega'), \tag{3.21}$$

$$\begin{cases} \varepsilon^{-1}\, e_{\alpha''1}(\bar{u}^{p\prime\prime}(\varepsilon)) \rightharpoonup 0 \\ \varepsilon^{-2}\, e_{11}(\bar{u}^{p\prime\prime}(\varepsilon)) \rightharpoonup -\dfrac{\lambda}{2\,\mu + \lambda}\, e_{\alpha''\alpha''}(\bar{u}^{p\prime\prime}(0)) \end{cases} \qquad weakly\ in\ L^2(\Omega''). \tag{3.22}$$

Some of these functions actually vanish. In fact,

LEMMA 3.5 : *We have*

$$\zeta_{\alpha'}^{p\prime} = 0, \quad \zeta_{\alpha''}^{p\prime\prime} = 0. \tag{3.23}$$

Proof : It follows from formulas (2.13) and (3.15) that

$$\int_{\Omega''} \bar{\bar{u}}_2^{p}{}''(\varepsilon)\, dx = 0\,, \tag{3.24}$$

from which we deduce that

$$\int_{\omega''} \zeta_2^{p}{}''(x_2, x_3)\, dx_2\, dx_3 = 0\,. \tag{3.25}$$

Furthermore, we get from the same formulas

$$\int_{\Omega''} W(\bar{\bar{u}}^{p}{}''(\varepsilon))\, dx = \begin{pmatrix} 0 & -a_3(u^{p}{}''(\varepsilon)) & a_2(u^{p}{}''(\varepsilon)) \\ a_3(u^{p}{}''(\varepsilon)) & 0 & 0 \\ -a_2(u^{p}{}''(\varepsilon)) & 0 & 0 \end{pmatrix}\,, \tag{3.26}$$

so that

$$\int_{\omega''} \partial_2\zeta_2^{p}{}''(x_2, x_3)\, dx_2\, dx_3 = \int_{\omega''} \partial_3\zeta_3^{p}{}''(x_2, x_3)\, dx_2\, dx_3 =$$

$$= \int_{\omega''} (\partial_3\zeta_2^{p}{}''(x_2, x_3) - \partial_2\zeta_3^{p}{}''(x_2, x_3))\, dx_2\, dx_3 = 0\,. \tag{3.27}$$

Now, if we define two-dimensional displacements on ω' and ω'' by

$$\begin{cases} \mathbf{u}^{p}{}' = (\zeta_1^{p}{}', \zeta_3^{p}{}') & \text{in } \omega'\,, \\ \mathbf{u}^{p}{}'' = (\zeta_2^{p}{}'', \zeta_3^{p}{}'') & \text{in } \omega''\,, \end{cases} \tag{3.28}$$

then, the same proof as in [11], Theorem 3.3, or as in [4], Proposition 2, Step 3, shows that these displacements satisfy

$$\int_{\omega'} \left(2\,\mu e_{\alpha'\beta'}(\mathbf{u}^{p}{}')\, e_{\alpha'\beta'}(\mathbf{u}^{p}{}') + \frac{2\,\mu\lambda}{2\,\mu + \lambda}\, e_{\alpha'\alpha'}(\mathbf{u}^{p}{}')\, e_{\beta'\beta'}(\mathbf{u}^{p}{}') \right) dx +$$

$$+ \int_{\omega''} \left(2\,\mu e_{\alpha''\beta''}(\mathbf{u}^{p}{}'')\, e_{\alpha''\beta''}(\mathbf{u}^{p}{}'') + \frac{2\,\mu\lambda}{2\,\mu + \lambda}\, e_{\alpha''\alpha''}(\mathbf{u}^{p}{}'')\, e_{\beta''\beta''}(\mathbf{u}^{p}{}'') \right) dx = 0\,. \tag{3.29}$$

Therefore, $e_{\alpha'\beta'}(\mathbf{u}^{p}{}') = 0$ in ω' and $e_{\alpha''\beta''}(\mathbf{u}^{p}{}'') = 0$ in ω''. As $\mathbf{u}^{p}{}'$ obeys a clamping condition on the left edge of the plate, it follows immediately that $\mathbf{u}^{p}{}' = 0$. Furthermore, $\mathbf{u}^{p}{}''$ is of the form

$$\mathbf{u}^{p}{}''(x_2, x_3) = \begin{pmatrix} -a(x_3 - 1/2) + b_1 \\ ax_2 + b_2 \end{pmatrix}\,. \tag{3.30}$$

The last junction condition in (3.20) implies that $b_2 = 0$, equation (3.25) that $b_1 = 0$ and equation (3.27) that $a = 0$, and the lemma is proved. $\square$

The weak limits of the eigenvectors are thus only determined by pairs $(\zeta_2^{p\prime}, \zeta_1^{p\prime\prime})$ which belong to the space $\mathcal{V}$ defined as

$$\mathcal{V} \overset{\text{def}}{=} \{(\xi', \xi'') \in H^2(\omega') \times H^2(\omega''), \xi'(1, x_3) = \partial_1 \xi'(1, x_3) = 0, \exists (a, b) \in \mathbb{R}^2$$

with $\xi'(0, x_3) = -a(x_3 - 1/2) + b, \xi''(0, x_3) =$
$$= 0, \partial_1 \xi'(0, x_3) = -\partial_2 \xi''(0, x_3)\} . \quad (3.31)$$

If we introduce the bending moments

$$\begin{cases} m'_{\alpha'\beta'}(\xi') = -\dfrac{E}{12(1 - \nu^2)} [(1 - \nu) \partial_{\alpha'\beta'} \xi' + \nu \Delta\xi' \delta_{\alpha'\beta'}], \\[4mm] m''_{\alpha''\beta''}(\xi'') = -\dfrac{E}{12(1 - \nu^2)} [(1 - \nu) \partial_{\alpha''\beta''} \xi'' + \nu \Delta\xi'' \delta_{\alpha''\beta''}], \end{cases} \quad (3.32)$$

where $E = \dfrac{\mu(3\lambda + 2\mu)}{\mu + \lambda}$ is the scaled Young modulus and $\nu = \dfrac{\lambda}{2(\mu + \lambda)}$ is Poisson's ratio, then we have :

THEOREM 3.1 : *The limits $\zeta_2^{p\prime}$, $\zeta_1^{p\prime\prime}$ and η_p^0 satisfy*

$$-\int_{\omega'} m'_{\alpha'\beta'}(\zeta_2^{p\prime}) \partial_{\alpha'\beta'} \xi' \, dx - \int_{\omega''} m''_{\alpha''\beta''}(\zeta_1^{p\prime\prime}) \partial_{\alpha''\beta''} \xi'' \, dx =$$

$$= \eta_p^0 \left[\int_{\omega'} \zeta_2^{p\prime} \xi' \, dx + \int_{\omega''} \zeta_1^{p\prime\prime} \xi'' \, dx + \frac{5}{12} \int_0^1 \partial_3 \zeta_2^{p\prime}(0, x_3) \, dx_3 \int_0^1 \partial_3 \xi'(0, x_3) \, dx_3 \right.$$

$$\left. + \int_0^1 \zeta_2^{p\prime}(0, x_3) \, dx_3 \int_0^1 \xi'(0, x_3) \, dx_3 \right], \quad (3.33)$$

for all $(\xi', \xi'') \in \mathcal{V}$.

Remark : The last two products may just as well be written as $\dfrac{5}{12} a(\zeta_2^{p\prime}) a(\xi') + b(\zeta_2^{p\prime}) b(\xi')$, granted that $a(\xi')$ and $b(\xi')$ denote the constants a and b in formula (3.31) for an element (ξ', ξ'') of $\mathcal{V}$. They represent the contribution of the vertical rigid motion of the free plate to the limit eigenvalue problem.

Proof : The proof is similar to the corresponding one in [11]-[12], and we thus only sketch it. The idea is the following: Given an arbitrary element (ξ', ξ'') of $\mathcal{V}$, approximate the corresponding Kirchhoff-Love displacement

$$v = \begin{cases} (- (x_2 - 1/2) \, \partial_1\xi', \xi', - (x_2 - 1/2) \, \partial_3\xi') & \text{in} \quad \Omega', \\ (\xi'', - (x_1 - 1/2) \, \partial_2\xi'', - (x_1 - 1/2) \, \partial_3\xi'') & \text{in} \quad \Omega'', \end{cases} \quad (3.34)$$

by a sequence of displacements $v(\varepsilon)$ that belong to $V(\varepsilon)$. This sequence is constructed so that the functions $\varepsilon^{-2} e_{22}(v'(\varepsilon))$, $\varepsilon^{-1} e_{\alpha'2}(v'(\varepsilon))$, $\varepsilon^{-2} e_{11}(v''(\varepsilon))$ and $\varepsilon^{-1} e_{\alpha''1}(v''(\varepsilon))$ converge strongly in L^2. Using these specific functions $v(\varepsilon)$ as test-functions in the variational equations (2.11), we can thus pass to the limit in their left-hand side, which gives the left-hand side of equation (3.33). In effect, it is not possible to follow exactly the above programme. In order that $v(\varepsilon)$ should satisfy the compatibility conditions (2.6) (and still keep the other properties listed above), it is necessary that $v''(\varepsilon)$ be of the form

$$v''(\varepsilon) = (\bar{\bar{v}}_1''(\varepsilon), \, \varepsilon^{-1}[b(\xi') - (x_3 - 1/2) \, a(\xi')] + \bar{\bar{v}}_2''(\varepsilon),$$

$$\varepsilon^{-1} x_2 \, a(\xi') + \bar{\bar{v}}_3''(\varepsilon)) \quad (3.35)$$

where it is $\bar{\bar{v}}''(\varepsilon)$ that actually approximates v''. This does not change the left-hand side, since only rigid displacements are added, but gives rise to the right-hand side of equation (3.33), by formulas (3.13), (3.14), (3.16) and (3.20). □

Remarks : The coefficients 5/12 and 1 of the last two terms are of course not universal. They are related to the geometry of the free plate. For an arbitrarily shaped plate ω'', we would have obtained

$$a(\zeta_2^p) \, a(\xi') \int_{\omega''} [(x_3 - 1/2)^2 + x_2^2] \, dx - (a(\zeta_2^p) \, b(\xi') +$$

$$+ b(\zeta_2^p) \, a(\xi')) \int_{\omega'} (x_3 - 1/2) \, dx + b(\zeta_2^p) \, b(\xi') \int_{\omega''} 1 \, dx . \quad (3.36)$$

Thus, the first coefficient is the trace of the inertia tensor of the free plate (with surface density 1) with respect to the center of the fold and the second coefficient is the area of the free plate. In general, these is also a cross-product term with coefficient $\int_{\omega''} (x_3 - 1/2) \, dx$ which has no influence on the results given below.

The fact that (3.33) determines the limit unknowns — i.e., that it is a well-posed eigenvalue problem — is not completely obvious, especially in view of its strong from. Let us set $\gamma = \{(0, 0, x_3) ; 0 < x_3 < 1\}$.

COROLLARY 3.1 : Any solution to equation (3.33) satisfies, at least formally, the following system :

$$\begin{cases} -\partial_{\alpha'\beta'} m_{\alpha'\beta'}(\zeta_2^{p\prime}) = \eta_p^0 \zeta_2^{p\prime} & in \quad \omega', \\ -\partial_{\alpha''\beta''} m_{\alpha''\beta''}(\zeta_1^{p\prime\prime}) = \eta_p^0 \zeta_1^{p\prime\prime} & in \quad \omega'', \end{cases} \tag{3.37}$$

$$\zeta_2^{p\prime} = -a(\zeta_2^{p\prime})(x_3 - 1/2) + b(\zeta_2^{p\prime}) \quad on \quad \gamma, \tag{3.38}$$

$$\zeta_1^{p\prime\prime} = 0, \quad \partial_1 \zeta_2^{p\prime} = -\partial_2 \zeta_1^{p\prime\prime} \quad on \quad \gamma, \tag{3.39}$$

$$m_{11}(\zeta_2^{p\prime}) = m_{22}(\zeta_1^{p\prime\prime}) \quad on \quad \gamma, \tag{3.40}$$

$$\begin{cases} \displaystyle\int_\gamma \partial_{\alpha'} m_{\alpha'1}(\zeta_2^{p\prime}) \, dx_3 = \eta_p^0 \, b(\zeta_2^{p\prime}), \\[2mm] \displaystyle\int_\gamma [(x_3 - 1/2) \partial_{\alpha'} m_{\alpha'1}(\zeta_2^{p\prime}) - m_{13}(\zeta_2^{p\prime})] \, dx_3 = \frac{5}{12} \eta_p^0 \, a(\zeta_2^{p\prime}), \end{cases} \tag{3.41}$$

plus the clamping and traction-free conditions on $\partial\omega'\backslash\gamma$, $\partial\omega''\backslash\gamma$.

Proof: Equations (3.37), (3.40) and (3.41) follow easily from formal integrations by parts in equation (3.33). $\square$

Remarks : Equations (3.37) are equivalent to the familiar plate eigenvalue equations

$$\begin{cases} \dfrac{E}{12(1-v^2)} \Delta^2 \zeta_2^{p\prime} = \eta_p^0 \, \zeta_2^{p\prime}, \\[3mm] \dfrac{E}{12(1-v^2)} \Delta^2 \zeta_1^{p\prime\prime} = \eta_p^0 \, \zeta_1^{p\prime\prime}, \end{cases} \tag{3.42}$$

although the boundary conditions are rather non standard. Problem (3.33) is nevertheless a well-posed eigenvalue problem, as we now proceed to show. The trick is to introduce the right function spaces. Let us thus define

$$\mathcal{H} = L^2(\omega') \times L^2(\omega'') \times \mathbb{R}^2, \tag{3.43}$$

with the norm

$$\| (\xi', \xi'', a, b) \|_{\mathcal{H}}^2 = \int_{\omega'} \xi'^2 \, dx + \int_{\omega''} \xi''^2 \, dx + \frac{5}{12} a^2 + b^2. \tag{3.44}$$

It is clear that $\mathcal{H}$ equipped with this norm is a Hilbert space. We endow $\mathcal{V}$ with its natural H^2 topology. Then, the imbedding

$$\begin{aligned} \mathcal{V} &\to \mathcal{H} \\ (\xi', \xi'') &\mapsto (\xi', \xi'', a(\xi'), b(\xi')) \end{aligned} \tag{3.45}$$

is obviously compact, and the following lemma is then just a straightforward consequence of the general spectral theory, see e.g. [16].

LEMMA 3.6 : *Problem* (3.33) *is a well-posed eigenvalue problem whose solutions* $(\xi^{p\prime}, \xi^{p\prime\prime}, \eta_p)$ *with* $\xi^{p\prime}, \xi^{p\prime\prime}$ *normalized in* $\mathscr{H}$ *are such that :*

$$0 < \eta_1 \leqslant \eta_2 \leqslant \cdots \leqslant \eta_p \leqslant \cdots \tag{3.46}$$

with $\eta_p \to \infty$ *as* $p \to \infty$, *and the family* $(\xi^{p\prime}, \xi^{p\prime\prime})$ *is an orthonormal basis of* $\mathscr{H}$ *and also an orthogonal basis of* $\mathscr{V}$ *equipped with the inner product defined by the left-hand side of equation* (3.33).

Proof : The only thing that remains to be proved is that the left-hand side of equation (3.33) defines a coercive bilinear from on $\mathscr{V}$. We refer the reader to [12] for such a proof. □

The following lemma is quite obvious.

LEMMA 3.7 : *We have*

$$((\zeta_2^{n\prime}, \zeta_1^{n\prime\prime}), (\zeta_2^{p\prime}, \zeta_1^{p\prime\prime}))_{\mathscr{H}} = \delta_{np} . \tag{3.47}$$

Proof : Pass to the limit in formula (2.12). □

Then, the same proof as in [4], Propositions 4 and 5, can be applied by observing that only weak-V convergences are needed to complete it, thus yielding the following theorem :

THEOREM 3.2 : *The sequence* η_p^0 *comprises all the eigenvalues of problem* (3.33), *i.e.,* $\eta_p^0 = \eta_p$ (*counting multiplicities*) *and the eigenvectors* $(\zeta_2^{p\prime}, \zeta_1^{p\prime\prime})$ *form an orthonormal basis of* $\mathscr{H}$ *and an orthogonal basis of* $\mathscr{V}$. *Moreover, for any* $p \geqslant 1$ *fixed, the whole family* $(\eta_p^\varepsilon)_{\varepsilon > 0}$ *converges to* η_p^0 *and, if* η_p^0 *is a simple eigenvalue of problem* (3.33), *the family* $(\zeta_2^{p\prime}, \zeta_1^{p\prime\prime})_{\varepsilon > 0}$ *converges to* $\pm (\xi^{p\prime}, \xi^{p\prime\prime})$.

Applying the techniques of [13], we next show easily that

PROPOSITION 3.1 : $(u^{p\prime}(\varepsilon), \bar{u}^{p\prime\prime}(\varepsilon)) \to (u^{p\prime}(0), \bar{u}^{p\prime\prime}(0))$ *strongly in* V *as* $\varepsilon \to 0$.

Proof : See [13]. □

Remarks : It is a consequence of the proof of Proposition 3.1 that the weak L^2-convergences of formulas (3.21) and (3.22) are in fact strong L^2-convergences. This fact will be used in Sectionc 4 to obtain the limits of the stresses.

Theorems 3.1 and 3.2 give a complete description of the limit behavior of the eigenvalues and eigenfunctions of the folded plates as $\varepsilon \to 0$, in terms of the well-posed 2d-2d model (3.33). In view of Lemma 3.6, it is clear that the limit eigenvalues also have the min-max characterization

$$\eta_p^0 = \min_{E_p \in \mathscr{W}_p} \max_{\substack{\xi \in E_p \\ \xi \neq 0}} R(\xi) , \tag{3.48}$$

where $\mathscr{W}_p$ is the set of all vector subspaces of $\mathscr{V}$ of dimension p, and $R(\xi)$ is the Rayleigh quotient :

$$R(\xi) = \frac{-\displaystyle\int_{\omega'} m'_{\alpha'\beta'}(\xi')\,\partial_{\alpha'\beta'}\,\xi'\,dx - \displaystyle\int_{\omega''} m''_{\alpha''\beta''}(\xi'')\,\partial_{\alpha''\beta''}\,\xi''\,dx}{\displaystyle\int_{\omega'} \xi'^2\,dx + \displaystyle\int_{\omega''} \xi''^2\,dx + \frac{5}{12}\,a(\xi')^2 + b\,(\xi')^2}\,. \tag{3.49}$$

4. CONVERGENCE OF THE STRESSES

In this section, we give a concise proof of the convergence of the scaled stresses which also applies to the static case of [11]-[12]. Actually the same proof provides an improvement of the result and a significant simplification of the proof of Destuynder [9] for the case of a single plate. Let us thus define the scaled stresses on Ω'

$$\begin{cases} \sigma^{p'}_{\alpha'\beta'}(\varepsilon) = 2\,\mu\,e_{\alpha'\beta'}(u^{p'}(\varepsilon)) + \lambda\,[e_{\alpha'\alpha'}(u^{p'}(\varepsilon)) + \varepsilon^{-2}\,e_{22}(u^{p'}(\varepsilon))]\,\delta_{\alpha'\beta'}\,, \\ \sigma^{p'}_{\alpha'2}(\varepsilon) = 2\,\mu\varepsilon^{-2}\,e_{\alpha'2}(u^{p'}(\varepsilon))\,, \\ \sigma^{p'}_{22}(\varepsilon) = (2\,\mu + \lambda)\,\varepsilon^{-4}\,e_{22}(u^{p'}(\varepsilon)) + \lambda\varepsilon^{-2}\,e_{\alpha'\alpha'}(u^{p'}(\varepsilon))\,, \end{cases} \tag{4.1}$$

and $\sigma^{p''}(\varepsilon)$ by analogous formulas on Ω''. These scaled stresses are related to the actual stresses $\sigma^{\varepsilon,p}$ in the original structures by the formulas

$$\begin{cases} \sigma^{p'}_{\alpha'\beta'}(\varepsilon) = \varepsilon\sigma^{\varepsilon,p}_{\alpha'\beta'}\circ\phi^{\varepsilon}\,, \\ \sigma^{p'}_{\alpha'2}(\varepsilon) = \sigma^{\varepsilon,p}_{\alpha'2}\circ\phi^{\varepsilon}\,, \\ \sigma^{p'}_{22}(\varepsilon) = \varepsilon^{-1}\,\sigma^{\varepsilon,p}_{22}\circ\phi^{\varepsilon}\,, \end{cases} \tag{4.2}$$

and their analogues on Ω''. The crucial property of the scaled stresses, which stems from equations (2.11), is that

$$\begin{cases} -\operatorname{div}\sigma^{p'}(\varepsilon) = \eta^{\varepsilon}_p \begin{pmatrix} \varepsilon^2\,u^{p'}_1(\varepsilon) \\ u^{p'}_2(\varepsilon) \\ \varepsilon^2\,u^{p'}_3(\varepsilon) \end{pmatrix} \quad \text{in } \Omega'\,, \\[3em] -\operatorname{div}\sigma^{p''}(\varepsilon) = \eta^{\varepsilon}_p \begin{pmatrix} u^{p''}_1(\varepsilon) \\ \varepsilon^2\,u^{p''}_2(\varepsilon) \\ \varepsilon^2\,u^{p''}_3(\varepsilon) \end{pmatrix} \quad \text{in } \Omega''\,, \end{cases} \tag{4.3}$$

in the sense of $H(\operatorname{div}, \Omega')$ and $H(\operatorname{div}, \Omega'')$. Moreover,

$$\begin{cases} \sigma^{p'}(\varepsilon)\,n' = 0 \quad \text{on } \omega'\,, \\ \sigma^{p''}(\varepsilon)\,n'' = 0 \quad \text{on } \omega''\,, \end{cases} \tag{4.4}$$

in the sense of $H^{-1/2}(\omega')$ and $H^{-1/2}(\omega'')$.

THEOREM 4.1 : *The scaled stresses converge as* $\varepsilon \to 0$ *according to*

$$\begin{cases} \sigma^{p'}_{\alpha'\beta'}(\varepsilon) \to \sigma^{p'}_{\alpha'\beta'}(0) & \text{strongly in } L^2(\Omega'), \\ \sigma^{p'}_{\alpha'2}(\varepsilon) \to \sigma^{p'}_{\alpha'2}(0) & \text{strongly in } H^1(0,1\,;H^{-1}(\omega')), \\ \sigma^{p'}_{22}(\varepsilon) \to \sigma^{p'}_{22}(0) & \text{strongly in } H^1(0,1\,;H^{-2}(\omega')), \end{cases} \qquad (4.5)$$

and similarly on Ω'', *where*

$$\begin{cases} \sigma^{p'}_{\alpha'\beta'}(0) = -\dfrac{E(x_2 - 1/2)}{(1 - \nu^2)}\,[(1 - \nu)\,\partial_{\alpha'\beta'}\zeta^{p'}_2 + \nu\,\Delta\zeta^{p'}_2\,\delta_{\alpha'\beta'}]\,, \\[4mm] \sigma^{p'}_{\alpha'2}(0) = \dfrac{Ex_2(x_2 - 1)}{2(1 - \nu^2)}\,\partial_{\alpha'}\,\Delta\zeta^{p'}_2\,, \\[4mm] \sigma^{p'}_{22}(0) = x_2(1 - x_2)(2x_2 - 1)\,\eta^0_p\,\zeta^{p'}_2\,, \end{cases} \qquad (4.6)$$

and analogous formulas hold for $\sigma^{p''}_{ij}(0)$.

Remarks : In Destuynder [9], it is shown that the convergence above hold true in general only in the strong $L^2(0,1\,;H^{-1}(\omega'))$ sense for the shear components $\sigma_{\alpha'2}(\varepsilon)$ and in the strong $L^2(0,1\,;H^{-2}(\omega'))$ sense for the normal component $\sigma_{22}(\varepsilon)$.

Proof: We can restrict ourselves to the plate Ω', as the argument we use is independent of the plate under consideration (it is also independent of the boundary conditions). First of all, the convergence of the components $\sigma^{p'}_{\alpha'\beta'}(\varepsilon)$ is fairly clear, see the Remark below Proposition 3.1. Then, using equation (4.3), we see that

$$\partial_2\sigma^{p'}_{\alpha'2}(\varepsilon) = -\partial_{\beta'}\,\sigma^{p'}_{\alpha'\beta'}(\varepsilon) - \varepsilon^2\,\eta^\varepsilon_p\,u^{p'}_{\alpha'}(\varepsilon)\,, \qquad (4.7)$$

and thus, $\sigma^{p'}_{\alpha'2}(\varepsilon) \in H^1(0,1\,;H^{-1}(\omega'))$ (the distinguished variable here is x_2). Now, $\varepsilon^2\,\eta^\varepsilon_p\,u^{p'}_{\alpha'}(\varepsilon) \to 0$ strongly in $L^2(\Omega')$, and $\partial_{\beta'}\sigma^{p'}_{\alpha'\beta'}(\varepsilon) \to \partial_{\beta'}\sigma^{p'}_{\alpha'\beta'}(0)$ strongly in the space $L^2(0,1\,;H^{-1}(\omega'))$. Therefore, since by equation (4.4),

$$\sigma^{p'}_{\alpha'2}(\varepsilon)(x_2) = \int_0^{x_2} \partial_2\sigma^{p'}_{\alpha'2}(\varepsilon)(t)\,dt\,, \qquad (4.8)$$

it follows that

$$\sigma^{p'}_{\alpha'2}(\varepsilon) \to \sigma^{p'}_{\alpha'2}(0) = -\int_0^{x_2}\partial_{\beta'}\,\sigma^{p'}_{\alpha'\beta'}(0)\,dt\ \text{strongly in } H^1(0,1\,;H^{-1}(\omega'))\,.$$

$$(4.9)$$

The second equation in formula (4.6) is obtained by replacing $\sigma^{p\prime}_{\alpha'\beta'}(0)$ in (4.9) by its expression in (4.6). Next, we have

$$\partial_2\sigma^{p\prime}_{22}(\varepsilon) = -\partial_{\beta'}\sigma^{p\prime}_{2\beta'}(\varepsilon) - \eta^{\varepsilon}_p u^{p\prime}_2(\varepsilon)\,, \qquad (4.10)$$

so that $\sigma^{p\prime}_{22}(\varepsilon) \in H^1(0, 1 ; H^{-2}(\omega'))$ and again, thanks to (4.4),

$$\sigma^{p\prime}_{22}(\varepsilon)(x_2) = \int_0^{x_2} \partial_2\sigma^{p\prime}_{22}(\varepsilon)(t)\, dt\,. \qquad (4.11)$$

Now, $\eta^{\varepsilon}_p u^{p\prime}_2(\varepsilon) \to \eta^0_p \zeta^{p\prime}_2$ strongly in $H^1(\Omega')$ and, by the preceding step,

$$\partial_{\alpha'}\sigma^{p\prime}_{2\alpha'}(\varepsilon) \to \partial_{\alpha'}\sigma^{p\prime}_{2\alpha'}(0) \quad \text{strongly in} \quad L^2(0, 1 ; H^{-2}(\omega'))\,.$$

Thus,

$$\sigma^{p\prime}_{22}(\varepsilon) \to \sigma^{p\prime}_{22}(0) = -\int_0^{x_2} \partial_{\alpha'}\sigma^{p\prime}_{2\alpha'}(0)(t)\, dt -$$
$$- x_2\, \eta^0_p \zeta^{p\prime}_2 \quad \text{strongly in} \quad H^1(0, 1 ; H^{-2}(\omega'))\,, \qquad (4.12)$$

and the last expression in formula (4.6) follows, since

$$-\int_0^{x_2} \partial_{\alpha'}\, \sigma^{p\prime}_{2\alpha'}(0)(t)\, dt = -\frac{E}{2(1-\nu^2)}\left[\frac{x_2^3}{3} - \frac{x_2^2}{2}\right]\Delta^2\zeta^{p\prime}_2\,, \qquad (4.13)$$

and since $\zeta^{p\prime}_2$ satisfies equation (3.42). $\square$

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