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L_∞-CONVERGENCE OF FINITE ELEMENT GALERKIN APPROXIMATIONS FOR PARABOLIC PROBLEMS (*)

by Joachim A. NITSCHE [1]

Abstract. — *Using weighted norms L_∞-error estimates of the Galerkin method for second order parabolic initial-boundary value problems are derived.*

0. INTRODUCTION

Let the model problem

$$\left.\begin{aligned} \dot{u}-\Delta u&=f \quad \text{in } \Omega\times(0, T], \\ u&=0 \quad \text{on } \partial\Omega\times(0, T], \\ u_{t=0}&=u_0 \quad \text{in } \Omega \end{aligned}\right\} \tag{1}$$

be given. With $S_h \subseteq \overset{\circ}{H}_1$ being a finite dimensional space — we will consider only finite elements — the standard Galerkin approximation $u_h = u_h(t) \in S_h$ is defined by

$$(\dot{u}_h, \chi)+D(u_h, \chi)=(f, \chi) \qquad \text{for} \quad \chi\in S_h \quad \text{and} \quad t\in(0, T] \tag{2}$$

with

$$u_h(0)=Q_h u_0. \tag{2'}$$

Here $(.,\ .)$ is the $L_2(\Omega)$-*scalar-product and* $D(.,\ .)$ the Dirichlet integral. Q_h may be any computable projection onto S_h. Substitution of f by $\dot{u}-\Delta u$ gives for the error $e=u-u_h$ the defining relation

$$(\dot{e}, \chi)+D(e, \chi)=0 \qquad \text{for} \quad \chi\in S_h. \tag{3}$$

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Because of the Hilbert-space setting error estimates in Sobolev-norms are available primarily. This part of the convergence analysis is solved now in a satisfactory way. In part (*b*) of the bibliography a number of papers dealing with this question is listed.

With the help of special techniques in one space dimension there are also results in the maximum-norm. We refer to Archer [1], Cavendish-Hall [4], Douglas-Dupont [6], Douglas-Dupont-Wheeler [7], Thomee [15], Wahlbin [16], and Wheeler [17]. Seemingly L_∞-estimates for general space dimensions are only treated by Bramble-Schatz-Thomee-Wahlbin [3]. The idea is to write (3) in the form

$$e + T_h \dot{e} = (I - R_h) u. \tag{4}$$

Here to any f the element $U_h = R_h \Delta^{-1} f = T_h f \in S_h$ is the Ritz approximation on $-\Delta^{-1} f$ defined by

$$D(U_h, \chi) = (f, \chi) \qquad \text{for} \quad \chi \in S_h. \tag{5}$$

In this way L_∞-estimates for the elliptic problem give rise to corresponding estimates for the parabolic problem. Using Sobolev-type embedding theorems Bramble *et al.* derive L_∞-estimates in terms of L_2-estimates of time derivatives of sufficiently high order depending on the dimension of Ω.

The aim of this paper is to give estimates the type

$$\| e \|_{L_\infty(L_\infty)} \leqq c h^m \{ \| u \|_{L_\infty(W_\infty^m)} + \| \dot{u} \|_{L_\infty(W_\infty^m)} + \| \ddot{u} \|_{L_2(W_\infty^m)} \}. \tag{6}$$

Here we consider only the case $u_h(0) = R_h u_0$. More general initial conditions and also the discretisation in time will be discussed in a forthcoming paper.

Similar to the elliptic case extensively we use weighted norms, *see* Natterer [8] and Nitsche [10] and [11]. The corresponding approximation properties of finite elements are derived in sections 2,3. A needed generalization of the boundedness of the L_2-projection is given in section 4 and the main error analysis in 5-7.

1. NOTATIONS, FINITE ELEMENTS

In the following $\Omega \subseteqq R^N$ denotes a bounded domain with boundary $\partial\Omega$ sufficiently smooth. For any $\Omega' \subseteqq R^N$ let $W_p^k(\Omega')$ be the Sobolev space of functions having L_p-integrable generalized derivatives up to order k. In case $p = 2$ we also adopt $H_k(\Omega') = W_2^k(\Omega')$. The norms are indicated by the corresponding subscripts. $\overset{\circ}{H}_1(\Omega')$ is the closure in $H_1(\Omega')$ of the functions with compact support.

For $T>0$ fixed the spaces $L_p(W_q^k(\Omega'))=L_p(0, T, W_q^k(\Omega'))$ consist of functions $u=u(t)\in W_q^k(\Omega')$ such that $\| u(t)\|_{W_q^k(\Omega')}$ is L_p-integrable (respective a. e. bounded for $p=\infty$) in (0, T) with the norms

$$\| u\|_{L_p(W_q^k(\Omega'))}=\left\{\int_0^T \| u(t)\|_{W_q^k(\Omega')}^p dt\right\}^{1/p}. \tag{1.1}$$

In case $\Omega'=\Omega$ we drop Ω, i. e. $H_k=H_k(\Omega)$, etc. If there is no confusion we will also simply write u instead of $u(t)$.

In addition we consider weighted semi-norms. Let $-|\cdot|$ denotes the euclidian distance in R^N :

$$\mu=|x-x_0|^2+\rho^2 \tag{1.2}$$

with $x_0\in\Omega$ and $\rho>0$. We define

$$\|\nabla^k v\|_{a.\Omega'}=\left\{\sum_{|\xi|=k}\int_{\Omega'}\int \mu^{-a}|D^\xi v|^2 dx\right\}^{1/2}. \tag{1.3}$$

$(.,.)_{a.\Omega'}$ is the corresponding bilinear form. According to above we drop Ω' in case $\Omega'=\Omega$. Furthermore, the $L_p(0, T)$-norm of $\| u\|_a=\| u(t)\|_{a.\Omega}$ is denoted by $\|.\|$ with subscript $L_p(a)$.

By Γ_h a subdivision of Ω into generalized simplices Δ_i is meant, i. e. Δ_i is a simplex if $\overline{\Delta}_i$ intersects $\partial\Omega$ in at most a finite number of points and ortherwise one of the faces may be curved. Γ_h is called $\varkappa$-regular if to any $\Delta_i\in\Gamma_h$ there are two spheres with radii $\varkappa^{-1}h$ and $\varkappa h$ such that Δ_i contains the one and is contained in the other.

The finite element spaces $S_h=S(\Gamma_h)$ have the following structure: Let the integer m be fixed. Any $\chi\in S_h$ is in $C^0(\Omega)$, i. e. continuous in Ω, and the restriction to $\Delta_i\in\Gamma_h$ is a polynomial of degree less than m. In the curved elements we use isoparametric modifications as discussed by Ciarlet-Raviart [5], Zlamal [18]. $\overset{o}{S}_h$ is the intersection of S_h and $\overset{o}{H}_1$.

By construction we have $S_h\subseteqq H_1$ but in general $S_h\nsubseteqq H_k$ for $k\geq 2$. It is useful to introduce the spaces $H_k'=H_k'(\Gamma_h)$ consisting of functions in L_2 the restriction of which to any $\Delta_i\in\Gamma_h$ is in $H_k(\Delta_i)$. Obviously $S_h\subseteqq H_k'$ for all k. Parallel to (1.3) we use the "broken" semi-norms

$$\|\nabla^k v\|_a'=\{\sum_{\Delta_i\in\Gamma_h}\|\nabla^k v\|_{a.\Delta_i}^2\}^{1/2}. \tag{1.4}$$

In order to avoid difficulties we will use three different letters for the "constants" in the estimates: k, γ, and c with the following distinctions:

(i) $k_1, k_2, \ldots$ denote numerical constants depending only on N and m (the space-dimension and the degree of the finite elements used);

(ii) the parameter ρ in (1.2) is independent of x but will change with h. Most of the lemmata and theorems are only valid if ρ is not too small compared with h. The corresponding conditions are formulated by "for $\gamma_i h \leqq \rho$" respective "let $\gamma_i \mathrm{h} = \rho$". Of course the γ's depend on N, m, the domain Ω and the regularity factor $\varkappa$ of Γ_h;

(iii) numerical constant with the same dependence as the γ's but entering directly the estimates are denoted by $c, c_1, c_2, \ldots$ Normally just c is used, it may differ at different locations. In order not to loose control in section 5 the constants c are numbered.

The case $m=2$, i.e. linear finite elements, need special treatment. Then logarithmic terms of h will appear in the error bounds, *see* [12] for the elliptic problem. In order not to overburden the paper we assume

$$m \geqq 3. \tag{1.5}$$

Furthermore we consider only regular subdivisions with some fixed $\varkappa$. Finally we remark the powers of μ for the weights μ^a are always within the limit $|a| \leqq 2N$.

2. APPROXIMATION PROPERTIES IN WEIGHTED NORMS

Let $\Delta_i \in \Gamma_h$ be any simplex as described in section 1. Then μ(1.2) does not change too fast if ρ is not too small compared with h :

LEMMA 1: *Let $\gamma_1 h < \rho$ with $\gamma_1 = 2\varkappa$. Then*

$$\sup_{x \in \Delta_i} \mu(x) \leqq 3 \inf_{x \in \Delta_i} \mu(x). \tag{2.1}$$

Proof: Let $\underline{x}$, $\bar{x} \in \overline{\Delta}_i$ be points with

$$\left.\begin{aligned} \underline{\mu} &= \mu(\underline{x}) = \inf\{\mu(x) \mid x \in \Delta_i\}, \\ \bar{\mu} &= \mu(\bar{x}) = \sup\{\mu(x) \mid x \in \Delta_i\}. \end{aligned}\right\} \tag{2.2}$$

Since in Δ_i :

$$|\nabla \mu| \leqq 2|x - x_0| \leqq 2\bar{\mu}^{1/2}, \tag{2.3}$$

we get

$$\bar{\mu} = \mu(\bar{x}) = \mu(\underline{x}) + (\bar{x} - \underline{x}).\nabla \mu \leqq \underline{\mu} + |\bar{x} - \underline{x}|\, 2\bar{\mu}^{1/2}. \tag{2.4}$$

We have $|\bar{x} - X| \leqq \varkappa h$ and therefore

$$\bar{\mu} \leqq \underline{\mu} + 2\varkappa h \bar{\mu}^{1/2}. \tag{2.5}$$

Schwartz's inequality in the form

$$2\varkappa h \mu^{1/2} \leqq \frac{1}{2}\mu + 2\varkappa^2 h^2$$

gives

$$\bar{\mu} \leqq 2\underline{\mu} + 4\varkappa^2 h^2. \tag{2.6}$$

Now – independent of x:

$$\mu \geqq \rho^2 \geqq \gamma_1^2 h^2 \tag{2.7}$$

and therefore the lemma is shown.

The approximability property

$$\| \nabla^k (v - \chi) \|_{L_2(\Delta_i)}^2 \leqq c h^{2(l-k)} \| \nabla^l v \|_{L_2(\Delta_i)}^2 \qquad (0 \leqq k \leqq l \leqq m) \tag{2.8}$$

with a proper interpolation resp. approximation $\chi \in S_h$ is well known. Because of lemma 1 we get from this

$$\| \nabla^k (v - \chi) \|_{a.\Delta_i}^2 \leqq 3^a h^{2(l-k)} \| \nabla^l v \|_{a.\Delta_i}^2 \tag{2.9}$$

and after summation over $\Delta_i \in \Gamma_h$.

LEMMA 2: *Let $\gamma_1 h \leqq \rho$. To any $v \in H_l'$ there is a $\chi \in S_h$ according to*

$$\| \nabla^k (v - \chi) \|_a' \leqq c h^{l-k} \| \nabla^l v \|_a' \qquad (0 \leqq k \leqq l \leqq m). \tag{2.10}$$

REMARK: Since (2.8) holds also for $v \in \mathring{H}_1$ with a $\chi \in \mathring{S}_h$ lemma 2 remains valid if H_l', S_h is replaced by $H_l' \cap \mathring{H}_1$ and $\mathring{S}_h$.

The proof of the next lemma follows the same lines and is omitted here.

LEMMA 3: *Let $\gamma_1 h \leqq \rho$. Then Bernstein-type inequalities hold: For any $\chi \in S_h$:*

$$\| \nabla^l \chi \|_a' \leqq c h^{k-l} \| \nabla^k \chi \|_a' \qquad (0 \leqq k \leqq l < m). \tag{2.11}$$

Multiplication of a function in S_h resp. $\mathring{S}_h$ gives no longer a function in these spaces. But still a certain "super-approximability" property of such functions is valid (*see* Nitsche-Schatz [13]):

LEMMA 4: *A function $\mu^{-b}\varphi$ with $\varphi \in S_h$ (resp. $\mathring{S}_h$) can be approximated by a $\chi \in S_h$ (resp. $\mathring{S}_h$) according to*

$$\| \nabla^k (\mu^{-b}\varphi - \chi) \|_a' \leqq c \{ h^{m-k} \| \varphi \|_{a+2b+m} + h^{2-k} \| \nabla\varphi \|_{a+2b+1} \}. \tag{2.12}$$

Before proving the lemma let us consider e. g. the case $a = -b$ and $k = 0$. Then (2.12) means

$$\| \mu^{-b}\varphi - \chi \|_{-b} \leqq c \{ h^m \| \varphi \|_{b+m} + h^2 \| \nabla\varphi \|_{b+1} \}. \tag{2.13}$$

Now using (2.11) with $l=1$, $k=0$ and the obvious inequality

$$\| \varphi \|_{b+b'} \leqq \rho^{-b'} \| \varphi \|_b \tag{2.14}$$

for $b' \geqq 0$ we get

$$\| \mu^{-b} \varphi - \chi \|_{-b} \leqq c(h/\rho) \| \varphi \|_b . \tag{2.15}$$

By choosing γ in $\gamma h \leqq \rho$ sufficiently large the bound on the right hand side becomes as small as wanted.

In order to prove lemma 4 we apply lemma 2 with $l=m$ and get

$$\| \nabla^k (\mu^{-b} \varphi - \chi) \|'_a \leqq c h^{m-k} \| \nabla^m (\mu^{-b} \varphi) \|'_a . \tag{2.16}$$

Since $\varphi \in S_h$ is piecewise a polynomial of degree $<m$ and because of

$$| D^{\xi} \mu^{-b} | \leqq c \mu^{-b-|\xi|/2} \tag{2.17}$$

Leibniz's rule gives

$$\| \nabla^m (\mu^{-b} \varphi) \|'_a \leqq c \sum_{n=0}^{m-1} \| \nabla^n \varphi \|'_{a+2b+m-n} . \tag{2.18}$$

The term with $n=0$ in connection with (2.16) leads to the first term of the right hand side in (2.12). Using lemma 3 and (2.14) we get for the rest

$$\begin{aligned} \sum_{n=1}^{m-1} \| \nabla^n \varphi \|'_{a+2b+m-n} &\leqq c \sum_{n=1}^{m-1} h^{1-n} \| \nabla \varphi \|_{a+2b+m-n} \\ &\leqq c h^{2-m} \| \nabla \varphi \|_{a+2b+1} \sum_{n=1}^{m-1} (h/\rho)^{m-1-n} . \end{aligned} \tag{2.19}$$

The last sum is bounded because of $h \leqq \rho$, thus the lemma is proved.

3. SHIFT THEOREMS, "A PRIORI" ESTIMATES

Solutions of boundary value problems obey certain shift theorems. Assume $u \in \mathring{H}_1$ and $k \geqq 0$. Then the norm of u in H_{k+2} is equivalent to that of Δu in H_k:

$$c^{-1} \| u \|_{H_{k+2}} \leqq \| \Delta u \|_{H_k} \leqq c \| u \|_{H_{k+2}} . \tag{3.1}$$

A direct consequence is :

LEMMA 5: *Let $k \geqq 2$ be an integer. Then for any $u \in \mathring{H}_1 \cap H_k$:*

$$\| \nabla^k u \|_a \leqq c \left\{ \sum_{n=0}^{k-2} \| \nabla^n \Delta u \|_{a+k-2-n} + \| \nabla u \|_{a+k-1} + \| u \|_{a+k} \right\} . \tag{3.2}$$

In order to prove the lemma the shift theorem (3.1) has to be applied to $\mu^{-b/2} u$ with $b=a$ resp. $a+1, a+2, \ldots$ and k resp. $k-1, k-2, \ldots$ The details are left.

There are some exceptions if a is an integer and one of the indices $a+k-2-n$ in the sum of (3.2) is zero. We will only need

LEMMA 5' : *Let* $w \in \overset{\circ}{H}_1 \cap H_3$. *Then*

$$\|\nabla^3 w\|_{-1} \leqq c\{\|\nabla \Delta w\|_{-1} + \|\Delta w\|\}, \tag{3.3}$$

$$\|\nabla^3 w\|_{-2} \leqq c\{\|\nabla \Delta w\|_{-2} + \|\Delta w\|_{-1} + \|\nabla w\|\}. \tag{3.4}$$

We will only give the proof of (3.3). We have

$$\|\nabla^3 w\|_{-1}^2 = \rho^2 \|\nabla^3 w\|^2 + \sum_{i=1}^{N} \iint (x_i - x_{0/i})^2 |\nabla^3 w|^2. \tag{3.5}$$

The shift theorem gives for the first term

$$\begin{aligned}\rho \|\nabla^3 w\| &\leqq c\rho\{\|\nabla \Delta w\| + \|\Delta w\|\} \\ &\leqq c\{\|\nabla \Delta w\|_{-1} + \|\Delta w\|_{-1}\} \leqq c\{\|\nabla \Delta w\|_{-1} + \|\Delta w\|\}.\end{aligned} \tag{3.6}$$

For the other terms we apply (3.1) with $k=1$ and $u=(x_i - x_{0/i})w$. Since $\nabla^3 u$ differs from $(x_i - x_{0/i})\nabla^3 w$ only by derivatives of w up to order 2 and the same is true for $\nabla \Delta u$ and $(x_i - x_{0/i})\nabla \Delta w$ we get

$$\iint (x_i - x_{0/i})^2 |\nabla^3 w|^2 \quad \leqq c \iint \{(x_i - x_{0/i})^2 |\nabla \Delta w|^2 + |\nabla^2 w|^2 + |\nabla w|^2\}. \tag{3.7}$$

The first integrand is bounded by $\|\nabla \Delta w\|_{-1}^2$ whereas the rest is bounded by $\|\Delta w\|^2$.

In general in (3.2) the terms with u and ∇u are present. But depending on a and k they may be interchangeable resp. can be dropped.

LEMMA 6: *Let* $u \in \overset{\circ}{H}_1 \cap H_2$. *Then:*

(i) *for* $b<0$ *the norms* $\|\nabla u\|_b$ *and* $\|u\|_{b+1}$ *are comparable modulo* $\|\Delta u\|_{b-1}$, *i.e.:*

$$\left.\begin{aligned}\|\nabla u\|_b &\leqq k\{\|u\|_{b+1} + \|\Delta u\|_{b-1}\}, \\ \|u\|_{b+1} &\leqq k\{\|\nabla u\|_b + \|\Delta u\|_{b-1}\}.\end{aligned}\right\} \tag{3.8}$$

(ii) *for* $0<b<(N/2)-1\,(N>2)$ *both terms are bounded by the last, i.e.:*

$$\|u\|_{b+1} + \|\nabla u\|_b \leqq k\|\Delta u\|_{b-1}. \tag{3.9}$$

(iii) *the case* $b=(N/2)-1$ *gives*

$$N(N-2)\rho^2 \|u\|_{b+2}^2+2\|\nabla u\|_b^2=2D(u, \mu^{-b}u). \tag{3.10}$$

(iv) *for arbitrary* b *the term with* ∇u *is always bounded by the others*

$$\|\nabla u\|_b \leqq k(\|u\|_{b+1}+\|\Delta u\|_{b-1}). \tag{3.11}$$

The relation

$$\|\nabla u\|_b^2=D(u, \mu^{-b}u)-\iint u\nabla u\nabla \mu^{-b} \tag{3.12}$$

is an identity which may be written also in the form

$$\|\nabla u\|_b^2=D(u, \mu^{-b}u)+\frac{1}{2}\iint u^2\Delta\mu^{-b}=(u, -\Delta u)_b+\frac{1}{2}\iint u^2\Delta\mu^{-b}. \tag{3.13}$$

Now direct differentiation gives $-r=|x-x_0|$:

$$\Delta\mu^{-b}=-2b\mu^{-b-2}(N\rho^2+(N-2b-2)r^2). \tag{3.14}$$

We prove only case (i) in detail, the other proofs follow the same lines. Now let $b<0$. Then $\Delta\mu^{-b}$ is positive and $\mu^{b+1}\Delta\mu^{-b}$ is bounded and bounded away from zero (3.13) then gives

$$\|\nabla u\|_b^2 \leqq (u, -\Delta u)_b+k\|u\|_{b+1}^2 \geqq (u, -\Delta u)_b+k^{-1}\|u\|_{b+1}^2. \tag{3.15}$$

Now the assertions of the lemma, part (i) follow from this and the obvious generalization of Schwarz's inequality $-b'$ being arbitrary:

$$(u, v)_b \leqq \|u\|_{b-b'}\|v\|_{b+b'}. \tag{3.16}$$

For the sake of completeness we note also

$$D(u, v) \leqq \|\nabla u\|_{-b'}\|\nabla v\|_{b'}. \tag{3.17}$$

In section 5 we will introduce to $\Phi \in \overset{\circ}{S}_h$ an auxiliary function w defined by

$$\left.\begin{aligned} -\Delta w&=\mu^{-\alpha-1}\Phi \quad \text{in } \Omega,\\ w&=0 \quad \text{on } \partial\Omega. \end{aligned}\right\} \tag{3.18}$$

Some of the needed estimates are handled here, the rest will be given in the appendix.

Because of $\overset{\circ}{S}_h \subseteqq \overset{\circ}{H}_1$ we have the regularity $w \in \overset{\circ}{H}_1 \cap H_3$. We will need a bound for the $(-\alpha)$-seminorm of the third derivatives. With the help of lemma 5 we get

$$\|\nabla^3 w\|_{-\alpha} \leqq c\{\|\nabla\Delta w\|_{-\alpha}+\|\Delta w\|_{-\alpha+1}+\|\nabla w\|_{-\alpha+2}+\|w\|_{-\alpha+3}\}. \tag{3.19}$$

First we have

$$\|\Delta w\|_{-\alpha+1}=\|\Phi\|_{\alpha+3}. \tag{3.20}$$

Next we get

$$\|\nabla(\Delta w)\|_{-\alpha}=\|\nabla(\mu^{-\alpha-1}\Phi)\|_{-\alpha}\leqq c\{\|\Phi\|_{\alpha+3}+\|\nabla\Phi\|_{\alpha+2}\}. \tag{3.21}$$

Lemma 3 and (2.14) give

$$\|\nabla\Delta w\|_{-\alpha}+\|\Delta w\|_{-\alpha+1}\leqq ch^{-2}(h/\rho)\|\Phi\|_{\alpha+1}. \tag{3.22}$$

In this way we have shown.

LEMMA 7: *Let w be defined by* (3.18) *with α arbitrary. Then*

$$\|\nabla^3 w\|_{-\alpha}\leqq c\{h^{-2}(h/\rho)\|\Phi\|_{\alpha+1}+\|\nabla w\|_{-\alpha+2}+\|w\|_{-\alpha+3}\}. \tag{3.23}$$

In deriving this lemma we have applied lemma 5. According to lemma 5′ there is the modification.

LEMMA 7′: *Let w be defined by* (3.18). *In case of the exceptional values $\alpha=1,2$ instead of* (3.23) *the estimates hold true*

$$\left.\begin{aligned}\|\nabla^3 w\|_{-1}&\leqq ch^{-2}(h/\rho)\|\Phi\|_2,\\ \|\nabla^3 w\|_{-2}&\leqq ch^{-2}(h/\rho)\|\Phi\|_3+\|\nabla w\|.\end{aligned}\right\} \tag{3.24}$$

4. L_2-PROJECTIONS

To any v the approximations $\chi\in S_h$ guaranted by lemma 2 may be replaced by $V_h:=P_h v\in S_h$ with the L_2-projector P_h defined by

$$(V_h,\chi)=(v,\chi)\qquad\text{for}\quad \chi\in S_h. \tag{4.1}$$

As a first result we mention :

THEOREM 1: *P_h is bounded with respect to any weighted norm, i.e. for a fixed there is a $\gamma_2\geqq\gamma_1$ depending only on N, m, $\varkappa$ and a such that for $\gamma_2 h\leqq\rho$:*

$$\|P_h v\|_a\leqq 2\|v\|_a. \tag{4.2}$$

This was presented at Second Conference on Finite Elements, Rennes 1975, and appeared in the proceedings of that conference, *see* [10]. *But those were distributed only in a limited number. With the above preparations the proof is rather short and will be reproduced here. Let* $\varphi=P_h v$ and $\chi\in S_h$ be arbitrary. Then with Schwarz's inequality (3.16):

$$\begin{aligned}\|\varphi\|_a^2&=(\varphi,\mu^{-a}\varphi)=(\varphi-v,\mu^{-a}\varphi-\chi)+(v,\varphi)_a\\ &\leqq\|\varphi-v\|_a\|\mu^{-a}\varphi-\chi\|_{-a}+\|v\|_a\|\varphi\|_a.\end{aligned} \tag{4.3}$$

The consequence (2.15) of lemma 4 gives

$$\|\varphi\|_a^2 \leqq c(h/\rho)\|\varphi\|_a^2 + (1+c(h/\rho))\|v\|_a\,\|\varphi\|_a. \tag{4.4}$$

Now we choose $\gamma_2 = \mathrm{Max}\,(\gamma_1,\, 3\,c)$ and get in case of $\gamma_2\, h \leqq \rho$:

$$\|\varphi\|_a \leqq \frac{1}{3}\|\varphi\|_a + \frac{4}{3}\|v\|_a. \tag{4.5}$$

A well-known consequence of theorem 1 is the "almost best" approximability

$$\|v - P_h v\|_a \leqq 3\,\inf\{\|v-\chi\|_a \mid \chi\in S_h\}. \tag{4.6}$$

In addition we have the property of simultaneous approximability of $P_h v$ on v which we formulate only in the way needed below:

COROLLARY 1: *With the assumptions of theorem* 1:

$$\|v - P_h v\|_a + h\|\nabla(v - P_h v)\|_a \leqq c\inf\{\|v-\chi\|_a + h\|\nabla(v-\chi)\|_a \mid \chi\in S_h\}. \tag{4.7}$$

Proof: Let again $\varphi = P_h v$ for abbreviation and let $\chi\in S_h$ be arbitrary. Then in using lemma 3 applied to $\varphi - \chi \in S_h$ we get

$$\begin{aligned} h\|\nabla(v-\varphi)\|_a &\leqq h\|\nabla(v-\chi)\|_a + h\|\nabla(\varphi-\chi)\|_a \\ &\leqq h\|\nabla(v-\chi)\|_a + c\|\varphi-\chi\|_a \\ &\leqq h\|\nabla(v-\chi)\|_a + c\|v-\varphi\|_a + c\|v-\chi\|_a \end{aligned} \tag{4.8}$$

and therefore with (4.6):

$$\|v-\varphi\|_a + h\|\nabla(v-\varphi)\|_a \leqq 3(1+c)\{\|v-\chi\|_a + h\|\nabla(v-\chi)\|_a\}. \tag{4.9}$$

Since $\chi\in S_h$ is arbitrary (4.9) is also correct with the infimum taken on the right hand side.

REMARK: All of the above statements hold true if S_h is replaced by $\mathring{S}_h$.

REMARK: If $v\in H_l'$ resp. $v\in \mathring{H}_1\cap H_l'$ then according to lemma 2 the right hand side of (4.7) is bounded by $ch^l\|\nabla^l v\|_a'$. This gives the simultaneous error estimates

$$\|\nabla^k(v - P_h v)\|_a \leqq ch^{l-k}\|\nabla^l v\|_a' \qquad (k=0,1). \tag{4.10}$$

For completeness we mention the result of Bramble-Scott [2] on simultaneous approximability which could be applied also here. But since the question of interpolation in weighted norms is not well-developed the direct proof is shorter. Another possibility would have been to apply the ideas of [9].

5. ESTIMATES IN WEIGHTED NORMS FOR FIXED TIME

In order to derive error estimates for the Galerkin method it is convenient to compare the Galerkin solution u_h with an appropriate approximation U_h on u in the subspace $\mathring{S}_h$. We will take the Ritz approximation $U_h = R_h u \in \mathring{S}_h$ defined by — *see* (5):

$$D(u - U_h, \chi) = 0 \qquad \text{for} \quad \chi \in \mathring{S}_h. \tag{5.1}$$

The error

$$e = u - u_h \tag{5.2}$$

can be splitted

$$e = (u - U_h) - (u_h - U_h) = \varepsilon - \Phi \tag{5.3}$$

with the effect that now Φ is an element of $\mathring{S}_h$. The defining relation for Φ is — *see* (3):

$$(\dot{\Phi}, \chi) + D(\Phi, \chi) = (\dot{\varepsilon}, \chi) \qquad \text{for} \quad \chi \in \mathring{S}_h. \tag{5.4}$$

Since estimates for ε, i.e. the error of the Ritz method, are available it will be sufficient to bound Φ in terms of ε resp. $\dot{\varepsilon}$. The aim of this section is the proof of

THEOREM 2: *Let $\alpha = N/2$ with $N \neq 3$ and let $\gamma_3\, h \leqq \rho$ with γ_3 properly chosen. Then*

$$\|\Phi\|_{\alpha+1}^2 + \|\nabla \Phi\|_\alpha^2 \leqq c_1 \rho^{-2} \|\dot{\varepsilon} - \dot{\Phi}\|_{\alpha-1}^2, \tag{5.5}$$

in case $N = 3$:

$$\|\Phi\|_2^2 + \|\nabla \Phi\|_1^2 \leqq c_1 \rho^{-1} \|\dot{\varepsilon} - \dot{\Phi}\|^2. \tag{5.6}$$

Firstly we will give the proof of (5.5) which is divided into three steps. In order to control the constants in this section they are numbered. c denotes in this section an upper bound of the constants in the previous sections. In *step* 1 we show the validity of

$$\|\nabla \Phi\|_\alpha^2 \leqq c_2 \{ \|\dot{\varepsilon} - \dot{\Phi}\|_{\alpha-1}^2 + \|\Phi\|_{\alpha+1}^2 \} \tag{5.7}$$

for α, N arbitrary. Using (5.4) and (3.13), (3.14) we get with $\chi \in \mathring{S}_h$ arbitrary

$$\begin{aligned} \|\nabla \Phi\|_\alpha^2 &\leqq D(\Phi, \mu^{-\alpha}\Phi) + c\|\Phi\|_{\alpha+1}^2, \\ &\leqq D(\Phi, \mu^{-\alpha}\Phi - \chi) - (\dot{\varepsilon} - \dot{\Phi}, \mu^{-\alpha}\Phi - \chi) + (\dot{\varepsilon} - \dot{\Phi}, \Phi)_\alpha + c\|\Phi\|_{\alpha+1}^2. \end{aligned} \tag{5.8}$$

Using Schwarz's inequality (3.16), (3.17) we derive

$$\begin{aligned} \|\nabla \Phi\|_\alpha^2 \leqq \frac{1}{4}\|\nabla \Phi\|_\alpha^2 &+ c_3 \{ \|\dot{\varepsilon} - \dot{\Phi}\|_{\alpha-1}^2 + \|\Phi\|_{\alpha+1}^2 \} \\ &+ \|\nabla(\mu^{-\alpha}\Phi - \chi)\|_{-\alpha}^2 + \|\mu^{-\alpha}\Phi - \chi\|_{-\alpha+1}^2. \end{aligned} \tag{5.9}$$

Now let χ be an appropriate approximation on u. Lemma 4 with $k=0, b=\alpha$, and $a=-\alpha+1$ gives

$$\|\mu^{-\alpha}\Phi-\chi\|_{-\alpha+1} \leqq c_4\{h^m\|\Phi\|_{\alpha+m+1}+h^2\|\nabla\Phi\|_{\alpha+2}\} \tag{5.10}$$

and because of (2.14) and $h/\rho<1$:

$$\|\mu^{-\alpha}\Phi-\chi\|_{-\alpha+1} \leqq c_4(h/\rho)\{\|\Phi\|_{\alpha+1}+\|\nabla\Phi\|_{\alpha}\}. \tag{5.11}$$

In the same way we come to

$$\|\nabla(\mu^{-\alpha}\Phi-\chi)\|_{-\alpha} \leqq c_5(h/\rho)\{\|\Phi\|_{\alpha+1}+\|\nabla\Phi\|_{\alpha}\}. \tag{5.12}$$

With the last two bounds (5.9) gives

$$\|\nabla\Phi\|_{\alpha}^2 \leqq \left\{\frac{1}{4}+2(c_4^2+c_5^2)(h/\rho)^2\right\}\|\nabla\Phi\|_{\alpha}^2 + c_6\{\|\dot{\varepsilon}-\dot{\Phi}\|_{\alpha-1}^2+\|\Phi\|_{\alpha+1}^2\}. \tag{5.13}$$

Now we choose $\gamma_3=\operatorname{Max}(\gamma_2,\ 4(c_4+c_5))$. Then obviously the coefficient of $\|\nabla\Phi\|_{\alpha}^2$ on the right hand side is less than $1/2$ and so (5.7) is shown.

In order to get an estimate for $\|\Phi\|_{\alpha+1}$ we introduce an auxiliary function w defined by

$$\left.\begin{aligned} -\Delta w &= \mu^{-\alpha-1}\Phi \quad \text{in } \Omega,\\ w &= 0 \quad \text{on } \partial\Omega. \end{aligned}\right\} \tag{5.14}$$

Then with any $\chi\in\mathring{S}_h$ we have

$$\|\Phi\|_{\alpha+1}^2=D(\Phi,\ w)=D(\Phi,\ w-\chi)-(\dot{\varepsilon}-\dot{\Phi},\ w-\chi)+(\dot{\varepsilon}-\dot{\Phi},\ w). \tag{5.15}$$

In *step* 2 of the proof of (5.5) we will show

$$\|\Phi\|_{\alpha+1}^2 \leqq c_7(h/\rho)\{\|\Phi\|_{\alpha+1}^2+\|\nabla\Phi\|_{\alpha}^2\} + \delta\|w\|_{-\alpha+1}^2+c_8(1+\delta^{-1})\|\dot{\varepsilon}-\dot{\Phi}\|_{\alpha-1}^2 \tag{5.16}$$

with $\delta>0$ arbitrary. The two terms with δ come from

$$(\dot{\varepsilon}-\dot{\Phi},\ w) \leqq \|\dot{\varepsilon}-\dot{\Phi}\|_{\alpha-1}\|w\|_{-\alpha+1} \leqq \delta\|w\|_{-\alpha+1}^2+\frac{1}{4\delta}\|\dot{\varepsilon}-\dot{\Phi}\|_{\alpha-1}^2. \tag{5.17}$$

With χ chosen properly next we have

$$D(\Phi,\ w-\chi) \leqq \|\nabla\Phi\|_{\alpha}\|\nabla(w-\chi)\|_{-\alpha} \leqq \|\nabla\Phi\|_{\alpha}ch^2\|\nabla^3 w\|_{-\alpha}. \tag{5.18}$$

Firstly let us consider the case $N>4$. Then we have to apply lemma 7. Since then $-\alpha+2=-N/2+2$ is negative part (i) of lemma 6 can be used. In this way we get

$$D(\Phi,\ w-\chi) \leqq c_9 \| \nabla \Phi \|_\alpha \{ (h/\rho) \| \Phi \|_{\alpha+1} + h^2 \| \nabla w \|_{-\alpha+2} \}. \qquad (5.19)$$

An essential aid is the next lemma the proof of which is given in the appendix:

LEMMA 8: *Let $N \geqq 4$ and $\alpha=N/2$. For w defined by* (5.14) *the* a priori *estimate*

$$\| \nabla w \|^2_{-\alpha+2} \leqq c_{10}\, \rho^{-4} \| \Phi \|^2_{\alpha+1} \qquad (5.20)$$

is valid.

Obviously the right hand side of (5.19) is bounded by that of (5.16).

For $N=4$ we have by lemma 7′ — note $\alpha=2$ in this case:

$$h^2 \| \nabla^3 w \|_{-\alpha} \leqq c (h/\rho) \| \Phi \|_{\alpha+1} + c h^2 \| \nabla w \|. \qquad (5.21)$$

Applying lemma 8 also here shows that the term $D(\Phi,\ w-\chi)$ is bounded by the right hand side of (5.16). Finally for $N=2$ lemma 7′ gives directly

$$D(\Phi,\ w-\chi) \leqq \| \nabla \Phi \|_\alpha c (h/\rho) \| \Phi \|_{\alpha+1} \leqq \frac{1}{2} c (h/\rho) \{ \| \nabla \Phi \|^2_\alpha + \| \Phi \|^2_{\alpha+1} \}. \qquad (5.22)$$

It remains to bound the middle term in (5.15).

We have

$$(\dot{\varepsilon}-\dot{\Phi},\ w-\chi) \leqq \| \dot{\varepsilon}-\dot{\Phi} \|_{\alpha-1} \| w-\chi \|_{-\alpha+1} \qquad (5.23)$$

and

$$\| w-\chi \|_{-\alpha+1} \leqq c h^3 \| \nabla^3 w \|_{-\alpha+1} \leqq c h^2 \| \nabla^3 w \|_{-\alpha}. \qquad (5.24)$$

With the help of the bounds given above for $\| \nabla^3 w \|_{-\alpha}$ we see that this term is bounded in the same way by the right hand side of (5.16).

In *step* 3 of the proof of (5.5) we apply a lemma which also is proved in the appendix.

LEMMA 9: *Let $N \geqq 2$, $\alpha=N/2$. Then for any $w \in \overset{\circ}{H}_1 \cap H_2$:*

$$\| w \|^2_{-\alpha+1} \leqq c_{11}\, \rho^{-2} \| \Delta w \|^2_{-\alpha-1}. \qquad (5.25)$$

For w defined by (5.14) this gives

$$\| w \|^2_{-\alpha+1} \leqq c_{11}\, \rho^{-2} \| \Phi \|^2_{\alpha+1}. \qquad (5.25)$$

Therefore we may rewrite (5.16):

$$\| \Phi \|^2_{\alpha+1} \leqq \{ c_7 (h/\rho) + c_{11}\, \delta \rho^{-2} \} \{ \| \Phi \|^2_{\alpha+1} + \| \nabla \Phi \|^2_\alpha \} + c_8 (1+\delta^{-1}) \| \dot{\varepsilon}-\dot{\Phi} \|^2_{\alpha-1} \qquad (5.27)$$

and compare this with (5.7). If

$$\{c_7(h/\rho)+c_{11}\,\delta\rho^{-2}\}\{1+c_2\}<1 \tag{5.28}$$

then $\|\Phi\|_{\alpha+1}$ and $\|\nabla\Phi\|_\alpha$ are bounded by $\|\dot\varepsilon-\dot\Phi\|_{\alpha-1}$. We may choose

$$\delta=\rho^2\{4\,c_{11}(1+c_2^2)\}^{-1} \tag{5.29}$$

and $\gamma_4\,h\leqq\rho$ with

$$\gamma_4=\operatorname{Max}(\gamma_3,\,4\,c_7(1+c_2)) \tag{5.30}$$

to guarantee this. In this way (5.5) is proved.

Now we turn over to (5.6) of theorem 2. We have already

$$\|\nabla\Phi\|_1^2\leqq c_2\{\|\dot\varepsilon-\dot\Phi\|^2+\|\Phi\|_2^2\} \tag{5.31}$$

since the power α was not restricted in step 1. Similar to above we define w by

$$\left.\begin{aligned}-\Delta w&=\mu^{-2}\,\Phi\quad\text{in }\Omega,\\ w&=0\quad\text{on }\partial\Omega,\end{aligned}\right\} \tag{5.32}$$

and get now — using lemma 7′:

$$\begin{aligned}\|\Phi\|_2^2&=D(\Phi,\,w-\chi)-(\dot\varepsilon-\dot\Phi,\,w-\chi)+(\dot\varepsilon-\dot\Phi,\,w)\\ &\leqq ch^2\|\nabla\Phi\|_1\,\|\nabla^3 w\|_{-1}+\|\dot\varepsilon-\dot\Phi\|\{\|w\|+h^3\|\nabla^3 w\|\}\\ &\leqq c_{12}(h/\rho)\|\nabla\Phi\|_1\,\|\Phi\|_2+c_{13}\|\dot\varepsilon-\dot\Phi\|\{\|w\|+(h/\rho)\|\Phi\|_2\}.\end{aligned} \tag{5.33}$$

In the analogue way (5.6) is then proved with the only difference that instead of (5.25) now the following lemma — *see* appendix — has to be applied.

LEMMA 9′: *Let* $N=3$. *Then for any* $w\in\mathring{H}_1\cap H_2$:

$$\|w\|^2\leqq c\,\rho^{-1}\|\Delta w\|_{-2}^2. \tag{5.34}$$

6. ERROR ESTIMATES IN WEIGHTED NORMS

Theorem 2 gives in case $N=2{,}3$:

$$\|\Phi\|_2\leqq c\,\rho^{-4+N}\ \{\|\dot\Phi\|^2+\|\dot\varepsilon\|^2\}. \tag{6.1}$$

Since by differentiation of (5.5):

$$(\ddot\Phi,\,\chi)+D(\dot\Phi,\,\chi)=(\ddot\varepsilon,\,\chi)\qquad\text{for}\quad\chi\in\mathring{S}_h \tag{6.2}$$

we get putting $\chi=\dot{\Phi}$ and integrating

$$\|\dot{\Phi}(t)\|^2 \leqq \|\dot{\Phi}(0)\|^2 + 2\int_0^t \|\ddot{\varepsilon}\| . \|\dot{\Phi}\| \, d\tau \tag{6.3}$$

and therefore by Gronwall's lemma

$$\|\dot{\Phi}(t)\|^2 \leqq c\left\{\|\dot{\Phi}(0)\|^2 + \int_0^t \|\ddot{\varepsilon}\|^2 \, d\tau\right\}. \tag{6.4}$$

Since our initial condition — *see* (5.3) and the remarks in the introduction regarding the choice of the initial value of u_h — is

$$\Phi(0)=0 \tag{6.5}$$

$\chi=\dot{\Phi}(0)$ in (5.4) gives

$$\|\dot{\Phi}(0)\|^2 = (\dot{\varepsilon}(0), \dot{\Phi}(0)) \leqq \|\dot{\varepsilon}(0)\|^2. \tag{6.6}$$

Therefore we can rewrite (6.4) in the form

$$\|\dot{\Phi}\|_{L_\infty(L_2)} \leqq c\{\|\dot{\varepsilon}\|_{L_\infty(L_2)} + \|\ddot{\varepsilon}\|_{L_2(L_2)}\}. \tag{6.7}$$

In connection with (6.1) we have shown — note that $L_\infty(a)$ is the $L_\infty(0, T)$ norm of $\|.\|_a$:

THEOREM 3': *Let $N=2,3$. Then*

$$\|\Phi\|^2_{L_\infty(2)} \leqq c\,\rho^{-4+N}\{\|\dot{\varepsilon}\|^2_{L_\infty(L_2)} + \|\ddot{\varepsilon}\|^2_{L_2(L_2)}\}. \tag{6.8}$$

In the case $N \geqq 4$ the $(\alpha-1)$-norm of Φ in (5.5) still is a weighted norm which has to be discussed further. The structure of the defining relation of Φ and $\dot{\Phi}$ is the same. Therefore we will work with Φ firstly and show

THEOREM 4: *Let $N \geqq 4$ and $\beta=(N/2)-1$. Then*

$$\|\Phi(t)\|^2_\beta \leqq \|\Phi(0)\|^2_\beta + c\int_0^t \|\dot{\varepsilon}\|^2_\beta \, d\tau. \tag{6.9}$$

Now we will apply this with Φ, $\dot{\varepsilon}$ replaced by $\dot{\Phi}$, $\ddot{\varepsilon}$. Further we have — the proof is given below.

LEMMA 10: *Let $\Phi(0)=0$ and a, N arbitrary. Then*

$$\|\dot{\Phi}(0)\|^2_a \leqq c\|\dot{\varepsilon}(0)\|^2_a. \tag{6.10}$$

With the help of (6.9), (6.10) theorem 2 leads to the counterpart of theorem 3'.

THEOREM 3: *Let* $N \geqq 4$, $\alpha = N/2$ *and* $\beta = \alpha - 1$. *Then*

$$\|\Phi\|^2_{L_\infty(\alpha+1)} \leqq c\,\rho^{-2}\{\|\dot{\varepsilon}\|^2_{L_\infty(\beta)} + \|\ddot{\varepsilon}\|^2_{L_2(\beta)}\}. \tag{6.11}$$

Proof of lemma 10 : We take

$$\chi = P_h(\mu^{-a}\dot{\Phi}(0)) \tag{6.12}$$

with P_h being the L_2-projector in (5.4):

$$\|\dot{\Phi}(0)\|^2_a = (\dot{\Phi}(0), \chi) = -D(\Phi(0), \chi) + (\dot{\varepsilon}(0), \chi) \leqq \|\dot{\varepsilon}(0)\|_a \|\chi\|_{-a}. \tag{6.13}$$

Because of theorem 1 we get

$$\|\chi\|_{-a} \leqq c\|\mu^{-a}\dot{\Phi}\|_{-a} = c\|\dot{\Phi}(0)\|_a. \tag{6.14}$$

Proof of theorem 4: We start with the identity $-\chi \in \mathring{S}_h$ is arbitrary

$$(\dot{\Phi}, \Phi)_\beta + D(\Phi, \mu^{-\beta}\Phi) = (\dot{\Phi}, \mu^{-\beta}\Phi - \chi) + D(\Phi, \mu^{-\beta}\Phi - \chi)$$
$$-(\dot{\varepsilon}, \mu^{-\beta}\Phi - \chi) + (\dot{\varepsilon}, \Phi)_\beta. \tag{6.15}$$

The choice $\chi = P_h(\mu^{-\beta}\Phi)$ causes that the first term on the right hand side disappears. Further in our case of β (3 : 10) gives

$$D(\Phi, \mu^{-\beta}\Phi) = \|\nabla\Phi\|^2_\beta + N(N-2)\frac{1}{2}\rho^2\|\Phi\|^2_{\beta+2}. \tag{6.16}$$

Therefore with the special χ:

$$(\dot{\Phi}, \Phi)_\beta + \|\nabla\Phi\|^2_\beta + k\rho^2\|\Phi\|^2_{\beta+2}$$
$$= D(\Phi, \mu^{-\beta}\Phi - \chi) - (\dot{\varepsilon}, \mu^{-\beta}\Phi - \chi) + (\dot{\varepsilon}, \Phi)_\beta. \tag{6.17}$$

Now lemma 4 with $b = -a = \beta$ and $k = 0$ resp. $k = 1$ in connection with theorem 1 gives

$$\|\nabla^k(\mu^{-\beta}\Phi - \chi)\|_{-\beta} \leqq ch^{-k}\{h^m\|\Phi\|_{\beta+m} + h^2\|\nabla\Phi\|_{\beta+1}\}$$
$$\leqq ch^{1-k}(h/\rho)\{\rho\|\Phi\|_{\beta+2} + \|\nabla\Phi\|_\beta\}. \tag{6.18}$$

In this way we get for the first two terms on the right hand side of (6.17):

$$D(\Phi, \mu^{-\beta}\Phi - \chi) + (\dot{\varepsilon}, \mu^{-\beta}\Phi - \chi)$$
$$\leqq c(h/\rho)\{\|\nabla\Phi\|_\beta + h\|\dot{\varepsilon}\|_\beta\}\{\|\nabla\Phi\|_\beta + \rho\|\Phi\|_{\beta+2}\}. \tag{6.19}$$

In the way analogue to the proof of theorem 2 – *see* especially (5.27) – we get with $\gamma_5 h \leqq \rho$ and $\gamma_5 \geqq \gamma_4$ chosen properly

$$(\Phi, \dot{\Phi})_\beta + \|\nabla\Phi\|^2_\beta + \rho^2\|\Phi\|^2_{\beta+2} \leqq c\{\|\dot{\varepsilon}\|^2_\beta + \|\Phi\|^2_\beta\} \tag{6.20}$$

respective

$$\frac{d}{dt}\|\Phi(t)\|_\beta^2=2(\Phi,\dot\Phi)_\beta\leqq c\{\|\dot\varepsilon\|_\beta^2+\|\Phi\|_\beta^2\}. \tag{6.21}$$

Then Gronwall's lemma gives (6.9).

7. POINTWISE ERROR ESTIMATES

Up to now we had conditions on ρ of the type $\gamma_i h\leqq\rho$. Now we fix $\rho=\gamma_5 h$. Let $t\in[0, T]$ be fixed. There is an $\hat x=\hat x_t\in\Omega$ such that

$$\Phi(\hat x, t)=\pm\|\Phi(t)\|_{L_\infty}. \tag{7.1}$$

We identify x_0 entering μ (1.2) with this $\hat x$. Further let $\Delta\in\Gamma_h$ be the simplex (or one of the simplices) with $\hat x\in\overline{\Delta}$.

The function Φ restricted to Δ is a polynomial of degree less than m, i.e. an element of a finite dimensional space. Therefore any two norms are equivalent. Because of the $\varkappa$-regularity of Δ there is a $k=k(N, m, \varkappa)$ such that

$$\|\Phi\|_{L_\infty(\Delta)}^2\leqq k\left\{h^{-N}\iint\Phi^2\,dx\right\}. \tag{7.2}$$

Since $x_0\in\overline{\Delta}$ we have in Δ:

$$\gamma_5^2 h^2\leqq\mu\leqq(\gamma_5^2+\varkappa^2)h^2 \tag{7.3}$$

and therefore with $\alpha=N/2$:

$$h^{-N}\int_\Delta\int\Phi^2\,dx\leqq c\,\rho^2\int_\Delta\int\mu^{-\alpha-1}\Phi^2\,dx\leqq c\,\rho^2\|\Phi\|_{\alpha+1}^2 \tag{7.4}$$

resp. combining (7.1), (7.2), (7.4):

$$\|\Phi(t)\|_{L_\infty}\leqq c\,\rho\,\|\Phi(t)\|_{\alpha+1} \tag{7.5}$$

With the help of theorem 3 we deduce for $N\geqq 4$ with $\beta=N/2-1$:

$$\|\Phi\|_{L_\infty(L_\infty)}\leqq c\{\|\dot\varepsilon\|_{L_\infty(\beta)}+\|\ddot\varepsilon\|_{L_2(\beta)}\}. \tag{7.6}$$

In case $N\leqq 3$ the same arguments give — *see* (7.4):

$$h^{-N}\int_\Delta\int\Phi^2\,dx\leqq c\,\rho^{4-N}\int_\Delta\int\mu^{-2}\Phi^2\,dx\leqq c\,\rho^{4-N}\|\Phi\|_2^2. \tag{7.7}$$

Because of theorem 3′ (7.6) is valid for $N \leqq 3$ with $\beta = 0$.

At the end the weighted norms may be replaced by L_p-norms. The factor $\mu^{-\beta}$ is L_q-integrable for $q < N/(N-2)$. Since then q' defined by $q^{-1} + q'^{-1} = 1$ is greater than $N/2$ for any $p > N$:

$$\| v \|_{\beta} \leqq c_p \| v \|_{L_p}. \tag{7.6}$$

In this way we get

THEOREM 5: *Let $p = 2$ for $N \leqq 3$ and $p > N$ for $N \geqq 4$. Then*

$$\| \Phi \|_{L_\infty(L_\infty)} \leqq c \{ \| \dot{\varepsilon} \|_{L_\infty(L_p)} + \| \ddot{\varepsilon} \|_{L_2(L_p)} \}. \tag{7.9}$$

Scott [14] and Nitsche [10] gave the error estimates for the Ritz-method

$$\| \varepsilon \|_{L_\infty} = \| u - R_h u \|_{L_\infty} \leqq c h^k \| u \|_{W_\infty^k} \tag{7.10}$$

for $k \leqq m$. Because of $e = u - u_h = \varepsilon - \Phi$ — *see* (5.3) — we have the final result:

THEOREM 6: *Assume the regularity of the solution u of the initial-boundary value problem* (1)*:*

(i) $u \in L_\infty(0, T, W_\infty^k(\Omega))$;

(ii) $\dot{u} \in L_\infty(0, T, W_\infty^k(\Omega))$;

(iii) $\ddot{u} \in L_2(0, T, W_\infty^k(\Omega))$.

Then the error $e = u - u_h$ between the exact solution u and the Galerkin approximation u_h defined by (2) is of order h^k with $k \leqq m$ — the order of the finite elements used.

REMARK: For $N \leqq 3$ the regularity assumptions on $\dot{u}$, $\ddot{u}$ can be lowered:

$$\dot{u} \in L_\infty(0, T, W_2^k(\Omega)), \qquad \ddot{u} \in L_2(0, T, W_2^k(\Omega))$$

is sufficient.

REMARK: Having theorem 5 in mind one would expect assumptions of the type:

(ii′) $\dot{u} \in L_\infty(0, T, W_p^k(\Omega))$;

(iii′) $\ddot{u} \in L_2(0, T, W_p^k(\Omega))$,

instead of (ii), (iii) of theorem 6. As was pointed out by Scott the estimates (7.10) togehter with the L_2-bounds

$$\| \varepsilon \|_{L_2} \leqq c h^k \| u \|_{W_2^k} \tag{7.11}$$

do not imply

$$\| \varepsilon \|_{L_p} \leqq c h^k \| u \|_{W_p^k}. \tag{7.12}$$

This is the reason for the formulation with L_∞-norms in theorem 6.

The convergence rate up to h^m is optimal with respect to the power of h.But in order to get this bounds for the second time derivative are needed. We can get from (6 . 9) a reduced convergence result but without needing $\ddot{\varepsilon}$. With $\Phi(0)=0$ we have

$$\|\Phi\|_{L_\infty(\beta)} \leqq c \|\dot{\varepsilon}\|_{L_2(\beta)}. \tag{7.13}$$

For $\beta = N/2-1$ now $c\|\Phi\|_\beta$ is an upper bound of $h\|\Phi\|_{L_\infty}$ if x_0 (1 . 2) is chosen properly. This gives

THEOREM 7: *Let $N \geqq 3$ and $p > N$. Then*

$$\|\Phi\|_{L_\infty(L_\infty)} \leqq c h^{-1} \|\dot{\varepsilon}\|_{L_\infty(L_p)}. \tag{7.14}$$

The counterpart of theorem 6 is then

THEOREM 8: *The error of the Galerkin approximation is of order h^{k-1} ($k \leqq m$) provided the regularity assumptions*

(i) $u \in L_\infty(0, T, W_\infty^{k-1}(\Omega))$;

(ii) $\dot{u} \in L_2(0, \mathrm{T}, W_\infty^k(\Omega))$,

hold.

8. APPENDIX : PROOF OF LEMMATA 8, 9

For bounded domains $\Omega' \subseteqq R^N$ let

$$\lambda(\Omega') = \sup\left\{ \frac{\|\nabla w\|^2_{-\alpha+2\,.\,\Omega'}}{\|\Delta w\|^2_{-\alpha-1\,.\,\Omega'}} \,\middle|\, w \in H_1(\Omega') \cap H_2(\Omega') \right\} \tag{8.1}$$

and

$$\Lambda(\Omega') = \sup\left\{ \frac{\|w\|^2_{-\alpha+3\,.\,\Omega'}}{\|\Delta w\|^2_{-\alpha-1\,.\,\Omega'}} \,\middle|\, w \in \mathring{H}_1(\Omega') \cap H_2(\Omega') \right\}. \tag{8.2}$$

Because of the definition of w (5 . 14) lemma 8 is proved if we can show $\lambda(\Omega) \leqq c\rho^{-4}$. Firstly we consider the case $N > 4$. Then $-\alpha+2$ is negative and lemma 6, (i) gives

$$\lambda(\Omega') \leqq k\{\Lambda(\Omega') + \rho^{-4}\}, \qquad \Lambda(\Omega') \leqq k\{\lambda(\Omega') + \rho^{-4}\} \tag{8.3}$$

with k independent of Ω'. Obviously Λ is monotone in Ω', i. e. $\Lambda(\Omega') \leqq \Lambda(\Omega'')$ for $\Omega' \subseteqq \Omega''$. Next let $K = K_R(x_0)$ be a sphere of radius $R = \operatorname{diam}(\Omega)$ with center x_0. Then $\Omega \subseteqq K$ and hence $\Lambda(\Omega) \leqq \Lambda(K)$. The supremum $\Lambda(K)$ is attained for a positive function w_K with $-\Delta w_K > 0$ because of the maximum principle, and w_K

solves the eigenvalue problem

$$\left.\begin{aligned} \Delta(\mu^{\alpha+1}\Delta w) &= \Lambda^{-1}\mu^{\alpha-3}w \quad \text{in } K, \\ w=\Delta w &= 0 \quad \text{on } \partial K. \end{aligned}\right\} \tag{8.4}$$

Without loss of generality we can assume $w_K = w_K(r)$ with $r=|x-x_0|$ since μ depends only on r, for otherwise the spherical average of w_K solves the same eigenvalue problem and is also positive. Therefore we can restrict the space of admissible functions without changing Λ:

$$\Lambda(K)=\sup\left\{\frac{\|w\|^2_{-\alpha+3.K}}{\|\Delta w\|_{-\alpha-1.K}}\,\middle|\, w\in V_K\right\} \tag{8.5}$$

with $V_K=\mathring{H}_1(K)\cap H_2(K)\cap\{w\,|\,w=w(r)\}$. Now with lemma 6, (i) we get

$$\lambda(\Omega)\leqq k\{\rho^{-4}+\Lambda(K)\}\leqq k\left\{\rho^{-4}+\sup\left\{\frac{\|\nabla w\|_{-\alpha+2.K}}{\|\Delta w\|_{-\alpha-1.K}}\,\middle|\, w\in V_K\right\}\right\}. \tag{8.6}$$

Functions $w\in V_K$ have the representation ($w'=dw/dr$):

$$w'=r^{1-N}\int_0^r s^{N-1}\Delta w\, ds. \tag{8.7}$$

Schwarz's inequality gives

$$|w'|^2\leqq r^{2-2N}f(r)\int_0^r s^{N-1}\mu^{\alpha+1}|\Delta w|^2\,ds \tag{8.8}$$

with

$$f(r)=\int_0^r s^{N-1}\mu^{-\alpha-1}\,ds\leqq c\left\{\begin{aligned} \rho^{-N-2}r^N & \text{ for } r\leqq\rho, \\ \rho^{-2} & \text{ for } r\geqq\rho, \end{aligned}\right\} \tag{8.9}$$

because of $\alpha=N/2$.

Therefore

$$\begin{aligned} \|\nabla w\|^2_{-\alpha+2.K} &= k\int_0^R r^{N-1}\mu^{\alpha-2}|w'|^2\,dr \\ &\leqq k\int_0^R r^{1-N}\mu^{\alpha-2}f(r)\,dr\int_0^r s^{N-1}\mu^{\alpha+1}|\Delta w|^2\,ds \\ &= k\int_0^R s^{N-1}\mu^{\alpha+1}|\Delta w|^2\,ds\int_s^R r^{1-N}\mu^{\alpha-2}f(r)\,dr \\ &\leqq k\|\Delta w\|^2_{-\alpha-1.K}\int_0^R r^{1-N}\mu^{\alpha-2}f(r)\,dr. \end{aligned} \tag{8.10}$$

The last integral is bounded by $c\,\rho^{-4}$. This completes the proof in case $N>4$. For $N=4$ without using lemma 6 we directly consider the supremum of $\|\nabla w\|^2/\|\Delta w\|^2_{-3}$ and get the same result with the same arguments.

Proof of lemma 9: The proof follows the above lines. In the definition of λ, Λ we replace the indices of $\|\nabla w\|$ resp. $\|w\|$ by $-\alpha$ resp. $-\alpha+1$. Then $-\alpha=N/2$ is negative. Up to formula (8.9) nothing is changed. But then

$$\|\nabla w\|^2_{-\alpha.K}=\int_0^R r^{N-1}\mu^\alpha|w'|^2\,dr \leqq \|\Delta w\|^2_{-\alpha-1.K}\int_0^R r^{1-N}\mu^\alpha f(r)\,dr \quad (8.11)$$

and the last integral is bounded by $c(1+R^2\rho^{-2})\leqq c'\rho^{-2}$.

The proof of lemma 9′ is analogue to the preceding one and is omitted here.

There is an interesting remark to be added. In (8.1) resp. (8.2) the $(-\alpha+2)$-norm of the first derivatives resp. the $(-\alpha+3)$-norm of the function itself is compared with the $(-\alpha-1)$-norm of the second derivatives. Roughly speaking each differentiation in weighted norms may be considered as reducing the weight-power by one. Then $\|\nabla w\|_{-\alpha+2}$ and $\|w\|_{-\alpha+3}$ would be something like $\|\Delta w\|_{-\alpha+1}$. Since this is compared with $\|\Delta w\|_{-\alpha-1}$ the behavior λ, $\Lambda\approx\rho^{-4}$ is "understandable". Of course this "rule" is only valid for special α and has to be checked in each case. Just lemma 9 is an example that it may be violated.

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