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FINITE ELEMENT APPROXIMATION OF NONLINEAR ELLIPTIC PROBLEMS WITH DISCONTINUOUS COEFFICIENTS (*)

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Abstract — *The paper presents a detailed theory of the finite element solution of second-order nonlinear elliptic equations with discontinuous coefficients in a general nonpolygonal domain* Ω *with nonhomogeneous mixed Dirichlet-Neumann boundary conditions In the discretization of the problem we proceed in the usual way the domain* Ω *is approximated by a polygonal one, conforming piecewise linear triangular elements are used and the integrals are evaluated by numerical quadratures We prove the solvability of the discrete problem and study the convergence of the method both in strongly monotone and pseudomonotone cases under the only assumption that the exact solution* $u \in H^1(\Omega)$ *Provided u is piecewise of class* H^2 *and the problem is strongly monotone, we get the error estimate* $O(h)$

Résume — *Dans cet article nous présentons une théorie détaillée des éléments finis pour la solution des équations elliptiques non linéaires de second ordre avec des coefficients discontinus, dans le domaine* Ω *général, avec les conditions aux limites de Dirichlet-Neumann non homogènes Nous discrétisons le problème de la façon habituelle le domaine* Ω *est remplacé par le domaine polygonal et on utilise les éléments finis linéaires conformes et l'intégration numérique Nous démontrons l'existence de la solution du problème discret et étudions la convergence de la méthode dans les cas strictement monotones ou pseudo-monotones dans l'hypothèse où la solution exacte* $u \in H^1(\Omega)$ *Supposé que u appartient dans la classe* H^2 *par morceaux et le problème est strictement monotone, nous obtenons l'estimation de l'erreur* $O(h)$

INTRODUCTION

A series of processes in technology and science is described by partial differential equations of the type

$$(0.1) \quad -\sum_{i=1}^{2} \frac{\partial}{\partial x_i} a_i(x, u(x), \nabla u(x)) + a_0(x, u(x), \nabla u(x)) = f(x), \ x \in \Omega.$$

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The coefficients a_i and right-hand side f usually depend on the properties of materials that form the device represented by the domain Ω. In general, a_i and f have different values and structures in particular subregions $\Omega_s \subset \Omega$, $s = 1, \dots, m$, made from different materials. Hence, a_i and f are discontinuous across the common boundaries of Ω_s, $s = 1, \dots, m$, where instead of equation (0.1) the so-called transition conditions are used.

As a typical example the stationary magnetic field in a plane domain $\Omega \subset R^2$ can be introduced. It is described by equation (0.1) of the form

$$(0.2) \qquad - \sum_{i=1}^{2} \frac{\partial}{\partial x_i} \left(\nu_i(x, |\nabla u|^2 (x)) \frac{\partial u}{\partial x_i}(x) \right) = j(x) .$$

Here $\nu_i = 1/\mu_i$, where μ_i is the permeability, u is the magnetic field potential and j represents the current density. Provided Ω consists e.g. of iron, copper and (holes of) air, then ν_i is discontinuous, since it is equal to different constants in copper and air and it is a nonlinear function of $|\nabla u|^2$ in iron. Also the right-hand side j can be discontinuous. Often, $j = 0$ in air and iron, and $j = \text{const.} \neq 0$ in copper wire conductors. (*Cf.* e.g. [10, 11, 14, 17].)

We get a similar situation in heat conductivity processes described by the equation for the absolute temperature u:

$$(0.3) \quad - \sum_{i=1}^{2} \frac{\partial}{\partial x_i} \left(k(x, u(x), \nabla u(x)) \frac{\partial u}{\partial x_i}(x) \right) = f(x) , \quad x \in \Omega .$$

If Ω consists of several different materials, then the heat conductivity coefficient k and the heat sources density f are discontinuous in general. Other examples can be found in nuclear physics.

The weak solvability of a problem with discontinuous coefficients can be proved by the methods and techniques treated in [16, 19]. Some results concerning the properties and numerical solution of problems with discontinuous coefficients can be found e.g. in [1, 13, 20, 21, 22].

In this paper we present a general theory of the finite element solution to nonlinear equation (0.1) with discontinuous coefficients in a bounded domain $\Omega \subset R^2$. We generalize here the methods and techniques from [6-9]. One of our starting points is also the work [12], where the finite element discretization of nonlinear problems with discontinuous coefficients in polygonal domains was studied and computer realization was carried out. Here we consider the problem in a general nonpolygonal domain.

In Section 1 we give the classical formulation of the problem and derive the generalized weak formulation. Section 2 is devoted to the discretization of the problem. We procede as it is usual in practice : the domain Ω is approximated by a polygonal domain Ω_h, which is triangulated in a suitable way. We use conforming piecewise linear finite elements. The integrals are evaluated by numerical quadratures. (By Strang [24] we commit basic variational crimes.) In paragraph 2.3 we prove the existence of approximate solutions. Paragraph 3.1 deals with their convergence in the space $H^1(\Omega)$ to an exact solution. As a by-product the solvability of the continuous problem in $H^1(\Omega)$ is obtained. No additional assumption on the regularity of the exact solution is needed.

Provided the problem is strongly monotone and the exact solution is piecewise of class H^2, i.e. $u \mid \Omega_s \in H^2(\Omega_s)$ for $s = 1, \dots, m$, we prove in paragraph 3.2 that the error is of order $O(h)$. We use here an improved version of the Green's theorem method. Near the boundary Γ_N, where the Neumann condition is considered, we use the « triple application of Green's theorem », proposed in [7] ([1]).

1. CONTINUOUS PROBLEM

1.1. Assumptions

1.1.1. *Assumptions concerning the domain and the boundary*

Let $\Omega, \Omega_1, \dots, \Omega_m \subset R^2$ be bounded domains with Lipschitz-continuous boundaries $\partial\Omega, \partial\Omega_1, \dots, \partial\Omega_m$ and let

$$(1.1) \qquad \bar{\Omega} = \bigcup_{s=1}^{m} \bar{\Omega}_s\,, \quad \Omega_s \cap \Omega_r = \emptyset \quad \text{for} \quad r, s = 1, \dots, m\,, \quad r \neq s\,,$$

$$\Omega_0 = \bigcup_{s=1}^{m} \Omega_s\,,$$

$$(1.2) \qquad \partial\Omega = \bar{\Gamma}_D \cup \bar{\Gamma}_N\,, \quad \Gamma_D \cap \Gamma_N = \emptyset\,, \quad \text{meas}_1\,(\Gamma_D) > 0\,.$$

$\bar{\Omega}, \bar{\Omega}_s, \bar{\Gamma}_D$ etc. denote the closures of $\Omega, \Omega_s, \Gamma_D$ etc., meas_1 denotes one-dimensional measure defined on $\partial\Omega, \partial\Omega_1$ etc. We set

$$(1.3) \qquad \bar{\Gamma}_{rs} = \bar{\Gamma}_{sr} = \partial\Omega_r \cap \partial\Omega_s\,, \quad r, s = 1, \dots, m\,, \quad r \neq s\,,$$

$$\bar{\Gamma}_{sD} = \bar{\Gamma}_D \cap \partial\Omega_s\,, \quad \bar{\Gamma}_{sN} = \bar{\Gamma}_N \cap \partial\Omega_s\,, \quad s = 1, \dots, m\,.$$

Let $\Gamma_D, \Gamma_N, \Gamma_{rs}$ be formed by a finite number of open arcs (i.e. arcs without their endpoints) or simple closed curves. It is evident that

$$(1.4) \quad \partial\Omega_s = \bar{\Gamma}_{sN} \cup \bar{\Gamma}_{sD} \cup \left(\bigcup_{\substack{r=1 \\ r \neq s}}^{m} \bar{\Gamma}_{rs} \right), \quad \bar{\Gamma}_N = \bigcup_{s=1}^{m} \bar{\Gamma}_{sN}\,, \quad \bar{\Gamma}_D = \bigcup_{s=1}^{m} \bar{\Gamma}_{sD}\,.$$

Of course, some of the sets $\bar{\Gamma}_{sr}, \bar{\Gamma}_{sN}, \bar{\Gamma}_{sD}$ can be empty. (See *fig.* 1.1.)

([1]) It should be noted that simultaneously with this paper and independently on it the same problem has been treated in [27]. The approach from [27] is quite different to our approach.

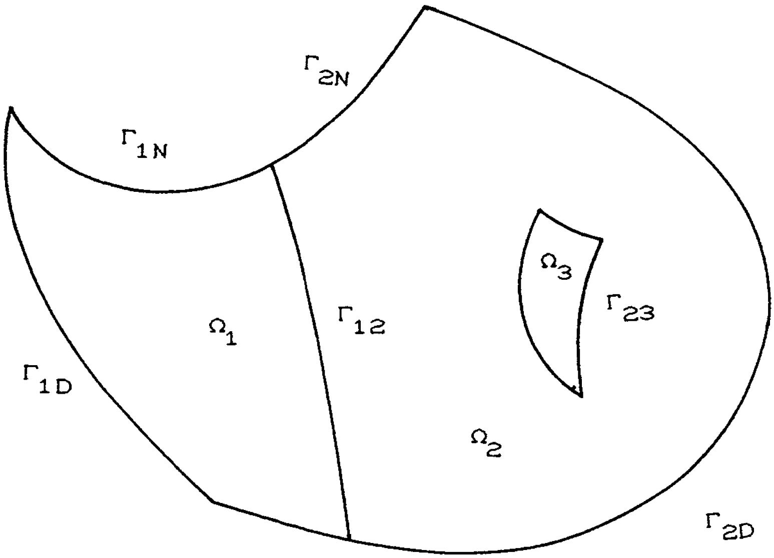

Figure 1.1.

In the discretization of the problem (see Section 2) we shall work with polygonal approximations Ω_h of Ω and Ω_{sh} of Ω_s for $h \in (0, h_0)$ $(h_0 > 0)$. Let Ω_s^* be bounded domains such that

$$\Omega_s^* \supset \Omega_s \cup \Omega_{sh} \quad \forall h \in (0, h_0), \quad s = 1, \ldots, m. \tag{1.5}$$

1.1.2. *Function spaces*

By the symbols $C^k(\bar{\Omega})$, $C^k(\bar{\Omega}_s)$, $L^p(\Omega)$, $L^p(\partial\Omega)$, $L^p(\Gamma_N)$, $W^{k,p}(\Omega)$, $H^k(\Omega)$, $W^{1,\infty}(\Omega)$, $W^{1,\infty}(\Omega_s^*)$ etc., etc. we shall denote the well-known spaces of continuously-differentiable functions and Lebesgue and Sobolev spaces of measurable functions, equipped with their usual norms (see e.g. [15, 18, 2]). We put $C(\bar{\Omega}) = C^0(\bar{\Omega})$. By $\|\cdot\|_{0,\Omega}$, $\|\cdot\|_{0,\partial\Omega}$, $\|\cdot\|_{0,p,\Omega}$, $\|\cdot\|_{0,p,\partial\Omega}$, $\|\cdot\|_{k,\Omega}$, $\|\cdot\|_{k,p,\Omega}$ we denote the norms in the spaces $L^2(\Omega)$, $L^2(\partial\Omega)$, $L^p(\Omega)$, $L^p(\partial\Omega)$, $H^k(\Omega)$ $(= W^{k,2}(\Omega))$, $W^{k,p}(\Omega)$, respectively. In $H^1(\Omega)$ beside the norm

$$\|u\|_{1,\Omega} = \left(\int_\Omega (u^2 + |\nabla u|^2)\, dx\right)^{1/2} \tag{1.6}$$

we shall use the seminorm

$$|u|_{1,\Omega} = \left(\int_\Omega |\nabla u|^2\, dx\right)^{1/2}. \tag{1.7}$$

(We set $\nabla u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2} \right)$.) The norm $\|\cdot\|_{1,\Omega}$ in $H^1(\Omega)$ is induced by the scalar product $(\cdot,\cdot)_{1,\Omega}$ defined on $H^1(\Omega) \times H^1(\Omega)$:

$$(u, v)_{1,\Omega} = \int_\Omega (uv + \nabla u \cdot \nabla v)\, dx . \tag{1.8}$$

We shall also consider the mentioned spaces over other open sets and use a similar notation.

By meas we shall denote the two-dimensional Lebesgue measure.

1.1.3. *Assumptions on the coefficients in equation* (0.1) *and on data*

(A) *a)* $f_s \in W^{1,\infty}(\Omega_s^*)$, $f : \Omega_0 \to R^1$ and $f \mid \Omega_s = f_s \mid \Omega_s$ $(s = 1, \ldots, m)$.

b) $\partial\Omega$ and $\partial\Omega_s$ $(s = 1, \ldots, m)$ are Lipschitz-continuous and piecewise of class C^3.

c) $q : \bar{\Gamma}_N \to R^1$, $q \in L^\infty(\Gamma_N)$, q is piecewise of class C^2 on $\bar{\Gamma}_N$.

d) $u_D : \bar{\Gamma}_D \to R^1$, $u_D = u^* \mid \bar{\Gamma}_D$, where $u^* \in W^{1,p}(R^2)$ with $p > 2$.

There exist functions $a_i^s : \bar{\Omega}_s^* \times R^3 \to R^1$ $(i = 0, 1, 2, \quad s = 1, \ldots, m)$, $a_i^s = a_i^s(x, \xi)$, $x = (x_1, x_2) \in \Omega_s^*$, $\xi = (\xi_0, \xi_1, \xi_2) \in R^3$, with the following properties:

(B) a_i^s $(i = 0, 1, 2)$ are continuous in $\bar{\Omega}_s^* \times R^3$; there exists a constant $c_0 > 0$ such that

$$|a_i^s(x, \xi)| \le c_0 \left(1 + \sum_{j=0}^{2} |\xi_j| \right) \quad \forall x \in \Omega_s^*, \quad \forall \xi \in R^3,$$
$$i = 0, 1, 2, \quad s = 1, \ldots, m .$$

(C) The derivatives $\frac{\partial a_i^s}{\partial \xi_j}$ are continuous and bounded in $\Omega_s^* \times R^3$:

$$\left| \frac{\partial a_i^s}{\partial \xi_j}(x, \xi) \right| \le c_0^* \quad \forall x \in \Omega_s^*, \quad \forall \xi \in R^3, \quad i, j = 0, 1, 2, \quad s = 1, \ldots, m .$$

(D_1) There exist constants $c_1 > 0$, $c_2 \ge 0$ such that

$$\sum_{i=0}^{2} a_i^s(x, \xi)\, \xi_i \ge c_1(\xi_1^2 + \xi_2^2) - c_2 \left(\sum_{i=0}^{2} |\xi_i| + 1 \right)$$
$$\forall x \in \Omega_s^*, \quad \forall \xi \in R^3, \quad s = 1, \ldots, m .$$

(D_2) There exists a constant $\alpha > 0$ such that

$$\sum_{i,j=1}^{2} \frac{\partial a_i^s}{\partial \xi_j}(x, \xi)\, \theta_i\, \theta_j \geq \alpha(\theta_1^2 + \theta_2^2)$$

$$\forall x \in \Omega_s^*, \quad \forall \xi \in R^3, \quad \forall \theta = (\theta_1, \theta_2) \in R^2, \quad s = 1, \dots, m.$$

(E) The derivatives $\dfrac{\partial a_i^s}{\partial x_j}$ are continuous in $\Omega_s^* \times R^3$ and

$$\left| \frac{\partial a_i^s}{\partial x_j}(x, \xi) \right| \leq c_0^{**}\left(1 + \sum_{k=0}^{2} |\xi_j|\right)$$

$$\forall x \in \Omega_s^*, \quad \forall \xi \in R^3, \quad i = 0, 1, 2, \quad j = 1, 2.$$

In Section 3.2 instead of (D_1) and (D_2) we shall consider the following assumption:

(D) There exists a constant $\alpha > 0$ such that

$$\sum_{i,j=0}^{2} \frac{\partial a_i^s}{\partial \xi_j}(x, \xi)\, \eta_i\, \eta_j \geq \alpha(\eta_1^2 + \eta_2^2)$$

$$\forall x \in \Omega_s^*, \quad \forall \xi \in R^3, \quad \forall \eta = (\eta_0, \eta_1, \eta_2) \in R^3, \quad s = 1, \dots, m.$$

(It is easy to prove that (D) and (B) $\Rightarrow$ (D_1) and (D_2) and (B), *cf.* [9].)

1.1.4. *Remark*

Assumption (A, d) says that the function u_D (from Dirichlet condition (1.11)), defined on the set $\Gamma_D \subset \partial\Omega$, has an extension to a function $u^* \in W^{1,p}(R^2)$. This is possible, if e.g. $u_D = \phi \mid \Gamma_D$ and the function $\phi : \partial\Omega \to R^1$ is obtained by integration of a function $\varphi : L^p(\partial\Omega)$ along $\partial\Omega$. This situation is often met in applications (we can remind stream function problems in fluid dynamics, *cf.* e.g. [5, 6]). The assumption $u^* \in H^2(R^2)$ usually used in the finite element analysis is rather strong and unrealistic in some cases.

We assume that the coefficients in (0.1) have the form

(1.9) $a_i(x, \xi) = a_i^s(x, \xi) \quad \forall x \in \Omega_s, \ \forall \xi \in R^3, \ i = 0, 1, 2, \ s = 1, \dots, m.$

Thus, the functions $a_i : \Omega_0 \times R^3 \to R^1$ and $f : \Omega_0 \to R^1$ can have discontinuities across Γ_{rs}.

1.2. Classical Formulation

If $u : \bar{\Omega} \to R^1$, then by u^s we denote an extension of $u | \Omega_s$ onto $\bar{\Omega}_s$. Let $\vec{n}^s(x) = (n_1^s(x), n_2^s(x))$ denote the unit outer normal to $\partial\Omega_s$. Obviously, $\vec{n}^s(x) = -\vec{n}^r(x)$ for $x \in \Gamma_{rs}$.

We shall study the following

1.2.1. *Boundary value problem*

Find $u : \bar{\Omega} \to R^1$ satisfying the equation (0.1) in Ω_0, i.e.

$$(1.10) \quad -\sum_{i=1}^{2} \frac{\partial}{\partial x_i} a_i(x, u(x), \nabla u(x)) + a_0(x, u(x), \nabla u(x)) = f(x), \quad x \in \Omega_0,$$

the boundary conditions

$$(1.11) \qquad u(x) = u_D(x), \quad x \in \Gamma_D \quad \text{(Dirichlet condition)},$$

$$(1.12) \qquad \sum_{i=1}^{2} a_i^s(x, u^s(x), \nabla u^s(x))\, n_i^s(x) = q(x),$$

$$x \in \Gamma_{sN}, \quad s = 1, \dots, m \quad \text{(Neumann condition)}$$

and the transition conditions

$$(1.13) \quad \sum_{i=1}^{2} a_i^s(x, u^s(x), \nabla u^s(x))\, n_i^s(x) = -\sum_{i=1}^{2} a_i^r(x, u^r(x), \nabla u^r(x))\, n_i^r(x),$$

$$x \in \Gamma_{rs}, \quad r, s = 1, \dots, m, \quad r \neq s.$$

It is obvious, how to define a classical solution of this problem.

1.2.2. *Definition*

We call $u : \bar{\Omega} \to R^1$ *a* classical solution *of problem* (1.10)-(1.13), *if* $u \in C(\bar{\Omega})$, $u^s \in C^2(\bar{\Omega}_s)$ *for* $s = 1, \dots, m$ *and* (1.10)-(1.13) *are satisfied.*

1.3. Generalized Weak Formulation

Let us put

$$(1.14) \qquad \mathscr{V} = \{v \in C^\infty(\bar{\Omega}), \operatorname{supp} v \subset \Omega \cup \Gamma_N\},$$

where supp v denotes the support of the function v, and define the space V as the closure of $\mathscr{V}$ in $H^1(\Omega)$:

$$(1.15) \qquad V = \bar{\mathscr{V}}^{H^1(\Omega)} = \{v \in H^1(\Omega),\ v | \Gamma_D = 0\}.$$

Since $\text{meas}_1(\Gamma_D) > 0$, the seminorm $|\cdot|_{1,\Omega}$ is a norm in V, equivalent to $\|\cdot\|_{1,\Omega}$:

$$\|v\|_{1,\Omega} \leq \hat{c}_3 |v|_{1,\Omega} \quad \forall v \in V \tag{1.16}$$

with a constant $\hat{c}_3 > 0$ independent of v.

Let us assume that u is a classical solution of problem (1.10)-(1.13). If we multiply equation (1.10) by an arbitrary $v \in \mathscr{V}$, integrate over Ω_0 and apply Green's theorem for each Ω_s, $s = 1, \dots, m$, then by (1.12), (1.13) and the fact that $\text{meas}(\Omega - \Omega_0) = 0$, we get the identity

$$\int_\Omega \left[\sum_{i=1}^{2} a_i(\cdot, u, \nabla u) \frac{\partial v}{\partial x_i} + a_0(\cdot, u, \nabla u) v \right] dx = \\ = \int_\Omega fv \, dx + \int_{\Gamma_N} qv \, ds \quad \forall v \in \mathscr{V}. \tag{1.17}$$

This leads us to the concept of a generalized weak solution. Let us denote

$$a(u, v) = \int_\Omega \left[\sum_{i=1}^{2} a_i(\cdot, u, \nabla u) \frac{\partial v}{\partial x_i} + a_0(\cdot, u, \nabla u) v \right] dx \tag{1.18}$$

for $u, v \in H^1(\Omega)$,

$$L^\Omega(v) = \int_\Omega fv \, dx, \quad L^\Gamma(v) = \int_\Gamma qv \, ds, \tag{1.19}$$

$$L(v) = L^\Omega(v) + L^\Gamma(v), \quad v \in H^1(\Omega). \tag{1.20}$$

1.3.1. *Definition*

We say that $u : \bar{\Omega} \to R^1$ *is a* weak solution *of problem* (1.10)-(1.13), *if*

$$\begin{aligned} &a) \; u \in H^1(\Omega), \\ &b) \; u - u^* \in V, \\ &c) \; a(u, v) = L(v) \quad \forall v \in V. \end{aligned} \tag{1.21}$$

1.3.2. *Properties of the forms* a, L^Ω, L^Γ, L

Under assumptions 1.1.3 (A), (B) there exists a constant $c > 0$ such that

$$|a(u, v)| \leq c(1 + \|u\|_{1,\Omega}) \|v\|_{1,\Omega} \quad \forall u, v \in H^1(\Omega), \tag{1.22}$$

$$|L(v)| \leq |L^\Omega(v)| + |L^\Gamma(v)| \leq c \|v\|_{1,\Omega} \quad \forall v \in H^1(\Omega). \tag{1.23}$$

Hence, for each $u \in H^1(\Omega)$ the functional $a(u, \cdot)$ and the functionals L^Ω, L^Γ, L are continuous and linear on $H^1(\Omega)$.

1.3.3. *Remark*

It is possible to show that any classical solution in the sense of Definition 1.2.1 is a weak solution. On the other hand, if u is a weak solution and $u^s \in C^2(\bar{\Omega}_s)$ for each $s = 1, \dots, m$, then u is a classical solution.

Weak problem (1.21, *a-c*) and its solvability can be treated under much weaker assumptions (*cf.* [1, 19]). Our strong assumptions will be necessary for the finite element analysis.

2. DISCRETE PROBLEM

In this section we shall suppose that assumptions (1.1), (1.2), (1.9) and 1.1.3 (A), (B) are satisfied.

2.1. Triangulations

Let us consider systems $\{\Omega_h\}_{h \in (0, h_0)}$ and $\{\Omega_{sh}\}_{h \in (0, h_0)}$, $s = 1, \dots, m$, $h_0 > 0$, of polygonal approximations of Ω and Ω_s, respectively, with the following properties:

(2.1) $\bar{\Omega}_h = \bigcup_{s=1}^{m} \bar{\Omega}_{sh}$, $\Omega_{sh} \cap \Omega_{rh} = \emptyset$ for $r \neq s$, $r, s = 1, \dots, m$.

(2.2) $\partial\Omega_h$ and $\partial\Omega_{sh}$ are formed by finite numbers of simple closed piecewise linear curves the vertices of which are lying on $\partial\Omega$ and $\partial\Omega_s$, respectively.

Let $\mathcal{T}_h$ and $\mathcal{T}_{sh}$ denote triangulations of Ω_h and Ω_{sh}, respectively, formed by finite numbers of closed triangles. We assume that

(2.3)
$$a) \quad \mathcal{T}_h = \bigcup_{s=1}^{m} \mathcal{T}_{sh},$$
$$b) \quad \bar{\Omega}_h = \bigcup_{T \in \mathcal{T}_h} T, \quad \bar{\Omega}_{sh} = \bigcup_{T \in \mathcal{T}_{sh}} T;$$

(2.4) if $T_1, T_2 \in \mathcal{T}_h$, $T_1 \neq T_2$, then either $T_1 \cap T_2 = \emptyset$ or $T_1 \cap T_2$ is a common vertex or $T_1 \cap T_2$ is a common side of T_1, T_2;

(2.5) if $T \in \mathcal{T}_{sh}$ ($s = 1, \dots, m$), then at most two vertices of T are lying on $\partial\Omega_s$.

We denote by $\sigma_h = \{P_1, \dots, P_N\}$ and σ_{sh} the set of all vertices of $\mathcal{T}_h$ and $\mathcal{T}_{sh}$, respectively, and let

(2.6) *a*) $\sigma_h \subset \bar{\Omega}$, $\sigma_{sh} \subset \bar{\Omega}_s$, $\sigma_h \cap \partial\Omega_h \subset \partial\Omega$, $\sigma_{sh} \cap \partial\Omega_{sh} \subset \partial\Omega_s$, $s = 1, \dots, m$,
b) $\bar{\Gamma}_D \cap \bar{\Gamma}_N \subset \sigma_h$,

c) the points from $\bigcup_{s=1}^{m} \partial\Omega_s$, where either the condition of C^3-smoothness of $\partial\Omega_s$ or the condition of C^2-smoothness of q are not satisfied, are elements of σ_h.

From the above assumptions it follows that

(2.7) *a*) to each $T \in \mathcal{T}_h$ there exists exactly one $s \in \{1, \dots, m\}$ such that $T \subset \bar{\Omega}_{sh}$, i.e. $T \in \mathcal{T}_{sh}$;

b) $\sigma_h = \bigcup_{s=1}^{m} \sigma_{sh}$;

c) $\partial\Omega \cap \bar{\Gamma}_{rs} \subset \sigma_h$ for $r \neq s$ and $\bar{\Gamma}_{r_1 s_1} \cap \bar{\Gamma}_{r_2 s_2} \subset \sigma_h$ for $\{r_1, s_1\} \neq \{r_2, s_2\}$, $r_1 \neq s_1$, $r_2 \neq s_2$.

By h_T and ϑ_T we shall denote the length of the maximal side and the magnitude of the minimal angle of $T \in \mathcal{T}_h$, respectively. We set

$$h = \max_{T \in \mathcal{T}_h} h_T, \quad \vartheta_h = \min_{T \in \mathcal{T}_h} \vartheta_T. \tag{2.8}$$

We shall assume that the system $\{\mathcal{T}_h\}_{h \in (0, h_0)}$ is *regular*. It means that there exists $\vartheta_0 > 0$ such that

$$\vartheta_h \geqslant \vartheta_0 > 0 \quad \forall h \in (0, h_0). \tag{2.9}$$

Further, by $\bar{\Gamma}_{Dh}$ and $\bar{\Gamma}_{Nh}$ we denote the parts of $\partial\Omega_h$ approximating $\bar{\Gamma}_D$ and $\bar{\Gamma}_N$, respectively. Similarly we define $\bar{\Gamma}_{sDh}$, $\bar{\Gamma}_{sNh}$ and $\bar{\Gamma}_{rsh}$ $(r \neq s)$ as the parts of $\partial\Omega_{sh}$ approximating $\bar{\Gamma}_{sD}$, $\bar{\Gamma}_{sN}$ and $\bar{\Gamma}_{rs}$.

2.2. Finite Element Discretization of the Problem

Approximate solutions to problem (1.21, *a-c*) will be sought in the finite-dimensional space of conforming piecewise linear elements $X_h \subset H^1(\Omega_h)$:

$$X_h = \{v_h ; v_h \in C(\bar{\Omega}_h),\ v_h \text{ is affine on each } T \in \mathcal{T}_h\}. \tag{2.10}$$

The space V will be approximated by

$$V_h = \{v_h \in X_h ; v_h | \Gamma_{Dh} = 0\} \equiv \{v_h \in X_h ; v_h(P_i) = 0 \ \forall P_i \in \sigma_h \cap \bar{\Gamma}_D\}. \tag{2.11}$$

In [26] it was proved that the seminorm $|\cdot|_{1,\Omega_h}$ is a norm on V_h, uniformly equivalent to $\|\cdot\|_{1,\Omega_h}$. It means that there exists a constant c_3 independent of

$v_h \in V_h$ and h such that

$$\|\cdot\|_{1,\Omega_h} \leq c_3 |\cdot|_{1,\Omega_h} \quad \forall v_h \in V_h, \quad \forall h \in (0, h_0) \tag{2.12}$$

(*cf.* also [6]).

Instead of the function $q : \bar{\Gamma}_N \to R^1$ we shall use its approximation $q_h : \bar{\Gamma}_{Nh} \to R^1$ defined in the same way as in [8, § 2.2].

Let $r_h : H^1(\Omega_h) \cap C(\bar{\Omega}_h) \to X_h$ be the operator of the Lagrange interpolation :

$$\begin{aligned} & r_h v \in X_h \quad \text{for} \quad v \in H^1(\Omega_h) \cap C(\bar{\Omega}_h), \\ & r_h v(P_j) = v(P_j) \quad \forall P_j \in \sigma_h . \end{aligned} \tag{2.13}$$

From 1.1.3 (A, *d*) and the imbedding theorem ([15, 18]) it follows that $u^* | \bar{\Omega}_h \in H^1(\Omega_h) \cap C(\bar{\Omega}_h)$. Let us set

$$u_h^* = r_h u^* . \tag{2.14}$$

It is evident that

$$u_h^*(P_j) = u_D(P_j) \quad \forall P_j \in \sigma_h \cap \bar{\Gamma}_D . \tag{2.15}$$

The forms a, L^Ω, L^Γ and L will be approximated by

$$\tilde{a}_h(u, v) = \sum_{s=1}^{m} \int_{\Omega_{sh}} \left[\sum_{s=1}^{m} a_i^s(., u, \nabla u) \frac{\partial v}{\partial x_i} + a_0^s(., u, \nabla u) v \right] dx , \tag{2.16}$$

$$u, v \in H^1(\Omega_h),$$

$$\tilde{L}_h^\Omega(v) = \sum_{s=1}^{m} \int_{\Omega_{sh}} f_s v \, dx , \quad v \in H^1(\Omega_h),$$

$$\tilde{L}_h^\Gamma(v) = \int_{\Gamma_{Nh}} q_h v \, ds , \qquad v \in H^1(\Omega_h),$$

$$\tilde{L}_h = \tilde{L}_h^\Omega + \tilde{L}_h^\Gamma .$$

2.2.1. *Discrete problem*

It can be written quite analogously as continuous problem (1.21, *a-c*) : find $\tilde{u}_h : \bar{\Omega}_h \to R^1$ such that

$$\begin{aligned} & a) \ \tilde{u}_h \in X_h , \\ & b) \ \tilde{u}_h - u_h^* \in V_h , \\ & c) \ \tilde{a}_h(\tilde{u}_h, v_h) = \tilde{L}_h(v_h) \quad \forall v_h \in V_h . \end{aligned} \tag{2.17}$$

2.2.2. *Numerical integration*

In practice the integrals in (2.16) are evaluated by numerical quadratures. We write

$$\text{(2.18)} \qquad a) \int_{\Omega_h} F\,dx = \sum_{T\in\mathcal{T}_h} \int_T F\,dx\,,$$

$$b) \int_T F\,dx \approx \text{meas}\,(T) \sum_{k=1}^{k_T} \omega_{T,k}\, F(x_{T,k}),\ \text{if } F \in C(T).$$

Here $x_{T,k} \in T$ and $\omega_{T,k} \in R^1$. We shall assume that

$$\text{(2.19)} \qquad a)\ \omega_{T,k} > 0\,, \quad b) \sum_{k=1}^{k_T} \omega_{T,k} = 1\,.$$

Similarly we evaluate integrals over Γ_{Nh}:

$$\text{(2.20)} \qquad a) \int_{\Gamma_{Nh}} F\,ds = \sum_{S\subset\bar\Gamma_{Nh}} \int_S F\,ds\,,$$

$$b) \int_S F\,ds \approx s(S) \sum_{j=1}^{k_S} \beta_{S,j}\, F(x_{S,j})\,, \quad \text{if} \quad F \in C\,(S)\,,$$

where $s(S)$ is the length of the side $S \subset \bar\Gamma_{Nh}$ (of a triangle $T \in \mathcal{T}_h$), $x_{S,j} \in S$ and $\beta_{S,j} \in R^1$. We assume that

(2.21) the degrees of precision of formulas (2.18, *b*) and (2.20, *b*) are ≥ 1.

If we approximate the forms $\tilde a_h$, $\tilde L_h^\Omega$ and $\tilde L_h^\Gamma$ by means of the formulas (2.18, *a-b*) and (2.20, *a-b*), we get

$$\text{(2.22)} \quad a_h(u_h, v_h) =$$

$$= \sum_{s=1}^{m} \sum_{T\in\mathcal{T}_h^s} \text{meas}\,(T) \left[\sum_{i=1}^{2} \frac{\partial v_h}{\partial x_i} \Big| T \sum_{j=1}^{k_T} \omega_{T,j}\, a_i^s(x_{T,j}, u_h(x_{T,j}), \nabla u_h | T) \right.$$

$$\left. + \sum_{j=1}^{k_T} \omega_{T,j}\, a_0^s(x_{T,j}, u_h(x_{T,j}), \nabla u_h | T)\, v_h(x_{T,j}) \right],$$

$$\text{(2.23)} \qquad L_h(v_h) = L_h^\Omega(v_h) + L_h^\Gamma(v_h)\,,$$

where

$$\text{(2.24)} \quad a)\ L_h^\Omega(v_h) = \sum_{s=1}^{m} \sum_{T\in\mathcal{T}_{sh}} \text{meas}\,(T) \sum_{j=1}^{k_T} \omega_{T,j}\, f_s(x_{T,j})\, v_h(x_{T,j}),$$

$$b)\ L_h^\Gamma(v_h) = \sum_{S\subset\bar\Gamma_{Nh}} s(S) \sum_{j=1}^{k_S} \beta_{S,j}\, q_h(x_{S,j})\, v_h(x_{S,j}).$$

Let us notice that if $x_{T,j} \in \sigma_h$, $x_{S,j} \in \sigma_h \cap \bar{\Gamma}_N$, then in practical calculations it is not necessary to extend the coefficients a_i from Ω_s onto Ω_s^* and to define the function q_h. Now we come to the definition of

2.2.3. *Discrete problem with the use of numerical integration*

Find $u_h : \bar{\Omega}_h \to R^1$ such that

$$\text{(2.25)} \qquad \begin{aligned} &a)\ \ u_h \in X_h\,, \\ &b)\ \ u_h - u_h^* \in V_h\,, \\ &c)\ \ a_h(u_h, v_h) = L_h(v_h) \quad \forall v_h \in V_h\,. \end{aligned}$$

2.3. Existence of Approximate Solutions

Let us consider assumptions (1.1), (1.2), (1.9), 1.1.3 (A), (B), (C), (D_1), (E) and assumptions from 2.1 and 2.2. (i.e., (2.1)-(2.6), (2.9), (2.19), (2.21)).

In the sequel the symbol c will denote a generic positive constant, independent of h, which can have different values at different places.

First let us draw our attention to the effect of numerical integration in the forms $\tilde{L}_h$ and $\tilde{a}_h$:

2.3.1. *Lemma*

There exists a constant $c > 0$ *such that*

$$\text{(2.26)} \qquad \left| L_h^{\Omega}(v) - \tilde{L}_h^{\Omega}(v) \right| \leq ch \, \| v \|_{1, \Omega_h}$$

$$\text{(2.27)} \qquad \left| L_h^{\Gamma}(v) - \tilde{L}_h^{\Gamma}(v) \right| \leq ch \, \| v \|_{1, \Omega_h}$$

$$\forall v \in X_h\,, \quad \forall h \in (0, h_0)\,,$$

$$\text{(2.28)} \qquad \left| a_h(u, v) - \tilde{a}_h(u, v) \right| \leq ch (1 + \| u \|_{1, \Omega_h}) \| v \|_{1, \Omega_h}$$

$$\forall u, v \in X_h\,, \quad \forall h \in (0, h_0)\,.$$

Proof of assertions (2.26) and (2.28) can be carried out on the basis of [8, Lemma 2.2.5] (which is a special case of [2, Theorem 4.1.5]) by a similar technique as in [8, Theorems 2.2.4 and 2.2.7]. E.g., in view of (2.16) and (2.22), we can write

$$\tilde{a}_h(u, v) - a_h(u, v) = I_1 + I_2\,,$$

where

$$I_1 = \sum_{s=1}^{m} \sum_{T \in \mathcal{T}_{sh}} \sum_{i=1}^{2} \frac{\partial v}{\partial x_i}\bigg|T\bigg\{ \int_T a_i^s(., u, \nabla u \,|\, T)\, dx - $$

$$- \text{meas }(T) \sum_{j=1}^{k_T} \omega_{T,j}\, a_i^s(x_{T,j}, u\,(x_{T,j}), \nabla u \,|\, T) \bigg\} ,$$

$$I_2 = \sum_{s=1}^{m} \sum_{T \in \mathcal{T}_{sh}} \bigg\{ \int_T a_0^s(., u, \nabla u \,|\, T)\, v\, dx - $$

$$- \text{meas }(T) \sum_{j=1}^{k_T} \omega_{T,j}\, a_0^s(x_{T,j}, u\,(x_{T,j}), \nabla u \,|\, T)\, v\,(x_{T,j}) \bigg\} .$$

Now we estimate the expressions in parenthesis in the same way as in [8, Theorem 2.2.7].

Concerning estimate (2.27), see [25, Theorem 5]. ■

Further, it is easy to prove the existence of a constant $c > 0$ such that

$$|\tilde{L}_h^{\Omega}(v)|, \ |\tilde{L}_h^{\Gamma}(v)|, \ |\tilde{L}(v)| \leq c \| v \|_{1, \Omega_h} \tag{2.29}$$

$$\forall v \in H^1(\Omega_h) , \quad \forall h \in (0, h_0) ,$$

$$|\tilde{a}_h(u, v)| \leq c(1 + \|u\|_{1, \Omega_h}) \|v\|_{1, \Omega_h} \tag{2.30}$$

$$\forall u, v \in H^1(\Omega_h) , \quad \forall h \in (0, h_0) ,$$

$$|L_h^{\Omega}(v)|, \ |L_h^{\Gamma}(v)|, \ |L(v)| \leq c \| v \|_{1, \Omega_h} \tag{2.31}$$

$$\forall v \in X_h , \quad \forall h \in (0, h_0) .$$

$$|a_h(u, v)| \leq c(1 + \|u\|_{1, \Omega_h}) \|v\|_{1, \Omega_h} \tag{2.32}$$

$$\forall u, v \in X_h , \quad \forall h \in (0, h_0) .$$

In the proof of these assertions we procede similarly as in [8, Lemma 3.2.2 and Theorem 3.1.2].

The proof of the solvability of discrete problems (2.17, *a-c*) and (2.25, *a-c*) is based on the following

2.3.2. *Lemma*

There exist constants $\tilde{c}$, $c > 0$ *such that*

$$\tilde{a}_h(u_h^* + v, v) - \tilde{L}_h(v) \geq \tag{2.33}$$

$$\geq c_1 c_3^{-2} \|v\|_{1, \Omega_h}^2 - \tilde{c}(1 + \|v\|_{1, \Omega_h} + \|u_h^*\|_{1, \Omega_h})(1 + \|u_h^*\|_{1, \Omega_h})$$

$$\forall v \in V_h , \quad \forall h \in (0, h_0)$$

and

(2.34) $a_h(u_h^* + v, v) - L_h(v) \geqslant$

$$\geqslant c_1 c_3^{-2} \|v\|_{1,\Omega_h}^2 - c(1 + \|v\|_{1,\Omega_h} + \|u_h^*\|_{1,\Omega_h})(1 + \|u_h^*\|_{1,\Omega_h})$$

$$\forall v \in V_h, \quad \forall h \in (0, h_0).$$

($u_h^* \in X_h$ *are functions defined by* (2.14); c_1 *and* c_3 *are constants from assumptions* 1.1.3 (D_1) *and* (2.12), *respectively*).

Proof: If we use assumptions 1.1.3 (B), (D_1), the inclusion $\Omega_{sh} \subset \Omega_s^*$ and write $\eta = (\vartheta + \eta) - \vartheta$, we easily prove that

(2.35) $$\sum_{i=0}^{2} a_i^s(x, \vartheta + \eta)\, \eta_i \geqslant c_1(\eta_1^2 + \eta_2^2) -$$

$$- c\left(1 + \sum_{i=0}^{2} (|\eta_i| + |\vartheta_i|)\right)\left(1 + \sum_{i=0}^{2} |\vartheta_i|\right)$$

$\forall s = 1, \dots, m, \quad \forall x \in \Omega_{sh}, \quad \forall \vartheta = (\vartheta_0, \vartheta_1, \vartheta_2), \quad \forall \eta = (\eta_0, \eta_1, \eta_2) \in R^3$

with a constant c depending on c_0, c_1 and c_2 from 1.1.3 only.

Now, let $v \in V_h$. Then, by (2.16) and (2.35),

(2.36) $$\tilde{a}_h(u_h^* + v, v) = \sum_{s=1}^{m} \int_{\Omega_{sh}} \left[\sum_{i=1}^{2} a_i^s(\cdot, u_h^* + v, \nabla(u_h^* + v)) \frac{\partial v}{\partial x_i} + \right.$$

$$\left. + a_0^s(\cdot, u_h^* + v, \nabla(u_h^* + v))\, v\right] dx$$

$$\geqslant c_1 \sum_{s=1}^{m} \left\{\int_{\Omega_{sh}} |\nabla v|^2\, dx - c I_s\right\},$$

where

$$\sum_{s=1}^{m} I_s = \int_{\Omega_h} [1 + |v| + |\nabla v| + |u_h^*| + |\nabla u_h^*|] \cdot [1 + |u_h^*| + |\nabla u_h^*|]\, dx.$$

Using the Cauchy inequality, we get

$$\sum_{s=1}^{m} I_s \leqslant c(1 + \|v\|_{1,\Omega_h} + \|u_h^*\|_{1,\Omega_h})(1 + \|u_h^*\|_{1,\Omega_h}).$$

This, (2.12), (2.29) and (2.36) immediately yield (2.33).

In proof of (2.34) we procede quite analogously. For $v \in V_h$ we have

(2.37) $$a_h(u_h^* + v, v) = \sum_{s=1}^{m} \sum_{T \in \mathcal{T}_{sh}} \text{meas}(T) \sum_{j=1}^{k_I} \omega_{T,j}\, G_{T,j}^s,$$

where

$$G^s_{T,j} = \sum_{i=1}^{2} a^s_i(x_{T,j}, (u^*_h + v)(x_{T,j}), \nabla(u^*_h + v)|T) \frac{\partial v}{\partial x_i}\Big|T + \\ + a^s_0(x_{T,j}, (u^*_h + v)(x_{T,j}), \nabla(u^*_h + v)|T)\, v(x_{T,j}) .$$

In virtue of (2.35),

$$(2.38) \quad G^s_{T,j} \geqslant c_1|\nabla v|^2_{1,T} - c[1 + |v(x_{T,j})| + |(\nabla v|T)| + |u^*_h(x_{T,j})| + \\ + |(\nabla u^*_h|T)|]\,[1 + |u^*_h(x_{T,j})| + |(\nabla u^*_h|T)|] \geqslant \\ \geqslant \tilde{G}_T := c_1|\nabla v|^2_{1,T} - c\Big[1 + \max_T |v| + |(\nabla v|T)| + \\ + \max_T |u^*_h| + |(\nabla u^*_h|T)|\Big]\Big[1 + \max_T |u^*_h| + |(\nabla u^*_h|T)|\Big].$$

Now, by (2.37), (2.38), (2.19, *a-b*), (2.31), the estimate

$$(2.39) \quad \max_T |v| \leqslant c(\text{meas }(T))^{-1/2}\,\|v\|_{0,T} \\ \forall v \in X_h, \quad \forall T \in \mathcal{T}_h, \quad \forall h \in (0, h_0)$$

valid with a constant c independent of v, T, h (see [8, Lemma 2.2.6]), the relations

$$(2.40) \quad \text{meas }(T)|(\nabla w|T)|^2 = |w|^2_{1,T}, \\ \text{meas }(T)|(\nabla w|T)| = \int_T |\nabla w|\,dx, \quad w \in X_h,$$

and the repeated application of the Cauchy inequality we come to (2.34). ■

2.3.3. *Lemma*

We have

$$(2.41) \quad a)\ |\tilde{a}_h(u_1, v) - \tilde{a}_h(u_2, v)| \leqslant c\|u_1 - u_2\|_{1,\Omega_h}\,\|v\|_{1,\Omega_h} \\ \forall u_1, u_2, v \in H^1(\Omega_h), \quad \forall h \in (0, h_0), \\ b)\ |a_h(u_1, v) - a_h(u_2, v)| \leqslant c\|u_1 - u_2\|_{1,\Omega_h}\,\|v\|_{1,\Omega_h} \\ \forall u_1, u_2, v \in X_h, \quad \forall h \in (0, h_0)$$

with a constant c independent of u_1, u_2, v and h.

Proof : Let us prove the second inequality. By (2.22), provided $v,\ u_1,\ u_2 \in X_h$,

$$(2.42)\quad a_h(u_1, v) - a_h(u_2, v) = \sum_{s=1}^{m} \sum_{T \in \mathcal{T}_{sh}} \text{meas}\,(T) \sum_{j=1}^{k_T} \omega_{T,j}\, \phi^s_{T,j}\,,$$

where

$$\phi^s_{T,j} = \sum_{i=1}^{2} \left[a^s_i(x_{T,j}, u_1(x_{T,j}), \nabla u_1 | T) - a^s_i(x_{T,j}, u_2(x_{T,j}), \nabla u_2 | T)\right] \frac{\partial v}{\partial x_i}\Big| T$$
$$+ \left[a^s_0(x_{T,j}, u_1(x_{T,j}), \nabla u_1 | T) - a^s_0(x_{T,j}, u_2(x_{T,j}), \nabla u_2 | T)\right] v(x_{T,j}).$$

In view of assumption 1.1.3 (C), we can apply the mean value theorem :

$$a^s_i(x, \eta) - a^s_i(x, \xi) = \sum_{j=0}^{2} \int_0^1 \frac{\partial a^s_i}{\partial \xi_j}(x, \xi + t(\eta - \xi))\, dt\, (\eta_j - \xi_j)$$

for all $x \in \Omega^*_s$ and $\xi,\ \eta \in R^3$ and get the estimate

$$|\phi^s_{T,j}| \leqslant 2\, c^*_0 \left(\max_T |v| + |(\nabla v | T)| \right) \times$$
$$\times \left(\max_T |u_1 - u_2| + |(\nabla(u_1 - u_2) | T)| \right).$$

Substituting into (2.42), using (2.39), (2.40) and the Cauchy inequality, we come to the desired result (2.41, *b*). The proof of (2.41, *a*) is analogous, but simpler. ∎

Finally, we come to the main result of this paragraph — the solvability theorem for the discrete problem.

2.3.4. *Theorem*

To each $h \in (0, h_0)$ *there exists at least one solution* $\tilde{u}_h$ *of problem* (2.17, *a-c*) *and at least one solution* u_h *of problem* (2.25, *a-c*). *Moreover, if*

$$(2.43)\qquad \|u^*_h\|_{1, \Omega_h} \leqslant c^* \quad \forall h \in (0, h_0)\,,$$

where c^* *is a constant independent of h, then there exists a constant* $c > 0$ *such that*

$$(2.44)\qquad \|\tilde{u}_h\|_{1, \Omega_h},\ \|u_h\|_{1, \Omega_h} \leqslant c \quad \forall h \in (0, h_0)\,.$$

Proof : Let us prove the existence of a solution u_h of problem (2.25, *a-c*). (The existence of $\tilde{u}_h$ as a solution to problem (2.17, *a-c*) can be proved in the same way.) We shall seek u_h in the form $u_h = u_h^* + z_h$, where $z_h \in V_h$. From (2.31) and (2.32) it follows that for each $z_h \in V_h$ the mapping

$$v_h \in V_h \to a_h(u_h^* + z_h, v_h) - L_h(v_h) \in R^1$$

is a continuous linear functional and hence, by the well-known Riesz theorem, we can write

$$a_h(u_h^* + z_h, v_h) - L_h(v_h) = (T_h(z_h), v_h)_{1,\Omega_h}, \tag{2.45}$$

where $(.,.)_{1,\Omega_h}$ is the scalar product in $H^1(\Omega_h)$ which induces the norm $\|\cdot\|_{1,\Omega_h}$ (compare with (1.8)) and $T_h(z_h) \in V_h$ with

$$\|T_h(z_h)\|_{1,\Omega_h} = \sup_{\substack{v_h \in V_h \\ v_h \neq 0}} |a_h(u_h^* + z_h, v_h) - L_h(v_h)| / \|v_h\|_{1,\Omega_h}. \tag{2.46}$$

Hence, $T_h : V_h \to V_h$ and the problem (2.25, *a-c*) is equivalent to the equation

$$T_h(z_h) = 0 \tag{2.47}$$

in the finite-dimensional space V_h. From (2.41, *b*) we see that the operator T_h is continuous. Moreover, by (2.34),

$$\begin{aligned}(T_h(v), v)_{1,\Omega_h} \geq c_1 c_3^{-2} \|v\|_{1,\Omega_h}^2 - \\ - c(1 + \|u_h^*\|_{1,\Omega_h}) \|v\|_{1,\Omega_h} - c(1 + \|u_h^*\|_{1,\Omega_h})^2,\end{aligned} \tag{2.48}$$

where the constant c is independent of h and $v \in V_h$. This yields the existence of a constant $K > 0$ such that $(T_h(v), v) \geq 0$ for all $v \in V_h$ with $\|v\|_{1,\Omega_h} = K$. Hence, by [16, Chap. 1, Lemma 4.3] equation (2.47) has at least one solution $z_h \in V_h$, which gives a solution $u_h = u_h^* + z_h$ of problem (2.25, *a-c*).

Now, let (2.43) be satisfied. Then, in view of (2.47) and (2.48),

$$0 = (T_h(z_h), z_h)_{1,\Omega_h} \geq p(\|z_h\|_{1,\Omega_h}) \quad \forall h \in (0, h_0),$$

where $p(t) = c_1 c_3^{-2} t^2 - c(1 + c^*) t - c(1 + c^*)^2$. As $c_1 c_3^{-2}$, $c(1 + c^*)$, $c(1 + c^*)^2 > 0$ are constants independent of h, there exists $\hat{c} > 0$ such that $\|z_h\|_{1,\Omega_h} \leq \hat{c}$ for all $h \in (0, h_0)$. Now it is evident that u_h satisfies (2.44) with $c = \hat{c} + c^*$. ■

2.3.5. *Remark*

The approximate finite element solutions $\tilde{u}_h$ or u_h to continuous problem (1.21, *a-c*) are obtained on the basis of the discretization process without or with the use of numerical integration, respectively. Therefore, in practical calculations we seek the solutions u_h. The solutions $\tilde{u}_h$ will have a theoretical importance in paragraph 3.2.

Now we shall deal with the uniqueness of the approximate solutions.

2.3.6. *Lemma*

Provided we consider assumption 1.1.3 (D) *instead of* 1.1.3 (D_1), *the forms* $\tilde{a}_h$ *and* a_h *are uniformly strongly monotone with respect to the seminorm* $|\cdot|_{1,\Omega_h}$:

$$\tilde{a}_h(u, u-v) - \tilde{a}_h(v, u-v) \geq \alpha |u-v|^2_{1,\Omega_h} \tag{2.49}$$

$$\forall u, v \in H^1(\Omega_h), \quad \forall h \in (0, h_0),$$

$$a_h(u, u-v) - a_h(v, u-v) \geq \alpha |u-v|^2_{1,\Omega_h} \tag{2.50}$$

$$\forall u, v \in X_h, \quad \forall h \in (0, h_0),$$

where α *is the constant from assumption* 1.1.3 (D).

Proof can be carried out similarly as in [8, Theorem 3.1.2] using the same technique as in the proof of Lemma 2.3.2. ■

2.3.7. *Theorem*

Provided we consider assumption 1.1.3 (D) *instead of* 1.1.3 (D_1), *the solutions* $\tilde{u}_h$ *and* u_h *to problems* (2.17, *a-c*) *and* (2.25, *a-c*), *respectively, are unique for each* $h \in (0, h_0)$.

Proof : If, e.g., $\tilde{u}_h^1$, $\tilde{u}_h^2$ are two solutions of (2.17, *a-c*), then by (2.17, *b*), $\tilde{u}_h^2 - \tilde{u}_h^1 \in V_h$ and thus, in view of (2.17, *c*), (2.12) and (2.49),

$$0 \leq \|\tilde{u}_h^2 - \tilde{u}_h^1\|_{1,\Omega_h} \leq c_3 |\tilde{u}_h^2 - \tilde{u}_h^1|_{1,\Omega_h} \leq 0 .$$

It means that $\tilde{u}_h^1 = \tilde{u}_h^2$. ■

3. CONVERGENCE

3.1. General Pseudomonotone Case

Let assumptions (1.1), (1.2), (1.9), 1.1.3 (A), (B), (C), (D_1), (D_2) and (E) and assumptions from 2.1 and 2.2 be satisfied. We shall use ideas from [6, 8, 9] based on the possibility to modify functions $v_h \in V_h$ in such a way that we get elements of the space V.

By the symbol $\mathcal{T}_h^{id}$ we denote the ideal triangulation of the domain Ω, associated with the triangulation $\mathcal{T}_h$ of Ω_h. If $T \in \mathcal{T}_h$ is a boundary element (i.e., two vertices of T are lying on $\partial\Omega$), then $T^{id} \in \mathcal{T}_h^{id}$ denotes the ideal element associated with the element T. (See [8, § 2.1.1].) Similarly we can speak about the ideal triangulation $\mathcal{T}_{sh}^{id}$ of the domain Ω_s, associated with $\mathcal{T}_{sh}$.

In order to simplify some our considerations we shall introduce the following *assumption* : if $S \subset \partial\Omega_h$ ($S \subset \partial\Omega_{sh}$) is a side of a boundary triangle $T \in \mathcal{T}_h$ ($T \in \mathcal{T}_{sh}$) and $\Sigma \subset \partial\Omega$ ($\Sigma \subset \partial\Omega_s$) is the corresponding curved side of the ideal element $T^{id} \in \mathcal{T}_h^{id}$ ($T^{id} \in \mathcal{T}_{sh}^{id}$) associated with T, then either $S = \Sigma$ or $S \cap \Sigma$ is formed by the common end-points of S and Σ, which are elements of σ_h (see *fig.* 3.1).

Let us set

$$\omega_h = \Omega - \bar{\Omega}_h\,, \quad \tau_h = \Omega_h - \bar{\Omega}\,, \tag{3.1}$$
$$\omega_{sh} = \Omega_s - \bar{\Omega}_{sh}\,, \quad \tau_{sh} = \Omega_{sh} - \bar{\Omega}_s\,.$$

In virtue of [7, Lemma 3.3.4],

$$\text{meas}\,(\tau_h \cup \omega_h)\,, \quad \text{meas}\,(\tau_{sh} \cup \omega_{sh}) \leq ch^2 \tag{3.2}$$
$$\forall h \in (0, h_0)\,, \quad s = 1, \ldots, m$$

with a constant c independent of h.

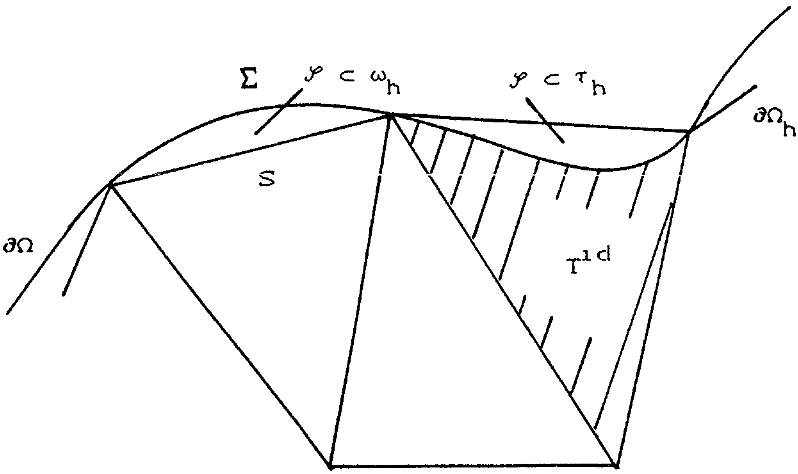

Figure 3.1.

By $\bar{v}_h$ we shall denote the natural extension of $v_h \in X_h$ onto $\bar{\Omega}_h \cup \bar{\Omega}$. It means that $\bar{v}_h \in C(\bar{\Omega}_h \cup \bar{\Omega})$, $\bar{v}_h = v_h$ on $\bar{\Omega}_h$ and $\bar{v}_h | T^{id} = p | T^{id}$ on $T^{id} \supset T$, where p is the polynomial of order ≤ 1 satisfying $p | T = v_h | T$. It is evident that $\bar{v}_h \in H^1(\Omega)$ (*cf.* [2, Theorem 2.1.1]).

3.1.1. *Lemma*

There exists a constant $c > 0$ such that

(3.3)
$$\begin{aligned} a)&\quad \|\bar{v}_h\|_{0,\tau_h \cup \omega_h} \leq ch \|v_h\|_{1,\Omega_h}, \\ b)&\quad \|\bar{v}_h\|_{1,\tau_h \cup \omega_h} \leq ch^{1/2} \|v_h\|_{1,\Omega_h}, \\ c)&\quad \|\bar{v}_h\|_{0,\tau_{sh} \cup \omega_{sh}} \leq ch \|v_h\|_{1,\Omega_{sh}}, \\ d)&\quad \|\bar{v}_h\|_{1,\tau_{sh} \cup \omega_{sh}} \leq ch^{1/2} \|v_h\|_{1,\Omega_{sh}}, \\ &\quad \forall v_h \in X_h, \quad \forall h \in (0, h_0) \quad (s = 1, \dots, m), \\ e)&\quad \|v\|_{0,\tau_{sh} \cup \omega_{sh}} \leq ch \|v\|_{1,\Omega_s^*} \\ &\quad \forall v \in H^1(\Omega_s^*), \quad \forall h \in (0, h_0) \quad (s = 1, \dots, m). \end{aligned}$$

Proof of *a*)-*b*). See [8, Lemma 3.3.12]; similarly we prove *c*) and *d*). Assertion *e*) follows from [8, Lemma 3.3.11]. ■

3.1.2. *Lemma*

To each $v_h \in V_h$ ($h \in (0, h_0)$) there exists a function $\hat{v}_h \in V$ such that

(3.4)
$$\|\hat{v}_h - \bar{v}_h\|_{1,\Omega} \leq ch \|v_h\|_{1,\Omega_h},$$

where c is a constant independent of h and v_h.

Proof: The function $\hat{v}_h$ can be chosen as the ideal interpolation of $\bar{v}_h$, defined in [9, § 5.1.1]. Then (3.4) follows from the proof of [9, Lemma 5.1.2]. ■

Now, for each $h \in (0, h_0)$ let us define a *function $u_h' \in H^1(\Omega)$ associated with the solution u_h* of problem (2.25, *a-c*) in the following way: if we express u_h in the form $u_h = u_h^* + z_h$ with $z_h \in V_h$ (*cf.* the proof of Theorem 2.3.4), then we set

(3.5)
$$u_h' = \bar{u}_h^* + \hat{z}_h .$$

Let us deal with the limit properties of $u_h^* = r_h u^*$, if $h \to 0^+$.

3.1.3. *Lemma*

It holds

(3.6)
$$\lim_{h \to 0^+} \|u_h^* - u^*\|_{1,\Omega_h} = 0 .$$

Proof: Let $\Omega^* \subset R^2$ be a domain such that $\Omega_h \subset \Omega^*$ for all $h \in (0, h_0)$. In view of assumption 1.1.3 (A, *d*), $u^* | \Omega^* \in W^{1,p}(\Omega^*)$ and $u^* | \Omega_h \in W^{1,p}(\Omega_h)$ for each $h \in (0, h_0)$. From [3, Theorem 6] (*cf.* also [2, Theorem 3.1.6] it follows that

$$\| v - r_h v \|_{1,p,\Omega_h} \leq c \| v \|_{1,p,\Omega_h} \quad \forall v \in W^{1,p}(\Omega^*), \quad \forall h \in (0, h_0)$$

with a constant $c > 0$ independent of h and v. Hence,

$$\| r_h v \|_{1,p,\Omega_h} \leq c' \| v \|_{1,p,\Omega_h} (\leq c' \| v \|_{1,p,\Omega^*}) \quad \forall v \in W^{1,p}(\Omega^*), \quad \forall h \in (0, h_0) \tag{3.7}$$

($c' = 1 + c$). Further, let us remind that provided $v \in W^{2,p}(\Omega^*)$,

$$\| v - r_h v \|_{1,p,\Omega_h} \leq ch \| v \|_{2,p,\Omega^*} \quad \forall h \in (0, h_0) \tag{3.8}$$

with a constant c independent of h and v (see [3, Theorem 6]).

Now let us consider an arbitrary $\varepsilon > 0$. In virtue of the density of $C^\infty(\bar{\Omega}^*)$ in $W^{1,p}(\Omega^*)$ ([18, Chap. 2, § 3]) we can choose $v \in C^\infty(\bar{\Omega}^*)$ such that

$$\| u^* - v \|_{1,p,\Omega_h} \leq \frac{\varepsilon}{3c'} . \tag{3.9}$$

From (3.7) it follows that

$$\| r_h(u^* - v) \|_{1,p,\Omega_h} \leq \frac{\varepsilon}{3} . \tag{3.10}$$

By (3.8), there exists $h_\varepsilon \in (0, h_0)$ such that

$$\| v - r_h v \|_{1,p,\Omega_h} \leq \frac{\varepsilon}{3} \quad \forall h \in (0, h_\varepsilon) . \tag{3.11}$$

Using (3.9)-(3.11) we come to

$$\| u^* - r_h u^* \|_{1,p,\Omega_h} \leq \| u^* - v \|_{1,p,\Omega_h} + \| v - r_h v \|_{1,p,\Omega_h} + \| r_h v - r_h u^* \|_{1,p,\Omega_h} \leq \varepsilon , \quad \forall h \in (0, h_\varepsilon) ,$$

which means that

$$\lim_{h \to 0^+} \| u_h^* - u^* \|_{1,p,\Omega_h} = 0 .$$

Finaly, from this and the Hölder inequality

$$\|\varphi\|_{0,\Omega_h} = \left(\int_{\Omega_h} \varphi^2\,dx\right)^{1/2} \leq (\text{meas}\,(\Omega_h))^{\frac{p-2}{2p}} \left(\int_{\Omega_h} \varphi^p\,dx\right)^{\frac{1}{p}} \leq$$

$$\leq c\,\|\varphi\|_{0,p,\Omega_h} \quad \forall \varphi \in L^p(\Omega_h)$$

(c independent of h and φ), applied to $u_h^* - u^*$ and $\frac{\partial}{\partial x_i}(u_h^* - u^*)$, we get (3.6). ■

Let us notice that (3.6) immediately implies (2.43).

3.1.4. *Lemma*

It holds

$$\lim_{h \to 0^+} \|\bar{u}_h^* - u^*\|_{1,\Omega} = 0\,. \tag{3.12}$$

Proof: We have

$$\|\bar{u}_h^* - u^*\|_{1,\Omega}^2 = \|u_h^* - u^*\|_{1,\Omega_h - \tau_h}^2 + \|\bar{u}_h^* - u^*\|_{1,\omega_h}^2$$

and thus,

$$\|\bar{u}_h^* - u^*\|_{1,\Omega} \leq c(\|u_h^* - u^*\|_{1,\Omega_h} + \|\bar{u}_h^*\|_{1,\omega_h} + \|u^*\|_{1,\omega_h}) \to 0\,,$$

as it follows from (3.6), (3.2), Lemma 3.1.1 and the absolute continuity of the Lebesgue integral. ■

3.1.5. *Lemma*

It holds

$$\left|\tilde{L}_h^{\Gamma}(v_h) - L^{\Gamma}(\bar{v}_h)\right| \leq ch^{\frac{3}{2}} \|v_h\|_{1,\Omega_h} \tag{3.13}$$

$$\forall v_h \in V_h\,, \quad \forall h \in (0, h_0)\,.$$

For the *proof* see [8, Lemma 3.3.13].

3.1.6. *Lemma*

There exists a constant $c > 0$ such that

$$\|u_h'\|_{1,\Omega'}\ \|\bar{u}_h\|_{1,\Omega} \leq c \quad \forall h \in (0, h_0)\,. \tag{3.14}$$

Proof : If we write $u_h = u_h^* + z_h$ and use the boundedness of u_h^* and u_h, then

$$\|z_h\|_{1,\Omega_h} \leqslant c \quad \forall h \in (0, h_0). \tag{3.15}$$

By Lemma 3.1.1 and (3.15), we get

$$\begin{aligned} \|\bar{z}_h\|_{1,\Omega} &= (\|z_h\|^2_{1,\Omega_h - \tau_h} + \|\bar{z}_h\|^2_{1,\omega_h})^{1/2} \\ &\leqslant c\|z_h\|_{1,\Omega_h} \leqslant c \quad \forall h \in (0, h_0). \end{aligned} \tag{3.16}$$

Similarly (or from (3.12)) we find out that

$$\|\bar{u}_h^*\|_{1,\Omega} \leqslant c \quad \forall h \in (0, h_0). \tag{3.17}$$

Further, by (3.5),

$$\begin{aligned} \|u_h'\|_{1,\Omega} &\leqslant \|\bar{u}_h^*\|_{1,\Omega} + \|\hat{z}_h\|_{1,\Omega} \\ &\leqslant \|\bar{u}_h^*\|_{1,\Omega} + \|\bar{z}_h\|_{1,\Omega} + \|\hat{z}_h - \bar{z}_h\|_{1,\Omega}. \end{aligned} \tag{3.18}$$

From (3.15)-(3.18) and (3.4) we immediately have the estimate $\|u_h'\|_{1,\Omega} \leqslant c$. The estimate $\|\bar{u}_h\|_{1,\Omega} \leqslant c$ follows from (3.16), (3.17) and the relation $\bar{u}_h = \bar{u}_h^* + \bar{z}_h$. ■

Let $G \subset \Omega_s^*$ be an open set and let $u, v \in H^1(G)$. If we denote

$$\tilde{a}_G^s(u, v) = \int_G \left[\sum_{i=1}^2 a_i^s(., u, \nabla u)\frac{\partial v}{\partial x_i} + a_0^s(., u, \nabla u)\, v\right] dx, \tag{3.19}$$

then, by 1.1.3 (B),

$$|\tilde{a}_G^s(u, v)| \leqslant c((\text{meas}\,(G))^{1/2} + \|u\|_{1,G})\|v\|_{1,G} \tag{3.20}$$

with c independent of G, u, v.

From (1.22) it follows that we can define the operator $A : H^1(\Omega) \to (H^1(\Omega))^*$ by the relation

$$\langle A(u), v\rangle = a(u, v) \quad u, v \in H^1(\Omega). \tag{3.21}$$

Here $(H^1(\Omega))^*$ is the dual to $H^1(\Omega)$ and $\langle .,. \rangle$ denotes the duality between $(H^1(\Omega))^*$ and $H^1(\Omega)$. The norm in $(H^1(\Omega))^*$ will be denoted by $\|\cdot\|_{-1,\Omega}$.

The proof of the convergence of the finite element approximations is based on the following fundamental properties of the operator A.

3.1.7. *Theorem*

a) *The operator A is Lipschitz-continuous : there exists a constant c such that*

(3.22) $$\|A(u_1) - A(u_2)\|_{-1,\Omega} \leq c\|u_1 - u_2\| \quad \forall u_1, u_2 \in H^1(\Omega),$$

and maps a bounded set into a bounded set : to each $\tilde{c} > 0$ there exists $c > 0$ such that

(3.23) $$\|A(v)\|_{-1,\Omega} \leq c \quad \forall v \in H^1(\Omega) \text{ with } \|v\|_{1,\Omega} \leq \tilde{c}.$$

b) *The operator A satisfies the* generalized condition (S) : *If*

(3.24) $$v_n \rightharpoonup v \quad \text{weakly in } V,$$

(3.25) $$w_n^* \to w^* \quad \text{in } H^1(\Omega),$$

(3.26) $$\langle A(w_n^* + v_n) - A(w^* + v), v_n - v\rangle \to 0,$$

then

(3.27) $$w_n = w_n^* + v_n \to w = w^* + v \quad \text{in } H^1(\Omega).$$

Proof : a) For u_1, u_2, $v \in H^1(\Omega)$, we have

$$\langle A(u_1) - A(u_2), v\rangle =$$
$$= \sum_{s=1}^{m} \int_{\Omega_s} \Big[\sum_{i=1}^{2} (a_i^s(\cdot, u_1, \nabla u_1) - a_i^s(\cdot, u_2, \nabla u_2)) \frac{\partial v}{\partial x_i} +$$
$$+ (a_0^s(\cdot, u_1, \nabla u_1) - a_0^s(\cdot, u_2, \nabla u_2))\, v\Big]\, dx.$$

Using the mean value theorem and assumption (C) from 1.1.3, we come to the estimate

$$|\langle A(u_1) - A(u_2), v\rangle| \leq$$
$$\leq c_0^* \sum_{s=1}^{m} \int_{\Omega_s} \left[\left(|u_1 - u_2| + \sum_{i=1}^{2} \left|\frac{\partial(u_1 - u_2)}{\partial x_i}\right|\right)\left(|v| + \sum_{j=1}^{2}\left|\frac{\partial v}{\partial x_j}\right|\right)\right] dx$$
$$\leq 3\, c_0^* \sum_{s=1}^{m} \|u_1 - u_2\|_{1,\Omega_s} \|v\|_{1,\Omega_s} \leq 3\, c_0^* \|u_1 - u_2\|_{1,\Omega} \|v\|_{1,\Omega'}$$

which implies (3.22).

Property (3.23) is a consequence of (3.22).

b) Let assumptions (3.24)-(3.26) be satisfied. We denote

$$\mathfrak{I}_n = a(w_n, v_n - v) - a(w, v_n - v),$$
$$I_n = a(w_n, w_n - w) - a(w, w_n - w),$$

where w_n, w are defined in (3.27). Then

$$\mathfrak{J}_n = a(w_n, v_n - v) - a(w, v_n - v) = I_n + K_n$$

with

$$K_n = a(w, w_n^* - w^*) - a(w_n, w_n^* - w^*) .$$

Similarly as in part *a*) of the proof, we find out that

$$|K_n| \leqslant 3\, c_0^* \| w - w_n \|_{1,\Omega} \| w_n^* - w^* \|_{1,\Omega} .$$

From this, the boundedness of the sequence $\{w - w_n\}$ and (3.25) we get $K_n \to 0$. Hence, by (3.26),

$$I_n \to 0 . \tag{3.28}$$

From (3.24), (3.25) and the compact imbedding $H^1(\Omega) \subset L^2(\Omega)$ it follows that

$$\| w_n - w \|_{0,\Omega} \to 0 . \tag{3.29}$$

Further, using again the mean value theorem, we obtain

$$\begin{aligned} I_n = \sum_{s=1}^{m} \int_{\Omega_s} \Bigg[\sum_{i=1}^{2} (a_i^s(\cdot, w_n, \nabla w_n) - a_i^s(\cdot, w, \nabla w)) \frac{\partial (w_n - w)}{\partial x_i} + \\ + (a_0^s(\cdot, w_n, \nabla w_n) - a_0^s(\cdot, w, \nabla w))(w_n - w) \Bigg] dx = \\ = \sum_{s=1}^{m} \int_{\Omega_s} \sum_{i,j=1}^{2} \int_0^1 \frac{\partial a_i^s}{\partial \xi_j} (\cdot, w + t(w_n - w), \nabla (w + t(w_n - w))) \times \\ \times \frac{\partial (w_n - w)}{\partial x_i} \frac{\partial (w_n - w)}{\partial x_j}\, dt\, dx + \sigma_n , \end{aligned} \tag{3.30}$$

where

$$\begin{aligned} \sigma_n = \sum_{s=1}^{m} \int_{\Omega_s} \Bigg\{ \sum_{i=1}^{2} \int_0^1 \frac{\partial a_i^s}{\partial \xi_0} (\cdot, w + t(w_n - w), \nabla (w + t(w_n - w))) \times \\ \times \frac{\partial (w_n - w)}{\partial x_i} (w_n - w)\, dt + (a_0^s(\cdot, w_n, \nabla w_n) \\ - a_0^s(\cdot, w, \nabla w))(w_n - w) \Bigg\} dx . \end{aligned} \tag{3.31}$$

From (3.30), (3.31) and assumptions 1.1.3 (B), (C) and (D_2) we derive the inequalities

$$I_n \geqslant \alpha | w_n - w |_{1,\Omega}^2 + \sigma_n ,$$

$$|\sigma_n| \leqslant c \| w_n - w \|_{0,\Omega} \| w_n - w \|_{1,\Omega} ,$$

where α is the constant from 1 1 3 (D_2) and c is a constant independent of w_n, w This, (3 28), (3 29) and the boundedness of the sequence $\{w_n\}$ immediately yield (3 27) ■

Now let us go back to the approximate solutions u_h and the functions u_h' defined in (3 5) In virtue of Lemma 3 1 6 and Theorem 3 1 7, there exists a constant $c > 0$ such that

$$\|u_h'\|_{1\,\Omega}, \quad \|A(u_h')\|_{1\,\Omega} \leq c \quad \forall h \in (0, h_0) \tag{3 32}$$

Let $\{h_m\} \subset (0, h_0)$, $h_m \to 0$ On the basis of (3 32) and the reflexivity of the space $H^1(\Omega)$ we can choose a subsequence $\{h_n\} \subset \{h_m\}$ such that

$$u_{h_n}' \rightharpoonup u \quad \text{weakly in } H^1(\Omega) \tag{3 33}$$

In the sequel we shall show that the weak limit u from (3 33) is a weak solution of the continuous problem

3 1 8 *Theorem*

Let $u_h' \in H^1(\Omega)$ be the function associated by (3 5) *with a solution $u_h \in X_h$ of the discrete problem* (2 25, *a-c*) *If $\{h_n\} \subset (0, h_0)$, $h_n \to 0$ and $u_{h_n}' \rightharpoonup u$ weakly in $H^1(\Omega)$, then $u_{h_n}' \to u$ strongly in $H^1(\Omega)$ and u is a solution of problem* (1 21, *a-c*)

Proof For simplicity we shall omit the subscript n at h and write $h = h_n \to 0$, $u_h' = u_{h_n}'$, $u_h' \rightharpoonup u$ etc

I) It is evident that u satisfies conditions (1 21, *a-b*) Actually, from $u_h' = \bar{u}_h^* + \hat{z}_h \rightharpoonup u$ in $H^1(\Omega)$ and (3 12) it follows that $u \in H^1(\Omega)$ and $\hat{z}_h \rightharpoonup u - u^*$ Since the space V is weakly closed, we see that $u - u^* \in V$

II) Now we prove the existence of $\chi \in (H^1(\Omega))^*$ such that

$$A(u_h') = A(u_{h_n}') \rightharpoonup \chi \quad \text{weakly in } (H^1(\Omega))^* \tag{3 34}$$

and

$$\langle \chi, v \rangle = L(v) \quad \forall v \in V \tag{3 35}$$

On the basis of the reflexivity of $(H^1(\Omega))^*$ we can choose a subsequence of $\{A(u_h')\}$ weakly convergent to an element $\chi \in (H^1(\Omega))^*$ In the following we shall prove that this χ satisfies (3 35) This fact immediately implies that the whole sequence $\{A(u_h')\}$ is weakly convergent to χ satisfying (3 35) Thus, it is sufficient to prove the implication (3 34) $\Rightarrow$ (3 35)

Let $v \in \mathscr{V}$ (see (1.14)). By $v_c \in H^2(R^2)$ we denote the Calderon extension in the space H^2 of v from $\bar{\Omega}$ onto R^2 (*cf.* [18, Chap. 2, § 3.7] and put $v_h = r_h v_c \in V_h$. From (3.8) with $p = 2$ we obtain

$$\|v_h - v_c\|_{1, \Omega_h} \to 0 , \quad \text{if } h \to 0 . \tag{3.36}$$

From this, (3.2), (3.3, *a-b*) and (3.4) we easily prove that

$$\hat{v}_h \to v , \quad \bar{v}_h \to v \quad \text{in } H^1(\Omega) \tag{3.37}$$

and

$$\|v_h\|_{1, \Omega_h} , \quad \|\bar{v}_h\|_{1, \Omega} , \quad \|\hat{v}_h\|_{1, \Omega} \leq c \quad \forall h \in (0, h_0) , \tag{3.38}$$

where c is a constant independent of h.

If we use (2.25, *c*), (1.18), (1.1), (2.1), (2.16) and (3.19), we can write

$$\begin{aligned} &a(u_h', \hat{v}_h) + [a(u_h', \bar{v}_h) - a(u_h', \hat{v}_h)] + [a(\bar{u}_h, \bar{v}_h) - a(u_h', \bar{v}_h)] + \\ &+ \sum_{s=1}^{m} [\tilde{a}^s_{\tau_{sh}}(u_h, v_h) - \tilde{a}^s_{\omega_{sh}}(\bar{u}_h, \bar{v})] + [a_h(u_h, v_h) - \tilde{a}_h(u_h, v_h)] = \\ &= L^\Gamma(\bar{v}_h) + \sum_{s=1}^{m} \left[\int_{\tau_{sh}} f_s v_h \, dx - \int_{\omega_{sh}} f_s \bar{v}_h \, dx \right] + [L_h^\Omega(v_h) - \tilde{L}_h^\Omega(v_h)] + \\ &+ L^\Omega(\bar{v}_h) + [\tilde{L}_h^\Gamma(v_h) - L^\Gamma(\bar{v}_h)] + [L_h^\Gamma(v_h) - \tilde{L}_h^\Gamma(v_h)] . \end{aligned} \tag{3.39}$$

In the following we show that

$$\lim_{h \to 0^+} a(u_h', \hat{v}_h) = \langle \chi, v \rangle , \tag{3.40}$$

$$\lim_{h \to 0^+} L^\Omega(\bar{v}_h) = L^\Omega(v) , \tag{3.41}$$

$$\lim_{h \to 0^+} L^\Gamma(\bar{v}_h) = L^\Gamma(v) \tag{3.42}$$

and that the expressions in square brackets in (3.39) tend to zero, if $h \to 0$. Then, from (3.39)-(3.42) we have $\langle \chi, v \rangle = L(v)$ for all $v \in \mathscr{V}$ and thus, by the density of $\mathscr{V}$ in V, we get (3.35).

a) We have

$$a(u_h', \hat{v}_h) = \langle A(u_h'), \hat{v}_h \rangle .$$

From this, (3.32), (3.34) and (3.37) we easily deduce (3.40).

b) Assertions (3.41) and (3.42) immediately follow from (3.37) and the continuity of functionals L^{Ω} and L^{Γ}.

c) Now let us show that the expressions in square brackets tend to zero. By (1.22), Lemmas 3.1.2, 3.1.6, and (3.38) we have

$$\left|a(u_h', \bar{v}_h) - a(u_h', \hat{v}_h)\right| \leqslant c(1 + \|u_h'\|_{1,\Omega}) \left\|\bar{v}_h - \hat{v}_h\right\|_{1,\Omega} \leqslant ch \to 0 .$$

Further, from (3.21), the Lipschitz-continuity of the operator A, (3.4), (3.38) and (3.15) we get

$$\begin{aligned}\left|a(\bar{u}_h, \bar{v}_h) - a(u_h', \bar{v}_h)\right| &\leqslant c\left\|\bar{u}_h - u_h'\right\|_{1,\Omega} \left\|\bar{v}_h\right\|_{1,\Omega} \\ &\leqslant c\left\|(\bar{u}_h^* + \bar{z}_h) - (\bar{u}_h^* + \hat{z}_h)\right\|_{1,\Omega} \to 0 .\end{aligned}$$

In view of (3.20), (3.3, *a*-*b*), (3.14) and (3.38),

$$\begin{aligned}&\left|\tilde{a}^{S}_{\tau_{sh}}(u_h, v_h) - \tilde{a}^{S}_{\omega_{sh}}(\bar{u}_h, \bar{v}_h)\right| \leqslant \\ &\leqslant c(1 + \|u_h\|_{1,\tau_{sh}})\|v_h\|_{1,\tau_{sh}} + c(1 + \|\bar{u}_h\|_{1,\omega_{sh}})\|\bar{v}_h\|_{1,\omega_{sh}} \\ &\leqslant c\left(1 + ch^{\frac{1}{2}}\right) h^{\frac{1}{2}} \to 0 .\end{aligned}$$

(2.28) and the boundedness of the sequences $\{u_h\}$ and $\{v_h\}$ imply that

$$\left|a_h(u_h, v_h) - \tilde{a}_h(u_h, v_h)\right| \to 0 .$$

Concerning the terms on the right-hand side of (3.39), we use analogous arguments. By Lemmas 2.3.1 and 3.1.5, we have

$$\left|L_h^{\Omega}(v_h) - \tilde{L}_h^{\Omega}(v_h)\right| , \quad \left|L_h^{\Gamma}(v_h) - \tilde{L}_h^{\Gamma}(v_h)\right| \to 0$$

and

$$\left|\tilde{L}_h^{\Gamma}(v_h) - L_h^{\Gamma}(\bar{v}_h)\right| \to 0 ,$$

respectively. Finally, by assumption 1.1.3 (A, a) and Lemma 3.1.1,

$$\left|\int_{\tau_{sh}} f_s v_h \, dx - \int_{\omega_{sh}} f_s \bar{v}_h \, dx\right| \leqslant \|f_s\|_{0,\tau_{sh}\cup\omega_{sh}} \left\|\bar{v}_h\right\|_{0,\tau_{sh}\cup\omega_{sh}} \leqslant ch \to 0 .$$

III) Let us put $z = u - u^*$ and prove that

$$\lim_{h\to 0} \left\langle A(u_h') - A(u), \hat{z}_h - z\right\rangle = 0 . \tag{3.43}$$

In virtue of the part I), $\hat{z}_h \rightharpoonup z$ and thus,

$$\left\langle A(u), \hat{z}_h - z\right\rangle \to 0 .$$

By (3.34) and (3.35),

$$\langle A(u_h'), z\rangle \to L(z).$$

Therefore, it remains to show that

$$\langle A(u_h'), \hat{z}_h\rangle \to L(z). \tag{3.44}$$

We proceed similarly, as in the part II). If we set $v_h := z_h$ in (2.25, *c*), use (1.18), (1.1), (2.1), (2.16) and (3.19), we obtain

$$\begin{aligned} &a(u_h', \hat{z}_h) + [a(u_h', \bar{z}_h) - a(u_h', \hat{z}_h)] + [a(\bar{u}_h, \bar{z}_h) - a(u_h', \bar{z}_h)] + \\ &+ \sum_{s=1}^{m} [\tilde{a}^s_{\tau_{sh}}(u_h, z_h) - \tilde{a}^s_{\omega_{sh}}(\bar{u}_h, \bar{z}_h)] + [a_h(u_h, z_h) - \tilde{a}_h(u_h, z_h)] = \\ &= L^{\Omega}(\hat{z}_h) + L^{\Omega}(\bar{z}_h - \hat{z}_h) + \sum_{s=1}^{m}\left[\int_{\tau_{sh}} f_s z_h\, dx - \int_{\omega_{sh}} f_s \bar{z}_h\, dx\right] + \\ &+ [L_h^{\Omega}(z_h) - \tilde{L}_h^{\Omega}(z_h)] + L^{\Gamma}(\hat{z}_h) + L^{\Gamma}(\bar{z}_h - \hat{z}_h) \\ &+ [\tilde{L}_h^{\Gamma}(z_h) - L^{\Gamma}(\bar{z}_h)] + [L_h^{\Gamma}(z_h) - \tilde{L}_h^{\Gamma}(z_h)]. \end{aligned} \tag{3.45}$$

From $\hat{z}_h \rightharpoonup z$ in $H^1(\Omega)$ and the continuity of the functionals L^{Ω} and L^{Γ} it follows that

$$L^{\Omega}(\hat{z}_h) \to L^{\Omega}(z), \quad L^{\Gamma}(\hat{z}_h) \to L^{\Gamma}(z).$$

Analogously, as in the part II), we can show that all other terms in (3.45) tend to zero, if $h \to 0$, except $a(u_h', \hat{z}_h) = \langle A(u_h'), \hat{z}_h\rangle$. Hence, we immediately get (3.44).

Finally, we apply Theorem 3.1.7*b*, where we substitute $v_n := \hat{z}_h$, $v := z$, $w_n^* := \bar{u}_h^*$, $w^* := u^*$. If we use (3.12), (3.33) and realize that (3.43) represents assumption (3.26), we obtain

$$u_h' \to u \quad \text{in } H^1(\Omega). \tag{3.46}$$

This and the Lipschitz-continuity of the operator A imply

$$A(u_h') \to A(u) \quad \text{in } (H^1(\Omega))^*. \tag{3.47}$$

From (3.47), (3.34) and (3.35) we see that

$$\langle A(u), v\rangle = L(v) \quad \forall v \in V,$$

which is equivalent to (1.21, *c*). Hence, u is a weak solution of the continuous problem (1.21, *a-c*). ■

As a corollary of Theorem 3.1.8 we get

3.1.9. *Theorem*

The sequence $\{u_{h_n}\}$ *from Theorem* 3.1.8 *satisfies*

$$\lim_{h_n \to 0^+} \|u_{h_n} - u_c\|_{1, \Omega_{h_n}} = 0 , \tag{3.48}$$

where $u_c \in H^1(R^2)$ *is the Calderon extension of* u.

Proof follows from (3.46), (2.44), (3.2) and (3.3, *a-b*). ■

3.1.10. *Remark*

Comparing our results with [9], we see that beside the generalization to the problem with discontinuous coefficients, we replaced the assumption $u^* \in H^2(\Omega)$ by a weaker one $u \in W^{1,p}(\Omega)$, $p > 2$ and moreover, we did not need the monotony of the sequence $\{h_n\}$ (i.e. $h_{n+1} < h_n$) supposed in [9].

3.1.11. *Remark*

The above results can also be adopted to the approximate solutions $\tilde{u}_h$ of the discrete problem derived without numerical integration. If we write $\tilde{u}_h = u_h^* + z_h$, $z_h \in V_h$, and set $\tilde{u}'_h = \bar{u}_h^* + \hat{z}_h \in H^1(\Omega)$, then by the same technique as above we prove that each weak limit u in $H^1(\Omega)$ of a sequence $\{\tilde{u}'_{h_n}\}$, with $h_n \to 0$, is a weak solution of the continuous problem and

$$\lim_{h_n \to 0^+} \|\tilde{u}_{h_n} - u_c\|_{1, \Omega_{h_n}} = 0 .$$

3.2. Strongly Monotone Case and Error Estimate

In this paragraph we shall consider assumptions (1.1), (1.2), 1.1.3 (A), (B), (C), (D), (E) and assumptions from paragraphs 2.1, 2.2. It means that we consider the same assumptions as in 3.1, except (D_1) and (D_2) that are replaced by (D).

In this case, by Theorem 2.3.7, the approximate solutions $\tilde{u}_h$ and u_h of problems (2.17, *a-c*) and (2.25, *a-c*), respectively, are unique. The same is valid for the solution of the continuous problem (1.21, *a-c*) :

3.2.1. *Theorem*

Problem (1.21, *a-c*) *has a unique solution.*

Proof follows immediately from the strong monotony of the form $a(u, v)$ with respect to the seminorm $|\cdot|_{1,\Omega}$:

$$a(u, u - v) - a(v, u - v) \geq \alpha |u - v|^2_{1, \Omega} \quad \forall u, v \in H^1(\Omega) \tag{3.49}$$

and inequality (1.16). Assertion (3.49) is a consequence of assumptions 1.1.3 (C) and (D). ■

Combining this result with Theorems 3.1.8 and 3.1.9, we find out that each sequence $\{u'_{h_n}\}$, with $\{h_n\} \subset (0, h_0)$, $h_n \to 0$, weakly convergent in $H^1(\Omega)$, converges strongly to the unique solution u of (1.21, *a-c*). Hence, we have

3.2.2. *Theorem*

It holds

(3.50) $$\lim_{h \to 0^+} u'_h = u \quad \text{in } H^1(\Omega)$$

and

(3.51) $$\lim_{h \to 0^+} \|u_h - u_c\|_{H^1(\Omega_h)} = 0 ,$$

where $u_c \in H^1(R^2)$ *is the Calderon extension of the solution* u *to problem* (1.21, *a-c*). ■

In the following we shall deal with the *error estimate, provided* u *is piecewise of class* H^2. It means that

(3.52) $$u^s = u \mid \Omega_s \in H^2(\Omega_s) , \quad s = 1, \dots, m .$$

We shall procede similarly as in [6, 7] and separate the discretization error from the error caused by numerical integration.

3.2.3. *Estimate of the discretization error*

Our further considerations are based on the following abstract error estimate.

3.2.4. *Theorem*

Let us assume that for every $h \in (0, h_0)$ *the following assumptions are satisfied :*

1) $X_h \subset H^1(\Omega_h)$ *is a finite-dimensional space,* $V_h \subset X_h$ *is its subspace,* $u_h^* \in X_h$,

(3.53) $$W_h = u_h^* + V_h = \{\phi_h = u_h^* + v_h \, ; v_h \in V_h\}$$

and $\tilde{L}_h$, $\ell_h : V_h \to R^1$ *are continuous linear functionals.*

2) $\tilde{a}_h : H^1(\Omega_h) \times H^1(\Omega_h) \to R^1$ *is a form satisfying conditions* (2.49) *and* (2.41, *a*).

3) $\tilde{u} \in H^1(\Omega_h)$ *is a function satisfying the condition*

$$\tilde{a}_h(\tilde{u}, v_h) = \tilde{L}_h(v_h) + \ell_h(v_h) \quad \forall v_h \in V_h , \tag{3.54}$$

$\tilde{u}_h \in W_h$ *is a solution of the equation*

$$\tilde{a}_h(\tilde{u}_h, v_h) = \tilde{L}_h(v_h) \quad \forall v_h \in V_h . \tag{3.55}$$

4) *The condition* (2.12) *is satisfied.*
Then there exist constants A_1, $A_2 > 0$ *independent of* h *such that*

$$\|\tilde{u} - \tilde{u}_h\|_{1, \Omega_h} \leq A_1 \|\ell_h\|^*_{1, \Omega_h} + A_2 \inf_{\phi_h \in W_h} \|\tilde{u} - \phi_h\|_{1, \Omega_h} , \tag{3.56}$$

where

$$\|\ell_h\|^*_{1, \Omega_h} = \sup_{\substack{v_h \in V_h \\ v_h \neq 0}} \frac{|\ell_h(v_h)|}{\|v_h\|_{1, \Omega_h}} . \tag{3.57}$$

For the *proof* see [7, Theorem 3.1.1]. ■

Let us extend the exact weak solution $u : \Omega \to R^1$ to $\tilde{u} : \Omega \cup \left(\bigcup_{h \in (0, h_0)} \Omega_h \right) \to R^1$ in such a way that on the part of $\Omega_h - \Omega$ adjacent to Ω_s we set $\tilde{u} = u_c^s$, where $u_c^s \in H^2(R^2)$ is the Calderon extension of $u^s = u \mid \Omega_s \in H^2(\Omega_s)$. Hence, we set

$$\tilde{u} = u \text{ on } \Omega , \quad \tilde{u} = u_c^s \text{ on } \Omega_{sh} - \Omega , \qquad s = 1, \ldots, m , \quad h \in (0, h_0) . \tag{3.58}$$

The *first fundamental result* of this paragraph is formulated as the following

3.2.5. *Theorem*

If the solution u *of the continuous problem satisfies condition* (3.52), *then there exists a constant* $c > 0$ *such that*

$$\|\tilde{u} - \tilde{u}_h\|_{1, \Omega_h} \leq ch \quad \forall h \in (0, h_0) . \tag{3.59}$$

Proof will be carried out in two steps.

I) First we shall prove that

$$\inf_{\phi_h \in W_h} \|\tilde{u} - \phi_h\|_{1, \Omega_h} \leq ch \quad \forall h \in (0, h_0) \tag{3.60}$$

with $c > 0$ independent of h.

Let us denote $\Omega_{sh}^* = \Omega_s \cup (\bar{\tau}_h \cap \Omega_{sh}) = \Omega_s \cup (\Omega_{sh} - \Omega)$. We have $\bar{\Omega} \cup \bar{\Omega}_h = \bigcup_{s=1}^{m} \bar{\Omega}_{sh}^*$,

$$\tilde{u} \in H^1(\Omega \cup \Omega_h) \quad \text{and} \quad \tilde{u} \,|\, \Omega_{sh}^* = u_c^s \,|\, \Omega_{sh}^* \in H^2(\Omega_{sh}^*) \subset C(\bar{\Omega}_{sh}^*)$$

for each $s = 1, \ldots, m$. Let $r, s \in (1, \ldots, m)$, $r \neq s$, $\Gamma_{rs} \neq \emptyset$. It is evident that $\Gamma_{rs} = \partial\Omega_{sh}^* \cap \partial\Omega_{rh}^*$. Since u^r and u^s have the same traces on Γ_{rs} equal to $u \,|\, \Gamma_{rs}$, we see that $\tilde{u}$ is continuous in $\bar{\Omega} \cup \bar{\Omega}_h$ (eventually after changing $\tilde{u}$ on $\mathfrak{M} \subset \bar{\Omega} \cup \bar{\Omega}_h$ with meas $(\mathfrak{M}) = 0$). Therefore, $r_h \tilde{u}$ has sense and, by (1.21, b), $r_h \tilde{u} \in W_h$. This implies that

$$\inf_{\phi_h \in W_h} \|\tilde{u} - \phi_h\|_{1, \Omega_h} \leq \|\tilde{u} - r_h \tilde{u}\|_{1, \Omega_h}. \tag{3.61}$$

Hence, it is sufficient to estimate $\|\tilde{u} - r_h \tilde{u}\|_{1, \Omega_h}$.

It holds

$$\|\tilde{u} - r_h \tilde{u}\|_{1, \Omega_h}^2 = \sum_{s=1}^{m} \|\tilde{u} - r_h \tilde{u}\|_{1, \Omega_{sh}}^2 \tag{3.62}$$

and

$$\|\tilde{u} - r_h \tilde{u}\|_{1, \Omega_{sh}}^2 = \sum_{T \in \mathcal{T}_{sh}} \|\tilde{u} - r_h \tilde{u}\|_{1, T}^2. \tag{3.63}$$

a) If $T \in \mathcal{T}_{sh}$ and $T \subset \bar{\Omega}_{sh}^*$, then $\tilde{u} = u_c^s$ on T and $u_c^s \,|\, T \in H^2(T)$. In virtue of [2, Theorem 3.1.6],

$$\|\tilde{u} - r_h \tilde{u}\|_{1, T} \leq ch \|u_c^s\|_{2, T} \quad \forall h \in (0, h_0) \tag{3.64}$$

with c independent of u^s, h, T.

b) Let $T \in \mathcal{T}_{sh}$, $T \not\subset \bar{\Omega}_{sh}^*$ (then, by (2.5), $T \subset \bar{\Omega}$) and $\mathcal{S}_{T,s} = \dot{T} - \bar{\Omega}_s \subset \Omega_r$, where $\dot{T}$ is the interior of T. See figure 3.2. Then $\partial\mathcal{S}_{T,s} = \Sigma_{T,s} \cup S_{T,s}$, where $S_{T,s} \subset \Gamma_{rsh}$ is a side of T approximating the arc $\Sigma_{T,s} \subset \Gamma_{rs}$. Further,

$$\text{(i) } \tilde{u} = u_c^s \ \text{ on } T \cap \bar{\Omega}_s, \quad \text{(ii) } u_c^s \,|\, T \in H^2(T), \quad \text{(iii) } r_h \tilde{u} \,|\, T = r_h u_c^s \,|\, T. \tag{3.65}$$

Again, by [2, Theorem 3.1.6] and (3.65, iii),

$$\|u_c^s - r_h \tilde{u}\|_{1, T} \leq ch \|u_c^s\|_{2, T} \quad \forall h \in (0, h_0) \tag{3.66}$$

with c independent of u^s, T, h. Further,

$$\|\tilde{u} - r_h \tilde{u}\|_{1, T} \leq \|\tilde{u} - u_c^s\|_{1, T} + \|u_c^s - r_h \tilde{u}\|_{1, T}. \tag{3.67}$$

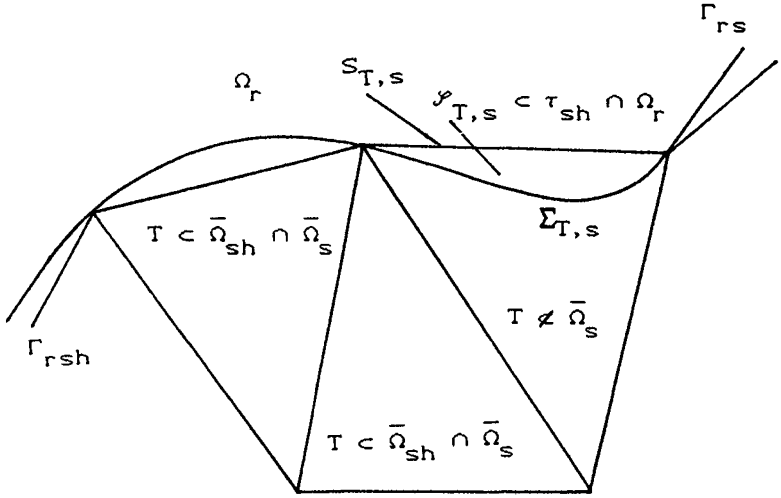

Figure 3.2.

In virtue of (3.65, i),

(3.68) $$\|\tilde{u} - u_c^s\|_{1,T} = \|\tilde{u} - u_c^s\|_{1,\mathscr{S}_{T,s}}.$$

Now, using (3.62)-(3.64), (3.66)-(3.68) and the relations

(3.69) $$\bigcup_{r=1}^{m}\left(\bigcup_{\substack{T\in\mathscr{T}_{sh}\\ T\cap\Omega_r\neq\varnothing}} \mathscr{S}_{T,s}\right) = \bigcup_{r=1}^{m}[(\Omega_{sh} - \bar{\Omega}_s)\cap\Omega_r] = \bigcup_{r=1}^{m}(\tau_{sh}\cap\Omega_r) = \tau_{sh}\cap\Omega$$

(the unions are disjoint), we get

(3.70) $$\|\tilde{u} - r_h\tilde{u}\|_{1,\Omega_h}^2 \leq \sum_{s=1}^{m}\left\{ch^2\sum_{T\in\mathscr{T}_{sh}}\|u_c^s\|_{2,T}^2 + \|\tilde{u} - u_c^s\|_{1,\tau_{sh}\cap\Omega}^2\right\}.$$

Moreover, from the fact that

$$(\tilde{u} - u_c^s)\,|\,\Omega_r \in H^2(\Omega_r) \quad \forall r, s = 1, \dots, m$$

and Lemma 3.2.6 (proved in the sequel) we obtain the estimate

$$\|\tilde{u} - u_c^s\|_{1,\tau_{sh}\cap\Omega} \leq ch\sum_{r=1}^{m}\|\tilde{u} - u_c^s\|_{2,\Omega_r},$$

which together with (3.70) and the equality $\tilde{u}\,|\,\Omega_r = u_c^r\,|\,\Omega_r$ gives

$$\|\tilde{u} - r_h\tilde{u}\|_{1,\Omega_h} \leq ch\sum_{s=1}^{m}\|u_c^s\|_{2,R^2} \leq ch.$$

This and (3.61) yield (3.60).

II) Now we shall deal with the estimate of $\|\ell_h\|^*_{1,\Omega_h}$. We set

$$(3.71)\qquad f_s^* = -\sum_{i=1}^{2} \frac{\partial}{\partial x_i} a_i^s(\cdot, \tilde{u}, \nabla \tilde{u}) + a_0^s(\cdot, \tilde{u}, \nabla \tilde{u}) - f_s$$

$$\text{in } \Omega_{sh}^* = \Omega_s \cup (\Omega_{sh} - \Omega) \quad \text{for} \quad s = 1, \dots, m$$

and define $f^* : \bigcup_{s=1}^{m} \Omega_{sh}^* \to R^1$ by

$$(3.72)\qquad f^* \mid \Omega_{sh}^* = f_s^*, \quad s = 1, \dots, m.$$

In view of (3.58) and 1.1.3 (B), (C), (E), we have $f^* \in L^2\left(\Omega \cup \left(\bigcup_{h \in (0, h_0)} \Omega_h\right)\right)$.

If we apply Green's theorem to identity (1.21, c) with suitable test functions v, we find out that

$$(3.73)\qquad a)\ f^* = 0 \quad \text{almost everywhere in } \Omega,$$

$$b)\ \sum_{i=1}^{2} a_i^s(x, u^s(x), \nabla u^s(x))\, n_i^s(x) = q(x)$$

$$\text{for } x \in \Gamma_{sN} - \mathfrak{M}_s, \quad \text{meas}_1 (\mathfrak{M}_s) = 0, \quad s = 1, \dots, m,$$

$$c)\ \sum_{i=1}^{2} a_i^s(x, u^s(x), \nabla u^s(x))\, n_i^s(x) =$$

$$= -\sum_{i=1}^{2} a_i^r(x, u^r(x), \nabla u^r(x))\, n_i^r(x)$$

$$\text{for } x \in \Gamma_{rs} - \mathfrak{M}_{rs}, \quad \text{meas}_1 (\mathfrak{M}_{rs}) = 0, \quad r, s = 1, \dots, m, \quad r \neq s.$$

Let us set $\tilde{\Omega}_{sh} = (\Omega_s \cap \Omega_{sh}) \cup (\Omega_{sh} - \Omega)$. It is evident that $\tilde{\Omega}_{sh}$ is a domain. With respect to (1.1), (2.1) and notation (3.1),

$$(3.74)\qquad \Omega_h = \bigcup_{s=1}^{m} \left[\tilde{\Omega}_{sh} \cup \left(\bigcup_{r=1}^{m} \tau_{sh} \cap \Omega_r \right) \right] \cup \mathfrak{M}, \quad \text{meas} (\mathfrak{M}) = 0,$$

where the unions are disjoint. Further, by the symbol $\mathcal{S}$ we shall denote components of the sets τ_h, ω_h, τ_{sh} and ω_{sh}. Let $v_h \in V_h$. Then, by (3.73, a) and (3.74),

$$(3.75)\qquad \int_{\tau_h} f^* v_h \, dx = \int_{\Omega_h} f^* v_h \, dx =$$

$$= \sum_{s=1}^{m} \int_{\tilde{\Omega}_{sh}} f^* v_h \, dx + \sum_{s=1}^{m} \sum_{r=1}^{m} \sum_{\mathcal{S} \subset \tau_{sh} \cap \Omega_r} \int_{\mathcal{S}} f^* v_h \, dx.$$

From (3.58) and 1.1.3 (B), (C), (E) we see that

$$(3.76)\quad \tilde{u}\,|\,\tilde{\Omega}_{sh} = u_c^s\,|\,\tilde{\Omega}_{sh} \in H^2(\tilde{\Omega}_{sh})\,,\quad a_\iota^s(.,\,\tilde{u},\nabla\tilde{u})\,|\,\tilde{\Omega}_{sh} \in H^1(\tilde{\Omega}_{sh})\,,$$

$$\tilde{u}\,|\,\mathcal{S} = u^r\,|\,\mathcal{S} \in H^2(\mathcal{S})\,,\quad a_\iota^r(.,\,\tilde{u},\nabla\tilde{u})\,|\,\mathcal{S} \in H^1(\mathcal{S})\quad \forall\,\mathcal{S} \subset \tau_{sh} \cap \Omega_r\,.$$

Now, if we use these results and again apply Green's theorem, we can write

$$(3.77)\quad \int_{\tau_h} f^*\, v_h\, dx =$$

$$= \sum_{s=1}^{m} \int_{\tilde{\Omega}_{sh}} \left[\sum_{\iota=1}^{2} a_\iota^s(.,\,\tilde{u},\nabla\tilde{u})\,\frac{\partial v_h}{\partial x_\iota} + a_0^s(.,\,\tilde{u},\nabla\tilde{u})\,v_h - f_s\,v_h\right] dx\, +$$

$$+ \sum_{s=1}^{m}\sum_{r=1}^{m}\sum_{\mathcal{S}\subset\tau_{sh}\cap\Omega_r} \int_{\mathcal{S}} \left[\sum_{\iota=1}^{2} a_\iota^r(.,\,\tilde{u},\nabla\tilde{u})\,\frac{\partial v_h}{\partial x_\iota} + a_0^r(.,\,\tilde{u},\nabla\tilde{u})\,v_h - f_r\,v_h\right] dx$$

$$- \sum_{s=1}^{m} \int_{\partial\tilde{\Omega}_{sh}} \sum_{\iota=1}^{2} a_\iota^s(.,\,\tilde{u},\nabla\tilde{u})\,n_\iota\,v_h\,ds$$

$$- \sum_{s=1}^{m}\sum_{r=1}^{m}\sum_{\mathcal{S}\subset\tau_{sh}\cap\Omega_r} \int_{\partial\mathcal{S}} \sum_{\iota=1}^{2} a_\iota^r(.,\,\tilde{u},\nabla\tilde{u})\,n_\iota\,v_h\,ds\,.$$

Comparing (3.77) with (2.16), we see that the sum of the first two terms in the right-hand side of (3.77) is equal to

$$\tilde{a}_h(\tilde{u}, v_h) - \tilde{L}_h^{\Omega}(v_h)\, +$$

$$+ \sum_{s=1}^{m}\sum_{r=1}^{m} \int_{\tau_{sh}\cap\Omega_r} \left\{\left[\sum_{\iota=1}^{2} a_\iota^r(.,\,\tilde{u},\nabla\tilde{u})\,\frac{\partial v_h}{\partial x_\iota} + a_0^r(.,\,\tilde{u},\nabla\tilde{u})\,v_h - f_r\,v_h\right]\right.$$

$$\left. - \left[\sum_{\iota=1}^{2} a_\iota^s(.,\,\tilde{u},\nabla\tilde{u})\,\frac{\partial v_h}{\partial x_\iota} + a_0^s(.,\,\tilde{u},\nabla\tilde{u})\,v_h - f_s\,v_h\right]\right\} dx\,.$$

The sum of line integrals in (3.77) along straight sides $S \subset \bar{\Gamma}_{rsh}$ of triangles $T \in \mathcal{T}_h$ is equal to zero, the line integrals along curved sides $\Sigma \subset \bar{\Gamma}_{rs}$ give

$$- \sum_{\substack{r,s=1\\ r<s}}^{m} \int_{\Gamma_{rs}} \left[\sum_{\iota=1}^{2} (a_\iota^s(.,\,\tilde{u},\nabla\tilde{u})\,n_\iota^s + a_\iota^r(.,\,\tilde{u},\nabla\tilde{u}))\,n_\iota^r\right] v_h\,ds\,,$$

which is equal to zero, in virtue of (3.73, c). The rest of the line integrals in (3.77) is equal to

$$- \sum_{s=1}^{m} \int_{\Gamma_{sNh}} \sum_{\iota=1}^{2} a_\iota^s(.,\,\tilde{u},\nabla\tilde{u})\,n_\iota^s\,v_h\,ds\,.$$

From the above considerations and (3.54) we obtain the relation (we use notation (3.19))

$$(3.78)\quad \ell_h(v_h) = \tilde{a}_h(\tilde{u}, v_h) - \tilde{L}_h^{\Omega}(v_h) - \tilde{L}_h^{\Gamma}(v_h) =$$

$$= \int_{\tau_h} f^* v_h\, dx + \sum_{s=1}^{m} \sum_{r=1}^{m} \left[(\tilde{a}^s_{\tau_{sh} \cap \Omega_r}(\tilde{u}, v_h) - \tilde{a}^r_{\tau_{sh} \cap \Omega_r}(\tilde{u}, v_h)) + \right.$$

$$\left. + \int_{\tau_{sh} \cap \Omega_r} (f_s - f_r)\, v_h\, dx \right]$$

$$+ \sum_{s=1}^{m} \int_{\Gamma_{sNh}} \sum_{i=1}^{2} a_i^s(., \tilde{u}, \nabla \tilde{u})\, n_i^s v_h\, ds - L^{\Gamma}(\bar{v}_h) + (L^{\Gamma}(\bar{v}_h) - \tilde{L}_h^{\Gamma}(v_h)) .$$

By (3.73, *b*) and (1.19),

$$(3.79)\quad L^{\Gamma}(\bar{v}_h) = \sum_{s=1}^{m} \int_{\Gamma_{sN}} q \bar{v}_h\, ds = \sum_{s=1}^{m} \int_{\Gamma_{sN}} a_i^s(., \tilde{u}, \nabla \tilde{u})\, n_i\, \bar{v}_h\, ds .$$

For the following considerations let us denote by $\omega_{sh}(N)$ ($\tau_{sh}(N)$) the part of ω_h (τ_h) adjacent to Γ_{sN}. On the basis of this notation we can write

$$\int_{\Gamma_{sNh}} \sum_{i=1}^{2} a_i^s(., \tilde{u}, \nabla \tilde{u})\, n_i\, v_h\, ds - \int_{\Gamma_{sN}} \sum_{i=1}^{2} a_i^s(., \tilde{u}, \nabla \tilde{u})\, n_i\, \bar{v}_h\, ds =$$

$$= \sum_{\mathcal{S} \subset \tau_{sh}(N)} \int_{\partial \mathcal{S}} \sum_{i=1}^{2} a_i^s(., \tilde{u}, \nabla \tilde{u})\, n_i\, v_h\, ds -$$

$$- \sum_{\mathcal{S} \subset \omega_{sh}(N)} \int_{\partial \mathcal{S}} \sum_{i=1}^{2} a_i^s(., \tilde{u}, \nabla \tilde{u})\, n_i\, \bar{v}_h\, ds .$$

If we realize that $\tilde{u} | \mathcal{S} = u_c^s | \mathcal{S} \in H^2(\mathcal{S})$ and thus, $a_i^s(., \tilde{u}, \nabla \tilde{u}) \in H^1(\mathcal{S})$ for all components $\mathcal{S}$ of $\omega_{sh}(N) \cup \tau_{sh}(N)$, and *apply Green's theorem the third time* in this proof, we get

$$\int_{\Gamma_{sNh}} \sum_{i=1}^{2} a_i^s(., \tilde{u}, \nabla \tilde{u})\, n_i\, v_h\, ds - \int_{\Gamma_{sN}} \sum_{i=1}^{2} a_i^s(., \tilde{u}, \nabla \tilde{u})\, n_i\, \bar{v}_h\, ds =$$

$$= \sum_{\mathcal{S} \subset \tau_{sh}(N)} \int_{\mathcal{S}} \sum_{i=1}^{2} \left[v_h \frac{\partial}{\partial x_i} a_i^s(., \tilde{u}, \nabla \tilde{u}) - a_i^s(., \tilde{u}, \nabla \tilde{u}) \frac{\partial v_h}{\partial x_i} \right] dx -$$

$$- \sum_{\mathcal{S} \subset \omega_{sh}(N)} \int_{\mathcal{S}} \sum_{i=1}^{2} \left[\bar{v}_h \frac{\partial}{\partial x_i} a_i^s(., \tilde{u}, \nabla \tilde{u}) - a_i^s(., \tilde{u}, \nabla \tilde{u}) \frac{\partial \bar{v}_h}{\partial x_i} \right] dx .$$

Now, from this, (3.78) and (3.79) it follows that

$$(3.80)\quad \ell_h(v_h) = \int_{\tau_h} f^* v_h\, dx + $$
$$+ \sum_{s=1}^{m}\sum_{r=1}^{m}\left[(\tilde{a}^s_{\tau_{sh}\cap\Omega_r}(\tilde{u}, v_h) - \tilde{a}^r_{\tau_{sh}\cap\Omega_r}(\tilde{u}, v_h)) + \int_{\tau_{sh}\cap\Omega_r}(f_s - f_r)\, v_h\, dx\right]$$
$$+ \sum_{s=1}^{m}\left\{\sum_{\mathcal{S}\subset\tau_{sh}(N)}\int_{\mathcal{S}}\sum_{i=1}^{2}\left[v_h\frac{\partial}{\partial x_i}a_i^s(., \tilde{u}, \nabla\tilde{u}) - a_i^s(., \tilde{u}, \nabla\tilde{u})\frac{\partial v_h}{\partial x_i}\right]dx\right.$$
$$\left. - \sum_{\mathcal{S}\subset\omega_{sh}(N)}\int_{\mathcal{S}}\sum_{i=1}^{2}\left[\bar{v}_h\frac{\partial}{\partial x_i}a_i^s(., \tilde{u}, \nabla\tilde{u}) - a_i^s(., \tilde{u}, \nabla\tilde{u})\frac{\partial\bar{v}_h}{\partial x_i}\right]dx\right\}$$
$$+ L^{\Gamma}(\bar{v}_h) - \tilde{L}^{\Gamma}_h(v_h)\,.$$

It is easy to find out that in view of assumptions 1.1.3, Lemma 3.1.1, (3.2) and (3.71)-(3.72), we have the estimate

$$(3.81)\quad \left|\int_{\tau_h} f^* v_h\, dx + \sum_{s=1}^{m}\sum_{r=1}^{m}\int_{\tau_{sh}\cap\Omega_r}(f_r - f_s)\, v_h\, dx\right| \leqslant$$
$$\leqslant c\left[\|f^*\|_{0,\Omega\cup\left(\bigcup_{h\in(0,h_0)}\Omega_h\right)} + \max_{s=1,\dots,m}\|f_s\|_{0,\Omega_s^*}\right]\|v_h\|_{0,\bigcup_{s=1}^{m}\tau_{sh}}$$
$$\leqslant ch\|v_h\|_{1,\Omega_h}\,.$$

Further, by (3.2), (3.20) and Lemmas 3.1.1, 3.2.6,

$$(3.82)\quad \left|\sum_{s=1}^{m}\sum_{r=1}^{m}[\tilde{a}^s_{\tau_{sh}\cap\Omega_r}(\tilde{u}, v_h) - \tilde{a}^r_{\tau_{sh}\cap\Omega_r}(\tilde{u}, v_h)]\right| \leqslant$$
$$\leqslant c\sum_{s=1}^{m}\sum_{r=1}^{m}[(\text{meas}\,(\tau_{sh}\cap\Omega_r))^{1/2} + \|\tilde{u}\|_{1,\tau_{sh}\cap\Omega_r}]\,\|v_h\|_{1,\tau_{sh}\cap\Omega_r}$$
$$\leqslant ch^{3/2}\left(1 + \sum_{r=1}^{m}\|u_c^r\|_{2,R^2}\right)\|v_h\|_{1,\Omega_h} \leqslant ch^{3/2}\|v_h\|_{1,\Omega_h}\,.$$

Similarly, taking into account 1.1.3 (B), we get the estimate

$$(3.83)\quad \sum_{s=1}^{m}\sum_{\mathcal{S}\subset\tau_{sh}(N)\cup\omega_{sh}(N)}\int_{\mathcal{S}}\sum_{i=1}^{2}\left|a_i^s(., \tilde{u}, \nabla\tilde{u})\frac{\partial\bar{v}_h}{\partial x_i}\right|dx \leqslant$$
$$\leqslant c\sum_{s=1}^{m}[(\text{meas}\,(\tau_{sh}\cup\omega_{sh}))^{1/2} + \|u_c^s\|_{1,\tau_{sh}\cup\omega_{sh}}]\,\|\bar{v}_h\|_{1,\tau_{sh}\cup\omega_{sh}},$$
$$\leqslant ch^{3/2}\left(1 + \sum_{s=1}^{m}\|u_c^s\|_{2,R^2}\right)\|v_h\|_{1,\Omega_h} \leqslant ch^{3/2}\|v_h\|_{1,\Omega_h}\,.$$

Finally,

$$I := \sum_{s=1}^{m} \sum_{\mathscr{S} \subset \tau_{sh}(N) \cup \omega_{sh}(N)} \int_{\mathscr{S}} \left| \left(\frac{\partial}{\partial x_i} a_i^s(., \tilde{u}, \nabla \tilde{u}) \right) \bar{v}_h \right| dx$$
$$\leq \sum_{s=1}^{m} \| a_i^s(., \tilde{u}, \nabla \tilde{u}) \|_{1, \tau_{sh} \cup \omega_{sh}} \| \bar{v}_h \|_{0, \tau_{sh} \cup \omega_{sh}} .$$

In virtue of $\tilde{u} = u_c^s$ on $\tau_{sh} \cup \omega_{sh}$, $u_c^s \in H^2(\Omega_1^*)$ and 1.1.3 (B), (C), (E), we have

$$a_i^s(., u_c^s, \nabla u_c^s) \in H^1(\Omega_s^*) ,$$
$$\| a_i^s(., \tilde{u}, \nabla \tilde{u}) \|_{1, \tau_{sh} \cup \omega_{sh}} \leq \| a_i^s(., u_c^s, \nabla u_c^s) \|_{1, \Omega_s^*} .$$

If we again use Lemma 3.1.1, we immediately get the estimate

$$(3.84) \qquad 0 \leq I \leq ch \sum_{s=1}^{m} \| a_i^s(., u_c^s, \nabla u_c^s) \|_{1, \Omega_s^*} \| v_h \|_{1, \Omega_h}$$
$$\leq ch \| v_h \|_{1, \Omega_h} .$$

Summarizing (3.80)-(3.84), we come to the inequality $| \ell_h(v_h) | \leq ch \| v_h \|_{1, \Omega_h}$ satisfied for all $v_h \in V_h$ and all $h \in (0, h_0)$ with a constant c independent of v_h and h. Hence,

$$(3.85) \qquad \| \ell_h \|_{1, \Omega_h}^* \leq ch \quad \forall h \in (0, h_0) .$$

Now, by (3.56), (3.60) and (3.85) we get the desired result (3.59). ∎

In order to complete the proof of Theorem 3.2.5, we must prove the following

3.2.6. *Lemma*

There exists a constant $c > 0$ such that

$$(3.86) \quad a) \quad \| v \|_{1, \tau_{sh} \cap \Omega_r} \leq ch \| v \|_{2, \Omega_r}$$
$$\forall v \in H^2(\Omega_r) , \quad \forall h \in (0, h_0) , \quad r, s = 1, ..., m ,$$
$$b) \quad \| v \|_{1, \tau_{sh} \cup \omega_{sh}} \leq ch \| v \|_{2, \Omega_s^*}$$
$$\forall v \in H^2(\Omega_s^*) , \quad \forall h \in (0, h_0) , \quad s = 1, ..., m .$$

Proof: We shall deal with the estimate (3.86, *a*) only. (The proof of (3.86, *b*) is similar.) It is sufficient to show that

$$(3.87) \qquad \| v \|_{0, \tau_{sh} \cap \Omega_r} \leq ch \| v \|_{1, \Omega_r} \quad \forall v \in H^1(\Omega_r) .$$

Then, provided $v \in H^2(\Omega_r)$, we combine (3.87) with this estimate applied to $\partial v/\partial x_i$, $i = 1, 2$ and get easily (3.86, a).

As $C^\infty(\bar{\Omega}_r)$ is dense in $H^1(\Omega_r)$, we can consider $v \in C^\infty(\bar{\Omega}_r)$ only. Let $\mathcal{S} \subset \tau_{sh} \cap \Omega_r$ be a component of τ_{sh}. We write $\partial\mathcal{S} = \Sigma \cup S$, where $\Sigma \subset \bar{\Gamma}_{rs}$, $S \subset \bar{\Gamma}_{rsh}$ is a side of a triangle $T \in \mathcal{T}_{sh}$ and approximates Σ. On $\mathcal{S}$ we introduce local Cartesian coordinates y_1-measured in the normal direction to S and y_2-measured along S. Then Σ can be expressed as the graph of a function $y_1 = \varphi(y_2)$, $y_2 \in [0, s]$, where s is the length of S. Let y_1 be oriented in such a way that $\varphi \geq 0$. Then $\mathcal{S} = \{(y_1, y_2)\,;\, 0 < y_1 < \varphi(y_2),\, y_2 \in (0, s)\}$ and

$$\int_{\mathcal{S}} v^2\, ds = \int_0^s \left(\int_0^{\varphi(y_2)} v^2(y_1, y_2)\, dy_1\right) dy_2\,. \tag{3.88}$$

By integrating and applying the Cauchy inequality,

$$\begin{aligned} v^2(y_1, y_2) &= \left[v(\varphi(y_2), y_2) - \int_{y_1}^{\varphi(y_2)} \frac{\partial v}{\partial y_1}(t, y_2)\, dt\right]^2 \leq \\ &\leq 2\left[v^2(\varphi(y_2), y_2) + (\varphi(y_2) - y_1)\int_{y_1}^{\varphi(y_2)} \left(\frac{\partial v}{\partial y_1}(t, y_2)\right)^2 dt\right]. \end{aligned} \tag{3.89}$$

If we integrate (3.89) over $\mathcal{S}$ and use the estimate $0 \leq \varphi(y_2) \leq ch^2$, where c is independent of h and y_2 (see [7, 3.3.2]), we obtain

$$\begin{aligned} \int_{\mathcal{S}} v^2\, dx \leq 2\Bigg[ch^2 \int_0^s v^2(\varphi(y_2), y_2)\, dy_2 + \\ + ch^4 \int_0^s \left(\int_0^{\varphi(y_2)} \left(\frac{\partial v}{\partial y_1}(t, y_2)\right)^2 dt\right) dy_2\Bigg]. \end{aligned}$$

Taking into account that

$$\int_0^s v^2(\varphi(y_2), y_2)\, dy_2 \leq \int_0^s v^2(\varphi(y_2), y_2)(1 + \varphi'(y_2)^2)^{1/2}\, dy_2 = \int_\Sigma v^2\, ds\,,$$

we have

$$\int_{\mathcal{S}} v^2\, dx \leq 2\, ch^2\left[\int_\Sigma v^2\, dS + \int_{\mathcal{S}} \left(\frac{\partial v}{\partial y_1}(t, y_2)\right)^2 dx\right]. \tag{3.90}$$

By the summation of (3.90) over all $\mathcal{S} \subset \tau_{sh} \cap \Omega_r$ and the use of the theorem on traces we get

$$\int_{\tau_{sh} \cap \Omega_r} v^2\, dx \leq 2\, ch^2\left[\int_{\partial\Omega_r} v^2\, dS + \int_{\tau_{sh} \cap \Omega_r} |\nabla v|^2\, dx\right] \leq ch^2 \|v\|^2_{1, \Omega_r}\,,$$

which gives (3.87). ■

3.2.7. *The effect of numerical integration*

We shall estimate $\|u_h - \tilde{u}_h\|_{1,\Omega_h}$ on the basis of the following abstract error estimate.

3.2.8. *Theorem*

Let for every $h \in (0, h_0)$ the following assumptions be fulfilled:

1) $X_h \subset H^1(\Omega_h)$ *is a finite-dimensional space,* V_h *is its subspace,* $u_h^* \in X_h$, $W_h = u_h^* + V_h$ *and* L_h, $\ell_h^I : V_h \to R^1$ *are continuous linear functions.*

2) $a_h = a_h(u_h, v_h) : X_h \times X_h \to R^1$ *is a function satisfying* (2.50).

3) u_h *and* $\tilde{u}_h \in W_h$ *are solutions of the equations*

$$a_h(u_h, v_h) = L_h(v_h) \quad \forall v_h \in V_h \tag{3.91}$$

and

$$a_h(\tilde{u}_h, v_h) = L_h(v_h) + \ell_h^I(v_h) \quad \forall v_h \in V_h , \tag{3.92}$$

respectively.

4) *Condition* (2.12) *is satisfied.*

Then there exists a constant $A_3 > 0$ such that

$$\|u_h - \tilde{u}_h\|_{1,\Omega_h} \leq A_3 \|\ell_h^I\|_{1,\Omega_h}^* \quad \forall h \in (0, h_0) . \tag{3.93}$$

Proof: See [7, Theorem 3.4.1]. ■

As an easy consequence of this theorem we get the *second fundamental result.*

3.2.9. *Theorem*

There exists a constant $c > 0$ such that

$$\|\tilde{u}_h - u_h\|_{1,\Omega_h} \leq ch \quad \forall h \in (0, h_0) . \tag{3.94}$$

Proof: As $\tilde{u}_h$ and u_h are solutions of problems (2.17, *a-c*) and (2.25, *a-c*), respectively, we see that conditions (3.91), (3.92) are satisfied with

$$\ell_h^I(v_h) = [a_h(\tilde{u}_h, v_h) - \tilde{a}_h(\tilde{u}_h, v_h)] - [L_h(v_h) - \tilde{L}_h(v_h)] .$$

Using Lemma 2.3.1 and the boundedness of approximate solutions $\tilde{u}_h$, we immediately get the estimate

$$|\ell_h^I(v_h)| \leq ch \|v_h\|_{1,\Omega_h} \quad \forall v_h \in V_h , \quad \forall h \in (0, h_0) .$$

This and (3.93) yield (3.94). ■

Combining Theorems 3.2.5 and 3.2.9, we get the *final result* for the strongly monotone case under assumption (3.52).

3.2.10. *Theorem*

There exists a constant $c > 0$ *such that*

$$\left\| u_h - \tilde{u} \right\|_{1, \Omega_h} \leqslant ch \quad \forall h \in (0, h_0),$$

where u_h is the approximate solution calculated with the use of numerical integration and $\tilde{u}$ is the extension of the exact weak solution defined by (3.58). ■

3.2.11. *Remark*

There is an interesting question, if the techniques applied in this paragraph also yield improved error estimates, provided the exact solution u is piecewise of class H^k $(k \geqslant 3)$ and is approximated by higher order isoparametric finite elements.

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