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Harmonic almost-complex structures

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0. Introduction

Let M be a smooth, orientable, even-dimensional ($\dim M = n = 2k$) manifold. An *almost-complex structure* on M is a field of endomorphisms $J_x: T_xM \to T_xM$ varying smoothly with $x \in M$, satisfying the anti-involution property $J^2 = -1$. In this paper we ask the question, 'Which are the best almost-complex structures on M?' The criterion which we propose may be summarized as follows. We assume that M comes equipped with a preferred Riemannian metric g, in which case we may sensibly restrict attention to the g-orthogonal almost-complex structures, otherwise known as *almost-Hermitian* structures. When $k > 1$ this still leaves a potentially vast choice. The totality of almost-Hermitian structures on (M, g) is parametrized by the manifold $\mathcal{C}(\pi)$ of smooth sections of the *twistor bundle* $\pi: Z(M) \to M$, by which we understand the fibre bundle with fibre $F = SO(n)/U(k)$ associated to the oriented orthonormal frame bundle of M via the usual left action of $SO(n)$ on F. The twistor space $Z(M)$ has a natural Riemannian metric, obtained by horizontally lifting g (horizontality being relative to the Levi–Civita connection), and supplementing with the metric on the fibres induced by an $SO(n)$-invariant metric on F. It is therefore possible to compute the *energy* of any section σ [11, 9], and to single-out from $\mathcal{C}(\pi)$ the critical points with respect to variations through sections. Further distinctions may be made on the basis of stability.

Some further remarks are in order.

(1) It is important to emphasize that we are studying the harmonic map problem *with constraints,* whose critical points are therefore not in general harmonic maps. This being understood, we will refer to a critical point σ of our variational problem as a *harmonic section*, and the corresponding J as a *harmonic almost-complex structure.*

(2) Rather than tackle the constrained harmonic map problem directly, it is natural to normalize the energy functional and consider instead the vertical variational theory of the *vertical energy functional.* Besides capturing all the essential geometry, this encourages a conceptual shift to the theory of harmonic maps $f: M \to F$; indeed, if the Levi–Civita connection is flat then a harmonic section is locally the graph of a harmonic map into the fibre. Our techniques may thus be considered adaptations/generalizations of those suited to harmonic maps into F; in particular, those of Lichnerowicz [20] and Xin [32].

(3) The zeroes of the vertical energy functional (and hence absolute minima of the energy functional) are the horizontal sections, which parametrize the Kähler structures

of (M, g). Therefore every Kähler structure is harmonic. More generally, recall that an almost-Hermitian structure is said to be *nearly-Kähler* if the 3-covariant tensor $\nabla\omega$ is totally antisymmetric, where ω is the Kähler 2-form; then every nearly-Kähler structure is also harmonic, by [31], Theorem 1 or Corollary 3.2 below. The converse is more complicated. Recall that an almost-Hermitian structure is said to be *almost-Kähler* (or *symplectic*) if $d\omega = 0$. For compact 4-dimensional almost-Kähler structures we obtain the following two contrasting results. On one hand we have a Liouville-type theorem (Corollary 3.5): on a compact Einstein 4-manifold, a harmonic almost-Kähler structure with constant $*$scalar curvature (see below) is necessarily Kähler. (We also show in Section 7 that if a Calabi–Eckmann structure on the product of odd-dimensional spheres is *stable* harmonic, then with the possible exception of $S^1 \times S^3$ and $S^3 \times S^3$ it is necessarily Kähler $S^1 \times S^1$; Kähler structures are of course topologically prohibited in higher dimensions. However, apart from the torus no Calabi–Eckmann structure is almost-Kähler). On the other hand, in Theorem 5.4 we deduce the existence of a stable harmonic almost-Kähler structure on Thurston's symplectic 4-manifold [26], another topologically non-Kähler compact space.

(4) Our criterion is not in general equivalent to minimality of the submanifold $\sigma(M) \subset Z(M)$; in fact, only when σ is horizontal does the induced metric coincide with g. The contrast is illustrated by a recent result of Calabi and Gluck [6] which states that $\sigma(S^6)$ has minimal volume if and only if σ parametrizes the standard almost-complex structure J on S^6; whereas in [31] it was shown that if J on S^6 is standard, then σ is certainly a harmonic section, but has a vertical energy index of at least 7.

(5) The Euler–Lagrange equations for a harmonic almost-Hermitian J are [31]:

$$[J, \nabla^*\nabla J] = 0 \tag{0.1}$$

where $\nabla^*\nabla$ is the covariant (or 'rough') Laplacian of (M, g), and $[,]$ is the commutator bracket for endomorphisms. It is noteworthy that (0.1) was obtained by Valli [28] as the geodesic equation in the infinite-dimensional gauge group (see Remark 4.2 below). Rather than attack (0.1) directly, we make use of the natural almost-complex structure on twistor space and confine attention to vertically holomorphic, or antiholomorphic, sections; in the terminology of [10], sections which are J_i-*holomorphic*, $i = 1, 2$. Such sections parametrize Hermitian (i.e. integrable) and $(1, 2)$-symplectic structures, respectively ([22], and Lemma 2.7 below), the latter class including nearly-Kähler and almost-Kähler structures; so the loss of generality still leaves many interesting examples.

(6) A variational principle for almost-Kähler structures was described by Blair and Ianus [4] (see also Remark 3.3). However, its critical points do not coincide with the harmonic almost-Kähler structures; indeed, it will be seen that the Abbena–Thurston almost-Kähler structure [1, 26] is harmonic, but not critical for [4].

The contents of the paper are organized as follows. In Section 1 we adapt techniques of [20] to study in a fairly general context the effect of vertical homotopies on the holomorphic and anti-holomorphic components of the vertical energy functional. In Section 2 this is specialized to the twistor bundle, yielding realizations of the Euler–Lagrange equations in which all second order terms are absorbed by the curvature tensor (Theorem 2.8). Also, when coupled to some elementary gauge theory, a precise measure is obtained of energy change along deformations of σ by 1-parameter subgroups of gauge transformations. Although this does not guarantee that J_i-holomorphic sections are local energy-minimizers (unlike [20]), it does provide a platform for proving stability in certain cases, described in Sections 4–6.

It emerges from Sections 1 and 2 that the following 2-form is of key significance in determining whether a given almost-Hermitian structure J is harmonic:

$$\phi = \mathcal{R}(\omega) \in \Omega^2(M) \tag{0.2}$$

where ω is the Kähler 2-form of J, and $\mathcal{R}$ is the curvature operator acting on 2-forms. This 2-form ϕ is a natural generalization of the Chern/Ricci form of a Kähler manifold, although in general it is not closed. Indeed, we show in the Appendix that the first Chern class of a general almost-Hermitian manifold is represented by a 2-form γ defined:

$$2\pi\gamma = \phi + \chi, \quad \text{where} \quad \chi(X,Y) = \tfrac{1}{4}\,\omega(\nabla_X J, \nabla_Y J). \tag{0.3}$$

(It was shown in [15] that γ represents the first Chern class of a nearly-Kähler manifold). Since ϕ is the Ricci-component of γ, we call ϕ the *Ricci form* of (M, g, J). In many cases (Theorem 2.8 gives the precise conditions), notably each of the four irreducible classes of Gray–Hervella [17] (see Theorem 3.1), harmonic J are characterized by the reducibility of ϕ:

$$\phi \in \Omega^{1,1}(M) \tag{0.4}$$

This condition is examined in Section 3. For example, it is automatically satisfied on any conformally flat (or, in dimension 4, conformally half-flat) manifold. Also, when the Ricci form is proportional to the Kähler form:

$$\phi = \tfrac{1}{n}\, s^*.\,\omega, \qquad s^*\colon M \to \mathbb{R}. \tag{0.5}$$

An almost-Hermitian manifold satisfying (0.5) is called *$*$Einstein* [27], and the function s^* is its *$*$scalar curvature*. In the special case $\phi = 0$, the corresponding section σ is an E-minimizer in its vertical homotopy class, provided either J is (1, 2)-symplectic, or integrable and cosymplectic (Theorem 2.8). For more general $*$Einstein manifolds, where it should be noted that unlike Einstein manifolds there is no guarantee that s^* is constant, we prove the following weaker stability result:

THEOREM 4.4. *Suppose J is a $*$Einstein structure. If J is (1, 2)-symplectic and $s^* \leqslant 0$, or J is cosymplectic Hermitian and $s^* \geqslant 0$, then J is stable harmonic.*

Since the standard almost-Hermitian structure on S^6 is nearly-Kähler (*a fortiori* (1, 2)-symplectic) and $*$Einstein with $s^* = 6$, the instability of the nearly-Kähler six-sphere [31], Theorem 2, is a partial converse to Theorem 4.4. Another instance of (0.4) is theoretically possible in 4-dimensions:

$$*\phi = -\phi \tag{0.6}$$

where $*$ is the Hodge duality operator acting on 2-forms. In contrast to (0.5), this anti-self-duality condition seems to have received little attention in the literature. In Section 5 we prove:

THEOREM 5.3. *An almost-Kähler structure with anti-self-dual Ricci form is stable harmonic.*

Also in Section 5, we analyze the almost-Kähler structure of Abbena [1] on the symplectic 4-manifold of Thurston [26] (Theorem 5.4), a structure which satisfies neither (0.5) nor (0.6). In Section 6 we study the Sasaki almost-Kähler structure on the tangent bundle of a Riemannian manifold, and show that it is harmonic precisely when the base manifold has *harmonic curvature* (Theorem 6.2). In Theorem 6.6 we prove that in the 4-dimensional case (i.e. the tangent bundle of a constant curvature surface) the only stable Sasaki structure is that of the flat torus. The realm of harmonic *complex* structures is contemplated briefly in Section 7. We show that the complex structure of a Calabi–Eckmann manifold is harmonic (Theorem 7.1), and then use techniques adapted from those of [32] to prove that apart from the torus, and the low-dimensional cases $S^1 \times S^3$ and $S^3 \times S^3$ where the technique is inconclusive, all are unstable (Theorem 7.5). There are also estimates on the index and nullity. Finally, in an Appendix, we discuss some of the differential geometric properties of the twistor fibration used in this paper, and its predecessor [31].

Conventions. Our curvature convention is: $R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$.

Local orthonormal frame fields on M will be denoted (E_i), and subject to the summation convention. If $\xi\colon \mathcal{F} \to M$ is a smooth fibre bundle, then the manifold of smooth sections will be denoted $\mathcal{C}(\xi)$ or $\mathcal{C}(\mathcal{F})$. All structures (manifolds, maps, etc.) are assumed smooth.

1. Vertical energy

Let $\pi\colon (N,h) \to (M,g)$ be any submersion of orientable Riemannian manifolds. The following vertical/horizontal decomposition is fundamental:

$$TN = \mathcal{V} \oplus \mathcal{H} \tag{1.1}$$

where $\mathcal{V} = \ker d\pi$ and $\mathcal{H}$ is the orthogonal complement. If $A \in TN$ we write

$$A = vA + hA$$

for the vertical and horizontal components. So, any $\sigma \in \mathcal{C}(\pi)$ has a vertical derivative $\mathrm{d}^v\sigma$ defined

$$\mathrm{d}^v\sigma(X) = v\big(\mathrm{d}\sigma(X)\big), \quad \forall X \in TM,$$

and the energy density $e(\sigma)$ [9, 11] has a vertical component

$$\mathrm{e}^v(\sigma) = \tfrac{1}{2}|\mathrm{d}^v\sigma|^2.$$

The *vertical energy functional* E^v is then defined

$$E^v(\sigma; U) = \int_U \mathrm{e}^v(\sigma)\mathrm{d}x, \qquad U \subset M \text{ relatively compact open,}$$

where $\mathrm{d}x$ is the Riemannian volume element. The first variation of E^v may be written in divergence form as follows:

$$\mathrm{d}E^v(V) = -\int_M h(\tau^v(\sigma), V)\mathrm{d}x \tag{1.2}$$

for all compactly supported vertical lifts V of σ. By analogy with harmonic map theory [11], we call $\tau^v(\sigma)$ the *vertical tension field* of σ. If π has totally geodesic (t.g.) fibres, it was shown in [30] that the vertical tension is

$$\tau^v(\sigma) = -\delta^v \mathrm{d}^v \sigma = \operatorname{Tr} \nabla^v \mathrm{d}^v \sigma \tag{1.3}$$

where ∇^v is the connection in the vector bundle $\mathcal{V} \to N$ obtained by horizontally projecting the Levi–Civita connection of (N, h). In fact, by attacking the vertical energy functional directly, the approach adopted in this paper obviates the need for Euler–Lagrange equations (1.3), with the exception of Section 7 where the second variation is studied. We note that if π is a *Riemannian submersion* (i.e. the restriction $\mathrm{d}\pi|\mathcal{H}$ is isometric) then E^v normalizes the energy functional E, because

$$E(\sigma) = E^v(\sigma) + \frac{n}{2}\mathrm{Vol}\,(M, g),$$

assuming for simplicity that M is compact. So in this case the constrained harmonic map problem for $\mathcal{C}(\pi)$ is equivalent to the vertical variational theory of E^v:

$$E(\sigma_t) - E(\sigma) = E^v(\sigma_t) - E^v(\sigma)$$

where σ_t is any 1-parameter variation of σ through sections.

Now suppose J^M is an almost-Hermitian structure for (M, g), and each fibre of π is an almost-Hermitian manifold in such a way that there is a smooth orthogonal almost-complex structure J^v in $\mathcal{V}$. With respect to decompositions $T^{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M$ and $\mathcal{V}^{\mathbb{C}} = \mathcal{V}^{1,0} \oplus \mathcal{V}^{0,1}$ the complexification of $\mathrm{d}^v\sigma$ splits into four components:

$$\partial^v\sigma\colon T^{1,0}M \to \mathcal{V}^{1,0}, \qquad \bar{\partial}^v\sigma\colon T^{0,1}M \to \mathcal{V}^{1,0}$$

and their conjugates. Therefore, with respect to the Hermitian norms on $T^{\mathbb{C}}M$ and $\mathcal{V}^{\mathbb{C}}$:

$$\mathrm{e}^v(\sigma) = \tfrac{1}{2}|(\mathrm{d}^v\sigma)^{\mathbb{C}}|^2 = |\partial^v\sigma|^2 + |\bar{\partial}^v\sigma|^2 = h^v(\sigma) + a^v(\sigma), \qquad \text{say.}$$

So the vertical energy splits into holomorphic and antiholomorphic components:

$$E^v(\sigma) = H^v(\sigma) + A^v(\sigma).$$

On the other hand, following [20] we define

$$k^v(\sigma) = h^v(\sigma) - a^v(\sigma) \quad \text{and} \quad K^v(\sigma) = H^v(\sigma) - A^v(\sigma).$$

To interpret K^v geometrically, let ω denote the Kähler 2-form:

$$\omega(X, Y) = g(J^M X, Y) \quad \forall X, Y \in \mathcal{C}(TM)$$

and extend the Kähler forms of the fibres to the following degenerate 2-form η on N:

$$\eta(A, B) = h(J^v(vA), vB) \quad \forall A, B \in \mathcal{C}(TN).$$

LEMMA 1.1. *For all $\sigma \in \mathcal{C}(\pi)$ we have* $k^v(\sigma) = g(\omega, \sigma^*\eta)$.

Proof. One computes:

$$2h^v(\sigma) = \mathrm{e}^v(\sigma) + g(\omega, \sigma^*\eta), \qquad 2a^v(\sigma) = \mathrm{e}^v(\sigma) - g(\omega, \sigma^*\eta).$$

□

In view of [20], it is natural to study the behaviour of K^v under vertical homotopies. Let P be the homotopy operator on the de Rham complex, defined as follows:

$$P\colon \Omega^q(M \times I) \to \Omega^{q-1}(M);\ \beta \mapsto \int_0^1 j_t^*(\beta \lrcorner \partial_t)\,\mathrm{d}t \tag{1.4}$$

where $I = [0,1] \subset \mathbb{R}$, $j_t\colon M \hookrightarrow M \times I; x \mapsto (x,t)$, ∂_t is the standard unit vertical vector field on $M \times I$, and $\lrcorner$ is interior product. The fundamental property of P is

$$j_1^*\beta - j_0^*\beta = P\mathrm{d}\beta + \mathrm{d}P\beta. \tag{1.5}$$

PROPOSITION 1.2. *Let $\sigma_t(x) = \Sigma(x,t)$ be a 1-parameter variation of $\sigma = \sigma_0$ through sections, and let δ denote the codifferential operator on $\Omega^*(M)$. Then*

$$K^v(\sigma_1) - K^v(\sigma_o) = \int_M \big(g(\delta\omega, P\Sigma^*\eta) + g(\omega, P\Sigma^*\mathrm{d}\eta)\big)\mathrm{d}x.$$

Proof. Application of (1.5) to $\beta = \Sigma^*\eta$ yields

$$\sigma_1^*\eta - \sigma_0^*\eta = P\mathrm{d}(\Sigma^*\eta) + \mathrm{d}(P\Sigma^*\eta)$$

and the result follows from the Lemma 1.1 and Stokes' Theorem. □

2. Harmonic sections of the twistor bundle

Let $\xi\colon SO(M) \to M$ denote the principal bundle of positively oriented orthonormal tangent frames. The twistor space of M may be constructed by taking the $U(k)$-quotient: $Z(M) = SO(M)/U(k)$ and $\pi\colon Z(M) \to M$ is the natural projection. Let $\zeta\colon SO(M) \to Z(M)$ be the quotient map; then $\pi \circ \zeta = \xi$. In conformance with the notation of Section 1 we abbreviate $Z(M) = N$. Let h be the Riemannian metric on N described in the Introduction (see also (A.1) in the Appendix). Then π is a Riemannian submersion with t.g. fibres [29], and $\mathcal{H}$ in (1.1) is the dζ-image of the horizontal distribution for the Levi–Civita connection of g. Moreover, the invariant Kähler structure of $SO(m)/U(k)$ induces a natural compatible almost-complex structure J^v in $\mathcal{V}$.

Now let $\mathcal{E} \to M$ denote the skew-symmetric subbundle of End (TM), and $\pi^*\mathcal{E} \to N$ its π-pullback. The differential geometry of π is facilitated by a canonical isometric vector bundle embedding $\iota\colon \mathcal{V} \hookrightarrow \pi^*\mathcal{E}$, allowing vertical vectors to be treated as tensors on M. (Nonetheless, we will always distinguish between $V \in \mathcal{V}$ and $\iota V \in \pi^*\mathcal{E}$.) Essentially, ι is a component of the Maurer–Cartan form of the Lie group $SO(n)$, transferred fibre-by-fibre to $Z(M)$; a precise formulation is given in the Appendix. Let $\kappa\colon TN \to \pi^*\mathcal{E}$ be the composition of ι with the horizontal projection of TN onto $\mathcal{V}$. In addition, there is a tautological almost-Hermitian structure $\mathcal{J}$ in the vector bundle $\pi^*TM \to N$; namely, if $y \in N$ then $\mathcal{J}_y \in \mathrm{End}\,(T_{\pi(y)}M)$ is the endomorphism of $T_{\pi(y)}M$ whose matrix with respect to any orthonormal frame in $\zeta^{-1}(y)$ is

$$J_o = \begin{pmatrix} 0 & -\mathbb{I}_k \\ \mathbb{I}_k & 0 \end{pmatrix} \tag{2.1}$$

where $\mathbb{I}_k$ is the $k \times k$ identity matrix. If $\mathcal{J}$ is viewed as a section of $\pi^*\mathcal{E}$, then J^v is characterized as follows (see the Appendix):

$$\iota \circ J^v = \mathcal{J} \circ \iota. \tag{2.2}$$

Further relevant basic differential geometric properties of π are summarized in the following sequence of Lemmas 2.1–2.4, proofs of which are all in the Appendix. The first is a characterization of κ.

LEMMA 2.1. *For all $A \in TN$ we have*

$$\kappa A = \tfrac{1}{2}\mathcal{J} \circ \nabla_A \mathcal{J}$$

where the covariant derivative is the π-pullback of the Levi–Civita connection of g.

PROPOSITION 2.2. *For all $X \in TM$ we have*

$$\iota\big(\mathrm{d}^v\sigma(X)\big) = \tfrac{1}{2}J \circ \nabla_X J.$$

Proof. By definition J is the σ-pullback of $\mathcal{J}$, and $\iota \circ \mathrm{d}^v\sigma = \kappa \circ \mathrm{d}\sigma$. The result therefore follows on pulling back Lemma 2.1 by σ, using $\pi \circ \sigma = \mathrm{id}$. □

We note that the Levi–Civita connection in the tensor algebra of (M, g) restricts to a connection in $\mathcal{E}$. Thus both bundles TN and $\pi^*\mathcal{E}$ have natural connections, which are intertwined by κ as follows.

LEMMA 2.3. *If $A \in TN$ and $B \in \mathcal{C}(TN)$ then*

$$\kappa(\nabla_A B) = -\tfrac{1}{2}\mathcal{J}[\mathcal{J}, \nabla_A(\kappa B)] + \tfrac{1}{4}\mathcal{J}[\mathcal{J}, R(\pi_*A, \pi_*B)]$$

where the covariant derivative on the left (respectively right) side is the Levi–Civita connection of h (respectively g, pulled back to $\pi^*\mathcal{E}$), and R is the Riemann curvature tensor of (M, g).

It follows from the fact that the fibres of π are t.g. Kähler submanifolds that $\nabla^v J^v(\mathcal{V}, \mathcal{V}) = 0$. In fact the holonomy invariance of J^v is slightly stronger.

LEMMA 2.4. *The almost-complex structure J^v in $\mathcal{V}$ satisfies* $\nabla^v J^v(TN, \mathcal{V}) = 0$.

We now have enough information to explicitly compute the variation of K^v, beginning with the exterior derivative of the degenerate 2-form η.

LEMMA 2.5. *For all $A, B, C \in \mathcal{C}(TN)$ we have*

$$\mathrm{d}\eta(A, B, C) = \mathfrak{C}\ \tfrac{1}{2}g\big([\mathcal{J}, R(\pi_*A, \pi_*B)], \kappa C\big)$$

where $\mathfrak{C}$ denotes cyclic summation over A, B, C.

Proof. By Lemma 2.4 the Levi–Civita covariant derivative is

$$\nabla_A \eta(B,C) = T(A,B,C) - T(A,C,B)$$

where

$$T(A,B,C) = h\big(\nabla_A(hB), J^v(vC)\big) = \tfrac{1}{4}g\big([\mathcal{J}, R(\pi_* A, \pi_* B)], \kappa C\big).$$

The second equality comes from (2.2) and Lemma 2.3, since ι is isometric and $\ker \kappa = \mathcal{H}$. Now take the cyclic sum. □

Now let ϕ be the Ricci form as defined in (0.2), and let Φ denote the associated skew-symmetric endomorphism field:

$$g(\Phi X, Y) = \phi(X,Y); \quad \Phi X = -\tfrac{1}{2}R(E_j, JE_j)X. \tag{2.3}$$

Also, let δJ denote the coderivative when J is viewed as a TM-valued 1-form:

$$g(\delta J, X) = \delta\omega(X). \tag{2.4}$$

For the twistor bundle, Proposition 1.2 assumes the following form.

PROPOSITION 2.6. *Let $\sigma \in \mathcal{C}(\pi)$ parametrize $J = J^M$. Let J_t be the variation of J parametrized by a variation σ_t of σ, and $V_t = \frac{\mathrm{d}}{\mathrm{d}t}\sigma_t$. Then*

$$K^v(\sigma_t) - K^v(\sigma) = \frac{1}{2}\int_0^t \int_M g\big(\nabla_{\delta J} J_s - [J_s, \Phi], \iota V_s\big)\mathrm{d}x\mathrm{d}s$$

Note. The almost-Hermitian structure J^M used to construct each $K^v(\sigma_t)$ is fixed throughout.

Proof. It follows from Proposition 2.2 that

$$j_t^*(\Sigma^*\eta \lrcorner \partial_t)(X) = \tfrac{1}{2}g(\iota V_t, \nabla_X J_t).$$

Therefore by definition of the homotopy operator (1.4):

$$g(\delta\omega, P\Sigma^*\eta) = P\Sigma^*\eta(\delta J) = \frac{1}{2}\int_0^1 g(\nabla_{\delta J} J_t, \iota V_t)\mathrm{d}t.$$

On the other hand, Lemma 2.5 implies

$$j_t^*(\Sigma^*\mathrm{d}\eta \lrcorner \partial_t)(X,Y) = \tfrac{1}{2}g\big([J_t, R(X,Y)], \iota V_t\big).$$

Therefore by the definition (2.3) of Φ:

$$g(\omega, P\Sigma^*\mathrm{d}\eta) = -\frac{1}{2}\int_0^1 g\big([J_t, \Phi], \iota V_t\big)\mathrm{d}t.$$

The result now follows from Proposition 1.2. □

Recall that an almost-Hermitian structure J is said to be *integrable* (respectively (1, 2)-*symplectic*) if its Nijenhuis tensor [18], vol. 2, p. 123 (respectively the (1, 2)-component of $d\omega^{\mathbb{C}}$) vanishes. The following characterizations are standard (see [13]):

(C1) J is integrable if and only if $\nabla_{JX}J(JY) = \nabla_X J(Y)$,

(C2) J is (1, 2)-symplectic if and only if $\nabla_{JX}J(JY) = -\nabla_X J(Y)$,

and lead to the following criteria for sections of the twistor bundle to be vertically holomorphic, or antiholomorphic.

LEMMA 2.7 (see also [22], Propositions 1.3 and 3.2).

(1) *J is integrable if and only if $\bar{\partial}^v\sigma = 0$.*

(2) *J is* (1, 2)*-symplectic if and only if $\partial^v\sigma = 0$.*

Proof. It follows from (2.2) and Proposition 2.2 that

$$\iota \circ d^v\sigma \circ J(X) = \tfrac{1}{2} J \circ \nabla_{JX} J \quad \text{and} \quad \iota \circ J^v \circ d^v\sigma(X) = -\tfrac{1}{2}\nabla_X J.$$

The equation $\bar{\partial}^v\sigma = 0$ is equivalent to $J^v \circ d^v\sigma = d^v\sigma \circ J$. Thus $\bar{\partial}^v\sigma = 0$ precisely when $\nabla J(JX, JY) = \nabla J(X, Y)$, and hence by (C1) when J is integrable. (2) goes similarly. □

With reference to (1.2), we define $\tau(J)$ to be the following skew-symmetric field of endomorphisms of TM:

$$\tau(J) = \iota \circ \tau^v(\sigma) \tag{2.5}$$

which we call the *tension field of* J. Thus J is harmonic if and only if $\tau(J) = 0$. Starting from (1.3), it was shown in [31] that

$$\tau(J) = -\tfrac{1}{4}[J, \nabla^*\nabla J].$$

Hence (0.1). However, more tractable geometric expressions for $\tau(J)$ are now available from Proposition 2.6, in certain special cases. It should be noted that since $K^v(\sigma)$ is usually not invariant under vertical homotopies, it cannot be inferred that the corresponding critical points of E^v are minimizers. However, there are exceptions (see also Theorem 5.3). Recall that J is said to be *cosymplectic* if $\delta J = 0$; by (C2) all (1, 2)-symplectic structures have this property. Furthermore, recall that a vector field X on M is *Kähler null* if $\nabla_X J = 0$. An almost-Hermitian manifold with $\phi \in \Omega^{1,1}$ will be said to have *reducible* Ricci form.

THEOREM 2.8. *Let J be the almost-complex structure of an almost-Hermitian manifold.*

1. *If J is integrable then*

$$\tau(J) = \frac{1}{2}\big([J, \Phi] - \nabla_{\delta J} J\big).$$

Therefore J is harmonic if and only if

$$[J, \Phi] = \nabla_{\delta J} J.$$

2. *If J is* (1, 2)*-symplectic then*

$$\tau(J) = -\frac{1}{2}[J, \Phi].$$

Therefore J is harmonic if and only if $[J, \Phi] = 0$.

3. *If J is integrable and δJ is Kähler null, or J is* (1, 2)*-symplectic, then J is harmonic if and only if its Ricci form is reducible.*
4. *If J is integrable and cosymplectic, or J is* (1, 2)*-symplectic, then σ minimizes energy in its vertical homotopy class if the Ricci form of J vanishes.*

Proof. If J is integrable then $A^v(\sigma) = 0$ by Lemma 2.7(1), and hence $\frac{d}{dt}\big|_{t=0} A^v(\sigma_t) = 0$ for any variation σ_t of σ. Therefore by Proposition 2.6

$$\frac{d}{dt}\bigg|_{t=0} E^v(\sigma_t) = \frac{d}{dt}\bigg|_{t=0} K^v(\sigma_t) = \frac{1}{2}\int_M g\big(\nabla_{\delta J} J - [J, \Phi], \iota V\big)dx,$$

where $V = \frac{d}{dt}\big|_{t=0}\sigma_t$, and the expression for $\tau(J)$ in (1) follows by comparison with (1.2). On the other hand, if J is (1,2)-symplectic then $\delta J = 0$ by (C2), and it follows from Lemma 2.7 (2) and Proposition 2.6 that

$$\frac{d}{dt}\bigg|_{t=0} E^v(\sigma_t) = -\frac{d}{dt}\bigg|_{t=0} K^v(\sigma_t) = \frac{1}{2}\int_M g\big([J, \Phi], \iota V\big)dx$$

which yields the expression for $\tau(J)$ in (2). If δJ is Kähler null then both characterizations of equations $\tau(J) = 0$ boil down to $[J, \Phi] = 0$, which is equivalent to $\phi \in \Omega^{1,1}$. Finally, the hypotheses $\delta J = 0 = \phi$ ensure that K^v is a vertical homotopy invariant, by Proposition 2.6. If J is integrable then since $A^v(\sigma) = 0$ we have

$$E(\sigma_t) - E(\sigma) = K^v(\sigma_t) - K^v(\sigma) + 2A^v(\sigma_t) = 2A^v(\sigma_t) \geqslant 0.$$

Similarly, if J is (1, 2)-symplectic then since $H^v(\sigma) = 0$ we have

$$E(\sigma_t) - E(\sigma) = 2H^v(\sigma_t) - \big(K^v(\sigma_t) - K^v(\sigma)\big) = 2H^v(\sigma_t) \geqslant 0.$$

□

3. Reducibility of the Ricci form

It turns out that all the examples of harmonic almost-complex structures considered in this paper can be deduced from part (3) of Theorem 2.8, where the characterization is reducibility of the Ricci form.

Remark. If J is integrable, or (1, 2)-symplectic, then it follows from (C1) and (C2) of Section 2 that $\chi \in \Omega^{1,1}$, where χ is defined in (0.3). Therefore, under the hypotheses of Theorem 2.8(3), if J is harmonic then the first Chern class of the almost-complex manifold (M, J) has a representative 2-form of type (1,1). This generalizes a cohomological property of Kähler manifolds.

Reducibility of ϕ may be expressed in a number of different ways. From certain points of view [4, 15, 27] it is more natural to work with the *$*$Ricci tensor*:

$$Ric^*(X, Y) = \phi(X, JY) = -\tfrac{1}{2}g(R(E_i, JE_i)X, JY). \tag{3.1}$$

It follows that

$$Ric^*(JX, JY) = Ric^*(Y, X)$$

but in general Ric^* is neither symmetric nor skew-symmetric. We also recall the decomposition of the curvature operator under the action of $SO(n)$ [3], Chapter 1:

$$\mathcal{R} = \frac{1}{n(n-1)} s1 + \frac{1}{n-2}\mathcal{Z} + \mathcal{W}$$

where s is the scalar curvature, $\mathcal{W}$ is the Weyl component, and $\mathcal{Z}$ is the trace-free Ricci component. More precisely, if $\lambda \in \Omega^2(M)$ and L is the associated skew-symmetric endomorphism field, obtained by raising an index (cf. (2.3)), then

$$\mathcal{Z}(\lambda)(X,Y) = Ric(LX,Y) - Ric(X,LY) - \frac{2s}{n}\lambda(X,Y) \tag{3.2}$$

where Ric is the Ricci curvature of (M,g). Further, on an almost-Hermitian manifold $\Omega^{1,1}$ splits under the action of $U(k)$:

$$\Omega^{1,1} = \Omega^{1,1}_{\mathrm{o}} \oplus \mathcal{C}(L_\omega) \tag{3.3}$$

as an orthogonal direct sum, where L_ω is the line subbundle spanned by ω. Elements λ of $\Omega^{1,1}_{\mathrm{o}}$ are characterized by the condition $\lambda(E_i, JE_i) = 0$; it therefore follows from (3.2) that $\mathcal{Z}(\omega) \in \Omega^{1,1}_{\mathrm{o}}$. In the light of these remarks it should be evident that all the following conditions are equivalent:

(R1) $\phi \in \Omega^{1,1}$,
(R2) $[J, \Phi] = 0$,
(R3) Ric^* is J-invariant,
(R4) Ric^* is symmetric,
(R5) $\mathcal{R}: \mathcal{C}(L_\omega) \to \Omega^{1,1}$,
(R6) $\mathcal{W}: \mathcal{C}(L_\omega) \to \Omega^{1,1}$.

We now list some examples of almost-Hermitian manifolds with reducible Ricci form.

(1) Those with the curvature identity

$$g(R(JX,JY)JZ,JW) = g(R(X,Y)Z,W) \tag{3.4}$$

which arises naturally from the representation theory of the unitary group on the space of curvature-like tensors [27], and is the weakest of the three curvature restrictions studied in [16]. Accordingly, if (3.4) holds we say that R is *weakly reducible.*

(2) The $*$Einstein manifolds (0.5), whose Ricci form satisfies the following equivalent strong reducibility conditions:

(S1) $\phi \in \mathcal{C}(L_\omega)$,
(S2) Φ is proportional to J: $\Phi = \frac{1}{n}s^* J$,
(S3) Ric^* is proportional to g: $Ric^* = \frac{1}{n}s^* g$,
(S4) $\mathcal{R}: \mathcal{C}(L_\omega) \to \mathcal{C}(L_\omega)$,
where
$s^* = -\frac{1}{2}g(R(E_i, JE_i)E_j, JE_j) = g(\phi, \omega)$.

Some criteria for an almost-Hermitian manifold with reducible Ricci form to be $*$Einstein appear in Corollary 3.4.

(3) Conformally flat manifolds ($\mathcal{W} = 0$), by (R6). When $n = 4$, since ω is self-dual, conformally half-flat ($\mathcal{W}_+ = 0$) suffices.

(4) When $n = 4$, those with anti-self-dual (a.s.d.) Ricci form. Recall that when $n = 4$ there is the $SO(4)$-splitting

$$\Omega^2(M) = \Omega^2_+(M) \oplus \Omega^2_-(M) \tag{3.5}$$

into rank 3 eigenbundles of the Hodge star operator, and a 2-form is said to be a.s.d. if it lives in Ω^2_-. A comparison of (3.5) with the $U(2)$-splitting yields [23], Proposition 7.1

$$\Omega^2_+ = \left(\Omega^{2,0} \oplus \Omega^{0,2}\right) \oplus \mathcal{C}(L_\omega), \qquad \Omega^2_- = \Omega^{1,1}_o. \tag{3.6}$$

Therefore, in the class of almost-Hermitian 4-manifolds with reducible Ricci form, the $*$Einstein manifolds are complementary to those with a.s.d. ϕ; indeed, the latter are characterized by $s^* = 0$. (A variational characterization of a.s.d. Ricci forms is given in Theorem 5.3.) For an almost-Kähler manifold, contraction of the curvature identity [16], Corollary 4.3 yields

$$|\nu|^2 = 32(s^* - s),$$

where ν is the Nijenhuis tensor [18], vol. 2, p. 123. Thus a necessary condition for the existence of an almost-Kähler structure with a.s.d. ϕ is $s \leqslant 0$. It follows from Corollary 3.5 below that the only compact almost-Kähler *Einstein* manifolds with a.s.d. ϕ are Ricci-flat Kähler surfaces.

It is possible to slightly refine the hypotheses of Theorem 2.8(3), to which end we now summarize the Gray–Hervella 'classification' of almost-Hermitian structures [17]. If $\mathfrak{W}$ denotes the space of all 3-covariant tensor fields with the same symmetries as $\nabla\omega$, then there is a splitting into $U(k)$-irreducible subspaces:

$$\mathfrak{W} = \mathfrak{W}_1 \oplus \mathfrak{W}_2 \oplus \mathfrak{W}_3 \oplus \mathfrak{W}_4. \tag{3.7}$$

Following the designations of [17], $\nabla\omega \in \mathfrak{W}_i$ (abbreviated $J \in \mathfrak{W}_i$) corresponds to J being nearly-Kähler, almost-Kähler and cosymplectic Hermitian, according as $i = 1, 2, 3$ respectively. Class $\mathfrak{W}_4$ is characterized by the following identity:

$$\begin{aligned}\nabla_X\omega(Y,Z) = \frac{1}{2-n}\big(&g(X,Y)\delta\omega(Z) - g(X,Z)\delta\omega(Y) + \\ &+ \omega(X,Y)\delta\omega(JZ) - \omega(X,Z)\delta\omega(JY)\big).\end{aligned} \tag{3.8}$$

Furthermore, $J \in \mathfrak{W}_1 \oplus \mathfrak{W}_2$ if and only if J is (1, 2)-symplectic, and $J \in \mathfrak{W}_3 \oplus \mathfrak{W}_4$ precisely when J is integrable. The cosymplectic structures form class $\mathfrak{W}_1 \oplus \mathfrak{W}_2 \oplus \mathfrak{W}_3$. Theorem 2.8 therefore characterizes the harmonic J in a total of seven out of the sixteen possible invariant classes (including the Kähler J, which are the zeroes of the vertical energy functional). That characterization is particularly simple in each of the four irreducible classes. The following expanded statement of Theorem 2.8(3) summarizes the situation.

THEOREM 3.1. *Suppose that* $J \in \mathfrak{W}_i$ $(i = 1, \ldots, 4)$, *or* $J \in \mathfrak{W}_1 \oplus \mathfrak{W}_2$. *Then* J *is harmonic if and only if its Ricci form is reducible. In particular,* J *is harmonic if* (M, g, J) *has any of the following:*

(H1) *weakly reducible curvature,*
(H2) *the* $*$*Einstein condition,*
(H3) (M, g) *conformally flat; or* $n = 4$ *and* (M, g) *conformally half-flat,*
(H4) $n = 4$ *and anti-self-dual Ricci form.*

Proof. The hypotheses of Theorem 2.8(3) are clearly satisfied for all the invariant classes mentioned, except possibly $\mathfrak{W}_4$. However it is a simple consequence of (3.8) that if $J \in \mathfrak{W}_4$ then δJ is Kähler null. The sufficiency of conditions (H1)–(H4) follows from the preceding discussion. □

Remark. When $n = 4$, the splitting (3.7) simplifies to $\mathfrak{W} = \mathfrak{W}_2 \oplus \mathfrak{W}_4$.

Since the curvature tensor of a nearly-Kähler manifold is weakly reducible [15], we obtain an alternative proof of the following result of [31].

COROLLARY 3.2. *If* $J \in \mathfrak{W}_1$ *(i.e. a nearly-Kähler structure) then* J *is harmonic.*

Corollary 3.2 raises the interesting question of which nearly-Kähler structures are stable; see the last paragraph of Section 4 for a conjecture. The endowment of S^6 with its nearly-Kähler structure may be generalized to arbitrary orientable 6-dimensional submanifolds $M \subset \mathbb{R}^8$ using the 3-fold vector cross products on $\mathbb{R}^8$ [14]. In case $M = S^2 \times \mathbb{R}^4$ one obtains a (1, 2)-symplectic manifold with condition (H1) (see also [16], Theorem 6.6), and hence by Theorem 3.1 a harmonic almost-complex structure. On the other hand, if $V^4 \subset \mathbb{R}^8$ is a linear subspace, and $\Sigma \subset V$ is a minimal surface, then the almost-Hermitian structure induced on $M = \Sigma \times V^\perp$ is in $\mathfrak{W}_3$, but also has (H1), and so by Theorem 3.1 is also harmonic. The best-known representatives of class $\mathfrak{W}_4$ are the Hopf manifolds, which satisfy (H3) and hence by Theorem 3.1 are harmonic.

Remark 3.3. In case $J \in \mathfrak{W}_2$ it is interesting to compare Theorem 3.1 with a result of Blair and Ianus [4], which states that the following functional

$$E(J, g) = \frac{1}{2} \int_M |\nabla J|^2 \mathrm{d}x$$

is stationary with respect to all ω-preserving variations of the *pair* (J, g) if and only if the Ricci curvature of (M, g) is J-invariant. The set of critical points for the Blair–Ianus variational problem therefore includes the almost-Kähler Einstein manifolds. Now Proposition 2.2 implies

$$E(J, g) = 4E^v(\sigma, g)$$

and by Theorem 3.1 the set of harmonic almost-complex structures includes the almost-Kähler $*$*Einstein* manifolds. This prompts the question of which almost-Hermitian structures on an Einstein manifold are harmonic. More generally, define the following 2-form ρ of type (1, 1):

$$\rho(X, Y) = \tfrac{1}{2}\big(Ric(JX, Y) - Ric(X, JY)\big)$$

and say that an almost-Hermitian manifold is (1, 1)-*Einstein* if ρ is proportional to ω. From (3.3), this is equivalent to the vanishing of the trace-free part ρ_o of ρ, and it follows from (3.2) that $\rho_o = \mathcal{Z}(\omega)$. Therefore an almost-Hermitian structure is (1,1)-Einstein precisely when $\mathcal{Z}(\omega) = 0$.

COROLLARY 3.4. *Let (M, g, J) be a 4-dimensional almost-Hermitian $(1,1)$-Einstein manifold, with $J \in \mathfrak{W}_2$ or $J \in \mathfrak{W}_4$. Then J is harmonic if and only if J is $*$Einstein.*

Proof. By Theorem 3.1 a harmonic J in either of these two classes may be characterized by (R6), which in 4-dimensions is strengthened by the self-duality of ω:

$$\mathcal{W} = \mathcal{W}_+ : \mathcal{C}(L_\omega) \to \Omega^{1,1} \cap \Omega^2_+ = \mathcal{C}(L_\omega) \quad \text{by (3.6).}$$

Since $\mathcal{Z}(\omega) = 0$, this implies the $*$Einstein condition (S4) of Section 3. □

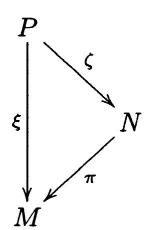
COROLLARY 3.5. *If (M, g, J) is a compact almost-Kähler Einstein 4-manifold with constant $*$scalar curvature, then J is harmonic if and only if J is Kähler.*

Proof. The result follows from Theorem 3.4 and [24], where it is shown that compact almost-Kähler Einstein 4-manifolds which are $*$Einstein with constant $*$scalar curvature are necessarily Kähler. □

4. Stability of $*$Einstein structures

Our aim in Section 4 is to generalize the stability result Theorem 2.8(4). We begin with some basic gauge theory, the notation and terminology of which is essentially that of [5].

The construction of the twistor fibration described in Section 2 may be summarized by the following commutative triangle of bundles:



where $P = SO(M)$, $N = Z(M)$, and the quotient map ζ is a principal $U(k)$-fibration. We abbreviate $G = SO(n)$ and $H = U(k)$, and let $\mathfrak{g}, \mathfrak{h}$ denote the corresponding Lie algebras:

$$\mathfrak{g} = \{\text{skew-symmetric } n \times n \text{ matrices}\}, \qquad \mathfrak{h} = \{a \in \mathfrak{g} : aJ_o = J_o a\}$$

with J_o defined in (2-1). The *gauge group* $\mathcal{G}$ of ξ is defined

$$\mathcal{G} = \{G\text{-equivariant} \mid \psi : P \to G\}$$

where G acts on itself by conjugation. Then $\mathcal{G}$ is naturally isomorphic to the group of orthogonal (1,1)-tensor fields on M. Now $\mathcal{G}$ acts on the left of P:

$$\psi p = p\psi(p) \quad \forall p \in P. \tag{4.1}$$

Since this action is equivariant and fibre-preserving, it descends to a fibre-preserving action on N. So there is a $\mathcal{G}$-action on the manifold $\mathcal{C}(\pi)$; indeed, if σ parametrizes the almost-complex structure J, then $\psi.\sigma$ parametrizes $\psi \circ J \circ \psi^{-1}$. This action is locally transitive; so to test the stability of a harmonic section σ it suffices to consider variations generated by 1-parameter subgroups of $\mathcal{G}$. We note that the isotropy subgroup of σ is

$$\mathcal{H}_\sigma = \{\psi \in \mathcal{G} \quad \text{s.t.} \quad \psi \mid Q: Q \to H\} \subset \mathcal{G}$$

where $Q = \sigma^* P \subset P$ is the total space of the principal H-subbundle of J-unitary frames.

Now let $\mathfrak{G}$ denote the *gauge algebra* of ξ:

$$\mathfrak{G} = \{G\text{-equivariant} \mid \alpha: P \to \mathfrak{g}\}$$

where G acts on $\mathfrak{g}$ via the adjoint representation. Then $\mathfrak{G}$ is naturally isomorphic to the Lie algebra of sections of the vector bundle $\mathcal{E} \to M$; i.e. skew-symmetric $(1,1)$-tensor fields on M. Since $\mathcal{G}$ acts on $\mathcal{C}(\pi)$, every $\alpha \in \mathfrak{G}$ induces a vector field $\hat{\alpha}$ on $\mathcal{C}(\pi)$:

$$\hat{\alpha}(\sigma) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left(\mathrm{e}^{t\alpha}.\sigma\right) \in T_\sigma \mathcal{C}(\pi)$$

where

$$\mathrm{e}^{t\alpha} = 1 + \alpha + \frac{1}{2!}\alpha^2 + \cdots$$

is the 1-parameter subgroup of $\mathcal{G}$ tangent to α. But $T_\sigma \mathcal{C}(\pi)$ is naturally isomorphic to $\mathcal{C}(\sigma^* \mathcal{V})$, so $\hat{\alpha}(\sigma)$ may be thought of as a vertical variation field for σ; namely, that of the variation $\sigma_t = e^{t\alpha}\sigma$. The following is proved in the Appendix.

LEMMA 4.1. *For all $\alpha \in \mathfrak{G}$ we have*

$$\iota \circ \hat{\alpha}(\sigma) = -\frac{1}{2} J[J, \alpha].$$

Now let $\mathfrak{m}$ denote the symmetric complement of $\mathfrak{h}$ in $\mathfrak{g}$:

$$\mathfrak{m} = \{a \in \mathfrak{g} : aJ_o = -J_o a\}.$$

Since $\mathfrak{h}$ and $\mathfrak{m}$ are both $Ad(H)$-stable, there exist vector bundles $\mathcal{E}_\mathfrak{h}, \mathcal{E}_\mathfrak{m} \to N$ associated to ζ, with fibres $\mathfrak{h}$ and $\mathfrak{m}$ respectively. Then $\pi^* \mathcal{E} = \mathcal{E}_\mathfrak{h} \oplus \mathcal{E}_\mathfrak{m}$ and

$$\mathcal{E}_\mathfrak{h} = \{\alpha \in \pi^* \mathcal{E} : \alpha \mathcal{J} = \mathcal{J} \alpha\}, \qquad \mathcal{E}_\mathfrak{m} = \{\alpha \in \pi^* \mathcal{E} : \alpha \mathcal{J} = -\mathcal{J} \alpha\}.$$

The gauge subalgebra

$$\mathfrak{H}_\sigma = \{\alpha \in \mathfrak{G} \quad \text{s.t.} \quad \alpha \mid Q: Q \to \mathfrak{h}\} \subset \mathfrak{G}$$

is isomorphic to the Lie algebra of sections of $\sigma^* \mathcal{E}_\mathfrak{h} \to M$, and comprises those elements of $\mathfrak{G}$ whose induced vector field on $\mathcal{C}(\pi)$ vanishes at σ. The infinitesimal vertical variations of σ are therefore generated by the following complementary subspace of $\mathfrak{G}$:

$$\mathfrak{M}_\sigma = \{\alpha \in \mathfrak{G} \quad \text{s.t.} \quad \alpha \mid Q: Q \to \mathfrak{m}\}$$

which is isomorphic to the space of sections of $\sigma^* \mathcal{E}_\mathfrak{m}$; i.e. skew-symmetric $(1,1)$-tensor fields on M which anticommute with J.

Remark 4.2. If M is compact, $\mathfrak{G}$ may be equipped with the following semi-definite metric:

$$\langle\!\langle A, B\rangle\!\rangle = \int_M g(\nabla A, \nabla B)\mathrm{d}x$$

and $\mathcal{G}$ endowed with the right-invariant semi-Riemannian metric obtained by right-translation of $\langle\!\langle,\rangle\!\rangle$. It is shown in [28] that the loop $\mathrm{e}^{tJ} = \cos t1 + \sin tJ$ is a closed geodesic precisely when (0.1) holds. Moreover, thinking of $J \in \mathcal{G}$, Equation (0.1) also characterizes those J which are *harmonic gauges*, where the latter are defined to be elements ψ of $\mathcal{G}$ which criticalize the following energy functional:

$$E(\psi, g) = \frac{1}{2}\int_M |\psi \circ \nabla \circ \psi^{-1} - \nabla|^2 \mathrm{d}x.$$

LEMMA 4.3. *Suppose* $\alpha \in \mathfrak{M}_\sigma$, *and let* $\sigma_t = e^{t\alpha}\sigma$. *If the almost-Hermitian structure* J *parametrized by* σ *is cosymplectic, with reducible Ricci form, then*

$$k^v(\sigma_t) - k^v(\sigma) = \operatorname{Tr} L_{\alpha,t} \quad \textit{where} \quad L_{\alpha,t} = J \circ \Phi \circ \sinh^2 t\alpha.$$

Proof. Since $\delta J = 0$ it follows from Proposition 2.6 that

$$k^v(\sigma_t) - k^v(\sigma) = -\frac{1}{2}\int_0^t g([J_s, \Phi], \iota V_s)\mathrm{d}s = -\frac{1}{2}\int_0^t \kappa_s \mathrm{d}s, \quad \text{say.}$$

Now $J_s = \mathrm{e}^{s\alpha} J \mathrm{e}^{-s\alpha}$ and $V_s = \hat{\alpha}(\sigma_s)$, so by Lemma 4.1

$$\kappa_s = -\tfrac{1}{2} g([J_s, \Phi], J_s[J_s, \alpha]) = g([J_s, \Phi], \alpha)$$

since α anticommutes with J. We now calculate κ_s in two ways. First, using only that α anticommutes with J, and is skew-symmetric:

$$\kappa_s = g(J\mathrm{e}^{-2s\alpha}\Phi - \Phi\mathrm{e}^{2s\alpha}J, \alpha) = -2g(\Phi, J\alpha\mathrm{e}^{-2s\alpha}).$$

Secondly, using (R2) of Section 3:

$$\kappa_s = g(\mathrm{e}^{2s\alpha}\Phi J - J\Phi\mathrm{e}^{-2s\alpha}, \alpha) = 2g(\Phi, J\alpha\mathrm{e}^{2s\alpha}).$$

Addition yields

$$\kappa_s = -2g(J\Phi, \alpha \sinh 2s\alpha)$$

and the result follows on integration. □

By a *stable* harmonic almost-complex structure we mean one for which all compactly supported vertical perturbations of the parametrizing section are locally energy-non-decreasing. The following result extends Theorem 2.8(4), which dealt with the *Ricci-flat case.

THEOREM 4.4. *Let* J *be a *Einstein structure. If* $J \in \mathfrak{W}_3$ *and* $s^* \geqslant 0$, *or* $J \in \mathfrak{W}_1 \oplus \mathfrak{W}_2$ *and* $s^* \leqslant 0$, *then* J *is stable harmonic.*

Proof. It follows from (S2) of Section 3 that

$$L_{\alpha,t} = -\frac{1}{n}s^* \sinh^2 t\alpha.$$

Lemma 4.3 therefore yields

$$k^v(\sigma_t) - k^v(\sigma) = -\frac{1}{n}s^* \mathrm{Tr}\, \sinh^2 t\alpha.$$

Since α is skew-symmetric, so is $\sinh t\alpha$, which therefore has imaginary eigenvalues. Hence $\sinh^2 t\alpha$ has negative trace, provided $\alpha \neq 0$. Using the expressions in the proof of Theorem 2.8(4) we conclude that $E(\sigma_t) - E(\sigma) \geqslant 0$. □

Remark. If in addition to the hypotheses of Theorem 4.4 it is assumed that J is *not* $*$Ricci-flat then the proof yields the strict inequality $E(\sigma_t) - E(\sigma) > 0$.

By [15], Theorem 5.1 all non-Kähler 6-dimensional nearly-Kähler manifolds are $*$Einstein, with constant $s^* > 0$. Also in [15], every non-Kähler 6-dimensional nearly-Kähler manifold is shown to be *strict*, which means that $\nabla_X J = 0$ only when $X = 0$. This non-degeneracy was needed in the proof of the instability of the nearly-Kähler six-sphere [31], Theorem 2; however, other parts of the proof relied specifically on spherical geometry. In the light of Theorem 4.4 it is nevertheless tempting to make the following:

Conjecture 4.5. A 6-dimensional nearly-Kähler structure J is stable harmonic if and only if J is Kähler.

5. Stability in Dimension 4

The first aim of Section 5 is to extend in another direction the $*$Ricci-flat stability Theorem 2.8(4), this time to almost-Hermitian structures with reducible Ricci form and $s^* = 0$. Recall that in 4-dimensions these are precisely the almost-Hermitian manifolds with a.s.d. Ricci form. Throughout Section 5 we assume (M, g, J) is a 4-dimensional *almost-Kähler* manifold; i.e. $J \in \mathfrak{W}_2$ (recall that $\mathfrak{W} = \mathfrak{W}_2 \oplus \mathfrak{W}_4$ in 4-dimensions). Our first result is a consequence of the twistor bundle of a 4-manifold having 2-dimensional fibres.

LEMMA 5.1. *Suppose J is harmonic. Then K^v is constant on a $C(\pi)$-neighbourhood of σ if and only if $[\alpha, \Phi] = 0$ for all $\alpha \in \mathfrak{M}_\sigma$.*

Proof. As in the proof of Lemma 4.3 we have

$$k^v(\sigma_t) - k^v(\sigma) = -\frac{1}{2}\int_0^t g([J_s, \Phi], \iota V_s) \mathrm{d}s.$$

Now define

$$\lambda_s = g([J_s, \Phi], \iota J^v V_s).$$

From (2.2) and Lemma 4.1 it follows that

$$\lambda_s = \tfrac{1}{2} g\big([J_s, \Phi], [J_s, \alpha]\big) = g(\Phi, \alpha + J\alpha J) = 2g(\Phi, \alpha).$$

Since J is harmonic, $[J, \Phi] = 0$ by Theorem 3.1. So Φ is orthogonal to α, and $\lambda_s = 0$. Since $\mathcal{V}$ is spanned by V_s and $J^v V_s$, a necessary and sufficient condition for K^v to be constant about σ is $[J_s, \Phi] = 0$; equivalently, $\Phi\alpha = \alpha\Phi$ for all $\alpha \in \mathfrak{M}_\sigma$. □

Another distinctive feature of 4-dimensions is the following non-faithfulness of the adjoint representation, whose verification is a straightforward matrix computation.

LEMMA 5.2. *If* $\rho: \mathfrak{h} \to \mathfrak{m}^* \otimes \mathfrak{m}$ *is the adjoint representation, then*

$$\ker\rho = \begin{cases} \langle J_o \rangle^\perp & \text{if } k = 2, \\ 0 & \text{if } k \geqslant 3. \end{cases}$$

THEOREM 5.3. *Suppose J is a 4-dimensional harmonic almost-Kähler structure. Then K^v is constant on a neighbourhood of σ if and only if ϕ is anti-self-dual, in which case J is stable.*

Proof. Since J is harmonic, $\phi \in \Omega^{1,1}$ by Theorem 3.1. From the Lemmas, K^v is constant about σ on condition that $g(\phi, \omega) = 0$, which by (3.6) is equivalent to $\phi \in \Omega^2_-$. The stability of J now follows as in the proof of Theorem 2.8(4). □

We now turn to the symplectic manifold M^4 of Thurston [26], endowed with the compatible almost-Kähler structure (J, g) of Abbena [1]. In [2], M is exhibited as a homogeneous space, and it is shown that

$$Ric^*(e_2, e_2) = -\tfrac{1}{4} = Ric^*(e_3, e_3), \qquad Ric^*(e_i, e_j) = 0 \quad \text{all other } i, j, \tag{5.1}$$

where $(e_1, \dots, e_4)$ is an appropriate global left-invariant orthonormal frame field. It follows that Ric^* is non-zero and degenerate, so J is not $*$Einstein. Moreover, $s^* = -\frac{1}{2} \neq 0$, so ϕ is not a.s.d. However, Ric^* is symmetric, so J is harmonic by Theorem 3.1. (On the other hand, it also follows from [2] that Ric is not J-invariant, so J is *not* a critical point for the Blair–Ianus variational principle mentioned in Remark 3.3).

THEOREM 5.4. *The Abbena–Thurston almost-Kähler structure is stable harmonic.*

Proof. Let $\sigma_t = e^{t\alpha}\sigma$ with $\alpha \in \mathfrak{M}_\sigma$. Since J satisfies the hypotheses of Lemma 4.3, and $h^v(\sigma) = 0$ by Lemma 2.7, it follows that

$$e(\sigma_t) - e(\sigma) = 2h^v(\sigma_t) - \big(k^v(\sigma_t) - k^v(\sigma)\big) = 2h^v(\sigma_t) - \operatorname{Tr} L_{\alpha,t}.$$

Now $\sinh^2 t\alpha$ is negative semi-definite, as in the proof of Theorem 4.4. Furthermore Ric^* is negative semi-definite by (5.1), hence $J\Phi$ is positive semi-definite. Therefore $L_{\alpha,t}$ has non-positive trace. □

6. Tangent bundles

Let (M', g') be an orientable k-dimensional Riemannian manifold, and let $M = TM'$. Denote by $\tau': M \to M'$ the bundle projection, and write

$$TM = \mathcal{W} \oplus \mathcal{K}$$

where $\mathcal{W} = \ker d\tau'$ and $\mathcal{K}$ is the horizontal distribution for the Levi–Civita connection of g'. For any $y \in M$ let $M_y \subset M$ denote the tangent space of M' containing y, and let $y^{\mathcal{W}}$ (respectively $y^{\mathcal{K}}$) denote the vertical (respectively, horizontal) lift of y to a vector field along M_y (see [8]). The Sasaki almost-Kähler structure on M is then defined as follows:

$$\begin{aligned} &g(y^{\mathcal{W}}, z^{\mathcal{W}}) = g'(y,z) = g(y^{\mathcal{K}}, z^{\mathcal{K}}), \qquad g(y^{\mathcal{W}}, z^{\mathcal{K}}) = 0 \\ &J(y^{\mathcal{K}}) = y^{\mathcal{W}}, \qquad J(y^{\mathcal{W}}) = -y^{\mathcal{K}}. \end{aligned} \tag{6.1}$$

The curvature of (M, g) was computed in [19], and the following is an easy consequence.

LEMMA 6.1. *The $*$Ricci curvature of the Sasaki almost-Kähler structure is:*

(a) $Ric^*\big(y^{\mathcal{W}}(x), z^{\mathcal{W}}(x)\big) = \tfrac{1}{2}Ric'(y,z) + \tfrac{1}{8}g'\big(R'(x,y), R'(x,z)\big)$
$\qquad = Ric^*\big(y^{\mathcal{K}}(x), z^{\mathcal{K}}(x)\big)$,

(b) $Ric^*\big(y^{\mathcal{W}}(x), z^{\mathcal{K}}(x)\big) = 0$

(c) $Ric^*\big(y^{\mathcal{K}}(x), z^{\mathcal{W}}(x)\big) = \tfrac{1}{2}g'(\delta' R'(x)y, z), \quad \forall y, z \in M_x.$

Recall that (M', g') is said to have *harmonic curvature* if $\delta' R' = 0$, a condition which generalizes the Einstein equations when $k \geqslant 3$; see [3], Chapter 16. In all dimensions a necessary condition for harmonic curvature is constant scalar curvature. By Theorem 3.1 a necessary and sufficient condition for J to be harmonic is the symmetry of Ric^*, which by Lemma 6.1 yields the following result.

THEOREM 6.2. *Let (M, g, J) be the tangent bundle of a Riemannian manifold (M', g'), equipped with the Sasaki almost-Kähler structure. Then J is harmonic if and only if (M', g') has harmonic curvature. In particular, if $k = 2$ then J is harmonic if and only if (M', g') has constant curvature.*

What about stability? The following result shows that the hypotheses of our stability Theorem 4.4 are rather strong in this instance. We recall that the Sasaki structure is Kähler precisely when (M', g') is flat [8], Section 5, Corollary.

PROPOSITION 6.3. *If the Sasaki almost-Kähler structure J is $*$Einstein then (M', g') is flat, and hence J is Kähler.*

Proof. It follows from Lemma 6.1 that for all $x \in M$

$$s^*(x) = s'\big(\tau'(x)\big) + \frac{1}{4}\sum_j |R'(x, e_j)|^2. \tag{6.2}$$

When J is $*$Einstein Lemma 6.1(a) implies that for all $y, z \in M_x$

$$\frac{1}{n}s^*(x)\, g'(y,z) = \tfrac{1}{2}Ric'(y,z) + \tfrac{1}{8}g'\big(R'(x,y), R'(x,z)\big)$$

from which

$$kRic'(x,x) = s^*(x)g'(x,x).$$

Therefore s^* is constant on M_x, and hence g' is an Einstein metric with $s' \circ \tau' = s^*$. It then follows from (6.2) that $|R'| \equiv 0$. □

Remark. Proposition 6.3 remains true with (1,1)-Einstein in place of $*$Einstein, and therefore in 4-dimensions is consistent with Corollary 3.4.

Since Thurston's M^4 is cohomologically non-Kähler, and the Abbena–Thurston J is *not* $*$Einstein, it is tempting to make the following '$*$Goldberg' conjecture (in [12] it was conjectured that every compact almost-Kähler Einstein manifold is Kähler).

Conjecture 6.4. Every (compact?) $*$Einstein almost-Kähler structure (with non-positive $*$scalar curvature?) is necessarily Kähler.

We now examine stability in the 4-dimensional case; ie when (M', g') is a surface of constant curvature, say c. It follows from (6.2) that ϕ is a.s.d. only when g' is flat, so the hypotheses of our stability Theorem 5.3 are also very strong. Suspicions of instability are confirmed by our next theorem, which requires more precision in the endomorphism field $L_{\alpha,t}$ (see Lemma 4.3) than hitherto, and hence the following:

LEMMA 6.5. *Let M be the tangent bundle of a surface of constant curvature c. Let $\alpha \in \mathfrak{M}_\sigma$ where σ parametrizes the Sasaki structure. At each point $x \in M$ we have*

$$\operatorname{Tr} L_{\alpha,t} = -\tfrac{1}{8} p_c(r) \operatorname{Tr} \sinh^2 t\alpha, \quad \textit{where } r = |x| \textit{ and } p_c(r) = c(4 + cr^2).$$

Proof. Recall the definition (Lemma 4.3):

$$L_{\alpha,t} = J \circ \Phi \circ \sinh^2 t\alpha.$$

Let (u, v) be a positively-oriented orthonormal basis of the vector space M_x. If $x \neq 0$ take $u = x/r$; then by Lemma 6.1 the matrix of Ric^* (hence $-J\Phi$) with respect to the unitary basis $\big(u^{\mathcal{K}}(x), v^{\mathcal{K}}(x), u^{\mathcal{W}}(x), v^{\mathcal{W}}(x)\big)$ of $T_x M$ is diagonal:

$$\tfrac{1}{4} c \operatorname{diag}(2, 2 + cr^2, 2, 2 + cr^2).$$

If $x = 0$ then the matrix of Ric^* with respect to any unitary frame is $c/2$ times the identity, and the same conclusion holds. Since $\sinh t\alpha$ is a skew-symmetric endomorphism which anti-commutes with J, its matrix with respect to any unitary frame is of the form

$$\begin{pmatrix} A & B \\ B & -A \end{pmatrix}$$

where A and B are skew-symmetric 2×2 matrices. Therefore $A^2 + B^2 = -a^2 \mathbb{I}_2$ for some $a \neq 0$, where necessarily $-4a^2$ is the trace of $\sinh^2 t\alpha$, and the matrix of $L_{\alpha,t}$ at x looks like

$$\frac{1}{4} ca^2 \begin{pmatrix} \Lambda & * \\ -* & \Lambda \end{pmatrix} \quad \text{where} \quad \Lambda = \begin{pmatrix} 2 & 0 \\ 0 & 2 + cr^2 \end{pmatrix}.$$

It follows that

$$\operatorname{Tr} L_{\alpha,t} = \tfrac{1}{2} p_c(r) a^2 = -\tfrac{1}{8} p_c(r) \operatorname{Tr} \sinh^2 t\alpha.$$

□

THEOREM 6.6. *The Sasaki almost-Kähler structure on the tangent bundle of an orientable surface (M', g') is unstable harmonic, unless g' is flat.*

Proof. Suppose $c \neq 0$. We assume M' is compact; otherwise our argument may be applied to a compact chunk $C' \subset M'$. We work with 1-parameter variations $\sigma_t = e^{t\alpha}\sigma$ where $\alpha \in \mathfrak{M}_\sigma$, and construct a subspace $\mathfrak{M}' \subset \mathfrak{M}_\sigma$ with the following property: for every $\alpha \in \mathfrak{M}'$ and $\varepsilon > 0$ there exists a real number $\rho = \rho(\alpha, \varepsilon) > 0$ such that if $R \geqslant \rho$ then

$$E(\sigma_t; T_R) - E(\sigma; T_R) < 0$$

whenever $0 < |t| < \varepsilon$, where T_R is the tubular neighbourhood of the zero section, of radius R. Then, multiplication of α by a smooth bump function:

$$f: M \to \mathbb{R}^+; \; f(x) = \begin{cases} 1, & \text{if } x \in T_\rho \\ 0, & \text{if } x \in M \backslash T_{\rho+\delta} \end{cases}$$

for sufficiently small $\delta > 0$, yields a compactly-supported energy-decreasing variation of σ. (In the non-compact case, we would also want $\operatorname{supp}(f) \subset TC'$.)

Since $h^v(\sigma) = 0$ by Lemma 2.7, and J satisfies the hypotheses of Lemma 4.3, it follows that

$$e(\sigma_t) - e(\sigma) = 2h^v(\sigma_t) - \left(k^v(\sigma_t) - k^v(\sigma)\right) = 2h^v(\sigma_t) - \operatorname{Tr} L_{\alpha,t}.$$

Let β be a nowhere-vanishing 2-form on M', viewed as a skew-symmetric endomorphism field, and lift β to the following tensor α on M:

$$\alpha(y^{\mathcal{K}}) = \beta(y)^{\mathcal{W}}, \qquad \alpha(y^{\mathcal{W}}) = \beta(y)^{\mathcal{K}}.$$

It follows from (6.1) that α is skew-symmetric and anti-commutes with J, thus $\alpha \in \mathfrak{M}_\sigma$. This defines $\mathfrak{M}'$. For any $\alpha \in \mathfrak{M}'$ define the following smooth non-negative functions on $M' \times \mathbb{R}$:

$$\lambda(x', t) = -\operatorname{Tr} \sinh^2 t\alpha(x), \qquad \mu(x', t) = 2h^v(\sigma_t)(x),$$

where $x \in M$ is any point with $\tau'(x) = x' \in M'$ and $\sigma_t = \mathrm{e}^{t\alpha}$. It follows from Lemma 6.5 that at x we have

$$\operatorname{Tr} L_{\alpha,t} = \tfrac{1}{8} p_c(r) \lambda(x', t).$$

Notice that $\lambda(x', t) = 0$ if and only if $t = 0$, since $\alpha(x) \neq 0$. Therefore μ/λ is defined and continuous on $M' \times \mathbb{R} \backslash \{0\}$. Furthermore,

$$\mu(x', 0) = \frac{\partial \mu}{\partial t}(x', 0) = 0 = \lambda(x', 0) = \frac{\partial \lambda}{\partial t}(x', 0)$$

but

$$\frac{\partial^2 \lambda}{\partial t^2}(x', 0) = -2\operatorname{Tr}\left(\alpha(x)^2\right) = -4\operatorname{Tr}\left(\beta(x')^2\right) > 0, \quad \text{since } \beta(x') \neq 0.$$

Thus μ/λ extends continuously to $M' \times \mathbb{R}$. Let $B = B(\varepsilon)$ be an upper bound on $M' \times [-\varepsilon, \varepsilon]$. Then $\mu(x', t) < B\lambda(x', t)$ whenever $0 < |t| < \varepsilon$, and integration along the fibres of τ' yields:

$$2H^v(\sigma_t; T_R) = \pi R^2 \int_{M'} \mu(x', t) \mathrm{d}x' < \pi R^2 B \int_{M'} \lambda(x', t) \mathrm{d}x'$$

and

$$K^v(\sigma_t;T_R) - K^v(\sigma;T_R) = q_c(R)\int_{M'}\lambda(x',t)\mathrm{d}x'$$

where $8q_c(r) = \pi r^2 c(4 + cr^2/2)$. Therefore

$$E(\sigma_t;T_R) - E(\sigma;T_R) < \left(\pi R^2 B - q_c(R)\right)\int_{M'}\lambda(x',t)\mathrm{d}x'$$

and the result is achieved by choosing $\rho^2 > 8(2B - c)/c^2$. □

7. Calabi–Eckmann structures

It was shown in Section 3 that the complex structure of a Hopf manifold is harmonic. We now extend that result to the Calabi–Eckmann manifolds $M = M_1 \times M_2$ where $M_1 = S^{2p+1} \subset \mathbb{C}^{p+1}$ and $M_2 = S^{2q+1} \subset \mathbb{C}^{q+1}$ are unit spheres with their standard metrics g_1 and g_2 (altering relative diameters will not affect our results). The complex structure J on M may be constructed as follows [17, 27]. If ν_i is the unit outward-pointing normal field on M_i, and J_1, J_2 are the standard complex structures on $\mathbb{C}^{p+1}, \mathbb{C}^{q+1}$, respectively, then

$$TM_i = \langle J_i\nu_i\rangle \bigoplus \mathcal{U}_i \tag{7.1}$$

where $\langle J_i\nu_i\rangle$ is the (real) line subbundle generated by the Hopf vector field $J_i\nu_i$, and $\mathcal{U}_i$ is a J_i-invariant subbundle. Accordingly, every $X \in TM$ may be decomposed

$$X = X_a + a(X)J_1\nu_1 + X_b + b(X)J_2\nu_2$$

where $X_a \in \mathcal{U}_1$, $X_b \in \mathcal{U}_2$ and $a, b\colon TM \to \mathbb{R}$ are smooth functions. Then

$$JX = J_1X_a - b(X)J_1\nu_1 + J_2X_b + a(X)J_2\nu_2 \tag{7.2}$$

defines an almost-complex structure which is orthogonal for $g = g_1 \times g_2$, and integrable. Apart from the torus, J is never cosymplectic; and $J \in \mathfrak{W}_4$ if and only if $p = 0$ or $q = 0$. These facts may be deduced from the following formula:

$$\begin{aligned}\nabla_X J(Y) \;=\; &\left(a(Y)X_a - b(Y)JX_a\right) - \left(g(X_a, Y_a) + g(JX_b, Y_b)\right)J_1\nu_1 + \\ &+ \left(b(Y)X_b - a(Y)JX_b\right) + \left(g(JX_a, Y_a) - g(X_b, Y_b)\right)J_2\nu_2.\end{aligned} \tag{7.3}$$

If we split $X = X_1 + X_2 \in TM_1 \oplus TM_2$ then the curvature tensor of (M, g) is

$$R(X,Y)Z = \sum_{i=1,2}\left(g_i(Z_i, Y_i)X_i - g_i(Z_i, X_i)Y_i\right). \tag{7.4}$$

THEOREM 7.1. *If (M, g, J) is a Calabi–Eckmann manifold, then J is harmonic.*

Proof. It follows from (7.3) that

$$\delta J = 2pJ_1\nu_1 + 2qJ_2\nu_2.$$

So by (7.3) again, δJ is Kähler null. Moreover, by (7.4) Ric^* is symmetric:

$$Ric^*(X,Y) = g(X_a,Y_a) + g(X_b,Y_b). \tag{7.5}$$

By (R4) of Section 3 the Ricci form is reducible, and the result follows from Theorem 2.8(3).

The vanishing of $\tau(J)$ is examined more closely in the proof of Lemma 7.4 below. It follows from (7.5) that no Calabi–Eckmann manifold is $*$Einstein, except $M = S^1 \times S^1$ in which case $Ric^* = 0$ (cf. Proposition 6.3). Neither does the 4-dimensional Hopf manifold have a.s.d. Ricci form. To pursue the stability question, we look at the second variation.

PROPOSITION 7.2 ([31], Proposition 2). *If σ is a harmonic section of a Riemannian submersion $\pi\colon N \to M$ with t.g. fibres, and $\sigma_{s,t}$ is a 2-parameter variation through sections, then*

$$\left.\frac{\partial^2}{\partial s\partial t}\right|_{(0,0)} E^v(\sigma_{s,t}) = \int_M h\big(\delta^v d^v U - Ric^v_\sigma(U), W\big)dx$$

where

$$U = \left.\frac{\partial \sigma_{s,t}}{\partial s}\right|_{(0,0)}, \qquad W = \left.\frac{\partial \sigma_{s,t}}{\partial t}\right|_{(0,0)} \quad \text{and} \quad Ric^v_\sigma(U) = \operatorname{Tr} R^N(U, d^v\sigma)d^v\sigma.$$

As in [31] the *vertical Jacobi operator* Jac^v_σ of σ is defined on sections U of $\sigma^*\mathcal{V}$ as follows:

$$Jac^v_\sigma(U) = \delta^v d^v U - Ric^v_\sigma(U).$$

Henceforward we restrict attention to the subclass of $\mathcal{C}(\sigma^*\mathcal{V})$ comprising variation fields $U = d^v\sigma(u)$ where u is a vector field on M. If we define

$$T_\sigma(u) = \nabla^v d^v\sigma(E_j, \nabla_{E_j} u) \tag{7.6}$$

then a calculation yields

$$\delta^v d^v U = -d^v\sigma(\operatorname{Tr}\nabla^2 u) + \nabla^*\nabla d^v\sigma(u) + 2T_\sigma(u)$$

where

$$\nabla^*\nabla d^v\sigma = -\operatorname{Tr}(\nabla^v)^2 d^v\sigma.$$

This rough Laplacian may be re-expressed by application of a Weitzenböck formula (see [9], (1.34)) to the $\sigma^*\mathcal{V}$-valued 1-form $d^v\sigma$:

$$\nabla^*\nabla d^v\sigma = \delta^v d^v d^v\sigma - d^v\big(\tau^v(\sigma)\big) + Ric^v_\sigma \circ d^v\sigma - d^v\sigma \circ Ric$$

where Ric is the Ricci tensor of M. It follows that if σ is a harmonic section then

$$Jac^v_\sigma(U) = \delta^v \mathrm{d}^v \mathrm{d}^v \sigma(u) + 2T_\sigma(u) - \mathrm{d}^v\sigma\big(\mathrm{Tr}\,\nabla^2 u + Ric(u)\big). \tag{7.7}$$

We now specialize to the Calabi–Eckmann manifolds, and write $U = U_1 + U_2$ where $U_i = \mathrm{d}^v\sigma(u_i)$ and u_i is a vector field on M_i. As in [31], usefulness of the vertical Jacobi operator depends on a pleasant realization of the unpleasant-looking term $\delta^v \mathrm{d}^v \mathrm{d}^v \sigma$.

LEMMA 7.3. *For all $u \in TM$ we have $\delta^v \mathrm{d}^v \mathrm{d}^v \sigma(u) = pU_1 + qU_2$.*

Proof. For any almost-Hermitian manifold it was shown in [31] Proposition 4 that

$$\iota \circ \delta^v \mathrm{d}^v \mathrm{d}^v \sigma(X) = -\tfrac{1}{2} J[J, \delta R(X)] - \tfrac{1}{4}\{\nabla_{E_j} J, \{J, R(E_j, X)\}\}$$

where $\{,\}$ is the anticommutator bracket. Since (M, g) is a Riemannian product of space forms it has harmonic curvature (see [3], Chapter 16). Evaluation of the anticommutators using (7.3) and (7.4) yields:

$$\begin{aligned} &2\big(\iota \circ \delta^v \mathrm{d}^v \mathrm{d}^v \sigma(u)\big) w \\ &\quad = p\big(b(w)u_a + a(w)Ju_a\big) + \big(pg(u_a, Jw_a) + qg(u_b, w_b)\big) J_1\nu_1 - \\ &\qquad - q\big(a(w)u_b - b(w)Ju_b\big) - \big(pg(u_a, w_a) - qg(u_b, Jw_b)\big) J_2\nu_2 \\ &\quad = pJ \circ \nabla_{u_1} J(w) + qJ \circ \nabla_{u_2} J(w) \quad \text{by (7.3).} \end{aligned}$$

The result now follows from Proposition 2.2. □

LEMMA 7.4. *If u is a sum of gradient fields then*

$$Jac^v_\sigma(U) = (1-p)U_1 + (1-q)U_2.$$

Proof. Suppose $u_i = \mathrm{grad}\, f_i$ where the f_i are the restrictions to M_i of linear 1-forms on $\mathbb{R}^{2p+2}$ and $\mathbb{R}^{2q+2}$ respectively, and the gradient is that of M_i. Then by [32]:

$$\nabla_X u = -f_1 X_1 - f_2 X_2. \tag{7.8}$$

Define $\tau_i(J)$ to be the image under ι of the trace of $\nabla^v \mathrm{d}^v \sigma$ over M_i (cf. (2.5) and (1.3)). It then follows from (7.6) and (7.8) that

$$\iota \circ T_\sigma(u) = -f_1\tau_1(J) - f_2\tau_2(J).$$

Although we know that $\tau(J)$ vanishes, it is not clear that the same is true of $\tau_i(J)$. A modification of the chain of argument in Section 2, using the facts that each $J_i\nu_i$ is Kähler null by (7.3), and $R(J_1\nu_1, J_2\nu_2)$ vanishes by (7.4), shows (cf. Theorem 2.8) that

$$\tau_i(J) = \tfrac{1}{2}\big([J, \Phi_i] - \nabla_{\delta_i J} J\big)$$

where ϕ_i (respectively $\delta_i J$) are the partial Ricci forms (respectively partial coderivatives) obtained by tracing over $\mathcal{U}_i$. These may be computed from (7.4) and (7.3) respectively; for example

$$\Phi_1(u) = Ju_a, \qquad \delta_1 J = 2pJ_1\nu_1.$$

Then (7.2) implies $[J, \Phi_i] = 0$, and (7.3) implies that $\delta_i J$ is Kähler null; so $\tau_i(J) = 0$. The remaining terms in (7.7) may be computed from (7.4) and (7.8), respectively:

$$Ric(u) = 2pu_1 + 2qu_2, \qquad -\mathrm{Tr}\,\nabla^2 u = u_1 + u_2.$$

The result now follows from Lemma 7.3. □

By the *index* $\mathrm{i}(J)$ of a harmonic almost-Hermitian structure J we mean the vertical energy index (that is, with respect to vertical variations) at the parametrizing section σ; the *nullity* of J is the dimension of the kernel of Jac^v_σ. As in [25], it is convenient to define a *reduced nullity* $\mathrm{rn}(J)$ by ignoring variations of J generated by isometries of M.

THEOREM 7.5. *If σ parametrizes the Calabi–Eckmann structure J, and σ_t is a deformation of σ with variation field $U = \mathrm{d}^v\sigma(u)$ where u is a sum of gradient fields, then*

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\bigg|_{t=0} E^v(\sigma_t) = (1-p)\int_M |U_1|^2 \mathrm{d}x + (1-q)\int_M |U_2|^2 \mathrm{d}x.$$

(i) *If $p, q > 1$ then $\mathrm{i}(J) \geqslant 2p + 2q + 4$,*
(ii) *If $p \leqslant 1$ and $q > 1$ then $\mathrm{i}(J) \geqslant 2q + 2$,*
(iii) *If $p, q \leqslant 1$ then $\mathrm{rn}(J) \geqslant 4p + 4q$.*

Proof. The formula for the second variation follows from Proposition 7.2 and Lemma 7.4, after observing that U_1 and U_2 are orthogonal. This observation comes from Proposition 2.2:

$$\iota U_i = \frac{1}{2} J \circ \nabla_{u_i} J$$

and Equation (7.3). The estimates of index and nullity require some care, because by (7.3) the vertical derivative drops rank:

$$\ker \mathrm{d}^v\sigma = \langle J_1\nu_1 \rangle \oplus \langle J_2\nu_2 \rangle.$$

Let $\{f_{1j} : 1 \leqslant j \leqslant 2p + 2\}$ and $\{f_{2k} : 1 \leqslant k \leqslant 2q + 2\}$ be bases of the dual vector spaces $(\mathbb{R}^{2p+2})^*$ and $(\mathbb{R}^{2q+2})^*$ respectively, and suppose there is a dependence relation between the U_{1j}, U_{2k} i.e. there exist linear combinations λ_1 of the u_{1j} and λ_2 of the u_{2k} such that for all $x \in M$

$$\mathrm{d}^v\sigma\big(\lambda_1(x) + \lambda_2(x)\big) = 0 \quad \text{and hence} \quad \lambda_1(x)_a = 0 = \lambda_2(x)_b.$$

But non-trivial gradient fields on odd-dimensional spheres (apart from the circle) have Hopf-horizontal component which is not identically zero. Thus if $p, q > 0$ then $\lambda_i = 0$, which by linear independence of the u_{1j} and u_{2k} implies that the dependence relation is trivial. Therefore the $2p + 2q + 4$ $(p, q > 0)$ variation fields U_{1j}, U_{2k} are linearly independent. □

Theorem 7.5 yields no information on the index of the Calabi–Eckmann structures on $S^1 \times S^3$ and $S^3 \times S^3$, and the question of stability in these cases remains open.

Appendix. Some differential geometric properties of the twistor fibration

For notation and terminology, we refer to the bundle machinery described in Section 4. The vector bundle map $\kappa: TN \to \pi^*\mathcal{E}$ may be constructed as follows. Let φ be the $\mathfrak{g}$-valued Levi–Civita connection 1-form (for g) on $P = SO(M)$. The component $\varphi_{\mathfrak{m}}$ is H-equivariant, vanishes on $\ker d\zeta$, and the restriction $\varphi_{\mathfrak{m}} | \ker d\xi$ is the $\mathfrak{m}$-component of the Maurer–Cartan form of G. Therefore $\varphi_{\mathfrak{m}}$ projects to an $\mathcal{E}_{\mathfrak{m}}$-valued 1-form on N, with kernel $\mathcal{H}$, whose restriction to $\mathcal{V}$ is a connection-independent vector bundle isomorphism. This is precisely κ.

Proof of Lemma 2.1. The pullback vector bundle $\pi^*\mathcal{E} \to N$ is associated to the principal G-bundle $\pi^*P \to N$, with fibre $\mathfrak{g}$. Let $\tilde{\mathcal{J}}: \pi^*P \to \mathfrak{g}$ denote the G-equivariant lift of the section $\mathcal{J}$ of $\pi^*\mathcal{E}$. Then $\zeta: P \to N$ is a principal H-subbundle of $\pi^*P \to N$, and $\tilde{\mathcal{J}}|P = J_o$. Let D denote the exterior covariant derivative for φ:

$$\mathrm{D}\tilde{\mathcal{J}} = \mathrm{d}\tilde{\mathcal{J}} + [\varphi, \tilde{\mathcal{J}}].$$

If $\tilde{A} \in TP$ is any lift of $A \in TN$ then

$$\mathrm{D}\tilde{\mathcal{J}}(\tilde{A}) = \mathrm{d}\tilde{\mathcal{J}}(\tilde{A}) + [\varphi(\tilde{A}), \tilde{\mathcal{J}}] = [\varphi_{\mathfrak{m}}(\tilde{A}), \tilde{\mathcal{J}}] = -2\tilde{\mathcal{J}}.\varphi_{\mathfrak{m}}(\tilde{A})$$

since $\tilde{\mathcal{J}}|P = J_o$. When factored through ζ this becomes $\nabla_A \mathcal{J} = -2\mathcal{J} \circ \kappa(A)$, and the result follows since $\mathcal{J}^2 = -1$. □

Concerning the vector bundle $\mathcal{E}_{\mathfrak{m}} \to N$, let $\langle , \rangle$ denote the usual G-invariant metric on $\mathfrak{g}$:

$$\langle a, b \rangle = \mathrm{Tr}\,(ab^t), \qquad \forall a, b \in \mathfrak{g},$$

and also the metric induced on the fibres of $\mathcal{E}_{\mathfrak{m}}$ by its restriction to $\mathfrak{m}$. Then h is characterized as follows:

$$h(A, B) = g(\pi_* A, \pi_* B) + \langle \kappa A, \kappa B \rangle, \quad \forall A, B \in C(TN). \tag{A.1}$$

Furthermore, since the component $\varphi_{\mathfrak{h}}$ is a connection 1-form for the H-bundle ζ, there is an associated connection ∇^c in $\mathcal{E}_{\mathfrak{m}}$, which it is natural to call the *canonical connection.* The triple $(\mathcal{E}_{\mathfrak{m}}, \langle , \rangle, \nabla^c)$ is a Riemannian vector bundle:

$$A.\langle \alpha, \beta \rangle = \langle \nabla^c_A \alpha, \beta \rangle + \langle \alpha, \nabla^c_A \beta \rangle, \quad \forall \alpha, \beta \in C(\mathcal{E}_{\mathfrak{m}}). \tag{A.2}$$

Finally, we define an $\mathcal{E}_{\mathfrak{m}}$-valued 2-form T^c on N as follows:

$$T^c(A, B) = \nabla^c_A(\kappa B) - \nabla^c_B(\kappa A) - \kappa[A, B]. \tag{A.3}$$

PROPOSITION. *The canonical connection* ∇^c *and torsion* T^c *are:*
(a) $\nabla^c_A \alpha = -\frac{1}{2}\mathcal{J}[\mathcal{J}, \nabla_A \alpha]$,

(b) $T^c(A,B) = -\frac{1}{2}\mathcal{J}[\mathcal{J}, R(\pi_* A, \pi_* B)]$
where ∇ is the Levi–Civita connection, pulled-back to $\pi^*\mathcal{E}$, and R is the Riemann tensor.

Proof. We first observe that if $\{,\}$ is the anticommutator in $\pi^*\mathcal{E}$, then the following identity gives the decomposition of $\beta \in \pi^*\mathcal{E}$ into its $\mathcal{E}_{\mathfrak{h}}$- and $\mathcal{E}_{\mathfrak{m}}$-components:

$$\beta = -\frac{1}{2}\mathcal{J}\{\mathcal{J}, \beta\} - \tfrac{1}{2}\mathcal{J}[\mathcal{J}, \beta]. \tag{A.4}$$

(a) Let $\tilde{\alpha}: \pi^*P \to \mathfrak{g}$ be the G-equivariant lift of α. Let D^c denote the exterior covariant derivative for the canonical connection. Then

$$\mathrm{D}^c\tilde{\alpha} = \mathrm{d}\tilde{\alpha} + [\varphi_{\mathfrak{h}}, \tilde{\alpha}] + [\tilde{\alpha}, \varphi_{\mathfrak{h}}], \qquad \mathrm{D}\tilde{\alpha} = \mathrm{d}\tilde{\alpha} + [\varphi, \tilde{\alpha}] + [\tilde{\alpha}, \varphi]$$

where D is the Levi–Civita exterior covariant derivative, and so

$$\mathrm{D}\tilde{\alpha} = \mathrm{D}^c\tilde{\alpha} + [\varphi_{\mathfrak{m}}, \tilde{\alpha}] + [\tilde{\alpha}, \varphi_{\mathfrak{m}}].$$

It follows that if $\tilde{A} \in TP$ is any lift of A then

$$\mathrm{D}\tilde{\alpha}(\tilde{A})_{\mathfrak{m}} = \mathrm{D}^c\tilde{\alpha}(\tilde{A})$$

because $\tilde{\alpha}|P$ is $\mathfrak{m}$-valued, and $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$. Therefore ∇^c is the $\mathcal{E}_{\mathfrak{m}}$-component of ∇, the expression for which follows from (A.4).

(b) We have $T^c = \mathrm{d}^c\kappa$ where d^c is the ∇^c exterior derivative for $\mathcal{E}_{\mathfrak{m}}$-valued differential forms. Therefore, since κ is the projection to N of $\varphi_{\mathfrak{m}}$, T^c is the projection of the horizontal component of $\mathrm{d}\varphi_{\mathfrak{m}}$. The $\mathfrak{m}$-component of the Structure Equation is

$$\mathrm{d}\varphi_{\mathfrak{m}} = \Omega_{\mathfrak{m}} - [\varphi, \varphi]_{\mathfrak{m}},$$

where Ω is the Levi–Civita curvature 2-form. Now $\Omega_{\mathfrak{m}}$ is horizontal, and since $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$ the horizontal component of $[\varphi, \varphi]_{\mathfrak{m}}$ vanishes. Therefore T^c is the $\mathcal{E}_{\mathfrak{m}}$-component of the $\pi^*\mathcal{E}$-valued 2-form π^*R, the expression for which follows from (A.4). □

Proof of Lemma 2.3. Suppose $A \in T_yN$, and $B \in \mathcal{C}(TN)$. If $C \in \mathcal{C}(\mathcal{V})$ then

$$\langle \kappa(\nabla_A B), \kappa C\rangle = h(\nabla_A B, C).$$

If A is extended to a local vector field, then using the well-known characterization of the Levi–Civita connection [18], Chapter 4, Proposition 2.3, it follows from (A.1) that

$$\begin{aligned} 2\langle \kappa(\nabla_A B), \kappa C\rangle - \;=\; & A\langle \kappa B, \kappa C\rangle + B\langle \kappa A, \kappa C\rangle - C\langle \kappa A, \kappa B\rangle - \\ & - \langle \kappa A, \kappa[B,C]\rangle - \langle \kappa B, \kappa[A,C]\rangle + \langle \kappa C, \kappa[A,B]\rangle - \\ & - Cg(\pi_* A, \pi_* B) - g(\pi_* A, \pi_*[B,C]) - g(\pi_* B, \pi_*[A,C]). \end{aligned}$$

We claim that each of the three terms involving g vanishes. This is clearly so if either A or B is vertical. If A and B are both horizontal, then since $\nabla_A B$ depends only on the values of B on a slice through y transverse to the fibres of π, by redefining B if necessary we may assume that both vector fields are *basic* (see [21]). The claim then follows from the fact that C is π-adapted to the zero vector field on M.

The remaining terms may be expanded using Liebniz rule (A.2) along with identity (A.3):

$$\begin{aligned} 2\langle \kappa(\nabla_A B), \kappa C\rangle &= \langle 2\nabla^c_A(\kappa B) - T^c(A,B), \kappa C\rangle + \\ &+ \langle T^c(A,C), \kappa B\rangle + \langle T^c(B,C), \kappa A\rangle. \end{aligned}$$

Since C is vertical, part (b) of the Proposition implies $T^c(A,C)$ and $T^c(B,C)$ both vanish. Application of the Proposition to the remaining terms completes the proof. □

The section $\mathcal{J}$ of $\pi^*\mathcal{E}$ restricts to an almost-complex structure in the fibres of $\mathcal{E}_{\mathfrak{m}}$; indeed, it is the almost-complex structure induced by the $Ad(H)$-equivariant linear endomorphism $I: \mathfrak{m} \to \mathfrak{m}; I(a) = J_o a$. Since I endows the symmetric space G/H with its canonical Kähler structure, characterization (2.2) of J^v is immediate. Furthermore, $\nabla^c \mathcal{J} = 0$.

Proof of Lemma 2.4. If $A \in TN$ and $V \in \mathcal{C}(\mathcal{V})$ then

$$\nabla^v_A J^v(V) = \nabla^v_A(J^v V) - J^v(\nabla^v_A V).$$

From Lemma 2.3 and part (a) of the Proposition it follows that

$$\begin{aligned} \iota(\nabla^v_A J^v(V)) &= \nabla^c_A\big(\iota J^v(V)\big) - \iota J^v(\nabla^v_A V) \\ &\underset{(2.2)}{=} \nabla^c_A\big(\mathcal{J}(\iota V)\big) - \mathcal{J}\big(\nabla^c_A(\iota V)\big) = \nabla^c_A \mathcal{J}(\iota V) = 0. \end{aligned}$$

□

Proof of Lemma 4.1. Define $V\big(\sigma(x)\big) = \hat{\alpha}(\sigma)(x)$. Then $V = d\zeta(\tilde{V})$ where

$$\tilde{V}(q) = \frac{d}{dt}\bigg|_{t=0} e^{t\alpha} q \cdot \underset{(4.1)}{=} \frac{d}{dt}\bigg|_{t=0} q\, e^{t\alpha(q)} \quad \forall q \in Q.$$

Therefore $\varphi_{\mathfrak{m}}\big(\tilde{V}(q)\big) = \alpha(q)_{\mathfrak{m}}$ since φ is a connection form. When factored through ζ this says that ιV is the $\mathcal{E}_{\mathfrak{m}}$-component of the section α of $\pi^*\mathcal{E}$, the expression for which follows from (A.4). □

Finally, we justify our expression (0.3) for the Chern form γ. The pullback by σ of the canonical connection $\varphi_{\mathfrak{h}}$ in ζ is a connection in the principal H-bundle $Q \to M$ of J-unitary frames. Denote by Ω^c its curvature form – an $\mathfrak{h}$-valued 2-form on Q. Comparison of the structure equations for φ and $\varphi_{\mathfrak{h}}$ yields the following relation on TQ:

$$\Omega^c = \Omega - \Omega_{\mathfrak{m}} - [\varphi_{\mathfrak{m}}, \varphi_{\mathfrak{m}}].$$

By Chern–Weil theory [7], the multiple $-2\pi i c_1(M)$ is represented by the push-down to M of the unitary trace of Ω^c, which since we are working with the real form of $U(k)$ is

$$-\tfrac{i}{2}\,\mathrm{Tr}\,(J_o \Omega^c).$$

Now the unitary trace of any element of $\mathfrak{m}$ (in particular $\Omega_{\mathfrak{m}}$) vanishes, and by the proof of Lemma 2.1 the push-down of $2\varphi_{\mathfrak{m}}$ is $J\nabla J$. Therefore $4\pi c_1(M)$ is represented by the following difference of 2-forms:

$$\mathrm{Tr}\,(J \circ R) - \tfrac{1}{4}\mathrm{Tr}\,J[J\nabla J, J\nabla J].$$

By (2.3) the first 2-form is 2ϕ, and the second is easily shown to be -2χ.

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