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## The Schwartz space of a general semisimple Lie group IV: elementary mixed wave packets

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### 1. Introduction

Suppose  $G$  is a connected semisimple Lie group. Then the tempered spectrum of  $G$  consists of families of representations induced unitarily from cuspidal parabolic subgroups. Each family is parameterized by the unitary characters of a Cartan subgroup. The Plancherel theorem expands Schwartz class functions on  $G$  in terms of the distribution characters of these tempered representations. Very roughly, for  $f$  in the Schwartz space  $\mathcal{C}(G)$ , we can write

$$f(x) = \sum_{H \in \text{Car}(G)} f_H(x), \quad x \in G \quad (1.1a)$$

where  $\text{Car}(G)$  denotes a complete set of representatives for conjugacy classes of Cartan subgroups of  $G$  and

$$f_H(x) = \int_{\hat{H}} \Theta(H : \chi)(R(x)f)m(H : \chi) d\chi. \quad (1.1b)$$

Here  $\Theta(H : \chi)$  denotes the distribution character of the representation  $\pi(H : \chi)$  corresponding to  $\chi \in \hat{H}$ ,  $R(x)f$  is the right translate of  $f$  by  $x \in G$ , and  $m(H : \chi) d\chi$  is the Plancherel measure corresponding to  $\pi(H : \chi)$ .

Suppose that  $G$  has finite center and that  $f \in \mathcal{C}(G)$  is  $K$ -finite where  $K$  is a maximal compact subgroup of  $G$ . Fix  $H \in \text{Car}(G)$ . In [HC2, 3, 4] Harish-Chandra used Eisenstein integrals to construct wave packets of matrix coefficients of the representations  $\pi(H : \chi)$ ,  $\chi \in \hat{H}$ . He showed that these wave packets are Schwartz class functions and that  $f_H$  is a finite sum of wave packets. In particular, this shows that  $f_H \in \mathcal{C}(G)$ .

Now suppose that  $G$  has infinite center  $Z_G$ . (For example,  $G$  could be the

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universal covering group of one of the non-compact simple Lie groups of hermitian type.) Let  $K$  be a maximal relatively compact subgroup. That is,  $Z_G \subseteq K$  and  $K/Z_G$  is a maximal compact subgroup of  $G/Z_G$ . Then there are no  $K$ -finite functions in  $\mathcal{C}(G)$ . However the set  $\mathcal{C}(G)_K$  of  $K$ -compact functions, those with  $K$ -types lying in a compact subset of  $\hat{K}$  is dense in  $\mathcal{C}(G)$  [H1]. Let  $H \in \text{Car}(G)$ . Then for every  $f \in \mathcal{C}(G)_K$ ,  $f_H$  again decomposes naturally as a finite sum of wave packets. A new feature of the infinite center case is that for  $f \in \mathcal{C}(G)_K$ ,  $f_H$  and the wave packets into which it decomposes are not necessarily Schwartz class functions. This is because of interference between different series of representations when a principal series representation decomposes as a sum of limits of discrete series. When  $G$  has infinite center, these limits of discrete series can be actual limits along continuous families of relative discrete series representations, and so occur in a non-trivial way in the Plancherel formula in the terms corresponding to different Cartan subgroups. This means that for  $f \in \mathcal{C}(G)$  there are matching conditions between the terms  $f_H$ ,  $H \in \text{Car}(G)$ , which are necessary in order that the sum be a Schwartz class function when the individual terms are not. These matching conditions generalize those of H. Kraljević and D. Miličić for the universal covering group of  $\text{SL}(2, \mathbf{R})$  [KM].

The purpose of this paper is to define and study “elementary mixed wave packets.” These are finite sums of wave packets which patch together to form Schwartz class functions. They should be thought of as the basic building blocks from which Schwartz class functions are formed in the infinite center case. In this paper we first study in detail the identities relating the characters of different families of tempered representations. These identities are used to give the matching conditions which are the main feature in the definition of elementary mixed wave packets. We then show that every  $f \in \mathcal{C}(G)_K$  is a finite sum of elementary mixed wave packets. Finally, we show that elementary mixed wave packets satisfy a condition which is necessary for them to be Schwartz class. The asymptotic analysis required to complete the proof that they are Schwartz class is deferred to another paper. This paper is a continuation of the study of the Plancherel theorem and Schwartz space for general reductive Lie groups in [H1, 2] and [HW1–5].

In order to explain the results of the paper more precisely and with a minimum of technical notation, we will assume for the remainder of this introduction that  $G$  is a simple, simply connected, non-compact real Lie group of hermitian type. Let  $K$  be a maximal relatively compact subgroup of  $G$ . Then  $K = K_1 \times V$  where  $K_1 = [K, K]$  is compact and  $V \cong \mathbf{R}$  is a one-dimensional vector group in the center of  $K$ . Then  $\{e^h: h \in \mathfrak{iv}^*\}$  gives a one-parameter family of one-dimensional characters of  $K$ . Now let  $P = MAN$  be a cuspidal parabolic subgroup of  $G$  and  $H = TA$  a Cartan subgroup of  $G$  with  $T \subseteq K$  a maximal relatively compact Cartan subgroup of  $M$ . The characters  $e^h$ ,  $h \in \mathfrak{iv}^*$ , give

characters of  $T$  by restriction. Thus each  $\chi \in \widehat{T}$  lies in a continuous family of characters of  $T$  of the form  $\{\chi \otimes e^h : h \in i\nu^*\}$ . Each character in the family corresponds to a relative discrete series or limit of discrete series representation  $\pi(M : h)$  of  $M$ . Let  $\lambda(h) \in i\mathfrak{t}^*$  denote the Harish-Chandra parameter of  $\pi(M : h)$ , let  $\mathcal{C}$  be a Weyl chamber of  $i\mathfrak{t}^*$ , and let  $\mathcal{D} = \{h \in i\nu^* : \lambda(h) \in \mathcal{C}\}$ . Then  $\mathcal{D}$  is an open interval and is unbounded just in case the representations  $\pi(M : h)$ ,  $h \in \mathcal{D}$ , are holomorphic or anti-holomorphic relative discrete series. Now

$$\{\pi(H : h : \nu) = \text{Ind}_{MAN}^G(\pi(M : h) \otimes e^{i\nu} \otimes 1) : h \in \mathcal{D}, \nu \in \mathfrak{a}^*\}$$

is called a continuous family of representations of  $G$  corresponding to  $H$ .

Let  $\Phi_M$  denote the set of roots for  $(\mathfrak{m}_C, \mathfrak{t}_C)$  and choose a set  $\Phi_M^+$  of positive roots so that there is a unique non-compact simple root  $\beta$ . We will use  $h \leftrightarrow \langle \beta, h \rangle$  to identify  $i\nu^* \cong \mathbf{R}$ . Fix  $\chi \in \widehat{H}$  and let

$$F_0 = \{\alpha \in \Phi_M^+ : \langle \alpha, \lambda(0) \rangle = 0\}.$$

If there is a compact root  $\alpha \in F_0$ , then  $\langle \lambda(h), \alpha \rangle = \langle \lambda(0), \alpha \rangle = 0$  for all  $h \in i\nu^*$  so that the Plancherel function  $m(H : h : \nu)$  corresponding to  $\pi(H : h : \nu)$  is zero for all  $h \in i\nu^*$ ,  $\nu \in \mathfrak{a}^*$ . In this case the family plays no role in the Plancherel formula. Thus we assume that  $F_0$  contains no compact roots. Then  $\lambda(h)$  is regular for small  $h \neq 0$  and so there are Weyl chambers  $\mathcal{C}^\pm$  of  $i\mathfrak{t}^*$  so that  $\lambda(h) \in \mathcal{C}^+$  for small  $h > 0$  and  $\lambda(h) \in \mathcal{C}^-$  for small  $h < 0$ . (Of course if  $F_0 = \emptyset$ , then  $\lambda(0) \in \mathcal{C}^+ = \mathcal{C}^-$ .)

Now each  $F \subseteq F_0$  is a strongly orthogonal family of non-compact roots of  $M$  and so corresponds to Cartan subgroups  $H_{M,F}$  of  $M$  and  $H_F = H_{M,F}A = T_F A_F$  of  $G$ . We identify roots of  $H_F$  with those of  $H$  via the Cayley transform  $c_F$  corresponding to  $F$ . Let  $P_F = M_F A_F N_F$  be a cuspidal parabolic subgroup corresponding to  $H_F$ . Then for each  $F \subseteq F_0$ ,  $T_F \subseteq T$  and we define  $\chi_F \in \widehat{T}_F$  to be the restriction of  $\chi$ . Let  $\pi(F : h)$  be the relative discrete series representation of  $M_F$  corresponding to  $\chi_F \otimes e^h$  and define

$$\pi(F : h : \nu_F) = \text{Ind}_{M_F A_F N_F}^G(\pi(M_F : h) \otimes e^{i\nu_F} \otimes 1), \quad h \in i\nu^*, \quad \nu_F \in \mathfrak{a}_F^*. \tag{1.2}$$

Let  $\Theta(F : h : \nu_F)$  be the character of  $\pi(F : h : \nu_F)$ . In Theorem 3.11 we prove the following character identities relating the characters  $\Theta(F : h : \nu_F)$ . Fix  $E \subseteq F_0$ . For every  $E \subseteq F \subseteq F_0$ ,  $\mathfrak{a}_E \subseteq \mathfrak{a}_F$  and we can identify  $\mathfrak{a}_F^* \cong \mathfrak{a}_E^* \oplus \mathbf{R}^{|F \setminus E|}$  by  $\nu_F \leftrightarrow (\nu_E, (\mu_\alpha)_{\alpha \in F \setminus E})$  where  $\nu_E$  is the restriction of  $\nu_F$  to  $\mathfrak{a}_E$  and  $\mu_\alpha = \langle \nu_F, \alpha \rangle$ ,  $\alpha \in F \setminus E$ . Write  $(\nu_E, 0)$  for the element  $(\nu_E, (\mu_\alpha)_{\alpha \in F \setminus E})$  with  $\mu_\alpha = 0$  for all  $\alpha \in E \setminus F$  and define a differential operator on  $\mathfrak{a}_F^*$  by  $D_{F \setminus E} = i \sum_{\alpha \in F \setminus E} \partial / \partial \mu_\alpha$ . For  $F \subseteq F_0$ , let  $F^c = F_0 \setminus F$ .

**THEOREM 1.3.** Fix  $E \subseteq F_0$ . Then for all  $k \geq 0$ ,

$$\begin{aligned} & \lim_{h \downarrow 0} (\partial/\partial h)^k \Theta(E : h : v_E) + (-1)^{|E|+1} \lim_{h \uparrow 0} (\partial/\partial h)^k \Theta(E : h : v_E) \\ &= \sum_{E \subset F \subseteq F_0} c_{|F \setminus E|} \left[ \lim_{h \downarrow 0} (\partial/\partial h - D_{F \setminus E})^k \Theta(F : h : (v_E, 0)) \right. \\ & \quad \left. (-1)^{|F|} \lim_{h \uparrow 0} (\partial/\partial h - D_{F \setminus E})^k \Theta(F : h : (v_E, 0)) \right] \end{aligned}$$

for all  $v_E \in \mathfrak{a}_E^*$ . Here for all  $p \geq 0$ ,  $c_p = (d/dx)^p \tanh(x/2)|_{x=0}$ .

Note that  $\lambda_{F_0}(0)$  is regular so that  $\lim_{h \downarrow 0} \Theta(F_0 : h : v_{F_0}) = \lim_{h \uparrow 0} \Theta(F_0 : h : v_{F_0})$  for all  $v_{F_0} \in \mathfrak{a}_{F_0}^*$ . Further,  $c_1 = \frac{1}{2}$ , so that when  $|F_0 \setminus E| = 1$  and  $k = 0$ , (1.3) is just Schmid's identity [S]:

$$\lim_{h \downarrow 0} \Theta(E : h : v_E) + \lim_{h \uparrow 0} \Theta(E : h : v_E) = \Theta(F_0 : 0 : (v_E, 0))$$

for all  $v_E \in \mathfrak{a}_E^*$ .

Wave packets of Eisenstein integrals corresponding to a continuous family  $\{\pi(H : h : v) : h \in \mathcal{D}, v \in \mathfrak{a}^*\}$  are defined as follows. Fix  $\tau_1, \tau_2 \in \hat{K}$  with the same  $Z_G$  character as  $\chi$  and let  $W$  be a finite-dimensional complex vector space on which  $K$  acts on the left and right by  $(\tau_1, \tau_2)$ . For  $h \in iv^*$ , let  $\tau_{i,h} = \tau_i \otimes e^h$ ,  $i = 1, 2$ . In [HW5] we defined Eisenstein integrals  $E(P) : v \in \mathfrak{a}^* \times \mathfrak{a}^* \times G \rightarrow W$  which are holomorphic in  $h$  and  $v$  and are  $(\tau_{1,h}, \tau_{2,h})$ -spherical functions of matrix coefficients of the representations  $\pi(H : h : v)$  when  $h \in \mathcal{D}$ ,  $v \in \mathfrak{a}^*$ . Then we defined wave packets of the form

$$\Phi(H : \mathcal{D} : x) = \int_{\mathcal{D} \times \mathfrak{a}^*} E(P : h : v : x) \alpha(h : v) m(H : h : v) dv dh \tag{1.4}$$

where  $m(H : h : v) dv dh$  is the Plancherel measure corresponding to  $\pi(H : h : v)$  and  $\alpha : \mathcal{D} \times \mathfrak{a}^* \rightarrow \mathbf{C}$  is a jointly smooth function of  $h$  and  $v$  which extends smoothly to  $cl(\mathcal{D}) \times \mathfrak{a}^*$  and is rapidly decaying at infinity in both variables. It was proven in [H1, 2, HW5] that every  $K$ -compact Schwartz function is a finite sum of wave packets of this type and that an individual wave packet  $\Phi(H : \mathcal{D})$  is Schwartz class if and only if  $\alpha(h : v)$  has zeros of infinite order at the finite endpoints of the interval  $\mathcal{D}$  and if  $\alpha(h : v) m(H : h : v)$  is jointly smooth on  $\mathcal{D} \times \mathfrak{a}^*$ . Finite endpoints of  $\mathcal{D}$  correspond to limits of discrete series and points  $(h, v) \in \mathcal{D} \times \mathfrak{a}^*$  at which  $m(H : h : v)$  fails to be jointly smooth correspond to reducible principal series representations which decompose into limits of discrete series which are actual limits along continuous families of relative discrete series representations.

Now elementary mixed wave packets are defined roughly as follows. (See (4.1)

for the precise definition.) Fix a matching family  $\{\pi(F:h:v_F): F \subseteq F_0\}$  as in (1.2) such that the Plancherel function  $m^*(\emptyset:h:v_\emptyset)$  defined in (4.5) is jointly smooth at  $(0,0) \in i\nu^* \times \alpha_\emptyset^*$ . Suppose for each  $F \subseteq F_0$  we have  $\Phi(F): i\nu^* \times \alpha_F^* \times G \rightarrow W$  satisfying the following conditions. First, let  $\Phi^\pm(F)$  denote the restriction of  $\Phi(F)$  to  $\mathcal{D}^\pm \times \alpha_F^* \times G$  where  $\mathcal{D}^\pm = \{h \in i\nu^*: \lambda(h) \in \mathcal{C}^\pm\}$ . Then there are finitely many Eisenstein integrals  $E_i^\pm(P_F)$  corresponding to the family  $\{\pi(F:h:v_F): h \in \mathcal{D}^\pm, v_F \in \alpha_F^*\}$  and smooth, rapidly decreasing functions  $\alpha_i^\pm$  as in (1.4) so that for all  $h \in \mathcal{D}^\pm, v_F \in \alpha_F^*, x \in G$ ,

$$\Phi^\pm(F:h:v_F:x) = \sum_i \alpha_i^\pm(h:v_F)E_i^\pm(P_F:h:v_F:x). \tag{1.5a}$$

Second, there are a small neighborhood  $U$  of  $0 \in i\nu^*$  and a compact subset  $\omega \subset U$  so that

$$\Phi(F:h:v_F:x) = 0 \quad \text{for all } v_F \in \alpha_F^*, x \in G, \text{ if } h \notin \omega. \tag{1.5b}$$

$U$  must be small enough that  $U \subset \mathcal{D}^+ \cup \mathcal{D}^- \cup \{0\}$  and  $m^*(\emptyset:h:v)$  is jointly smooth on  $U \times \alpha_\emptyset^*$ . Finally, the functions  $\Phi(F)$  must satisfy the matching conditions of (1.3). That is, fix  $E \subseteq F_0$ . Then for all  $k \geq 0$ ,

$$\begin{aligned} & \lim_{h \downarrow 0} (\partial/\partial h)^k \Phi(E:h:v_E:x) + (-1)^{|E|+1} \lim_{h \uparrow 0} (\partial/\partial h)^k \Phi(E:h:v_E:x) \\ &= \sum_{E \subset F \subseteq F_0} c_{|F \setminus E|} \left[ \lim_{h \downarrow 0} (\partial/\partial h - D_{F \setminus E})^k \Phi(F:h:(v_E, 0):x) \right. \\ & \quad \left. + (-1)^{|F|} \lim_{h \uparrow 0} (\partial/\partial h - D_{F \setminus E})^k \Phi(F:h:(v_E, 0):x) \right] \end{aligned} \tag{1.5c}$$

for all  $v_E \in \alpha_E^*, x \in G$  where the  $D_{F \setminus E}, c_{|F \setminus E|}$  are defined as in (1.3). Then we say that

$$\Phi(x) = \sum_{F \subseteq F_0} \int_{i\nu^*} \int_{\alpha_F^*} \Phi(F:h:v_F:x) m(F:h:v_F) dh dv_F \tag{1.5d}$$

is an elementary mixed wave packet. If  $w \in W^*$  we say that

$$\phi(x) = \langle \Phi(x), w^* \rangle \tag{1.5e}$$

is a scalar-valued elementary mixed wave packet.

Note that if  $F_0 = \emptyset$ , then  $\Phi$  is a single series wave packet of the type defined in (1.4) and is Schwartz class since we can assume the neighborhood  $U$  of  $0 \in i\nu^*$  containing the support of  $\Phi(\emptyset)$  is small enough that  $U \subset \mathcal{D}^+ = \mathcal{D}^-$  and that  $m(\emptyset:h:v_\emptyset)$  is jointly smooth in  $U \times \alpha_\emptyset^*$ .

In this paper we will prove the following theorems.

**THEOREM 1.6.** *Every  $f \in \mathcal{C}(G)_K$  is the sum of finitely many scalar-valued elementary mixed wave packets.*

Suppose  $\Phi(x)$  is defined as in (1.5d), and for  $h \in iv^*$  define

$$\Phi(h : x) = \sum_{F \subseteq F_0} \int_{\mathfrak{a}_F^*} \Phi(F : h : v_F : x) m(F : h : v_F) dv_F. \tag{1.7}$$

**THEOREM 1.8.** *Let  $\Phi(x)$  be an elementary mixed wave packet. Then  $(h, x) \rightarrow \Phi(h : x)$  is jointly smooth on  $iv^* \times G$ .*

This is the first step in proving that  $\Phi$  is a Schwartz class function on  $G$  because of the following proposition which is proven in Section 2. Let  $A_0$  be the split part of the Iwasawa–Cartan subgroup so that  $G = K cl(A_0^+)K$  is the Cartan decomposition of  $G$ . Define  $\sigma$  and  $\Xi$  as in (2.2).

**PROPOSITION 1.9.** *Suppose  $F : iv^* \times G \rightarrow W$  is  $(\tau_{1,h}, \tau_{2,h})$ -spherical and define*

$$F(x) = \int_{iv^*} F(h : x) dh.$$

*Then  $F(x)$  is a Schwartz class function on  $G$  if and only if*

$$(h, x) \rightarrow F(h : x) \text{ is jointly smooth on } iv^* \times G$$

*and*

$$\sup_{h \in iv^*, a \in cl(A_0^+)} \Xi(a)^{-1} (1 + \sigma(a))^r (1 + |h|)^r \|F(h; D : D_1; a; D_2)\| < \infty$$

*for all  $r \geq 0$ , constant coefficient differential operators  $D$  on  $iv^*$  and  $D_1, D_2 \in \mathcal{U}(\mathfrak{g}_\mathbb{C})$ .*

Theorem 1.6 is proven roughly as follows. Recall the Plancherel theorem gives us a decomposition of  $f \in \mathcal{C}(G)_K$  as

$$f(x) = \sum_{H \in \text{Car}(G)} f_H(x).$$

We can define a similar decomposition of elementary mixed wave packets even though we do not know they are Schwartz class. Let

$$\Phi(x) = \sum_{F \subseteq F_0} \int_{iv^*} \int_{\mathfrak{a}_F^*} \Phi(F : h : v_F : x) m(F : h : v_F) dh dv_F,$$

$$\phi(x) = \langle \Phi(x), w^* \rangle, \quad w \in W^*,$$

as in (1.5). For  $H \in \text{Car}(G)$  let

$$S(H) = \{F \subseteq F_0 : H_F \text{ is conjugate to } H\}.$$

Then we have

$$\phi(x) = \sum_{H \in \text{Car}(G)} \phi_H(x)$$

where

$$\phi_H(x) = \sum_{F \in S(H)} \int_{iv^*} \int_{\alpha_F^*} \langle \Phi(F : h : v_F : x), w^* \rangle m(F : h : v_F) dh dv_F.$$

Note that for all  $F \subseteq F_0$ ,  $\dim A_F = \dim A_\emptyset + |F|$  so that  $S(H) = \emptyset$  and  $\phi_H(x) = 0$  if  $\dim A < \dim A_\emptyset$  or if  $\dim A = \dim A_\emptyset$  and  $H$  is not conjugate to  $H_\emptyset$ .

Order  $\text{Car}(G) = \{H_1, \dots, H_k\}$  so that  $0 = \dim A_1 \leq \dots \leq \dim A_k$ . Now  $H_1$  is relatively compact, so for all  $(\lambda, \chi) \in X(T_1)$ ,  $m^*(H_1 : h : v)$  is jointly smooth. Let  $f \in \mathcal{C}(G)_K$ . We start by defining finitely many elementary mixed wave packets  $\phi^{1,i}(x)$ ,  $i \in I_1$ , corresponding to elements  $(\lambda^{1,i}, \chi^{1,i}) \in X(T_1)$  so that

$$f_{H_1}(x) = \sum_{i \in I_1} \phi_{H_1}^{1,i}(x).$$

Assume that for  $1 \leq d \leq p \leq k-1$  we have constructed finitely many elementary mixed wave packets  $\phi^{d,i}(x)$ ,  $i \in I_d$ , corresponding to elements  $(\lambda^{d,i}, \chi^{d,i}) \in X(T_d)$  so that for all  $1 \leq d \leq p$ ,

$$f_{H_d}(x) = \sum_{1 \leq d' \leq p} \sum_{i \in I_{d'}} \phi_{H_d}^{d',i}(x).$$

Then we show that there are finitely many elementary mixed wave packets  $\phi^{p+1,i}(x)$ ,  $i \in I_{p+1}$ , corresponding to  $(\lambda^{p+1,i}, \chi^{p+1,i}) \in X(T_{p+1})$  so that

$$f_{H_{p+1}}(x) = \sum_{1 \leq d' \leq p+1} \sum_{i \in I_{d'}} \phi_{H_{p+1}}^{d',i}(x).$$

Let  $1 \leq d \leq p$ . Since  $\dim A_d \leq \dim A_{p+1}$  and  $H_d$  is not conjugate to  $H_{p+1}$ ,  $\phi_{H_d}^{p+1,i}(x) = 0$  for all  $i \in I_{p+1}$  and so we also have

$$f_{H_d}(x) = \sum_{1 \leq d' \leq p+1} \sum_{i \in I_{d'}} \phi_{H_d}^{d',i}(x).$$



Now by induction we have a finite collection of elementary mixed wave packets

$$\phi^{d,i}(x), i \in I_d, 1 \leq d \leq k$$

so that for all  $1 \leq d \leq k$ ,

$$f_{H_d}(x) = \sum_{1 \leq d' \leq k} \sum_{i \in I_{d'}} \phi^{d',i}(x).$$

Thus

$$f(x) = \sum_{1 \leq d \leq k} \sum_{i \in I_d} \phi^{d,i}(x).$$

Let  $\Phi(x)$  be an elementary mixed wave packet and write

$$\Phi(h : x) = \sum_{F \subseteq F_0} \int_{\mathfrak{a}_F^*} \Phi(F : h : v_F : x) m(F : h : v_F) dv_F$$

as in (1.7). In order to prove Theorem 1.8 we must show that for each  $F \subseteq F_0$ ,

$$\Phi_F(h : x) = \int_{\mathfrak{a}_F^*} \Phi(F : h : v_F : x) m(F : h : v_F) dv_F$$

is smooth on  $cl(\mathcal{D}^\pm) \times G$  and compute

$$D\Phi_F^\pm(0 : x) = \lim_{h \rightarrow 0, h \in \mathcal{D}^\pm} D\Phi_F(h : x)$$

for any differential operator  $D$  on  $iv^* \times G$ . Then we must show that

$$\sum_{F \subseteq F_0} D\Phi_F^+(0 : x) = \sum_{F \subseteq F_0} D\Phi_F^-(0 : x). \tag{1.10}$$

In order to do this we need to study the Plancherel functions  $m^*(F : h : v_F)$  which for  $F \neq \emptyset$  are not jointly smooth at  $(0, 0) \in iv^* \times \mathfrak{a}_F^*$ . For  $h \in iv^*$ ,  $v_F \in \mathfrak{a}_F^*$ , write

$$p_F(h : v_F) = \prod_{\alpha \in F} \frac{2}{\langle \alpha, \alpha \rangle} (\langle v, \alpha \rangle + ih).$$

What we prove in Section 5 is that there are functions  $g(F : h : v_F : x)$ ,  $F \subseteq F_0$ , which are jointly smooth on  $cl(\mathcal{D}^\pm) \times \mathfrak{a}_F^* \times G$  and satisfy matching conditions similar to those satisfied by the  $\Phi(F : h : v_F : x)$ ,  $F \subseteq F_0$ , and a constant  $c \neq 0$

independent of  $F$  so that

$$\Phi_F(h : x) = c(\pi i)^{-|F|} \int_{\mathfrak{a}_F^*} \frac{g(F : h : \nu_F : x)}{p_F(h : \nu_F)} d\nu_F. \tag{1.11}$$

Now it is a calculus exercise to compute  $D\Phi_F^\pm(0 : x)$ . Formula (1.10) follows from the matching conditions satisfied by the functions  $g(F)$ ,  $F \subseteq F_0$ . Formula (1.11) will also be needed for the proof that  $\Phi(h : x)$  satisfies the estimates of (1.9).

The organization of the paper is as follows.

In Section 2 we review definitions and theorems from [H1, 2, HW5], prove (1.9) in Proposition 2.8, and improve the a priori estimates for Eisenstein integrals given in [HW5] in Theorem 2.21.

In Section 3 we prove the character identity (1.3) in Theorem 3.11 and derive some consequences which will be needed to prove (1.8).

In Section 4 we define elementary mixed wave packets in (4.1) and prove (1.6) in Theorem 4.2.

In Section 5 we study the Plancherel functions  $m(H : h : \nu)$  and prove (1.11) in Theorem 5.3.

In Section 6 we prove some technical results about Plancherel functions for the universal covering groups of symplectic groups which are needed in Section 5.

In Section 7 we prove the calculus result (Theorem 7.2) which is needed to prove (1.8) once the elementary mixed wave packets are written in the form given by Theorem 5.3.

## 2. Preliminaries

Suppose  $G$  is a connected reductive Lie group. Fix a Cartan involution  $\theta$  as in [W] and let  $K$  denote the fixed point set of  $\theta$ . Then the center  $Z_G$  of  $G$  is contained in  $K$ , and  $K$  is the full inverse image of a maximal compact subgroup of the linear group  $G/Z_G$ . The following structural result was proven in [HW5].

**PROPOSITION 2.1.**  *$K$  has a unique maximal compact subgroup  $K_1$  and has a closed normal vector subgroup  $V$  such that  $K = K_1 \times V$  and  $Z = Z_G \cap V$  is co-compact in both  $V$  and  $Z_G$ .*

Let  $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$  be the  $\pm 1$  eigenspace decomposition under  $\theta$ . (For any Lie group  $G$  we will use the corresponding lower case German letter  $\mathfrak{g}$  to denote the real Lie algebra of  $G$ .) Choose a maximal abelian subspace  $\mathfrak{a}_0 \subset \mathfrak{p}$  and a positive restricted root system  $\Phi^+ = \Phi^+(\mathfrak{g}, \mathfrak{a}_0)$ . Let  $\rho = 1/2 \sum_{\alpha \in \Phi^+} m(\alpha)\alpha$  where  $m(\alpha)$  is the dimension of the root space of  $\mathfrak{g}$  corresponding to  $\alpha$ . For  $x \in G$ , define  $H(x) \in \mathfrak{a}_0$  using the Iwasawa decomposition,  $x \in K \exp(H(x))N_0$ . Then the zonal spherical

function on  $G$  for  $0 \in \mathfrak{a}_0^*$  is

$$\Xi(x) = \int_{K/Z} e^{-\rho(H(xk))} d(kZ). \tag{2.2a}$$

Now decompose  $x \in G$  as  $x = v(x)k_1(x) \exp \zeta(x)$  where  $v(x) \in V$ ,  $k_1(x) \in K_1$ , and  $\zeta(x) \in \mathfrak{p}$ . Polynomial growth in  $G$  is determined by the function

$$\tilde{\sigma}(x) = \sigma_V(x) + \sigma(x) \tag{2.2b}$$

where  $\sigma_V(x) = \|v(x)\|$  and  $\sigma(x) = \|\zeta(x)\|$ . Let  $W$  be a Banach space and  $f \in C^\infty(G: W)$ . If  $D_1, D_2 \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}})$  and  $r \geq 0$ , define

$${}_{D_1} \|f\|_{r, D_2} = \sup_{x \in G} (1 + \tilde{\sigma}(x))^r \Xi(x)^{-1} \|f(D_1 x; D_2)\|_W. \tag{2.2c}$$

The Schwartz space is

$$\mathcal{S}(G: W) = \{f \in C^\infty(G: W) : {}_{D_1} \|f\|_{r, D_2} < \infty \text{ for all } D_1, D_2 \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}}), r \geq 0\}. \tag{2.2d}$$

We write  $\mathcal{S}(G) = \mathcal{S}(G: \mathbb{C})$ .

For  $f \in \mathcal{S}(G: W)$ ,  $h \in i\mathfrak{v}^*$ ,  $x \in G$ , define

$$\hat{f}(h: x) = \int_V f(vx) e^{-h(v)} dv. \tag{2.3}$$

Let  $G_1 = \{x \in G : x = k \exp(\xi) \text{ for some } k \in K_1, \xi \in \mathfrak{p}\}$ . For any finite-dimensional real vector space  $E$ , let  $D(E)$  denote the constant coefficient differential operators on  $E$ .

**PROPOSITION 2.4.**  $f \in \mathcal{S}(G: W)$  if and only if  $\hat{f} \in C^\infty(i\mathfrak{v}^* \times G: W)$  and for all  $r \geq 0$ ,  $D \in D(i\mathfrak{v}^*)$ ,  $D_1, D_2 \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ ,

$$\sup_{h \in i\mathfrak{v}^*, x \in G_1} \Xi^{-1}(x) (1 + \sigma(x))^r (1 + |h|)^r \|\hat{f}(h; D: D_1 x; D_2)\| < \infty.$$

*Proof.* This follows from [H1, 2.14] using the duality of  $\mathcal{S}(i\mathfrak{v}^*)$  and  $\mathcal{S}(V)$  via the Fourier transform. □

**LEMMA 2.5.** Let  $A_0^+$  be the positive Weyl chamber for  $A_0$ . Then

$$G_1 = \{vk_1 a k_2 v^{-1} : v \in V, k_1, k_2 \in K_1, a \in cl(A_0^+)\}.$$

*Proof.* Let  $x \in G_1$ . Since  $G = K cl(A_0^+) K$  and  $K = VK_1$ , we can write

$x = v_1 k_1 a v_2 k_2$  where  $v_i \in V$ ,  $k_i \in K_1$ ,  $a \in cl(A_0^+)$ . But now

$$x = v_1 k_1 v_2 k_2 (v_2 k_2)^{-1} a v_2 k_2 = (v_1 v_2)(k_1 k_2)((v_2 k_2)^{-1} a v_2 k_2)$$

where  $v_1 v_2 \in V$ ,  $k_1 k_2 \in K_1$ , and  $(v_2 k_2)^{-1} a v_2 k_2 \in \exp \mathfrak{p}$ . Now since  $x \in G_1$ ,  $v_1 v_2 = 1$   $\square$

Let  $\tau_1, \tau_2 \in \hat{K}$ . For  $i = 1, 2$ , let  $\tau_i^1 \in \hat{K}_1$  denote the restriction of  $\tau_i$  to  $K_1$  and let  $h_i \in i\mathfrak{v}^*$  so that  $\tau_i(vk) = e^{h_i(v)} \tau_i(k)$  for all  $v \in V$ ,  $k \in K$ . Let  $W = W(\tau_1 : \tau_2)$  be the finite-dimensional subspace of  $L^2(K_1 \times K_1)$  on which  $K_1$  acts on the left and right by  $(\tau_1^1, \tau_2^1)$ . The action of  $K_1$  on  $W$  extends to an action  $(\tau_1, \tau_2)$  of  $K$  by

$$\tau_1(v_1 k_1) w \tau_2(v_2 k_2) = e^{h_1(v_1)} e^{h_2(v_2)} \tau_1^1(k_1) w \tau_2^1(k_2) \quad (2.6a)$$

for  $v_1, v_2 \in V$ ,  $k_1, k_2 \in K_1$ ,  $w \in W$ . For any  $h \in \mathfrak{v}_{\mathbb{C}}^*$ , write  $\tau_{i,h} = \tau_i \otimes e^h$ . Then  $(\tau_{1,h}, \tau_{2,h})$  is a double unitary representation of  $K$  on  $W$  for all  $h \in i\mathfrak{v}^*$ . We will say  $F : i\mathfrak{v}^* \times G \rightarrow W$  is  $(\tau_{1,h}, \tau_{2,h})$ -spherical if for all  $k_1, k_2 \in K$ ,  $x \in G$ ,  $h \in i\mathfrak{v}^*$ ,

$$F(h : k_1 x k_2) = \tau_{1,h}(k_1) F(h : x) \tau_{2,h}(k_2). \quad (2.6b)$$

LEMMA 2.7. Suppose  $F$  is a  $(\tau_{1,h}, \tau_{2,h})$ -spherical function. Let  $D \in D(i\mathfrak{v}^*)$ . Then for all  $D_1, D_2 \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}})$  there are finite subsets  $F_1, F_2 \subset \mathcal{U}(\mathfrak{g}_{\mathbb{C}})$  such that

$$\|F(h; D : D_1; v k_1 a k_2 v^{-1}; D_2)\| \leq \sum_{D' \in F_1, D'' \in F_2} \|F(h; D : D'; a; D'')\|$$

for all  $k_1, k_2 \in K_1$ ,  $v \in V$ ,  $a \in A_0$ .

*Proof.* For all  $h \in i\mathfrak{v}^*$  we can write

$$\begin{aligned} F(h; D_1; v k_1 a k_2 v^{-1}; D_2) &= \tau_1(v k_1) e^{h(v)} F(h; D_1^{(v k_1)^{-1}}; a; D_2^{k_2 v^{-1}}) \tau_2(k_2 v^{-1}) e^{h(v^{-1})} \\ &= \tau_1(v k_1) F(h; D_1^{(v k_1)^{-1}}; a; D_2^{k_2 v^{-1}}) \tau_2(k_2 v^{-1}). \end{aligned}$$

Thus for all  $D \in D(i\mathfrak{v}^*)$  we have

$$F(h; D : D_1; v k_1 a k_2 v^{-1}; D_2) = \tau_1(v k_1) F(h; D : D_1^{(v k_1)^{-1}}; a; D_2^{k_2 v^{-1}}) \tau_2(k_2 v^{-1}).$$

Now there are finite subsets  $F_1, F_2 \subset \mathcal{U}(\mathfrak{g}_{\mathbb{C}})$  and continuous functions  $a_i, b_i$  on  $K/Z$  so that  $D_1^k = \sum_{D'_i \in F_1} a_i(k) D'_i$  and  $D_2^k = \sum_{D''_i \in F_2} b_i(k) D''_i$  for all  $k \in K$ . Let

$$C = \sup_{k_1, k_2 \in K, i, j} |a_i(k_1) b_j(k_2)|.$$

Then we have

$$\|F(h; D : D_1; vk_1ak_2v^{-1}; D_2)\| \leq C \sum_{D' \in F_1, D'' \in F_2} \|F(h; D : D'; a; D'')\|. \quad \square$$

**PROPOSITION 2.8.** *Suppose  $F \in C^\infty(iv^* \times G : W)$  is  $(\tau_{1,h}, \tau_{2,h})$ -spherical, and suppose for all  $r \geq 0$ ,  $D \in D(iv^*)$ ,  $D_1, D_2 \in \mathcal{U}(\mathfrak{g}_\mathbb{C})$  that*

$$\sup_{h \in iv^*, a \in \mathfrak{a}(\mathcal{A}_\delta^+)} \Xi^{-1}(a)(1 + \sigma(a))^r(1 + |h|)^r \|F(h; D : D_1; a; D_2)\| < \infty.$$

Then if

$$F(x) = \int_{iv^*} F(h : x) dh,$$

$F \in \mathcal{C}(G : W)$ .

*Proof.* Combine Lemmas 2.4, 2.5, and 2.7. □

When  $K$  is non-compact there are no  $K$ -finite functions in  $\mathcal{C}(G)$ . The appropriate generalization in this case is the notion of a  $K$ -compact function defined as follows. For  $\tau \in \widehat{K}$ , let

$$\delta(\tau) = \deg(\tau^*) \text{trace}(\tau^*) \tag{2.9a}$$

denote the normalized character of the contragredient  $\tau^*$  of  $\tau$ . We say  $f \in \mathcal{C}(G)$  is  $K$ -compact if there is a compact subset  $\Omega$  of  $\widehat{K}$  so that for  $\tau \in \widehat{K}$ ,

$$\delta(\tau^*) *_K f = 0 = f *_K \delta(\tau), \quad \tau \notin \Omega. \tag{2.9b}$$

It was proven in [H1, 2.12] that the space  $\mathcal{C}(G)_K$  of  $K$ -compact functions is dense in  $\mathcal{C}(G)$ .

The Plancherel theorem expands functions in  $\mathcal{C}(G)_K$  as finite sums of wave packets as follows. (For details, see [H1, §3]. Some definitions have been changed slightly for convenience.)

Let  $H = TA$  be a  $\theta$ -stable Cartan subgroup of  $G$  and let  $P = MAN$  be a parabolic subgroup associated to  $H$ . Let  $\Phi_M = \Phi(\mathfrak{m}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$  denote the roots of  $\mathfrak{m}_\mathbb{C}$  with respect to  $\mathfrak{t}_\mathbb{C}$ ,  $\Phi_M^+$  a choice of positive roots. Let  $\rho_M$  denote the half sum over  $\Phi_M^+$ . For  $h \in iv^* = \{h \in i\mathfrak{t}^* : h(\mathfrak{k}_1) = 0\}$ , set  $h_M(h) = h|_{\mathfrak{t}}$ . Let

$$\Phi_{M,1} = \{\alpha \in \Phi_M : \langle \alpha, h_M(h) \rangle = 0 \text{ for all } h \in iv^*\}. \tag{2.10a}$$

Let

$$\Lambda_{M,1} = \{\lambda \in i\mathfrak{t}^* : \lambda - \rho_M \text{ is integral and } \lambda \text{ is } \Phi_{M,1} \text{ non-singular}\}. \tag{2.10b}$$

For  $\lambda \in \Lambda_{M,1}$  set

$$X(\lambda) = \{\chi \in Z_M(M^0)^\wedge : \chi|_{Z_{M^0}} \text{ is a multiple of } e^{\lambda - \rho_M}|_{Z_{M^0}}\} \quad (2.10c)$$

and let

$$X(T) = \{(\lambda, \chi) \in \Lambda_{M,1} \times Z_M(M^0)^\wedge : \chi \in X(\lambda)\}. \quad (2.10d)$$

Then for  $(\lambda, \chi) \in X(T)$ ,  $h \in i\nu^*$ , let  $\lambda(h) = \lambda + h_M(h)$  and  $\chi(h) = \chi \otimes e^h|_{Z_M(M^0)}$ . Then if  $\lambda(h)$  is regular we will write  $\pi(h)$  for the relative discrete series representation of  $M^0$  with Harish-Chandra parameter  $\lambda(h)$ . For  $\nu$  in  $\mathfrak{a}^*$  we set

$$\pi(H : \lambda : \chi : h : \nu) = \text{Ind}_{Z_M(M^0)M^0AN}^G(\chi(h) \otimes \pi(h) \otimes e^{i\nu} \otimes 1) \quad (2.10e)$$

and let

$$\Theta(H : \lambda : \chi : h : \nu) \text{ be the character of } \pi(H : \lambda : \chi : h : \nu). \quad (2.10f)$$

$Z$  is a central subgroup of  $Z_M(M^0)$  so that each  $\chi \in Z_M(M^0)^\wedge$  has a  $Z$ -character  $\zeta(\chi)$ . Let

$$\widehat{K}(\chi) = \{\tau \in \widehat{K} : \tau(kz) = \zeta(\chi : z)\tau(k) \text{ for all } k \in K, z \in Z\}. \quad (2.10g)$$

Then all  $K$ -types of the representation  $\pi(H : \lambda : \chi : h : \nu)$  lie in  $\widehat{K}(\chi \otimes e^h) = \{\tau_h = \tau \otimes e^h : \tau \in \widehat{K}(\chi)\}$ .

For  $(\lambda, \chi) \in X(T)$ ,  $\tau_1, \tau_2 \in \widehat{K}(\chi)$ ,  $x \in G$ , and  $f \in \mathcal{C}(G)$ , define

$$f(H : \lambda : \chi : \tau_1 : \tau_2 : x) = \int_{i\nu^*} \int_{\mathfrak{a}^*} \widehat{f}(H : \lambda : \chi : \tau_1 : \tau_2 : h : \nu : x) m(H : \lambda : \chi : h : \nu) \, d\nu \, dh \quad (2.11a)$$

where

$$\widehat{f}(H : \lambda : \chi : \tau_1 : \tau_2 : h : \nu : x) = \delta(\tau_{1,h}^* *_{K/Z} \Theta(H : \lambda : \chi : h : \nu : R(x)f) *_{K/Z} \delta(\tau_{2,h})). \quad (2.11b)$$

Here  $R(x)f$  is the right translate of  $f$  by  $x$  and  $m(H : \lambda : \chi : h : \nu)$  is the Plancherel function corresponding to  $\pi(H : \lambda : \chi : h : \nu)$ . (See (4.5) for the definition.) We call  $f(H : \lambda : \chi : \tau_1 : \tau_2)$  a wave packet associated to  $f$ . The Plancherel theorem will expand  $f$  in terms of these wave packets.

Let  $\text{Car}(G)$  denote a complete set of representatives for the  $\theta$ -stable Cartan

subgroups of  $G$ . Fix  $H = TA \in \text{Car}(G)$ . Write  $\mathfrak{t} = \mathfrak{t}_1 \oplus \mathfrak{t}_2$  where  $\mathfrak{t}_1 = \mathfrak{t} \cap \mathfrak{k}_1$  and  $\mathfrak{t}_2$  is the orthogonal complement of  $\mathfrak{t}_1$  with respect to the Killing form. Let  $\Lambda_{M,0} = \{\lambda \in \Lambda_{M,1} : \lambda|_{\mathfrak{t}_2} = 0\}$ . Then for  $\lambda \in \Lambda_{M,1}$ , there is a unique  $\lambda_0 \in \Lambda_{M,0}$  such that  $\{\lambda + h_M(h) : h \in i\nu^*\} = \{\lambda_0 + h_M(h) : h \in i\nu^*\}$ . Let  $X_1 = \{h \in i\nu^* : h_M(h) = 0\}$ . For  $\lambda_0 \in \Lambda_{M,0}$ , define an equivalence relation on  $X(\lambda_0)$  by  $\chi \sim \chi'$  if  $\chi' = \chi \otimes e^h$  for some  $h \in X_1$ . Let  $[X(\lambda_0)/X_1]$  denote a complete set of representatives for the equivalence classes of  $X(\lambda_0)$  with respect to  $\sim$ . Set  $X_0(T) = \{(\lambda, \chi) : \lambda \in \Lambda_{M,0}, \chi \in [X(\lambda_0)/X_1]\}$ . Define an equivalence relation on  $X_0(T)$  by  $(\lambda, \chi) \sim (\lambda', \chi')$  if there is  $w \in W(G, H) = N_G(H)/H$  such that  $\lambda' = w\lambda$  and  $\chi' \sim w\chi$ . Let  $X_0(T)/W_H$  denote a complete set of representatives for these equivalence classes. For  $\chi \in Z_M(M^0)^\wedge$  define an equivalence relation on  $\hat{K}(\chi)$  by  $\tau \sim \tau'$  if  $\tau' = \tau \otimes e^h$  for some  $h \in S_T = \{h \in i\nu^* : e^h|_T = 1\}$ . Let  $[\hat{K}(\chi)/S_T]$  denote a complete set of representatives for the corresponding equivalence classes.

**THEOREM 2.12.** *Let  $f \in \mathcal{C}(G)_K$ ,  $x \in G$ . Then*

$$f(x) = \sum_{H=TA \in \text{Car}(G)} \sum_{(\lambda, \chi) \in X_0(T)/W_H} \sum_{\tau_1 \in \hat{K}(\chi)} \sum_{\tau_2 \in [\hat{K}(\chi)/S_T]} f(H : \lambda : \chi : \tau_1 : \tau_2 : x).$$

*Proof.* This is essentially the result proven in [H1, 3.6]. The only differences are that first, we have eliminated listing wave packets more than once by summing over  $X_0(T)/W_H$  instead of  $X_0(T)$ . Second, for convenience of notation, we have absorbed all constants into the definition of  $m(H : \lambda : \chi : h : \nu)$ . (See (5.10).) □

Let  $\Omega$  be a compact subset of  $\hat{K}$ . Write  $\mathcal{C}(G : \Omega)_K$  for the set of all Schwartz functions with  $K$ -types contained in  $\Omega$ . For  $H = TA \in \text{Car}(G)$ , let  $\|\cdot\|$  be the norm on  $i\mathfrak{t}^*$  coming from the Killing form. Let  $B$  be a Cartan subgroup of  $K$ . For  $\tau \in \hat{K}$ , let  $\|\tau\| = \|\mu\|$  where  $\mu \in i\mathfrak{b}^*$  is the highest weight of  $\tau$ . For  $m \geq 0$ , let

$$X_0^m(T) = \{(\lambda, \chi) \in X_0(T) : \|\lambda\| \leq m\}$$

and, for  $\chi \in Z_M(M^0)^\wedge$ , let

$$\hat{K}^m(\chi) = \{\tau \in \hat{K}(\chi) : \|\tau\| \leq m\}.$$

**LEMMA 2.13.** *Let  $\Omega$  be a compact subset of  $\hat{K}$ . Then there is  $m \geq 0$  so that for all  $f \in \mathcal{C}(G : \Omega)_K$ ,  $H = TA \in \text{Car}(G)$ ,  $(\lambda, \chi) \in X_0(T)$ ,  $\tau_1, \tau_2 \in \hat{K}(\chi)$ ,*

$$\hat{f}(H : \lambda : \chi : \tau_1 : \tau_2 : h : \nu : x) = 0$$

*for all  $\nu \in \mathfrak{a}^*$ ,  $x \in G$  unless  $(\lambda, \chi) \in X_0^m(T)$ ,  $\tau_1, \tau_2 \in \hat{K}^m(\chi)$ ,  $|h| \leq m$ ,  $\|\lambda(h)\| \leq m$ , and  $\|\tau_{j,h}\| \leq m$ ,  $j = 1, 2$ . Further, for all  $m \geq 0$ ,  $\chi \in Z_M(M^0)^\wedge$ ,  $X_0^m(T)$  and  $\hat{K}^m(\chi)$  are finite sets.*

*Proof.* This follows from [H1, 3.11]. □

Wave packets associated to Schwartz functions can be identified with wave packets of Eisenstein integrals as follows. (For details see [H1].) First, we extend the wave packets associated to  $f$  to be vector-valued. For  $\tau_1, \tau_2 \in \hat{K}$ ,  $x \in G$ ,  $h \in i\mathfrak{v}^*$ ,  $f \in \mathcal{C}(G)$ , define

$$f(\tau_1 : \tau_2 : h : x) = \delta(\tau_{1,h}^*) *_{\mathbf{K}} f(x) *_{\mathbf{K}/Z} \delta(\tau_{2,h}); \quad (2.14a)$$

$$F(f : \tau_1 : \tau_2 : h : x)(k_1 : k_2) = f(\tau_1 : \tau_2 : h : k_1^{-1} x k_2^{-1}), \quad k_1, k_2 \in \mathbf{K}_1. \quad (2.14b)$$

Then  $F(h : x) = F(f : \tau_1 : \tau_2 : h : x)$  takes values in  $W = W(\tau_1 : \tau_2)$  and is  $(\tau_{1,h}, \tau_{2,h})$ -spherical. Now for  $(\lambda, \chi) \in X(T)$ , set

$$\hat{F}(f : H : \lambda : \chi : \tau_1 : \tau_2 : h : v : x) = \Theta(H : \lambda : \chi : h : v : R(x)F(h)); \quad (2.14c)$$

$$\begin{aligned} F(f : H : \lambda : \chi : \tau_1 : \tau_2 : x) \\ = \int_{i\mathfrak{v}^*} \int_{\mathfrak{a}^*} \hat{F}(f : H : \lambda : \chi : \tau_1 : \tau_2 : h : v : x) m(H : \lambda : \chi : h : v) dv dh. \end{aligned} \quad (2.14d)$$

Then

$$\hat{f}(H : \lambda : \chi : \tau_1 : \tau_2 : h : v : x) = \hat{F}(f : H : \lambda : \chi : \tau_1 : \tau_2 : h : v : x)(1 : 1)$$

and

$$f(H : \lambda : \chi : \tau_1 : \tau_2 : x) = F(f : H : \lambda : \chi : \tau_1 : \tau_2 : x)(1 : 1). \quad (2.15b)$$

Now let  $\mathcal{D}$  be a connected component of  $\{h \in i\mathfrak{v}^* : \langle \lambda(h), \alpha \rangle \neq 0 \text{ for all } \alpha \in \Phi_M^+\}$ . Holomorphic families of spherical functions of matrix coefficients of the representations  $\{\chi(h) \otimes \pi(h) : h \in \mathcal{D}\}$  of  $M^\dagger = Z_M(M^0)M^0$  are defined as follows. For  $h \in \mathcal{D}$ , let  $S(M^\dagger : W : h)$  be the set of all  $\Psi(h) : M^\dagger \rightarrow W$  such that

$$\begin{aligned} \Psi(h : k_1 x k_2) &= \tau_{1,h}(k_1) \Psi(h : x) \tau_{2,h}(k_2) \quad \text{for all } k_1, k_2 \in \mathbf{K}_M^\dagger \\ &= \mathbf{K} \cap M^\dagger, \quad x \in M^\dagger \end{aligned} \quad (2.16a)$$

and for each  $w^* \in W^*$ ,

$$x \rightarrow \langle \Psi(h : x), w^* \rangle \text{ is a finite sum of matrix coefficients of } \chi(h) \otimes \pi(h). \quad (2.16b)$$

Now let  $\mathcal{S}(M^\dagger : W) = \mathcal{S}(M^\dagger : \lambda : \chi : \mathcal{D} : W)$  be the set of all  $\Psi \in C^\infty(\mathfrak{v}_\mathbb{C}^* \times M^\dagger : W)$  such that

$$\Psi(h) \in S(M^\dagger : W : h) \quad \text{for all } h \in \mathcal{D}, \quad (2.17a)$$

$$h \rightarrow \Psi(h : m) \text{ is holomorphic on } \mathfrak{v}_\mathbb{C}^* \quad \text{for all } m \in M^\dagger, \quad (2.17b)$$



and

$$\Psi \text{ satisfies a moderate growth condition.} \tag{2.17c}$$

(See [H1, 5.2] for details.)

Now for  $\Psi \in \mathcal{S}(M^\dagger : W)$ , we extend  $\Psi$  to  $G$  by

$$\Psi(x) = \tau_{1,h}(\kappa(x))\Psi(h : \mu(x)), \quad x = \kappa(x)\mu(x)\exp(H_P(x))n(x) \in KM^\dagger AN \tag{2.18a}$$

and define the Eisenstein integral  $E(P : \Psi) : \mathfrak{v}\mathfrak{k}^* \times \mathfrak{a}\mathfrak{k}^* \times G \rightarrow W$  by

$$E(P : \Psi : h : v : x) = \int_{K/Z} \Psi(h : xk)\tau_{2,h}(k^{-1})e^{(iv - \rho_P)H_P(xk)} d(kZ). \tag{2.18b}$$

Let  $P(iv^* \times \mathfrak{a}^*)$  denote the set of all polynomial coefficient differential operators on  $iv^* \times \mathfrak{a}^*$ . For  $\alpha \in C^\infty(cl(\mathcal{D}) \times \mathfrak{a}^*)$  and  $D \in P(iv^* \times \mathfrak{a}^*)$  define

$$\|\alpha\|_D = \sup_{(h,v) \in \mathcal{D} \times \mathfrak{a}^*} |D\alpha(h : v)|. \tag{2.19a}$$

Then let

$$\mathcal{C}(\mathcal{D} \times \mathfrak{a}^*)_0 = \{\alpha \in C^\infty(cl(\mathcal{D}) \times \mathfrak{a}^*) : \|\alpha\|_D < \infty \text{ for all } D \in P(iv^* \times \mathfrak{a}^*)\}. \tag{2.19b}$$

Now the following theorem was proven in [H1, 8.3].

**THEOREM 2.20.** *Let  $f \in \mathcal{C}(G)$ . Then there are finitely many  $\Psi_i \in \mathcal{S}(M^\dagger : W)$ ,  $\alpha_i \in \mathcal{C}(\mathcal{D} \times \mathfrak{a}^*)_0$  so that*

$$\hat{F}(f : H : \lambda : \chi : \tau_1 : \tau_2 : h : v : x) = \sum_i \alpha_i(h : v)E(P : \Psi_i : h : v : x)$$

for all  $(h, v, x) \in cl(\mathcal{D}) \times \mathfrak{a}^* \times G$ .

Finally, we need to improve the growth estimates for the Eisenstein integral which were proven in [HW5] to give bounds which do not blow up along the boundary of  $\mathcal{D}$ .

**THEOREM 2.21.** *Let  $\Psi \in \mathcal{S}(M^\dagger : W)$ . For all  $D \in D(iv^* \times \mathfrak{a}^*)$ ,  $g_1, g_2 \in \mathcal{U}(\mathfrak{g}_\mathbb{C})$ , there exist constants  $C, r \geq 0$  so that*

$$\|E(P : \Psi : h : v : D : g_1 ; x ; g_2)\| \leq C\Xi(x)(1 + |h|)^r(1 + |v|)^r(1 + \tilde{\sigma}(x))^r$$

for all  $h \in \mathcal{D}, v \in \mathfrak{a}^*, x \in G$ .

Suppose  $\Phi(H : h : v : x) = \alpha(h : v)E(P : \Psi : h : v : x)$  for some  $\alpha \in \mathcal{C}(\mathcal{D} \times \mathfrak{a}^*)_0, \Psi \in \mathcal{S}(M^\dagger : W)$ .

**COROLLARY 2.22.**  $(h, v, x) \rightarrow \Phi(H:h:v:x)$  has a  $C^\infty$  extension to  $cl(\mathcal{D}) \times \mathfrak{a}^* \times G$ . Further, for all  $D \in D(iv^* \times \mathfrak{a}^*)$ ,  $r \geq 0$ ,  $g_1, g_2 \in \mathcal{U}(\mathfrak{g}_\mathbb{C})$ , there are constants  $C, s \geq 0$  so that for all  $x \in G, h \in \mathcal{D}, v \in \mathfrak{a}^*$ ,

$$\|\Phi(H:h:v; D:g_1; x; g_2)\|(1+|v|)^r(1+|h|)^r \leq C\Xi(x)(1+\tilde{\sigma}(x))^s.$$

*Proof.* We know from [HW5, 6.7] that  $E(P:\Psi:h:v:x)$  extends to be a jointly smooth function of  $(h, v, x) \in iv^* \times \mathfrak{a}^* \times G$ . By definition, each  $\alpha \in \mathcal{C}(\mathcal{D} \times \mathfrak{a}^*)_0$  extends to be smooth on  $cl(\mathcal{D}) \times \mathfrak{a}^*$ . Thus the first part of the corollary is clear. Now

$$\|\Phi(H:h:v; D:g_1; x; g_2)\|(1+|v|)^r(1+|h|)^r$$

can be bounded by a finite number of terms of the form

$$|\alpha(h:v; D')| \|E(P:\Psi:h:v; D''; g_1;x; g_2)\|(1+|h|)^r(1+|v|)^r$$

where  $D', D'' \in D(iv^* \times \mathfrak{a}^*)$  depend on  $D$ . But using Theorem 2.21 there are  $C', s \geq 0$  so that this term can be bounded by

$$|\alpha(h:v; D')| C' \Xi(x)(1+|h|)^{r+s}(1+|v|)^{r+s}(1+\tilde{\sigma}(x))^s.$$

Finally, by definition of  $\mathcal{C}(\mathcal{D} \times \mathfrak{a}^*)_0$ , there is  $C$  so that this last term is bounded by  $C\Xi(x)(1+\tilde{\sigma}(x))^s$ . □

In order to prove Theorem 2.21 we need the following result about the function  $\sigma$  controlling growth in  $G/Z$ .

**THEOREM 2.23.** Let  $P=MAN$  be a parabolic subgroup of  $G$ . For any  $x \in G$ , decompose  $x=kman$  with respect to the decomposition  $G=KMAN$ . Then there exists a constant  $C$  so that  $\sigma(m) \leq C(1+\sigma(x))$  for all  $x \in G$ .

Since  $\sigma$  factors through  $G/Z$ , it is enough to prove the theorem in the case that  $G$  is a semisimple group of adjoint type. Thus we can assume that there is  $n$  so that  $G \subset SL(n, \mathbb{C})$  and  $K \subset SO(n)$ . Further we can assume that  $P$  is contained in a standard block upper diagonal parabolic subgroup so that

$$P = MAN \subset \begin{pmatrix} A_1 & * & * \\ 0 & \ddots & * \\ 0 & 0 & A_k \end{pmatrix}$$

$$MA \subset \begin{pmatrix} A_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & A_k \end{pmatrix}$$

$$N \subset \begin{pmatrix} I & * & * \\ 0 & \ddots & * \\ 0 & 0 & I \end{pmatrix}$$

$$A \subset \begin{pmatrix} \alpha_1 I & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \alpha_k I \end{pmatrix}$$

where  $\alpha_i$  are positive real numbers. For  $X = (x_{ij}) \in M(n, \mathbf{C})$ , define  $\tau'(X) = \max |x_{ij}|$  and  $\tau(X) = \log(1 + \tau'(X))$ .

LEMMA 2.24.  $\tau(XY) \leq \tau(X) + \tau(Y) + \log n$  for all  $X, Y \in M(n, \mathbf{C})$ .

*Proof.* This follows easily from the fact that  $\tau'(XY) \leq n\tau'(X)\tau'(Y)$ . □

LEMMA 2.25. There are constants  $C_1$  and  $C_2$  so that  $\tau(x) \leq C_1(1 + \sigma(x))$  and  $\sigma(x) \leq C_2(1 + \tau(x))$  for all  $x \in G \subset SL(n, \mathbf{C})$ .

*Proof.* Write  $G = KA_0K$  where all elements of  $A_0$  are diagonal matrices with positive real entries. For all  $k \in K \subset SO(n)$ ,  $\tau'(k) \leq 1$  so that  $\tau(k) \leq \log 2$ . Thus for all  $x \in G$ ,  $k_1, k_2 \in K$ , using Lemma 2.24,

$$\tau(k_1 x k_2) \leq 2 \log n + \tau(k_1) + \tau(x) + \tau(k_2) \leq 2 \log n + 2 \log 2 + \tau(x).$$

Thus there is  $C$  so that for all  $x = k_1 a k_2 \in KA_0K$ ,  $\tau(x) \leq C + \tau(a)$  and  $\tau(a) = \tau(k_1^{-1} x k_2^{-1}) \leq C + \tau(x)$ . But  $\sigma(x) = \sigma(a)$ . Thus it is enough to prove the lemma when  $x = a \in A_0$ . Suppose  $a$  is the diagonal matrix with entries  $e^{\lambda_1}, \dots, e^{\lambda_n}$  where the  $\lambda_i$  are real numbers and  $\sum_{i=1}^n \lambda_i = 0$ . Reorder so that  $\lambda_1 \leq \dots \leq \lambda_k \leq 0 \leq \lambda_{k+1} \leq \dots \leq \lambda_n$ . Now  $\max \lambda_i = \lambda_n$  and  $\max |\lambda_i| = \max\{\lambda_n, |\lambda_1|\}$ . Further,

$$n\lambda_n \geq \lambda_n + \dots + \lambda_{k+1} = |\lambda_k| + \dots + |\lambda_1| \geq |\lambda_1|.$$

Thus

$$\max |\lambda_i|/n \leq \max \lambda_i \leq \max |\lambda_i|.$$

Thus

$$e^{\max |\lambda_i|/n} \leq \tau'(a) = \max e^{\lambda_i} = e^{\max \lambda_i} \leq e^{\max |\lambda_i|}.$$

Further, for all  $x \geq 0$ ,  $x \leq \log(1 + e^x) \leq x + \log 2$ . Thus

$$\tau(a) = \log(1 + \tau'(a)) \leq \log(1 + e^{\max |\lambda_i|}) \leq \log 2 + \max |\lambda_i|.$$

Further,

$$\max|\lambda_i| \leq \log(1 + e^{\max|\lambda_i|}) \leq \log(1 + \tau'(a)^n) \leq n \log(1 + \tau'(a)) = n\tau(a).$$

But there are constants  $C$  and  $C'$  so that  $\sigma(a) \leq C \max|\lambda_i|$  and  $\max|\lambda_i| \leq C'\sigma(a)$ . Thus  $\tau(a) \leq \log 2 + C'\sigma(a)$  and  $\sigma(a) \leq Cn\tau(a)$ . □

LEMMA 2.26. *There is a constant  $C$  so that for all  $a \in A$  and  $m \in M$ ,*

$$\tau(a^{-1}) \leq C(1 + \tau(ma)).$$

*Proof.* Write

$$ma = \begin{pmatrix} A_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & A_k \end{pmatrix} \quad \text{and} \quad a = \begin{pmatrix} \alpha_1 I & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \alpha_k I \end{pmatrix}$$

where the diagonal blocks are of size  $n_i$ ,  $1 \leq i \leq k$ . Reorder so that  $\max|\alpha_i^{-1}| = |\alpha_1^{-1}|$ . Now  $\det a = 1$  so that  $\alpha_1^{-1} = \alpha_2^{n_2/n_1} \dots \alpha_k^{n_k/n_1}$ . Further, for each  $1 \leq i \leq k$ ,  $|\alpha_i| = |\det A_i|^{1/n_i}$  and

$$|\det A_i| \leq n_i! \max|a_{rs}|^{n_i} \leq n_i! \tau'(ma)^{n_i}$$

where  $a_{rs}$  are the entries of  $A_i$ . Thus there are constants  $C'$  and  $r$  so that  $|\alpha_1^{-1}| \leq C' \tau'(ma)^r$ . Thus there are constants  $C''$  and  $C$  so that

$$\tau(a^{-1}) = \log(1 + |\alpha_1^{-1}|) \leq \log(1 + C' \tau'(ma)^r) \leq C'' + r \log(1 + \tau'(ma)) \leq C(1 + \tau(ma)).$$

□

PROOF OF THEOREM 2.23. Since  $\sigma(kman) = \sigma(man)$ , it is enough to prove the theorem when  $x = man \in P$ . Further, by Lemma 2.25, it is enough to prove that  $\tau(m) \leq C(1 + \tau(x))$  for all  $x = man \in P$ . But writing

$$ma = \begin{pmatrix} A_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & A_k \end{pmatrix} \quad \text{and} \quad man = \begin{pmatrix} A_1 & * & * \\ 0 & \ddots & * \\ 0 & 0 & A_k \end{pmatrix},$$

we see that  $\tau'(ma) \leq \tau'(man)$  so that  $\tau(ma) \leq \tau(man)$ . Finally, using Lemmas 2.24 and 2.26,

$$\begin{aligned} \tau(m) &= \tau(maa^{-1}) \leq \log n + \tau(ma) + \tau(a^{-1}) \\ &\leq \log n + \tau(ma) + C(1 + \tau(ma)). \end{aligned}$$

□

We now begin the proof of Theorem 2.21. Let  $\omega$  be a relatively compact neighborhood of  $0 \in iv^*$  and write  $\mathcal{D}_C = \mathcal{D} + i\omega$ . Write  $h \in \mathcal{D}_C$  as  $h = h_R + ih_I$  where  $h_R, h_I \in iv^*$  and write  $v \in \mathfrak{a}^*$  as  $v = v_R + iv_I$  where  $v_R, v_I \in \mathfrak{a}^*$ . We will prove the following theorem.

**THEOREM 2.27.** *Let  $\Psi \in \mathcal{S}(M^1:W)$ . For all  $D \in D(iv^* \times \mathfrak{a}^*)$ ,  $g_1, g_2 \in \mathcal{U}(\mathfrak{g}_C)$ , there exist constants  $C, C_0, r \geq 0$  so that*

$$\begin{aligned} & \|E(P: \Psi: h: v; D: g_1; x; g_2)\| \\ & \leq C \Xi(x) (1 + |h|)^r (1 + |v|)^r (1 + \tilde{\sigma}(x))^r e^{|h_I| \sigma_{v_I}(x)} e^{C_0 |v_I| \sigma(x)} \end{aligned}$$

for all  $h \in \mathcal{D}_C, v \in \mathfrak{a}^*, x \in G$ .

**LEMMA 2.28.** *Let  $\Phi: \mathfrak{v}^* \times G \rightarrow W$  be any smooth family of  $(\tau_{1,h}, \tau_{2,h})$ -spherical functions. Then given  $D_0 \in D(iv^*)$  and  $g_1, g_2 \in \mathcal{U}(\mathfrak{g}_C)$ , there are finite subsets  $S$  of  $D(iv^*)$  and  $S'$  of  $\mathcal{U}(\mathfrak{g}_C)$  and an  $r \geq 0$  so that*

$$\|\Phi(h; D_0: g_1; x; g_2)\| \leq (1 + |h|)^r \sum_{D \in S, g \in S'} \|\Phi(h; D: g; x)\|$$

for all  $(h, x) \in \mathfrak{v}^* \times G$ .

*Proof.* A similar estimate is proven in [HC1, Lemma 17] in the case that the parameter  $h$  does not occur. In extending the estimates to our situation it is only necessary to observe that terms of the form  $\|D(d\tau_h(\kappa))\|$  where  $D \in D(iv^*)$  and  $\kappa \in \mathcal{U}(\mathfrak{k}_C)$  grow polynomially in  $h$ .  $\square$

Write  $\Psi_v(h: x) = \Psi(h: x) e^{(iv - \rho_P)H_P(x)}$ . For any  $x \in G$ , decompose  $x$  as  $x = \kappa(x)\mu(x)a(x)n(x)$  using  $G = KMAN$ . Also write  $\log a(x) = H_P(x)$ .

**LEMMA 2.29.** *Let  $D \in D(iv^*)$  and  $g \in \mathcal{U}(\mathfrak{g}_C)$ . Then there are finite subsets  $S$  of  $D(iv^*)$  and  $S'$  of  $\mathcal{U}(\mathfrak{m}_C)$  and constants  $C, C_0, r \geq 0$  so that*

$$\begin{aligned} & \|E(P: \Psi: h; D: v: g; x)\| \leq C (1 + |h|)^r (1 + |v|)^r e^{C_0 |v_I| \sigma(x)} \Xi(x) \\ & \times \sum_{D', D'' \in S, v \in S'} \sup_{k \in K} |D' e^h(\kappa(kxk^{-1}))| \sup_{k \in K} \{\Xi_M(\mu(xk))^{-1} \|\Psi(h; D'': v; \mu(xk))\|\}. \end{aligned}$$

*Proof.* Let  $\omega_K$  be a compact subset of  $K$  such that for any continuous function on  $K/Z$ ,

$$\int_{K/Z} f(kZ) d(kZ) = \int_{\omega_K} f(kZ) dk.$$

Then we can write

$$\|E(P: \Psi: h; D: v: g; x)\| \leq \int_{\omega_K} \|D(\Psi_v(h: g; xk)\tau_{2,h}(k^{-1}))\| dk.$$

But

$$\begin{aligned} & \Psi_v(h; g_i; xk)\tau_{2,h}(k^{-1}) \\ &= e^h(k^{-1}\kappa(xk))\tau_1(\kappa(xk))\Psi_v(h; g^{\kappa(xk)^{-1}}; \mu(xk)a(xk)\tau_2(k^{-1})), \end{aligned}$$

so there are a finite subset  $S_1$  of  $D(iv^*)$  and a finite subset  $S_2$  of  $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$  so that

$$\begin{aligned} & \|D(\Psi_v(h; g_i; xk)\tau_{2,h}(k^{-1}))\| \\ & \leq \sum_{D', D'' \in S_1, g \in S_2} |D' e^h(\kappa(k^{-1}xk))| \|\Psi_v(h; D''; g_i; \mu(xk)a(xk))\|. \end{aligned}$$

Now as in [HW5, 9.5], we write  $g' = \kappa v v' b$  where  $\kappa \in \mathcal{U}(\mathfrak{k}_{\mathbb{C}})$ ,  $v \in \mathcal{U}(\mathfrak{m}_{\mathbb{C}})$ ,  $v' \in S(\mathfrak{a}_{\mathbb{C}})$ , and  $b \in \mathcal{U}(\mathfrak{n}_{\mathbb{C}})$ . Let  $S'$  denote the set of all  $v$  which occur in the decompositions of  $g' \in S_2$ . Then for all  $m \in M$ ,  $a \in A$ ,

$$\Psi_v(h; g_i; ma) = d\tau_{1,h}(\kappa)P(v)\Psi_v(h; v_i; ma)$$

where  $P$  is a polynomial in  $v$ . Now  $d\tau_{1,h}(\kappa)$  and all its derivatives in  $h$  grow polynomially in  $h$ , so that there are  $S$  as above and  $C, r \geq 0$  so that for all  $D'' \in S_1$ ,

$$\|\Psi_v(h; D''; g_i; ma)\| \leq C(1 + |h|)^r(1 + |v|)^r \sum_{D \in S} \|\Psi_v(h; D; v_i; ma)\|.$$

Now

$$\Psi_v(h; D; v_i; ma) = e^{(iv - \rho_P)(\log a)}\Psi(h; D; v_i; m).$$

Enlarge  $S$  so that  $S_1 \subset S$ . Thus,

$$\begin{aligned} & \|E(P; \Psi; h; D; v; g_i; x)\| \leq C(1 + |h|)^r(1 + |v|)^r \\ & \times \sum_{D_1, D_2 \in S, v \in S'} \int_{\omega_x} |D_1 e^h(\kappa(k^{-1}xk))| e^{|v_1 H_P(xk)|} e^{-\rho_P H_P(xk)} \|\Psi(h; D_2; v_i; \mu(xk))\| dk. \end{aligned}$$

Now as in [HW5, 9.6], there is  $C_0 \geq 0$  so that  $|v_1 H_P(xk)| \leq C_0 |v_1| \sigma(x)$  for all  $k \in K$ . Further,

$$\int_{K/Z} e^{-\rho_P H_P(xk)} \Xi(\mu(xk)) d(kZ) = \Xi(x) \quad \square$$

LEMMA 2.30. Let  $D \in D(iv^*)$ . Then there are constants  $C, c, r \geq 0$  so that for all  $x \in G, h \in \mathfrak{v}_\mathbb{C}^*$

$$\sup_{k \in K} |De^h(\kappa(k^{-1}xk))| \leq C(1 + \sigma_V(x))^r e^{|h_I|(c + \sigma_V(x))}.$$

*Proof.* By [HW5, 9.7], there are constants  $C, c$  so that

$$|e^h(\kappa(k^{-1}xk))| \leq C e^{|h_I|(\sigma_V(x) + c)}$$

for all  $x, k, h$ . But  $h \rightarrow e^h(\kappa(k^{-1}xk))$  is a holomorphic function of  $h$ . If we estimate its derivatives in  $h$  as in [HW5, 9.10] using radius  $(1 + \sigma_V(x))^{-1}$ , this gives the required estimate.  $\square$

When we decompose  $x = \kappa(x)\mu(x)a(x)n(x)$  the components  $\kappa(x)$  and  $\mu(x)$  are not unique. Write  $M$  in its Cartan decomposition as  $M = K_M cl(A_M^+)K_M$ . We will assume that the decomposition is chosen so that  $\mu(x) = a_M(x)\kappa_M(x)$  where  $a_M(x) \in cl(A_M^+)$  and  $\kappa_M(x) \in K_M$  has  $\sigma_V(\kappa_M(x))$  bounded independent of  $x \in G$ .

LEMMA 2.31. Let  $D \in D(iv^*), v \in \mathcal{U}(\mathfrak{m}_\mathbb{C})$ . Then there are constants  $C, r \geq 0$  so that for all  $x \in G, k \in K, h \in \mathcal{D}_\mathbb{C}$ ,

$$\Xi_M(\mu(xk))^{-1} \|\Psi(h; D : v; \mu(xk))\| \leq C(1 + |h|)^r (1 + \sigma(x))^r.$$

*Proof.*

$$\Xi_M(\mu(xk))^{-1} \Psi(h; v; \mu(xk)) = \Xi_M(a_M(xk))^{-1} \Psi(h; v; a_M(xk)) \tau_{2,h}(\kappa_M(xk)).$$

Thus it is enough to bound terms of the form  $\Xi_M(a_M(xk))^{-1} \|\Psi(h; D_1 : v; a_M(xk))\|$  and  $|D_2 e^h(\kappa_M(xk))|$  where  $D_1, D_2 \in D(iv^*)$ . But

$$|D_2 e^h(\kappa_M(xk))| \leq C(1 + \sigma_V(\kappa_M(xk)))^r e^{|h_I|\sigma_V(\kappa_M(xk))}$$

is bounded since  $\sigma_V(\kappa_M(xk))$  is bounded and  $|h_I|$  is bounded in  $\mathcal{D}_\mathbb{C}$ . Now for any  $a \in cl(A_M^+)$  and  $h \in \mathcal{D}_\mathbb{C}$ ,

$$\|\Psi(h; D_1 : v; a)\| \leq C(1 + |h|)^r (1 + \sigma(a))^r e^{-\rho_M(a)}.$$

This is proven for the case  $D_1 = 1$  in [HW5, 5.12]. In fact the same proof works for any  $D_1$  since the proof just reduces to the case that  $M$  is simple, connected, non-compact, and of hermitian type. In this case the estimates were proven in [HW3, 4] with derivatives in the  $h$  variable included. Now for any  $a \in cl(A_M^+)$ ,  $\Xi_M(a)^{-1} e^{-\rho_M(a)} \leq 1$ . Thus

$$\Xi_M(a_M(xk))^{-1} \|\Psi(h; D_1 : v; a_M(xk))\| \leq C(1 + |h|)^r (1 + \sigma(a_M(xk)))^r.$$

But  $\sigma(a_M(xk)) = \sigma(\mu(xk)) \leq C(1 + \sigma(xk))$  using Theorem 2.23. Finally,  $\sigma(xk) = \sigma(x)$  for all  $k$ .  $\square$

**PROOF OF THEOREM 2.27.** Combining Lemmas 2.28–2.31, for  $D \in D(iv^*)$  and  $g_1, g_2 \in \mathcal{U}(\mathfrak{g}_\mathbb{C})$  we have constants  $C, r, c, c_0$  so that for all  $h \in \mathcal{D}_\mathbb{C}$ ,  $v \in \mathfrak{a}_\mathbb{C}$ , and  $x \in G$ ,

$$\begin{aligned} & \|E(P; \Psi; h; D; v; g_1; x; g_2)\| \\ & \leq C(1 + |h|)^r(1 + |v|)^r(1 + \sigma_\nu(x))^r(1 + \sigma(x))^r \Xi(x) e^{c_0|v_I|\sigma(x)} e^{|h_I|(c + \sigma_\nu(x))}. \end{aligned}$$

Since  $|h_I|$  is bounded on  $\mathcal{D}_\mathbb{C}$ , the term  $e^{|h_I|c}$  is bounded by a constant. Since  $E(P; \Psi; h; D; v; g_1; x; g_2)$  is holomorphic in  $v$  we can estimate derivatives in  $v$  using the same method as in [HW5, 9.10]. This gives us additional factors of  $(1 + \sigma(x))$ . Finally, the terms with  $\sigma(x)$  and  $\sigma_\nu(x)$  can be combined and bounded by a term of the form  $(1 + \tilde{\sigma}(x))^r$ .  $\square$

### 3. Character identities

Let  $H = TA$  be a  $\theta$ -stable Cartan subgroup of  $G$ ,  $P = MAN$  a parabolic subgroup associated to  $H$ . Fix  $(\lambda, \chi) \in X(T)$  as in (2.10). Let  $\mathcal{C}$  be a Weyl chamber of  $it^*$  with respect to  $\Phi_M$ . Then we will write  $\pi(h; \mathcal{C})$  for the relative discrete series representation of  $M^0$  with Harish-Chandra parameter  $\lambda(h)$  if  $\lambda(h) \in \mathcal{C}$ , and for the limit of relative discrete series representation from  $\mathcal{C}$  corresponding to  $\lambda(h)$  if  $\lambda(h)$  is a boundary point of  $\mathcal{C}$ . Now for  $\nu$  in  $\mathfrak{a}^*$  we set  $\pi(H; h; \mathcal{C}; \nu) = \text{Ind}_{Z_M(M^0)M^0AN}^\mathbb{G}(\chi(h) \otimes \pi(h; \mathcal{C}) \otimes e^{i\nu} \otimes 1)$  and let  $\Theta(H; h; \mathcal{C}; \nu)$  be the character of  $\pi(H; h; \mathcal{C}; \nu)$ . By coherent continuation, we can extend the definition of  $\Theta(H; h; \mathcal{C}; \nu)$  to allow  $h$  to be any element of  $iv^*$ . However it is the character of a tempered representation of  $G$  only when  $\lambda(h) \in \text{cl}(\mathcal{C})$ .

Let  $\Phi(\lambda) = \{\alpha \in \Phi_M : \langle \alpha, \lambda \rangle = 0\}$ .

**LEMMA 3.1.**  $\Phi(\lambda)$  is a subroot system of  $\Phi_M$  of type  $A_1^k$  for some  $k \geq 0$  and every root in  $\Phi(\lambda)$  is non-compact. Further, if  $\mathcal{C}$  is any Weyl chamber with  $\lambda \in \text{cl}(\mathcal{C})$ , then for any  $\alpha \in \Phi(\lambda)$ , either  $\alpha$  or  $-\alpha$  is a simple root for the positive system of  $\Phi_M$  corresponding to  $\mathcal{C}$ .

*Proof.*  $\Phi(\lambda)$  is clearly a subroot system of  $\Phi_M$ . But since  $\lambda \in \Lambda_{M,1}$ , and every compact root of  $\Phi_M$  is in  $\Phi_{M,1}$ , every root in  $\Phi(\lambda)$  is non-compact. But if the sum of two non-compact roots is a root, it is compact. Thus the sum of two roots in  $\Phi(\lambda)$  cannot be a root, so  $\Phi(\lambda)$  is of type  $A_1^k$ . Now fix  $\mathcal{C}$  with  $\lambda \in \text{cl}(\mathcal{C})$ . Let  $\Delta$  denote the simple roots for the positive system  $\Phi_M^+$  of  $\Phi_M$  corresponding to  $\mathcal{C}$ . Then  $\alpha \in \Phi_M^+$  is in  $\Phi(\lambda)$  just in case  $\alpha$  is a sum of simple roots in  $\Delta \cap \Phi(\lambda)$ . But as above, no non-trivial sum of roots in  $\Phi(\lambda)$  is a root. Thus  $\Phi(\lambda) \cap \Phi_M^+ = \Phi(\lambda) \cap \Delta$ .  $\square$



Let  $F_0 = \Phi(\lambda) \cap \Phi_M^+$ . Then any subset  $F$  of  $F_0$  is a strongly orthogonal system of non-compact roots in  $\Phi_M$ . Let  $H_{M,F}$  denote the corresponding Cartan subgroup of  $M$ . That is, the complexified Lie algebra of  $H_{M,F}$  is obtained from that of  $T$  by Cayley transforms corresponding to the roots in  $F$ . Then  $H_F = H_{M,F}A = T_F A_F$  is a Cartan subgroup of  $G$ . Let  $P_F = M_F A_F N_F$  be a parabolic subgroup with split component  $A_F$ . Set  $M_F^* = M_F \cap M^0$ .

LEMMA 3.2. For any  $F \subseteq F_0$ ,  $Z_{M_F}(M_F^0) = Z_M(M^0)Z_{M_F^*}(M_F^0)$ . Further,  $Z_{M_F^*}(M_F^0) \subseteq T^0$  and  $Z_M(M^0) \cap Z_{M_F^*}(M_F^0) \subseteq Z_{M^0}$ .

*Proof.* Let  $\mathfrak{b}$  be a fundamental Cartan subalgebra of  $\mathfrak{g}$  and let  $SOS(H)$  denote the set of strongly orthogonal non-compact roots of  $(\mathfrak{g}_\mathbb{C}, \mathfrak{b}_\mathbb{C})$  used to define the Cayley transform  $c$  with  $c(\mathfrak{b}_\mathbb{C}) = \mathfrak{h}_\mathbb{C}$ . Then, as in [H1, §10], since  $F \subseteq \Phi_M \setminus \Phi_{M,1}$ ,  $SOS(H_F) = SOS(H) \cup c^{-1}F$ . Now  $T_F \subseteq T$  and the first statement of the lemma follows using the argument in [H1, 10.14]. But

$$Z_{M_F^*}(M_F^0) \subseteq T_F \cap M^0 \subseteq T \cap M^0 = T^0.$$

Finally,

$$Z_M(M^0) \cap Z_{M_F^*}(M_F^0) \subseteq Z_M(M^0) \cap M^0 = Z_{M^0}. \quad \square$$

Let  $F \subseteq F_0$ . Because of Lemma 3.2 we can define data for tempered representations of  $G$  corresponding to  $H_F$  as follows. Let  $\lambda_F = \lambda|_{\mathfrak{t}_F}$ . As in [H1, §9], an ordering  $\Phi_{M_F}^+$  can be chosen so that  $\rho_{M_F} = \rho_M|_{\mathfrak{t}_F}$ . Thus  $\lambda_F - \rho_{M_F} = (\lambda - \rho_M)|_{\mathfrak{t}_F}$ . Now since  $T_F \subseteq T$ ,  $\lambda_F - \rho_{M_F}$  is integral. Further, by (3.4) below,  $\lambda_F$  is  $\Phi_{M_F,1}$  non-singular. Thus  $\lambda_F \in \Lambda_{M_F,1}$ . Define  $\chi_F^0 = e^{\lambda - \rho_M}|_{Z_{M_F^*}(M_F^0)}$  and then define a representation of  $Z_F = Z_{M_F}(M_F^0) = Z_M(M^0)Z_{M_F^*}(M_F^0)$  by  $\chi_F = \chi \otimes \chi_F^0$ . Then  $(\lambda_F, \chi_F) \in X(T_F)$ . For  $h \in \mathfrak{iv}^*$ , set  $\lambda_F(h) = \lambda_F + h_{M_F}(h)$ ,  $\chi_F(h) = \chi_F \otimes e^h$ . Then for any chamber  $\mathcal{C}_F$  of  $\mathfrak{it}_F^*$ ,  $\pi_F(h: \mathcal{C}_F)$  denotes the relative discrete series or limit from  $\mathcal{C}_F$  of discrete series representation of  $M_F^0$  with Harish-Chandra parameter  $\lambda_F(h)$ . When  $\lambda_F(h) \in cl(\mathcal{C}_F)$ ,  $v \in \mathfrak{a}_F^*$ , we define

$$\pi(H_F: h: \mathcal{C}_F: v) = \text{Ind}_{Z_F M_F^0 A_F}(\chi_F(h) \otimes \pi_F(h: \mathcal{C}_F) \otimes e^{iv} \otimes 1). \quad (3.3a)$$

Finally, we let

$$\Theta(H_F: h: \mathcal{C}_F: v) \text{ be the character of } \pi(H_F: h: \mathcal{C}_F: v) \quad (3.3b)$$

when  $\lambda_F(h) \in \mathcal{C}_F$  and the coherent continuation of the character for arbitrary  $h \in \mathfrak{iv}^*$ .

Fix Cayley transforms  $c_F: \mathfrak{h}_\mathbb{C} \rightarrow \mathfrak{h}_{F,\mathbb{C}}$ . We will use these isomorphisms to identify linear functions on  $\mathfrak{h}_{F,\mathbb{C}}$  for any  $F \subseteq F_0$ . Write  $F^c = F_0 \setminus F$ .

LEMMA 3.4.  $F^c = \{\alpha \in \Phi_{M_F}^+ : \langle \alpha, \lambda_F \rangle = 0\}$ .

*Proof.*  $\Phi_{M_F}$  can be identified with  $\{\alpha \in \Phi_M : \alpha \perp F\}$ . Further, using this identification, for  $\alpha \perp F$ ,  $\langle \alpha, \lambda \rangle = \langle \alpha, \lambda_F \rangle$ .  $\square$

Given any chamber  $\mathcal{C}$  of  $it^*$  and  $\alpha \in \Phi_M$ , set  $\varepsilon_\alpha(\mathcal{C}) = \text{sign}\langle \tau, \alpha \rangle$ ,  $\tau \in \mathcal{C}$ . Now let  $\mathcal{C} \in C(\lambda)$ , the set of all chambers with  $\lambda \in cl(\mathcal{C})$ . Then for all  $\alpha \in \Phi_M^+ \setminus F_0$ ,  $\varepsilon_\alpha(\mathcal{C}) = \text{sign}\langle \lambda, \alpha \rangle$ . Thus there is a bijection between  $C(\lambda)$  and

$$\Sigma = \{(\varepsilon_\alpha)_{\alpha \in F_0} : \varepsilon_\alpha = \pm 1 \text{ for all } \alpha \in F_0\} \quad (3.5a)$$

so that  $\varepsilon \in \Sigma \leftrightarrow \mathcal{C} = \mathcal{C}(\varepsilon)$  if  $\varepsilon_\alpha = \varepsilon_\alpha(\mathcal{C})$  for all  $\alpha \in F_0$ . Similarly for any  $F \subseteq F_0$  there is a unique chamber  $\mathcal{C}_F(\varepsilon)$  with  $\lambda_F \in cl(\mathcal{C}_F)$  and  $\varepsilon_\alpha(\mathcal{C}_F) = \varepsilon_\alpha$  for all  $\alpha \in F^c$ . Given  $\varepsilon \in \Sigma$ ,  $h \in iv^*$ ,  $v \in \alpha_F^*$ , set

$$\Theta(F : h : \varepsilon : v) = \Theta(H_F : h : \mathcal{C}_F(\varepsilon) : v). \quad (3.5b)$$

Note for all  $\alpha \in F_0$ ,  $\langle \lambda(h), \alpha \rangle = \langle h_M(h), \alpha \rangle$ . Let

$$\mathcal{H}_\alpha = \{h \in iv^* : \langle h_M(h), \alpha \rangle = 0\}. \quad (3.5c)$$

Define an equivalence relation on  $F_0$  by  $\alpha \sim \beta$  if  $\mathcal{H}_\alpha = \mathcal{H}_\beta$ . Define  $\varepsilon_\alpha(h) = \text{sign}\langle h_M(h), \alpha \rangle \in \{1, -1, 0\}$ .

LEMMA 3.6. *The positive system  $\Phi_M^+$  can be chosen so that  $\alpha \sim \beta$  if and only if  $\varepsilon_\alpha(h) = \varepsilon_\beta(h)$  for all  $h \in iv^*$ .*

*Proof.* Decompose  $\Phi_M = \Phi_1 \cup \dots \cup \Phi_k$  into simple factors. Assume that  $1 \leq i \leq p$  are the indices such that the subgroup  $M_i$  of  $M^0$  corresponding to  $\Phi_i$  is non-compact, simply connected, and of hermitian type. Then  $F_0 \subset \Phi_1 \cup \dots \cup \Phi_p$ . Assume for  $1 \leq i \leq p$  that  $\Phi_i^+$  is chosen so that there is a unique non-compact simple root. Then for any non-compact root  $\alpha \in \Phi_i^+$ ,  $\langle \alpha, h_M(h) \rangle = n(\alpha)\langle \beta, h_M(h) \rangle$  where  $\beta$  is the non-compact simple root and  $n(\alpha) > 0$  is the coefficient of  $\beta$  in the expansion of  $\alpha$  in terms of the simple roots. Thus  $\varepsilon_\alpha(h) = \varepsilon_\beta(h)$  for all  $h \in iv^*$ . Let  $F_i = \Phi(\lambda) \cap \Phi_i^+$ . Then each equivalence class is a union of certain of the  $F_i$  and  $\varepsilon_\alpha(h) = \varepsilon_i(h)$  is independent of  $\alpha \in F_i$ . Suppose  $F_1 \cup \dots \cup F_r$  is an equivalence class. Fix an ordering of  $\Phi_1^+$  as above. Now for  $2 \leq i \leq r$ ,  $\varepsilon_i(h) = 0$  if and only if  $\varepsilon_1(h) = 0$ . Thus there is  $\sigma = \pm 1$  such that  $\varepsilon_i(h) = \sigma \varepsilon_1(h)$  for all  $h$ . But if  $\sigma = -1$ , we can replace  $\Phi_i^+$  by  $-\Phi_i^+$ .  $\square$

Write  $F_0 = F_0^1 \cup \dots \cup F_0^m$  where the  $F_0^i$  are the distinct equivalence classes in  $F_0$ . Let  $1 \leq i \leq m$  and define  $\mathcal{H}_i = \mathcal{H}_\alpha$ ,  $\alpha \in F_0^i$ . Fix  $h_i \in iv^*$  such that  $\alpha(h_i) > 0$  for all  $\alpha \in F_0^i$ . For any smooth function  $f$  on  $iv^*$ , define

$$\partial/\partial h_i f(h) = d/dt|_{t=0} f(h + th_i). \quad (3.7a)$$

Now for all  $\alpha \in F_0^i$ , pick  $\mu_\alpha \in \mathfrak{a}_{F_0}^*$  such that  $\mu_\alpha|_{\mathfrak{a}} = 0$ ,  $\langle \mu_\alpha, c_{F_0}\alpha \rangle = \langle h_i, \alpha \rangle$ ,  $\langle \mu_\alpha, c_{F_0}\beta \rangle = 0$  for all  $\beta \in F_0$ ,  $\beta \neq \alpha$ . For any  $F$  we can consider  $\mu_\alpha \in \mathfrak{a}_F^*$  by restriction from  $\mathfrak{a}_{F_0}$  to  $\mathfrak{a}_F$ . Now for any smooth function  $f$  on  $\mathfrak{a}_F^*$ , define

$$\partial/\partial\mu_\alpha f(v) = d/dt|_{t=0} f(v + t\mu_\alpha). \tag{3.7b}$$

Given  $\alpha \in F_0$  and  $\varepsilon \in \Sigma$ , define  $s_\alpha \varepsilon \in \Sigma$  by  $(s_\alpha \varepsilon)_\beta = \varepsilon_\beta$ ,  $\beta \neq \alpha$ ,  $(s_\alpha \varepsilon)_\alpha = -\varepsilon_\alpha$ . For  $F \subset F_0$  and  $\alpha \in F^c$ , let  $F(\alpha) = F \cup \{\alpha\}$ . Given  $v_F \in \mathfrak{a}_F^*$ , let  $(v_F, 0) \in \mathfrak{a}_{F(\alpha)}^*$  be defined by  $(v_F, 0)|_{\mathfrak{a}_F} = v_F$ ,  $\langle (v_F, 0), c_{F(\alpha)}\alpha \rangle = 0$ . Then [H1, 10.18] can be written as follows.

LEMMA 3.8. Fix  $F \subset F_0$ ,  $\alpha \in F^c \cap F_0^i$ ,  $\varepsilon \in \Sigma$ . Then for all  $k \geq 0$ ,

$$\begin{aligned} & (\partial/\partial h_i)^k (\Theta(F:h_0:\varepsilon:v_F) + \Theta(F:h_0:s_\alpha \varepsilon:v_F)) \\ &= (\partial/\partial h_i \pm i\partial/\partial \mu_\alpha)^k \Theta(F(\alpha):h_0:\varepsilon:(v_F, 0)) \end{aligned} \tag{3.8}$$

for all  $v_F \in \mathfrak{a}_F^*$ ,  $h_0 \in \mathcal{H}_i$ .

In general, the terms in Lemma 3.8 are not derivatives along continuous families of tempered representations. The problem is that for an arbitrary  $\varepsilon \in \Sigma$ , there may be no  $h \in i\mathfrak{v}^*$  with  $\lambda(h) \in \mathcal{C}(\varepsilon)$ . For  $\varepsilon \in \Sigma$ , write

$$\mathcal{D}(\varepsilon) = \{h \in i\mathfrak{v}^* : \varepsilon_\alpha(h) = \varepsilon_\alpha, \alpha \in F_0\}. \tag{3.9a}$$

Let

$$\Sigma_0 = \{\varepsilon \in \Sigma : \mathcal{D}(\varepsilon) \neq \emptyset\}. \tag{3.9b}$$

Now if for any  $\varepsilon \in \Sigma$ ,  $F \subseteq F_0$  we set  $\mathcal{D}_F(\varepsilon) = \{h \in i\mathfrak{v}^* : \varepsilon_\alpha(h) = \varepsilon_\alpha \text{ for all } \alpha \in F_0 \setminus F\}$ , then  $h \in \mathcal{D}_F(\varepsilon)$  if and only if  $\lambda_F(h) \in \mathcal{C}_F(\varepsilon)$ , and  $\mathcal{D}(\varepsilon) = \mathcal{D}_\emptyset(\varepsilon)$ . Let  $\varepsilon \in \Sigma_0$ . We will say that  $h_0 \in \mathcal{H}_i \cap cl(\mathcal{D}(\varepsilon))$  is semiregular if  $h_0 \notin \mathcal{H}_j$  for  $1 \leq j \leq m$ ,  $j \neq i$ . We will say that  $\mathcal{H}_i$  is a wall of  $\mathcal{D}(\varepsilon)$  if there are semiregular elements in  $\mathcal{H}_i \cap cl(\mathcal{D}(\varepsilon))$ . Write  $\Sigma_i$  for the set of all  $\varepsilon \in \Sigma_0$  such that  $\mathcal{H}_i$  is a wall of  $\mathcal{D}(\varepsilon)$ . For any  $h \in \mathcal{D}_F(\varepsilon)$ , write  $\Theta(F:h:v_F)$  for the tempered character  $\Theta(F:h:\varepsilon:v_F)$ . Now if  $\varepsilon \in \Sigma_i$ , for any semiregular  $h_0 \in \mathcal{H}_i \cap cl(\mathcal{D}_F(\varepsilon))$  we can interpret  $(\partial/\partial h_i)^k \Theta(F:h_0:\varepsilon:v_F)$  as the limit of  $(\partial/\partial h_i)^k \Theta(F:h:v_F)$  as  $h \rightarrow h_0$ ,  $h \in \mathcal{D}_F(\varepsilon)$ , thus as an actual limit of derivatives along a continuous family of tempered characters. Now the problem is that even if  $\varepsilon \in \Sigma_i$ ,  $s_\alpha \varepsilon \in \Sigma_i$  only if the equivalence class of  $F_0$  containing  $\alpha$  has no other elements.

For any  $\varepsilon \in \Sigma$ ,  $1 \leq i \leq m$ , define  $\varepsilon^\pm(i) \in \Sigma$  by

$$\varepsilon^\pm(i)_\alpha = \begin{cases} \varepsilon_\alpha, & \text{if } \alpha \in F_0 \setminus F_0^i; \\ \pm 1, & \text{if } \alpha \in F_0^i. \end{cases} \tag{3.10}$$

Now for any  $1 \leq i \leq m$  and  $\varepsilon \in \Sigma_i$ , both of  $\varepsilon^\pm(i) \in \Sigma_i$ ,  $\varepsilon$  is equal to one of  $\varepsilon^\pm(i)$ , and  $\mathcal{D}(\varepsilon^+(i))$  and  $\mathcal{D}(\varepsilon^-(i))$  are separated only by the wall  $\mathcal{H}_i$ . Fix  $E \subseteq F_0$  and a conjugacy class  $F_0^i$  of  $F_0$ . Write  $E(i) = E \cup F_0^i$ . For any  $F$  such that  $E \subseteq F \subseteq E(i)$  define  $(v_E, 0) \in \alpha_F^*$  by  $(v_E, 0)|_{\alpha_E} = v_E$ ,  $\langle (v_E, 0), c_F \alpha \rangle = 0$  for all  $\alpha \in F \setminus E$ .

**THEOREM 3.11.** Fix  $E \subseteq F_0$ ,  $1 \leq i \leq m$ ,  $\varepsilon \in \Sigma_i$ . Then for all  $k \geq 0$ ,

$$\begin{aligned} & (\partial/\partial h_i)^k (\Theta(E: h_0: \varepsilon^+(i): v_E) + (-1)^{|E(i) \setminus E|+1} \Theta(E: h_0: \varepsilon^-(i): v_E)) \\ &= \sum_{E \subset F \subseteq E(i)} c_{|F \setminus E|} \left( \partial/\partial h_i + i \sum_{\alpha \in F \setminus E} \pm \partial/\partial \mu_\alpha \right)^k \\ & \quad \times (\Theta(F: h_0: \varepsilon^+(i): (v_E, 0)) + (-1)^{|E(i) \setminus F|} \Theta(F: h_0: \varepsilon^-(i): (v_E, 0))) \end{aligned}$$

for all  $v \in \alpha_E^*$ ,  $h_0 \in \mathcal{H}_i \cap \text{cl}(\mathcal{D}_E(\varepsilon))$ . Here for all  $p \geq 0$ ,  $c_p = (d/dx)^p \tanh(x/2)|_{x=0}$ .

The remainder of this section is devoted to the proof of this theorem together with some consequences which will be needed in Section 7.

In order to prove Theorem 3.11 it is necessary to iterate Lemma 3.8. First, it will be convenient to work not with  $\Theta(F: h: \varepsilon: v_F)$ , but with

$$\tilde{\Theta}(F: h: \varepsilon: v_F) = \sigma_F(\varepsilon) \Theta(F: h: \varepsilon: v_F) \quad (3.12a)$$

where

$$\sigma_F(\varepsilon) = \prod_{\alpha \in F^c} \varepsilon_\alpha. \quad (3.12b)$$

Now if  $\alpha \in F^c \cap F_0^i$  and  $\varepsilon \in \Sigma$  such that  $\varepsilon_\alpha = 1$ , then  $\sigma_F(s_\alpha \varepsilon) = -\sigma_F(\varepsilon)$  and  $\sigma_{F(\alpha)}(\varepsilon) = \sigma_F(\varepsilon)$ . Thus in this case, Lemma 3.8 can be rewritten as

$$\begin{aligned} & (\partial/\partial h_i)^k (\tilde{\Theta}(F: h_0: \varepsilon: v_F) - \tilde{\Theta}(F: h_0: s_\alpha \varepsilon: v_F)) \\ &= (\partial/\partial h_i \pm i \partial/\partial \mu_\alpha)^k \tilde{\Theta}(F(\alpha): h_0: \varepsilon: (v_F, 0)). \end{aligned} \quad (3.13)$$

Further, if  $E \subseteq F_0$ ,  $1 \leq i \leq m$ , then for  $E \subseteq F \subseteq E(i)$ ,  $F^c = (F_0 \setminus E(i)) \cup (E(i) \setminus F)$  where  $(F_0 \setminus E(i)) \subseteq (F_0 \setminus F_0^i)$  and  $(E(i) \setminus F) \subseteq F_0^i$ . Thus  $\sigma_F(\varepsilon^+(i)) = \prod_{\alpha \in F_0 \setminus E(i)} \varepsilon_\alpha$  and  $\sigma_{F(\varepsilon^-(i))} = (-1)^{|E(i) \setminus F|} \prod_{\alpha \in F_0 \setminus E(i)} \varepsilon_\alpha$ . Thus Theorem 3.11 can be rewritten as

$$\begin{aligned} & (\partial/\partial h_i)^k (\tilde{\Theta}(E: h_0: \varepsilon^+(i): v_E) - \tilde{\Theta}(E: h_0: \varepsilon^-(i): v_E)) \\ &= \sum_{E \subset F \subseteq E(i)} c_{|F \setminus E|} \left( \partial/\partial h_i + i \sum_{\alpha \in F \setminus E} \pm \partial/\partial \mu_\alpha \right)^k \\ & \quad \times (\tilde{\Theta}(F: h_0: \varepsilon^+(i): (v_E, 0)) + \tilde{\Theta}(F: h_0: \varepsilon^-(i): (v_E, 0))). \end{aligned} \quad (3.14)$$

Suppose as above we write  $\tilde{\Theta}(F:h:v_F) = \tilde{\Theta}(F:h:\varepsilon:v_F)$  when  $h \in \mathcal{D}(\varepsilon)$ . The point is that on  $H_F$ ,  $\tilde{\Theta}(F:h:v_F)$  extends to a smooth function of  $h \in iv^*$ . On other conjugacy classes of Cartan subgroups of course it does not. Now in the case that the equivalence class of  $F_0$  containing  $\alpha$  has no other elements, (3.13) can be interpreted as giving the jumps of  $\tilde{\Theta}(F:h:v_F)$  and its normal derivatives as  $h$  crosses the wall  $\mathcal{H}_\alpha$ . In the general case, (3.14) gives a formula for jumps of  $\tilde{\Theta}(E:h:v_E)$  and its normal derivatives as  $h$  crosses the wall  $\mathcal{H}_i$  from  $\mathcal{D}(\varepsilon^+(i))$  to  $\mathcal{D}(\varepsilon^-(i))$  in terms of derivatives of families  $\tilde{\Theta}(F:h:v_F)$  where  $E \subset F \subseteq E(i)$ . These are now matching conditions involving only characters of tempered representations and their derivatives along continuous families of tempered representations. Thus they will give matching conditions for the Fourier transforms of Schwartz class functions.

Fix  $E \subseteq F_0$ , a conjugacy class  $F_0^i$  of  $F_0$ , elements  $h_0 \in \mathcal{H}_i$ ,  $v_E \in \mathfrak{a}_E^*$ , and  $k \geq 0$ . For any  $F$  such that  $E \subseteq F \subseteq E(i)$  and  $\varepsilon \in \Sigma$ , define

$$d(F:\varepsilon) = (\partial/\partial h_i + \sum_{\alpha \in F \setminus E} \pm i\partial/\partial \mu_\alpha)^k \tilde{\Theta}(F:h_0:\varepsilon:(v_E, 0)). \tag{3.15}$$

LEMMA 3.16. *Let  $E \subseteq F \subseteq E(i)$ . Then for all  $\alpha \in E(i) \setminus F$  and  $\varepsilon \in \Sigma$  such that  $\varepsilon_\alpha = 1$ ,*

$$d(F:\varepsilon) - d(F:s_\alpha \varepsilon) = d(F(\alpha):\varepsilon).$$

*Proof.* Write

$$\left( \partial/\partial h_i + \sum_{\beta \in F \setminus E} \pm i\partial/\partial \mu_\beta \right)^k = \sum_{p=0}^k \binom{k}{p} \left( \sum_{\beta \in F \setminus E} \pm i\partial/\partial \mu_\beta \right)^{k-p} (\partial/\partial h_i)^p$$

and

$$\left( \partial/\partial h_i + \sum_{\beta \in F(\alpha) \setminus E} \pm i\partial/\partial \mu_\beta \right)^k = \sum_{p=0}^k \binom{k}{p} \left( \sum_{\beta \in F \setminus E} \pm i\partial/\partial \mu_\beta \right)^{k-p} (\partial/\partial h_i \pm i\partial/\partial \mu_\alpha)^p.$$

By (3.13), we have for any  $p \geq 0$ ,

$$\begin{aligned} & (\partial/\partial h_i)^p (\tilde{\Theta}(F:h_0:\varepsilon:v_F) - \tilde{\Theta}(F:h_0:s_\alpha \varepsilon:v_F)) \\ &= (\partial/\partial h_i \pm i\partial/\partial \mu_\alpha)^p \tilde{\Theta}(F(\alpha):h_0:\varepsilon:(v_F, 0)) \end{aligned}$$

for all  $v_F \in \mathfrak{a}_F^*$ . Thus, differentiating both sides with respect to

$(\sum_{\beta \in F \setminus E} \pm i\partial/\partial\mu_\beta)^{k-p}$  and evaluating at  $v_F = (v_E, 0)$ , for any  $0 \leq p \leq k$ ,

$$\begin{aligned} & \left( \sum_{\beta \in F \setminus E} \pm i\partial/\partial\mu_\beta \right)^{k-p} (\partial/\partial h_i)^p (\tilde{\Theta}(F: h_0: \varepsilon: (v_E, 0)) - \tilde{\Theta}(F: h_0: s_\alpha \varepsilon: (v_E, 0))) \\ &= \left( \sum_{\beta \in F \setminus E} \pm i\partial/\partial\mu_\beta \right)^{k-p} (\partial/\partial h_i \pm i\partial/\partial\mu_\alpha)^p \tilde{\Theta}(F(\alpha): h_0: \varepsilon: (v_E, 0)). \quad \square \end{aligned}$$

For any  $E \subseteq F \subseteq E(i)$ ,  $\varepsilon \in \Sigma$ , define  $F^\pm(i) = F \cup \{\alpha \in E(i): \varepsilon_\alpha = \pm 1\}$ .

LEMMA 3.17. For any  $\varepsilon \in \Sigma$ ,  $E \subseteq F \subseteq E(i)$ ,  $d(F: \varepsilon) = \sum_{F \subseteq F' \subseteq F^+(i)} d(F': \varepsilon^-(i))$ .

*Proof.* Write  $F^+(i) \setminus F = \{\alpha_1, \dots, \alpha_r\}$ . Then the proof will be by induction on  $r$ . If  $r = 0$ , then  $F = F^+(i)$  and  $d(F: \varepsilon) = d(F: \varepsilon^-(i))$  since  $\varepsilon_\alpha = \varepsilon^-(i)_\alpha$  for all  $\alpha \in F_0 \setminus F$ . Assume  $r \geq 1$ . Then we can write

$$\begin{aligned} d(F: \varepsilon) - d(F: \varepsilon^-(i)) &= \sum_{i=0}^{r-1} (d(F: s_{\alpha_i} \cdots s_{\alpha_1} \varepsilon) - d(F: s_{\alpha_{i+1}} \cdots s_{\alpha_1} \varepsilon)) \\ &= \sum_{i=0}^{r-1} d(F(\alpha_{i+1}): s_{\alpha_i} \cdots s_{\alpha_1} \varepsilon) \end{aligned}$$

using Lemma 3.16 since  $(s_{\alpha_i} \cdots s_{\alpha_1} \varepsilon)_{\alpha_{i+1}} = \varepsilon_{\alpha_{i+1}} = 1$ . But

$$\{\alpha \in F(\alpha_{i+1})(i) \setminus F(\alpha_{i+1}): (s_{\alpha_i} \cdots s_{\alpha_1} \varepsilon)_\alpha = 1\} = \{\alpha_{i+2}, \dots, \alpha_r\}.$$

Thus using the induction hypothesis,

$$d(F(\alpha_{i+1}): s_{\alpha_i} \cdots s_{\alpha_1} \varepsilon) = \sum_{F(\alpha_{i+1}) \subseteq F' \subseteq F(\alpha_{i+1}) \cup \{\alpha_{i+2}, \dots, \alpha_r\}} d(F': \varepsilon^-(i)).$$

But now the lemma follows because for any

$$F \subseteq F' \subseteq F^+(i) = F \cup \{\alpha_1, \dots, \alpha_r\}, \quad F' \neq F,$$

let  $i+1$  be the smallest index such that  $\alpha_{i+1} \in F'$ . Then  $F(\alpha_{i+1}) \subseteq F' \subseteq F(\alpha_{i+1}) \cup \{\alpha_{i+2}, \dots, \alpha_r\}$ , but for  $j < i$ ,  $F(\alpha_{j+1}) \not\subseteq F'$ , while for  $j > i$ ,  $F' \not\subseteq F(\alpha_{j+1}) \cup \{\alpha_{j+2}, \dots, \alpha_r\}$ . Thus each term occurs exactly once.  $\square$

COROLLARY 3.18. For any  $\varepsilon \in \Sigma_i$ ,  $E \subseteq F \subseteq E(i)$ ,

$$d(F: \varepsilon^+(i)) = \sum_{F \subseteq F' \subseteq E(i)} d(F': \varepsilon^-(i)).$$

COROLLARY 3.19. For any  $\varepsilon \in \Sigma_i$ ,  $E \subseteq F \subseteq E(i)$ ,

$$d(F : \varepsilon^-(i)) = \sum_{F' \subseteq E(i)} (-1)^{|F' \setminus F|} d(F' : \varepsilon^+(i)).$$

*Proof.* Define  $d'(F : \varepsilon)$  as in (3.15) using  $\Theta$  instead of  $\tilde{\Theta}$ . Then there is a constant  $c = \pm 1$  so that  $d'(F : \varepsilon^+(i)) = cd(F : \varepsilon^+(i))$  and  $d'(F : \varepsilon^-(i)) = (-1)^{|E(i) \setminus F|} cd(F : \varepsilon^-(i))$  for all  $E \subseteq F \subseteq E(i)$ . Thus (3.18) can be rewritten as

$$d'(F : \varepsilon^+(i)) = \sum_{F' \subseteq E(i)} (-1)^{|E(i) \setminus F'|} d'(F' : \varepsilon^-(i)).$$

Now statements regarding  $d'$  are symmetric with respect to interchanging  $\varepsilon^+(i)$  and  $\varepsilon^-(i)$ , so we have

$$d'(F : \varepsilon^-(i)) = \sum_{F' \subseteq E(i)} (-1)^{|E(i) \setminus F'|} d'(F' : \varepsilon^+(i)).$$

Now translate back to  $d$  to obtain the result. □

LEMMA 3.20. For any  $E \subseteq F \subseteq E(i)$ ,

$$d(F : \varepsilon^+(i)) - d(F : \varepsilon^-(i)) = \sum_{F' \subseteq E(i)} c_{|F' \setminus F|} (d(F' : \varepsilon^+(i)) + d(F' : \varepsilon^-(i)))$$

where the constants  $c_p$ ,  $p \geq 0$ , are given by  $c_{2p} = 0$ ,  $p \geq 0$ , and

$$c_{2p+1} = 1/2 - 1/2 \sum_{q=0}^{p-1} \binom{2p+1}{2q+1} c_{2q+1}, \quad p \geq 0.$$

*Proof.* The proof is by induction on  $|E(i) \setminus F|$ . If  $F = E(i)$ , then  $d(F : \varepsilon^+(i)) = d(F : \varepsilon^-(i))$  and both sides of the equation are 0. Assume that  $|E(i) \setminus F| = n \geq 1$ . Combining (3.18) and (3.19) we can write

$$\begin{aligned} d(F : \varepsilon^+(i)) - d(F : \varepsilon^-(i)) &= \\ 1/2 \sum_{F' \subseteq E(i)} &((-1)^{|F' \setminus F|+1} d(F' : \varepsilon^+(i)) + d(F' : \varepsilon^-(i))). \end{aligned}$$

Now for  $F \subset F' \subseteq E(i)$ ,  $|F' \setminus F|$  odd,

$$(-1)^{|F' \setminus F|+1} d(F' : \varepsilon^+(i)) + d(F' : \varepsilon^-(i)) = d(F' : \varepsilon^+(i)) + d(F' : \varepsilon^-(i))$$

while for  $F \subset F'' \subseteq E(i)$ ,  $|F'' \setminus F|$  even,

$$(-1)^{|F'' \setminus F|+1} d(F'' : \varepsilon^+(i)) + d(F'' : \varepsilon^-(i)) = -(d(F'' : \varepsilon^+(i)) - d(F'' : \varepsilon^-(i))).$$

But since  $|E(i) \setminus F''| < n$ , we can use the induction hypothesis to write

$$d(F'' : \varepsilon^+(i)) - d(F'' : \varepsilon^-(i)) = \sum_{F'' \subset F' \subseteq E(i)} c_{|F' \setminus F''|} (d(F' : \varepsilon^+(i)) + d(F' : \varepsilon^-(i)))$$

where  $c_{|F' \setminus F''|} = 0$  if  $|F' \setminus F''|$  is even. Thus we can write

$$\begin{aligned} & d(F : \varepsilon^+(i)) - d(F : \varepsilon^-(i)) \\ &= 1/2 \sum_{F \subset F' \subseteq E(i), |F' \setminus F| \text{ odd}} \left( 1 - \sum_{F \subset F'' \subset F'} c_{|F' \setminus F''|} \right) (d(F' : \varepsilon^+(i)) + d(F' : \varepsilon^-(i))). \end{aligned}$$

Thus  $c_{|F' \setminus F|} = 0$  if  $|F' \setminus F|$  is even, and when  $|F' \setminus F|$  is odd,

$$c_{|F' \setminus F|} = 1/2 - 1/2 \sum_{F \subset F'' \subset F'} c_{|F' \setminus F''|}.$$

Suppose  $F \subset F'$  with  $|F' \setminus F| = 2p + 1$ . Then for  $0 \leq q \leq p - 1$ , there are  $\binom{2p+1}{2q+1}$  subsets  $F''$  with  $F \subset F'' \subset F'$  and  $|F' \setminus F''| = 2q + 1$ . Thus

$$c_{2p+1} = 1/2 - 1/2 \sum_{q=0}^{p-1} \binom{2p+1}{2q+1} c_{2q+1}. \quad \square$$

LEMMA 3.21. Suppose  $c_p, p \geq 0$ , are given by  $c_{2p} = 0, p \geq 0$ , and

$$c_{2p+1} = 1/2 - 1/2 \sum_{q=0}^{p-1} \binom{2p+1}{2q+1} c_{2q+1}, \quad p \geq 0.$$

Then for all  $p \geq 0, c_p = (d/dx)^p \tanh(x/2)|_{x=0}$ .

*Proof.* Using the recursion relation we can write

$$\begin{aligned} \sum_{p=0}^{\infty} c_p \frac{x^p}{p!} &= \frac{1}{2} \sum_{p=0}^{\infty} \frac{x^{2p+1}}{(2p+1)!} - \frac{1}{2} \sum_{p=0}^{\infty} \sum_{q=0}^{p-1} \frac{c_{2q+1} x^{2p+1}}{(2(p-q))!(2q+1)!} \\ &= \frac{1}{2} \sinh x - \frac{1}{2} \sum_{q=0}^{\infty} \frac{c_{2q+1} x^{2q+1}}{(2q+1)!} \sum_{p=q+1}^{\infty} \frac{x^{2(p-q)}}{2(q-p)!} \\ &= \frac{1}{2} \sinh x - \frac{1}{2} (\cosh x - 1) \sum_{p=0}^{\infty} c_p \frac{x^p}{p!}. \end{aligned}$$

Thus

$$\sum_{p=0}^{\infty} c_p \frac{x^p}{p!} = \frac{\sinh x}{\cosh x + 1} = \tanh(x/2). \quad \square$$



LEMMA 3.22. Suppose for  $p \geq 0$ ,  $c_p = (d/dx)^p \tanh(x/2)|_{x=0}$ . Then for every  $p \geq 0$ ,  $\sum_{q=0}^p 2^{2q+1} \binom{2p+1}{2q+1} c_{2q+1} = 1$ .

*Proof.* Write

$$\begin{aligned} & \sum_{p=0}^{\infty} \left( \sum_{q=0}^p 2^{2q+1} \binom{2p+1}{2q+1} c_{2q+1} \right) \frac{x^{2p+1}}{(2p+1)!} = \sum_{q=0}^{\infty} \frac{c_{2q+1} 2^{2q+1} x^{2q+1}}{(2q+1)!} \sum_{p=0}^{\infty} \frac{x^{2p}}{(2p)!} \\ & = \tanh(2x/2) \cosh x = \sinh x = \sum_{p=0}^{\infty} \frac{x^{2p+1}}{(2p+1)!}. \end{aligned}$$

Thus  $\sum_{q=0}^p 2^{2q+1} \binom{2p+1}{2q+1} c_{2q+1} = 1$  for all  $p$ . □

LEMMA 3.23. Fix  $E \subseteq F_0$  and suppose for each  $E \subseteq F \subseteq E(i)$  we have complex numbers  $a^{\pm}(F)$  satisfying the following condition. For each  $E \subseteq F \subseteq E(i)$ ,

$$a^+(F) - a^-(F) = \sum_{F \subset F' \subseteq E(i)} c_{|F' \setminus F|} (a^+(F') + a^-(F')).$$

Then

$$\sum_{E \subseteq F \subseteq E(i)} 2^{-|F|} ((-1)^{|F \setminus E|} a^+(F) - a^-(F)) = 0.$$

*Proof.* Write

$$\begin{aligned} & \sum_{E \subseteq F \subseteq E(i)} 2^{-|F|} ((-1)^{|F \setminus E|} a^+(F) - a^-(F)) \\ & = \sum_{E \subset F' \subseteq E(i), |F' \setminus E| \text{ odd}} -2^{-|F'|} (a^+(F') + a^-(F')) \\ & \quad + \sum_{E \subseteq F \subseteq E(i), |F \setminus E| \text{ even}} -2^{-|F|} (a^+(F) - a^-(F)). \end{aligned}$$

But in this second term,

$$a^+(F) - a^-(F) = \sum_{F \subset F' \subseteq E(i)} c_{|F' \setminus F|} (a^+(F') + a^-(F')).$$

But since  $c_{|F' \setminus F|} = 0$  unless  $|F' \setminus F|$  and hence  $|F' \setminus E|$  are odd,

$$\begin{aligned} & \sum_{E \subseteq F \subseteq E(i)} 2^{-|F|} ((-1)^{|F \setminus E|} a^+(F) - a^-(F)) \\ & = \sum_{E \subset F' \subseteq E(i), |F' \setminus E| \text{ odd}} 2^{-|F'|} (a^+(F') + a^-(F')) \left( -1 + \sum_{E \subseteq F \subset F', |F' \setminus F| \text{ odd}} 2^{|F' \setminus F|} c_{|F' \setminus F|} \right). \end{aligned}$$

But suppose  $|F' \setminus E| = 2p + 1$ . Then there are  $\binom{2p+1}{2q+1}$  sets  $F$  with  $E \subseteq F \subset F'$ ,  $|F' \setminus F| = 2q + 1$ . Thus

$$\sum_{E \subseteq F \subset F', |F' \setminus F| \text{ odd}} 2^{|F' \setminus F|} c_{|F' \setminus F|} = \sum_{q=0}^p \binom{2p+1}{2q+1} 2^{2q+1} c_{2q+1} = 1$$

by (3.22). □

#### 4. Elementary mixed wave packets

Fix  $H = TA$  a  $\theta$ -stable Cartan subgroup and  $(\lambda, \chi) \in X(T)$ ,  $\tau_1, \tau_2 \in \widehat{K}(\chi)$ . Let  $U(0)$  be a neighborhood of 0 in  $iv^*$  which is small enough that no  $h \in U(0)$  is more singular than  $h = 0$ . (See (4.6) for the precise definition.) We assume that the Plancherel function  $m^*(H : \lambda : \chi : h : \nu)$  defined in (4.5) corresponding to  $(\lambda, \chi)$  is jointly smooth as a function of  $(h, \nu) \in U(0) \times \mathfrak{a}^*$ . As in (3.3) we define  $F_0$  and  $H_F = T_F A_F$ ,  $(\lambda_F, \chi_F) \in X(T_F)$  for every  $F \subseteq F_0$ . Note that for any  $F \subset F_0$ ,  $\chi$  and  $\chi_F$  have the same  $Z$ -character so that  $\widehat{K}(\chi) = \widehat{K}(\chi_F)$ . Let  $\tau_1, \tau_2 \in \widehat{K}(\chi)$ ,  $W = W(\tau_1 : \tau_2)$ . Suppose for each  $F \subseteq F_0$  we have a function

$$\Phi(F) : iv^* \times \mathfrak{a}_F^* \times G \rightarrow W.$$

Then we will say that

$$\Phi(x) = \sum_{F \subseteq F_0} \int_{iv^*} \int_{\mathfrak{a}_F^*} \Phi(F : h : \nu_F : x) m(H_F : \lambda_F : \chi_F : h : \nu_F) d\nu_F dh \tag{4.1a}$$

is a ( $W$ -valued) elementary mixed wave packet if the functions  $\Phi(F)$  satisfy the following conditions. First, there is a compact subset  $\omega \subset U(0)$  so that for all  $F \subseteq F_0$ ,  $\nu_F \in \mathfrak{a}_F^*$ ,  $x \in G$ ,  $h \in iv^*$ ,

$$\Phi(F : h : \nu_F : x) = 0, \quad h \notin \omega. \tag{4.1b}$$

Second, let

$$W_F(\lambda, \chi) = \{w \in W(G, H_F) : w\lambda_F = \lambda_F, w\chi_F = \chi_F\}.$$

Then for all  $w \in W_F(\lambda, \chi)$ ,  $\nu_F \in \mathfrak{a}_F^*$ ,  $x \in G$ ,  $h \in iv^*$ ,

$$\Phi(F : h : w\nu_F : x) = \Phi(F : h : \nu_F : x). \tag{4.1c}$$

Third, for each  $F \subseteq F_0$ ,  $\varepsilon \in \Sigma_0$ , let  $\Phi(F : \varepsilon)$  denote the restriction of  $\Phi(F)$  to

$\mathcal{D}_F(\varepsilon) \times \mathfrak{a}_F^* \times G$  where  $\Sigma_0, \mathcal{D}_F(\varepsilon)$  are defined as in (3.9). Then, using the notation of (2.17), (2.18), there are finitely many functions  $\Psi_i \in \mathcal{S}(M_F^\dagger: \lambda_F: \chi_F: \mathcal{D}_F(\varepsilon): W)$ ,  $\alpha_i \in \mathcal{C}(\mathcal{D}_F(\varepsilon) \times \mathfrak{a}_F^*)_0$  so that

$$\Phi(F: \varepsilon: h: v_F: x) = \sum_i \alpha_i(h: v_F) E(P_F: \Psi_i: h: v_F: x) \tag{4.1d}$$

for all  $(h, v_F, x) \in \mathcal{D}_F(\varepsilon) \times \mathfrak{a}_F^* \times G$ . Finally, we require that the functions  $\Phi(F: h: v_F: x)$  satisfy the matching conditions of (3.20). That is, fix  $E \subseteq F_0, k \geq 0, 1 \leq i \leq m, \varepsilon \in \Sigma_i$ . For  $E \subseteq F \subseteq E(i) = E \cup F_0^i, v_E \in \mathfrak{a}_E^*, h_0 \in \mathcal{H}_i \cap cl(\mathcal{D}_E(\varepsilon)), x \in G$ , write

$$A^\pm(F: h_0: v_E: x) = \left( \partial/\partial h_i - i \sum_{\alpha \in F \setminus E} \partial/\partial \mu_\alpha \right)^k \sigma_{F(\varepsilon^\pm(i))} \Phi(F: \varepsilon^\pm(i): h_0: (v_E, 0): x).$$

Then we require that for all  $E \subseteq F \subseteq E(i)$ ,

$$\begin{aligned} & A^+(F: h_0: v_E: x) - A^-(F: h_0: v_E: x) \\ &= \sum_{F \subset F' \subseteq E(i)} c_{F \setminus F'} (A^+(F': h_0: v_E: x) + A^-(F': h_0: v_E: x)) \end{aligned} \tag{4.1e}$$

where  $c_k, k \geq 0$ , are defined as in (3.11). Finally, if  $\Phi$  is a  $W$ -valued elementary mixed wave packet and  $w^* \in W^*$ , we say that

$$\phi(x) = \langle \Phi(x), w^* \rangle \tag{4.1f}$$

is a scalar-valued elementary mixed wave packet.

**THEOREM 4.2.** *Every  $f \in \mathcal{C}(G)_K$  is the sum of finitely many scalar-valued elementary mixed wave packets.*

Suppose  $\Phi(x)$  is defined as in (4.1a) and for  $h \in iv^*$  set

$$\Phi(h: x) = \sum_{F \subseteq F_0} \int_{\mathfrak{a}_F^*} \Phi(F: h: v_F: x) m(H_F: \lambda_F: \chi_F: h: v_F) dv_F. \tag{4.3a}$$

Clearly  $\Phi(h: x)$  is  $(\tau_{1,h}, \tau_{2,h})$ -spherical and

$$\Phi(x) = \int_{iv^*} \Phi(h: x) dh. \tag{4.3b}$$

In (7.3) we will prove the following theorem.

**THEOREM 4.4.** *Let  $\Phi(x)$  be a  $W$ -valued elementary mixed wave packet. Then  $(h, x) \rightarrow \Phi(h: x)$  is jointly smooth on  $iv^* \times G$ .*

Because of Proposition 2.8, this is the first step in proving that  $\Phi \in \mathcal{C}(G: W)$ . The estimates needed to complete the proof will be deferred to another paper. The remainder of this section is devoted to the proof of Theorem 4.2.

Let  $H = TA$  be a  $\theta$ -stable Cartan subgroup of  $G$ ,  $(\lambda, \chi) \in X(T)$ . The Plancherel function  $m(h: v) = m(H: \lambda: \chi: h: v)$  is defined as follows. Let  $\Phi^+$  denote a set of positive roots for  $\Phi = \Phi(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$  and let  $\Phi_R^+ = \{\alpha \in \Phi^+ : \alpha \text{ takes real values on } \mathfrak{h}\}$ . For  $\alpha \in \Phi_R^+$ , let  $\Phi_\alpha^+ = \{\beta \in \Phi^+ : \beta|_{\mathfrak{a}} = c\alpha \text{ for some } c \neq 0\}$ . Let  $\Phi_R^+ = \bigcup_{\alpha \in \Phi_R^+} \Phi_\alpha^+$  and let  $\mathcal{D}$  be a connected component of  $\{h \in iv^* : \langle \lambda(h), \alpha \rangle \neq 0 \text{ for all } \alpha \in \Phi_M^+\}$ . Then there is a constant  $c(H: \lambda: \chi: \mathcal{D})$  so that for  $h \in \mathcal{D}$ ,  $v \in \mathfrak{a}^*$ , we have

$$m(h: v) = c(H: \lambda: \chi: \mathcal{D})m^*(h: v) \tag{4.5a}$$

where

$$m^*(h: v) = m^*(H: \lambda: \chi: h: v) = \prod_{\beta \in \Phi^+ \setminus \Phi_R^+} \langle \lambda(h) + iv, \beta \rangle \prod_{\alpha \in \Phi_R^+} m_\alpha^*(h: v) \tag{4.5b}$$

and

$$\begin{aligned} m_\alpha^*(h: v) &= m_\alpha^*(H: \lambda: \chi: h: v) \\ &= \prod_{\beta \in \Phi_\alpha^+} \langle \lambda(h) + iv, \beta \rangle \cdot \frac{\sinh \pi v_\alpha}{\cosh \pi v_\alpha - \varepsilon_\alpha(h)}, \quad \alpha \in \Phi_R^+. \end{aligned} \tag{4.5c}$$

Here for

$$\alpha \in \Phi_R^+, \quad v_\alpha = \frac{2\langle v, \alpha \rangle}{\langle \alpha, \alpha \rangle} \quad \text{and} \quad \varepsilon_\alpha(h)$$

is a continuous function of  $h$  defined as follows. For  $\alpha \in \Phi_R^+$ , let  $H_\alpha^* \in \mathfrak{a}$  be dual to  $2\alpha/\langle \alpha, \alpha \rangle$  under the Killing form. Let  $X_\alpha, Y_\alpha$  be elements of the root spaces  $\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}$  respectively so that  $\theta(X_\alpha) = Y_\alpha$  and  $[X_\alpha, Y_\alpha] = H_\alpha^*$ . Write  $Z_\alpha = X_\alpha - Y_\alpha$  and set  $\gamma_\alpha = \exp \pi Z_\alpha$ . Then  $\gamma_\alpha \in Z_M(M^0)$ . Let  $\rho_\alpha = 1/2 \sum_{\beta \in \Phi_\alpha^+} \beta(H_\alpha^*)$ . Then

$$\varepsilon_\alpha(h) = (-1)^{\rho_\alpha} \frac{\text{trace}[\chi(h: \gamma_\alpha) + \chi(h: \gamma_\alpha^{-1})]}{2 \det \chi(h)} \tag{4.5d}$$

For  $\alpha \in \Phi_R^+$ ,  $m_\alpha^*(h: v)$  is clearly jointly smooth on  $iv^* \times \mathfrak{a}^*$  except possibly at points  $(h, v)$  where  $v_\alpha = 0$  and  $\varepsilon_\alpha(h) = 1$ .

Now fix  $h_0 \in iv^*$ . We want to define a neighborhood  $U(h_0)$  of  $h_0$  which is small enough that no point of  $U(h_0)$  is more singular than  $h_0$  in the sense that

matching conditions satisfied by the characters  $\Theta(H : \lambda : \chi : h : \nu)$  at  $h_0$  will give all possible matching conditions in  $U(h_0)$ . For any  $h \in iv^*$ , let  $\Phi_M^+(h) = \{\beta \in \Phi_M^+ : \langle \lambda(h), \beta \rangle = 0\}$ . Define  $F_0 = \Phi_M^+(h_0)$  and for  $F \subseteq F_0$  define  $H_F$ ,  $\lambda(h_0)_F$ ,  $\chi(h_0)_F$  as in (3.3). For each  $F \subseteq F_0$  define  $\Phi_{F,R}^+$  and  $m_\alpha^*(F : h : \nu_F)$ ,  $\varepsilon_\alpha(F : h)$ ,  $\alpha \in \Phi_{F,R}^+$  as in (4.5) for  $H_F$  and  $(\lambda(h_0)_F, \chi(h_0)_F)$ . Let  $\Phi_{F,R}^+(1 : h) = \{\alpha \in \Phi_{F,R}^+ : \varepsilon_\alpha(F : h) = 1\}$ .

For  $\varepsilon > 0$ , let  $U(h_0) = U_\varepsilon(h_0) = \{h \in iv^* : |h - h_0| < \varepsilon\}$ . We will assume that  $\varepsilon$  is small enough that for all  $h \in U(h_0)$ ,

$$\Phi_M^+(h) \subseteq \Phi_M^+(h_0); \tag{4.6a}$$

$$\Phi_{F,R}^+(1 : h) \subseteq \Phi_{F,R}^+(1 : h_0) \quad \text{for all } F \subseteq F_0; \tag{4.6b}$$

$$|\varepsilon_\alpha(F : h) - \varepsilon_\alpha(F : h_0)| < 1 \quad \text{for all } F \subseteq F_0, \alpha \in \Phi_{F,R}^+. \tag{4.6c}$$

LEMMA 4.7. Define  $U(h_0)$  as in (4.6), and for  $F \subseteq F_0$ , let

$$\Phi_{M_F}^+(h) = \{\alpha \in \Phi_{M_F}^+ : \langle \alpha, \lambda_F(h) \rangle = 0\}.$$

Then for all  $h \in U(h_0)$ ,  $\Phi_{M_F}^+(h) \subseteq \Phi_{M_F}^+(h_0)$ . Further, let  $\alpha \in \Phi_{F,R}^+$ ,  $\nu_0 \in \alpha^*$ . Then if  $m_\alpha^*(F : h : \nu)$  is jointly smooth at  $(h_0, \nu_0)$ , it is jointly smooth at  $(h, \nu_0)$  for all  $h \in U(h_0)$ .

*Proof.* Using (3.4) we see that we can identify  $\Phi_{M_F}^+(h)$  with  $\{\alpha \in \Phi_M^+ : \alpha \perp F \text{ and } \langle \alpha, \lambda(h) \rangle = 0\} = \Phi_{M_F}^+ \cap \Phi_M^+(h)$ . Thus for all  $h \in U(h_0)$ , using (4.6a),  $\Phi_{M_F}^+(h) = \Phi_{M_F}^+ \cap \Phi_M^+(h) \subseteq \Phi_{M_F}^+ \cap \Phi_M^+(h_0) = \Phi_{M_F}^+(h_0)$ .

Now fix  $\alpha \in \Phi_{F,R}^+$  and assume that  $m_\alpha^*(F : h : \nu)$  is jointly smooth at  $(h_0, \nu_0)$ . Then one of the following possibilities occurs. First, if  $\varepsilon_\alpha(F : h_0) \neq 1$ , then by definition of  $U(h_0)$ ,  $\varepsilon_\alpha(F : h) \neq 1$  for every  $h \in U(h_0)$ , so  $m_\alpha^*(F : h : \nu)$  is jointly smooth at  $(h, \nu_0)$  for every  $h \in U(h_0)$ . Second, it is possible that  $\varepsilon_\alpha(F : h) = 1$  for all  $h \in iv^*$ . In this case,

$$m_\alpha^*(F : h : \nu) = \frac{i \langle \alpha, \alpha \rangle}{2} \prod_{\beta \in \Phi_\alpha^+ \setminus \{\alpha\}} \langle \lambda(h) + i\nu, \beta \rangle \cdot \frac{\nu_\alpha \sinh \pi \nu_\alpha}{\cosh \pi \nu_\alpha - 1}$$

is jointly smooth on  $iv^* \times \alpha^*$ . Finally, suppose  $\varepsilon_\alpha(F : h_0) = 1$ , but  $\varepsilon_\alpha(F : h)$  is not a constant function of  $h \in iv^*$ . Then, as in [H1, 10.3],  $\alpha$  must be a root of a simple factor  $\mathfrak{g}_1$  of  $\mathfrak{g}$  with  $G_1$  simply connected, non-compact, and of hermitian type. Let  $\mathfrak{b}$  be a fundamental Cartan subalgebra of  $\mathfrak{g}$  and let  $SOS(H_F)$  denote the set of strongly orthogonal non-compact roots of  $(\mathfrak{g}_C, \mathfrak{b}_C)$  used to define the Cayley transform  $c$  with  $c(\mathfrak{b}_C) = \mathfrak{h}_{F,C}$ . Then  $\alpha' = c^{-1}\alpha \in SOS(H_F)$ . Set

$$h_\alpha = \frac{2 \langle h, \alpha' \rangle}{\langle \alpha', \alpha' \rangle}.$$

Then  $h \rightarrow h_\alpha$  is a non-trivial linear function on  $i\mathfrak{v}^*$  and  $\varepsilon_\alpha(F:h) = \cos \pi(h-h_0)_\alpha$ . Now since  $m_\alpha^*(F:h:v)$  is jointly smooth at  $(h_0, v_0)$ ,  $v_\alpha^2 + (h-h_0)_\alpha^2$  divides  $\prod_{\beta \in \Phi_\alpha^+} \langle \lambda(h) + iv, \beta \rangle$ . Thus there are  $\gamma, \bar{\gamma} \in \Phi_\alpha^+$  and a constant  $c$  so that  $\langle \lambda(h) + iv, \gamma \rangle = c((h-h_0)_\alpha + iv_\alpha)$  and  $\langle \lambda(h) + iv, \bar{\gamma} \rangle = \pm c((h-h_0)_\alpha - iv_\alpha)$ . But now for  $h \in U(h_0)$ , if  $\varepsilon_\alpha(F:h) \neq 1$ , then  $m_\alpha^*(F:h:v)$  is jointly smooth. But if  $\varepsilon_\alpha(F:h) = \cos \pi(h-h_0)_\alpha = 1$ , then  $(h-h_0)_\alpha \in 2\mathbb{Z}$ . Suppose  $|(h-h_0)_\alpha| \geq 2$ . Then there is  $h' \in U(h_0)$  with  $|(h'-h_0)_\alpha| = 1$  so that  $\varepsilon_\alpha(F:h') = -1$ . But this contradicts the assumption that  $|\varepsilon_\alpha(F:h') - \varepsilon_\alpha(F:h_0)| < 1$  for all  $h' \in U(h_0)$ . Thus  $(h-h_0)_\alpha = 0$  so that again  $m_\alpha^*(F:h:v)$  is jointly smooth at  $(h, v_0)$ .  $\square$

Fix  $\alpha \in \Phi_R^+$  such that  $\varepsilon_\alpha(0) = 1$ , but  $\varepsilon_\alpha(h)$  is not identically 1, and use the notation in the proof of (4.7). Let  $H' = T'A'$  be the Cartan subgroup of  $G$  with  $SOS(H') = SOS(H) \setminus \{c^{-1}\alpha\}$ . Thus  $c_\alpha(\mathfrak{h}'_C) = \mathfrak{h}_C$  where  $c_\alpha$  is the Cayley transform corresponding to  $\alpha$ . Write  $\alpha' = c_\alpha^{-1}\alpha \in \Phi(\mathfrak{g}, \mathfrak{h}')$  and let  $P' = M'A'N'$  be a parabolic subgroup corresponding to  $H'$ .

LEMMA 4.8. *There is a unique  $\lambda' \in (i\mathfrak{t}')^*$  such that  $\lambda' - \rho_{M'}$  is integral,  $\lambda'|_{\mathfrak{t}} = \lambda$ , and  $\langle \lambda', \alpha' \rangle = 0$ . Further,  $\lambda' \in \Lambda_{M',1}$  if and only if  $(h, v) \rightarrow m_\alpha^*(h:v)$  is not jointly smooth at  $(0, 0)$ .*

*Proof.* The existence of  $\lambda'$  is proven in [H1, 10.13]. The uniqueness is clear since restriction to  $\mathfrak{t}$  gives a bijection between  $\{\lambda' \in (i\mathfrak{t}')^* : \langle \lambda', \alpha' \rangle = 0\}$  and  $i\mathfrak{t}^*$ .

Now suppose that  $m_\alpha^*(h:v)$  is jointly smooth at  $(0, 0)$ . Then as in the proof of (4.7) there are  $\gamma \in \Phi_\alpha^+$ ,  $c \in \mathbb{R}$ , such that  $\langle \lambda(h) + iv, \gamma \rangle = c(h_\alpha + iv_\alpha)$ . In particular,  $\langle \lambda, \gamma \rangle = 0$ . Write  $\gamma' = c_\alpha^{-1}\gamma$ . Note  $\gamma' \in \Phi_{M'}^+$  since the restriction of  $\gamma$  to  $\mathfrak{a}$  is a multiple of  $\alpha$ . But

$$\langle \lambda', \gamma' \rangle = \langle \lambda, \gamma \rangle + \frac{\langle \lambda', \alpha' \rangle}{\langle \alpha', \alpha' \rangle} \langle \gamma', \alpha' \rangle = 0.$$

Thus  $\gamma' \in \Phi_{M'}^+(0)$ . But  $\alpha' \in \Phi_{M'}^+(0)$  also and  $\langle \gamma', \alpha' \rangle \neq 0$ . Thus by (3.1),  $\lambda' \notin \Lambda_{M',1}$ .

Conversely, suppose  $\lambda' \notin \Lambda_{M',1}$ . Since  $\lambda' - \rho_{M'}$  is integral, there is  $\gamma' \in \Phi_{M',1}$  so that  $\langle \lambda', \gamma' \rangle = 0$ . Hence  $\langle \lambda, \gamma \rangle = 0$  where  $\gamma = c_\alpha \gamma'$ . Since  $\lambda \in \Lambda_{M,1}$ ,  $\gamma \notin \Phi_{M,1} = \{c_\alpha \beta' : \beta' \in \Phi_{M',1}, \langle \beta', \alpha' \rangle = 0\}$ . Thus  $\langle \gamma', \alpha' \rangle \neq 0$  so  $\gamma \in \Phi_\alpha^+$ . But

$$\langle v, \gamma \rangle = \frac{\langle \gamma, \alpha \rangle}{\langle \alpha, \alpha \rangle} \langle v, \alpha \rangle = \frac{\langle \gamma, \alpha \rangle}{2} v_\alpha$$

and

$$\langle \lambda(h), \gamma \rangle = \langle h_M(h), \gamma \rangle = \langle h_M(h), \gamma' \rangle - \frac{\langle h_M(h), \alpha' \rangle}{\langle \alpha', \alpha' \rangle} \langle \alpha, \gamma \rangle = -\frac{\langle \gamma, \alpha \rangle}{2} h_\alpha$$

since  $\gamma' \in \Phi_{M',1}$  means that  $\langle h_{M'}(h), \gamma' \rangle = 0$ . Thus

$$\langle \lambda(h) + iv, \gamma \rangle = \frac{-\langle \gamma, \alpha \rangle}{2} (h_\alpha - iv_\alpha).$$

Now  $\prod_{\beta \in \Phi_\alpha^+ \setminus \{\alpha\}} \langle \lambda(h) + iv, \beta \rangle$  is real-valued, so that this implies that  $h_\alpha^2 + v_\alpha^2$  divides  $\prod_{\beta \in \Phi_\alpha^+} \langle \lambda(h) + iv, \beta \rangle$ . Thus  $m_\alpha^*(h : v)$  is jointly smooth at  $(0, v)$  for all  $v \in \mathfrak{a}^*$ . □

**LEMMA 4.9.** *Let  $H = TA$  be a  $\theta$ -stable Cartan subgroup,  $(\lambda, \chi) \in X(T)$ , and define  $F_0, H_F, \lambda_F, \chi_F, F \subseteq F_0$  as in (3.3). Suppose  $F \neq \emptyset$  and let  $\alpha' \in F, \alpha = c_F \alpha' \in \Phi_{\mathfrak{R}}^+(\mathfrak{g}, \mathfrak{h}_F)$ . Then  $m_\alpha^*(H_F : \lambda_F : \chi_F : h : v_F)$  is not jointly smooth at  $(0, 0)$ . In particular,  $\varepsilon_\alpha(F : 0) = 1$ .*

*Proof.* We must first show that  $\varepsilon_\alpha(F : 0) = 1$  if  $\alpha' \in F$ . Since  $\alpha' \in F, \alpha$  must come from a simple factor of  $G$  which is non-compact, simply connected, and of hermitian type. Thus  $\gamma_\alpha$  is central in  $Z_{M_F}(M_F^0)$ . Let  $\Gamma_\alpha$  be the central subgroup of  $Z_{M_F}(M_F^0)$  generated by  $\gamma_\alpha$  and let  $\zeta$  be the  $\Gamma_\alpha$ -character of  $\chi_F(0)$  so that  $\chi_F(h : \gamma) = e^h(\gamma)\zeta(\gamma)\chi_F(0 : 1)$  for all  $\gamma \in \Gamma_\alpha, h \in i\mathfrak{v}^*$ . Then

$$\varepsilon_\alpha(F : h) = (-1)^{\rho_\alpha} \frac{e^h(\gamma_\alpha)\zeta(\gamma_\alpha) + e^h(\gamma_\alpha^{-1})\zeta(\gamma_\alpha^{-1})}{2}.$$

Thus  $\varepsilon_\alpha(F : 0) = 1$  just in case  $\zeta(\gamma_\alpha) = \zeta(\gamma_\alpha^{-1}) = (-1)^{\rho_\alpha}$ . Now since  $\gamma_\alpha \in Z_{M_F}(M_F^0) \subseteq T^0$ , using (3.3),  $\zeta(\gamma_\alpha) = e^{\lambda - \rho_M(\gamma_\alpha)} = e^{-\rho_M(\gamma_\alpha)}$  since  $\gamma_\alpha = \exp(\pi i H_\alpha^*)$  and  $\langle \lambda, \alpha' \rangle = 0$ . Thus to prove that  $\varepsilon_\alpha(F : 0) = 1$  it suffices to show that  $e^{-\rho_M(\gamma_\alpha)} = (-1)^{\rho_\alpha}$ . Now this is proven in [H1, 10.13] in the case that  $F = F' = \{\alpha'\}$ . Thus  $\varepsilon_\alpha(F' : 0) = 1$ . But for general  $F$  such that  $\alpha' \in F$ , it is proven in (5.5) that  $\varepsilon_\alpha(F : 0) = \varepsilon_\alpha(F' : 0)$ .

Now as in (4.7) we have  $\varepsilon_\alpha(F : h) = \cos \pi h_\alpha$  so that

$$m_\alpha^*(H_F : \lambda_F : \chi_F : h : v_F) = \prod_{\beta \in \Phi_\alpha^+} \langle \lambda_F(h) + iv_F, \beta \rangle \cdot \frac{\sinh \pi v_\alpha}{\cosh \pi v_\alpha - \cos \pi h_\alpha}.$$

Suppose  $m_\alpha^*(h : v_F)$  is jointly smooth at  $(0, 0)$ . Then as in (4.7) there are  $\gamma \in \Phi_\alpha^+, c \in \mathbf{R}$ , so that  $\langle \lambda_F(h) + iv_F, \gamma \rangle = c(h_\alpha + iv_\alpha)$ . But then as in the proof of (4.8),  $c_F^{-1}\gamma \in F_0 = \Phi_{M'}^+(0)$ . But  $\alpha' \in F_0$  also and  $\langle \alpha', c_F^{-1}\gamma \rangle \neq 0$ . This contradicts (3.1). □

**THEOREM 4.10.** *Suppose  $H = TA$  is a  $\theta$ -stable Cartan subgroup of  $G$  and  $(\lambda, \chi) \in X(T)$ . Then there are a  $\theta$ -stable Cartan subgroup  $H' = T'A'$  of  $G$ ,  $(\lambda', \chi') \in X(T')$ , and subset  $F \subseteq F_0 = \Phi_{M'}^+(0)$  such that  $H = H'_F, \lambda = \lambda'_F, \chi = \chi'_F$  and  $m^*(H' : \lambda' : \chi' : h : v')$  is jointly smooth at  $(0, v')$  for all  $v' \in (\mathfrak{a}')^*$ . Further,  $H'$  is unique up to conjugacy and once a choice of  $H'$  has been made,  $\lambda', \chi'$ , and  $F$  are unique.*

In order to prove the theorem we will need some preparation. Write  $\Phi_R^+(0) = \{\alpha \in \Phi_R^+ : m_\alpha^*(H : h : v) = m_\alpha^*(H : \lambda : \chi : h : v) \text{ is not jointly smooth at } (0, 0)\}$ . In particular,  $\varepsilon_\alpha(H : 0) = 1$ , but  $\varepsilon_\alpha(H : h)$  is not identically 1, for all  $\alpha \in \Phi_R^+(0)$ . Fix  $\alpha \in \Phi_R^+(0)$  and define  $H_1, \lambda_1 \in it^*$  as in (4.8). Define  $\chi_1 = \chi|_{Z_{M_1}(M_1^?)}$ . Identify linear functionals on  $\mathfrak{h}$  and  $\mathfrak{h}_1$  via  $c_\alpha$ . Then the set  $\Phi_{1,R}^+$  of positive real roots of  $H_1$  is identified with  $\{\beta \in \Phi_R^+ : \langle \beta, \alpha \rangle = 0\}$  and  $\alpha_1^*$  is identified with  $\{v \in \alpha^* : \langle v, \alpha \rangle = 0\}$ . For every  $\beta \in \Phi_{1,R}^+$ , write  $m_\beta^*(H_1 : h : v) = m_\beta^*(H_1 : \lambda_1 : \chi_1 : h : v)$  and  $\varepsilon_\beta(H_1 : h)$  for the term  $\varepsilon_\beta(h)$  occurring in  $m_\beta^*(H_1 : h : v)$ .

LEMMA 4.11. *Let  $\beta \in \Phi_{1,R}^+$  and suppose that  $m_\beta^*(H : h : v)$  is jointly smooth at  $(0, 0)$ . Then  $m_\beta^*(H_1 : h : v)$  is jointly smooth at  $(0, 0)$ .*

*Proof.* Suppose  $\alpha, \beta$  are both long roots in the same simple factor of  $\Phi_R$ . Then the result is proven in (6.1). Otherwise, by (5.5),  $\varepsilon_\beta(H_1 : h) = \varepsilon_\beta(H : h)$ . Thus the lemma is obvious if  $\varepsilon_\beta(H : 0) \neq 1$  or if  $\varepsilon_\beta(H : h) = 1$  for all  $h \in iv^*$ . Thus we assume that  $\varepsilon_\beta(H : h) = \cos \pi h_\beta$  is not identically 1. Now since  $m_\beta^*(H : h : v)$  is jointly smooth at  $(0, 0)$  there is  $\gamma \in \Phi_\beta^+$  so that  $\langle \lambda(h) + iv, \gamma \rangle = c(h_\beta + iv_\beta)$ . Now  $\langle \gamma, \alpha \rangle = 0$  so that  $\gamma \in \Phi_{1,\beta}^+ = \{\gamma \in \Phi^+ : \gamma|_{\alpha_1^*} = c\beta \text{ for some } c \neq 0\}$  and  $\langle \lambda_1(h) + iv_1, \gamma \rangle = \langle \lambda(h) + iv_1, \gamma \rangle = c(h_\beta + i(v_1)_\beta)$  for all  $v_1 \in \alpha_1^*$ . Thus  $m_\beta^*(H_1 : h : v)$  is jointly smooth at  $(0, 0)$ . □

Suppose now that  $H, (\lambda, \chi)$  are as in the theorem. Recall that  $c^{-1}\Phi_R^+(0) \subseteq SOS(H)$  and that every root in  $\Phi_R^+(0)$  is a long root in a simple factor  $\Phi(\alpha)$  of  $\Phi_R$  of type  $A_1$  or  $C_n$ . (When  $\Phi(\alpha)$  is of type  $A_1$  we consider all roots to be long.) We will define an equivalence relation on  $\Phi_R^+(0)$  as follows. First, we set  $\alpha \sim \beta$  if  $\alpha, \beta \in \Phi_R^+(0)$  are in the same simple factor of  $\Phi_R$ , that is if  $\Phi(\alpha) = \Phi(\beta)$ . Now let  $\alpha, \beta \in \Phi_R^+(0)$  such that  $\Phi(\alpha) \neq \Phi(\beta)$ . Then  $\alpha \sim \beta$  if and only if there are  $\gamma, \bar{\gamma} \in \Phi$  such that

$$\langle \gamma, \bar{\gamma} \rangle = 0; \tag{4.12a}$$

$$\gamma + \bar{\gamma} \in \text{span}(\alpha, \beta); \tag{4.12b}$$

$$s_\gamma s_{\bar{\gamma}} \alpha = \beta; \tag{4.12c}$$

$$\gamma - \bar{\gamma} \in it^* \quad \text{and} \quad \langle \gamma - \bar{\gamma}, \lambda \rangle = 0. \tag{4.12d}$$

LEMMA 4.13. *The relation  $\sim$  is an equivalent relation on  $\Phi_R^+(0)$ .*

*Proof.* Clearly  $\alpha \sim \alpha$  and  $\alpha \sim \beta$  if and only if  $\beta \sim \alpha$ . Now suppose  $\alpha \sim \beta$  and  $\beta \sim \delta$ . If  $\Phi(\alpha) = \Phi(\beta) = \Phi(\delta)$ , then obviously  $\alpha \sim \delta$ . Suppose that  $\Phi(\alpha) = \Phi(\beta) \neq \Phi(\delta)$ . Then  $\Phi(\alpha)$  is of type  $C_n$  and  $\alpha, \beta$  are long roots. Let  $w$  be the reflection in the Weyl group of  $C_n$  that exchanges  $\alpha$  and  $\beta$  and fixes all other long roots. Let  $\gamma, \bar{\gamma}$  be defined as in (4.12) for  $\beta$  and  $\delta$ . Let  $\gamma' = w\gamma, \bar{\gamma}' = w\bar{\gamma}$ . Then, since  $w\delta = \delta$  and  $w$  acts trivially on  $it^*$  it is easy to check that  $\gamma', \bar{\gamma}'$  satisfy (4.12) for  $\alpha, \delta$ . Thus  $\alpha \sim \delta$ . Finally, suppose  $\Phi(\alpha) \neq \Phi(\beta)$  and  $\Phi(\beta) \neq \Phi(\delta)$ . If  $\Phi(\alpha) = \Phi(\delta)$  then



$\alpha \sim \delta$ . Thus we can assume that  $\Phi(\alpha) \neq \Phi(\beta)$ . In particular,  $\alpha, \beta, \delta$  are mutually orthogonal. Define  $\gamma_1, \bar{\gamma}_1$  as in (4.12) for  $\alpha, \beta$  and  $\gamma_2, \bar{\gamma}_2$  as in (4.12) for  $\beta, \delta$ . Define  $w_1 = s_{\gamma_1} s_{\bar{\gamma}_1}$  and set  $\gamma_3 = w_1 \gamma_2, \bar{\gamma}_3 = w_1 \bar{\gamma}_2$ . Then  $\gamma_3, \bar{\gamma}_3$  satisfy (4.12) for  $\alpha, \delta$  since  $w_1 \alpha = \beta, w_1 \beta = \alpha, w_1 \delta = \delta, w_1(it^*) = it^*$ , and  $w_1 \lambda = \lambda$ . Thus we have  $\alpha \sim \delta$ .  $\square$

LEMMA 4.14. *Let  $\alpha \neq \beta \in \Phi_R^+(0)$  and define  $H_1, (\lambda_1, \chi_1) \in X(T_1)$  as above with respect to  $\alpha$ . Then  $\alpha \sim \beta$  if and only if  $m_\beta^*(H_1 : h : v)$  is jointly smooth at  $(0, 0)$ .*

*Proof.* Suppose first that  $\Phi(\alpha) = \Phi(\beta)$ . Then  $\alpha \sim \beta$ . But it follows from (6.1) that  $\varepsilon_\beta(H_1 : 0) = -\varepsilon_\beta(H : 0) = -1$  so that  $m_\beta^*(H_1 : h : v)$  is always jointly smooth at  $(0, 0)$ .

Assume now that  $\Phi(\alpha) \neq \Phi(\beta)$ . Then  $\varepsilon_\beta(H_1 : h) = \varepsilon_\beta(H : h)$  for all  $h \in iv^*$  so that we can apply (4.8) to  $H_1, (\lambda_1, \chi_1)$  and  $\beta$  to obtain  $H_2, (\lambda_2, \chi_2)$ . That is,  $SOS(H_2) = SOS(H_1) \setminus \{c^{-1}\beta\} = SOS(H) \setminus \{c^{-1}\alpha, c^{-1}\beta\}$ . Note that we would obtain the same  $H_2, \lambda_2, \chi_2$  if we had started by defining  $H'_1, \lambda'_1, \chi'_1$  with  $SOS(H'_1) = SOS(H) \setminus \{c^{-1}\beta\}$  and then had applied (4.8) to  $H'_1, (\lambda'_1, \chi'_1)$  and  $\alpha$ . Again, we identify linear functions on  $\mathfrak{h}, \mathfrak{h}_1, \mathfrak{h}'_1,$  and  $\mathfrak{h}_2$  via the Cayley transforms  $c_\alpha$  and  $c_\beta$ .

Assume first that  $\alpha \sim \beta$ . Define  $\gamma, \bar{\gamma}$  as in (4.12) with respect to  $\alpha, \beta$  and let  $\Psi = \Phi \cap \text{span}(\alpha, \beta, \gamma, \bar{\gamma})$ . Then since  $it_2^* = it^* \oplus \text{span}(\alpha, \beta), \Psi \subseteq \Phi \cap it_2^* = \Phi_{M_2}$ . Further,  $\langle \lambda_2, \delta \rangle = 0$  for all  $\delta \in \Psi$ . But  $\langle \gamma, \alpha \rangle \neq 0$ . Thus by (3.1),  $\lambda_2 \notin \Lambda_{M_2,1}$  so by (4.8),  $m_\beta^*(H_1 : h : v)$  is jointly smooth at  $(0, 0)$ .

Conversely, suppose that  $m_\beta^*(H_1 : h : v)$  is jointly smooth at  $(0, 0)$ . Then again by (4.8) there is  $\gamma \in \Phi_{M_2,1}$  so that  $\langle \gamma, \lambda_2 \rangle = 0$ . Suppose that  $\langle \gamma, \beta \rangle = 0$ . Then  $\gamma \in \Phi_{M_1,1}$  and  $\langle \lambda_1, \gamma \rangle = 0$ . But  $\lambda_1 \in \Lambda_{M_1,1}$  so this is impossible. Thus  $\langle \gamma, \beta \rangle \neq 0$ . Similarly,  $\langle \gamma, \alpha \rangle \neq 0$  since  $\lambda'_1 \in \Lambda_{M_1,1}$ . Define

$$\Psi = \Phi \cap \text{span}(\alpha, \beta, \gamma) \subseteq \Phi_{M_2}(\lambda_2) = \{\delta \in \Phi_{M_2} : \langle \lambda_2, \delta \rangle = 0\}.$$

Thus  $\Psi$  is a simple root system of rank 3. Since  $\alpha$  and  $\beta$  come from a simple factor of  $G$  which is non-compact and of hermitian type,  $\Psi$  is of type  $A_3, C_3$  or  $D_3$ . We may as well assume that  $\langle \gamma, \alpha \rangle > 0$ . If  $\langle \gamma, \beta \rangle > 0$ , replace  $\gamma$  by  $s_\beta \gamma$ . Thus we can assume that  $\langle \gamma, \beta \rangle < 0$ . Define  $\bar{\gamma} = -s_\alpha s_\beta \gamma \in \Psi$ . Then  $\gamma + \bar{\gamma} \in \text{span}(\alpha, \beta)$  and  $\gamma - \bar{\gamma} \in it^*$ . Suppose  $\langle \gamma, \bar{\gamma} \rangle > 0$ . Then  $\gamma - \bar{\gamma} \in \Psi \cap it^* = \{\delta \in \Phi_M : \langle \delta, \lambda \rangle = 0\}$ . In particular,  $\gamma - \bar{\gamma}$  is a root in  $\Psi$  which is orthogonal to both  $\alpha$  and  $\beta$ . Suppose  $\Psi$  is of type  $A_3$  or  $D_3$ . Then there are not 3 mutually orthogonal roots so this cannot happen. If  $\Psi$  is of type  $C_3$ , since  $\alpha$  and  $\beta$  are non-compact roots of  $\Phi_{M_2}$  and cannot both be long,  $\gamma - \bar{\gamma} \in \Phi_{M_2,1} \cap \Phi_M = \Phi_{M,1}$ . This contradicts the assumption that  $\lambda \in \Lambda_{M,1}$ . Thus  $\langle \gamma, \bar{\gamma} \rangle \leq 0$ . Suppose  $\langle \gamma, \bar{\gamma} \rangle > 0$ . Then

$$\gamma + \bar{\gamma} \in \Psi \cap \text{span}(\alpha, \beta) = \Phi_R \cap \text{span}(\alpha, \beta) = \{\pm\alpha, \pm\beta\}$$

since  $\alpha$  and  $\beta$  are in different simple factors of  $\Phi_R$ . But  $\langle \gamma + \bar{\gamma}, \alpha \rangle = 2\langle \gamma, \alpha \rangle \neq 0$

and  $\langle \gamma + \bar{\gamma}, \beta \rangle = 2\langle \gamma, \beta \rangle \neq 0$ . Thus  $\langle \gamma, \bar{\gamma} \rangle = 0$ . Now

$$s_\gamma s_{\bar{\gamma}} \alpha = \alpha - \frac{2\langle \alpha, \gamma \rangle}{\langle \gamma, \gamma \rangle} (\gamma + \bar{\gamma}) \in \Psi \cap \text{span}(\alpha, \beta) = \{\pm \alpha, \pm \beta\}.$$

But since  $\langle \alpha, \gamma \rangle > 0$  and  $\langle \beta, \gamma \rangle < 0$ ,  $\langle s_\gamma s_{\bar{\gamma}} \alpha, \beta \rangle > 0$  so that  $s_\gamma s_{\bar{\gamma}} \alpha = \beta$ . Thus  $\gamma, \bar{\gamma}$  satisfy (4.12) for  $\alpha, \beta$  so that  $\alpha \sim \beta$ .  $\square$

**PROOF OF THEOREM 4.10.** The proof of existence of  $H', \lambda', \chi'$ , and  $F$  is by induction on  $\dim A$ . If  $H$  is fundamental so that  $\dim A = d_0$  is minimal, then  $\Phi_R^+ = \emptyset$  so that  $m^*(h: v)$  must be jointly smooth. Thus we can take  $H' = H, \lambda' = \lambda, \chi' = \chi$ , and  $F = \emptyset$ . This is the only possible choice because of (4.9). Now suppose  $\dim A > d_0$ . If there are no roots  $\alpha \in \Phi_R^+$  such that  $m_\alpha^*(h: v)$  is not jointly smooth at  $(0, 0)$ , then again we can take  $H' = H$ . If not, fix  $\alpha \in \Phi_R^+$  such that  $m_\alpha^*(h: v)$  is not jointly smooth at  $(0, 0)$ . Define  $H_1$  so that  $\text{SOS}(H_1) = \text{SOS}(H) \setminus \{c^{-1}\alpha\}$  and choose  $\lambda_1 \in \Lambda_{M_1, 1}$  corresponding to  $\lambda$  as in (4.8). Let  $\chi_1$  be the restriction of  $\chi$  to  $Z_{M_1}((M_1)^0)$ . Then  $(\lambda_1, \chi_1) \in X(T_1)$  and  $\dim A_1 = \dim A - 1$  so by the induction hypothesis there are  $H', \lambda', \chi'$  and  $F_1 \subseteq F_0$  so that  $H_1 = H'_F, \lambda_1 = \lambda'_{F_1}, \chi_1 = \chi'_{F_1}$  and  $m^*(H': \lambda': \chi': h: v)$  is jointly smooth at  $(0, v')$  for all  $v' \in (\alpha')^*$ . But now  $\alpha' = c_{F_1}^{-1} c_\alpha^{-1} \alpha \in F_0$  and if we set  $F = F_1 \cup \{\alpha'\}$ , then  $H = H'_F, \lambda = \lambda'_F$  and  $\chi = \chi'_F$ . This proves the existence of  $H', \lambda', \chi', F$ . For fixed  $H', \chi' = \chi|_{Z_M((M)^0)}$  and  $F = \text{SOS}(H) \setminus \text{SOS}(H')$  are unique. Further,  $\lambda'$  is unique since restriction to  $\mathfrak{t}$  gives a bijection between  $\{\lambda' \in (\mathfrak{t}')^* : \langle \lambda', \alpha \rangle = 0 \text{ for all } \alpha \in F\}$  and  $\mathfrak{t}^*$ .

To see that  $H'$  is unique up to conjugacy we proceed as follows. Define  $\Phi_R^+(0)$  and the equivalence relation  $\sim$  as in (4.12). Let  $\Psi$  be a complete set of representatives for the equivalence classes and let  $H'(\Psi)$  be the Cartan subgroup of  $G$  with  $\text{SOS}(H'(\Psi)) = \text{SOS}(H) \setminus \{c^{-1}\alpha : \alpha \in \Psi\}$ . We claim that the conjugacy class of  $H'(\Psi)$  is independent of the choice of  $\Psi$ . Suppose  $\alpha$  and  $\beta$  represent the same conjugacy class in  $\Phi_R^+(0)$ . Then either both are long roots in a simple factor of type  $C_n$  or there are  $\gamma, \bar{\gamma}$  defined as in (4.12) corresponding to  $\alpha, \beta$ . In the first case, let  $w$  be the reflection in the Weyl group of  $C_n$  interchanging  $\alpha$  and  $\beta$  and fixing all other long roots. In the second case, let  $w = s_\gamma s_{\bar{\gamma}}$ . In both cases  $w$  represents an element of  $W(G, H)$  which interchanges  $\alpha$  and  $\beta$  and fixes every other root in  $c\text{SOS}(H)$ . Thus any two choices  $\Psi_1$  and  $\Psi_2$  are conjugate by an element of  $W(G, H)$  so that  $H'(\Psi_1)$  is  $G$ -conjugate to  $H'(\Psi_2)$ .

Now we claim that if  $H'$  is as in the theorem, it must be of the form  $H'(\Psi)$  for some  $\Psi$  as above. Thus suppose we have  $H', \lambda', \chi'$  with  $H = H'_F, \lambda = \lambda'_F, \chi = \chi'_F$ . Let  $\Psi = c_F(F) \subseteq \Phi_R^+$ . Then by (4.9),  $\Psi \subseteq \Phi_R^+(0)$ . Suppose that  $\alpha, \beta \in \Psi$  with  $\alpha \sim \beta$ . Let  $F_2 = F \setminus c_F^{-1}\{\alpha, \beta\}$ . Then as in (4.14),  $\lambda'_{F_2} \notin \Lambda_{M_{F_2}, 1}$ . Now suppose that there is  $\alpha \in \Phi_R^+(0)$  such that  $\alpha \not\sim \beta$  for all  $\beta \in \Psi$ . Then applying (4.14) and a simple induction argument,  $m_\alpha^*(H': \lambda': \chi': h: v')$  is not jointly smooth at  $(0, 0)$ . This contradicts the assumption that  $m(H': \lambda': \chi': h: v')$  is assumed to be jointly

smooth at  $(0, 0)$ . Thus  $\Psi$  is of the type considered above, so that  $H'$  is unique up to conjugacy.  $\square$

Suppose for every  $\theta$ -stable Cartan subgroup  $H = TA$ ,  $(\lambda, \chi) \in X(T)$ ,  $\tau_1, \tau_2 \in \hat{K}(\chi)$  we have a function

$$\beta(H : \lambda : \chi : \tau_1 : \tau_2) : iv^* \times \mathfrak{a}^* \times G \rightarrow W = W(\tau_1 : \tau_2)$$

satisfying the following conditions. First, for all  $h_0, h \in iv^*, v \in \mathfrak{a}^*, x \in G$ ,

$$\begin{aligned} \beta(H : \lambda(h_0) : \chi(h_0) : \tau_{1, h_0} : \tau_{2, h_0} : h : v : x) \\ = \beta(H : \lambda : \chi : \tau_1 : \tau_2 : h + h_0 : v : x). \end{aligned} \quad (4.15a)$$

Second, suppose for  $i = 1, 2$  we have  $\theta$ -stable Cartan subgroups  $H_i = T_i A_i$ ,  $(\lambda_i, \chi_i) \in X(T_i)$ , so that there is  $k \in K$  with  $H_1 = H_2^k$ ,  $\lambda_1 = \lambda_2^k$ ,  $\chi_1 = \chi_2^k$ . Then  $\hat{K}(\chi_1) = \hat{K}(\chi_2)$  and for all  $\tau_1, \tau_2 \in \hat{K}(\chi_1)$ ,  $h \in iv^*, v_2 \in \mathfrak{a}_2^*, x \in G$ ,

$$\beta(H_1 : \lambda_1 : \chi_1 : \tau_1 : \tau_2 : h : v_2^k : x) = \beta(H_2 : \lambda_2 : \chi_2 : \tau_1 : \tau_2 : h : v_2 : x). \quad (4.15b)$$

Third, we assume that the collection

$$\{\beta(H : \lambda : \chi : \tau_1 : \tau_2)\} \text{ satisfies all possible matching conditions.} \quad (4.15c)$$

That is, fix  $H = TA$  a  $\theta$ -stable Cartan subgroup,  $(\lambda, \chi) \in X(T)$ . Assume that  $m^*(H : \lambda : \chi : h : v)$  is jointly smooth in  $U(0) \times \mathfrak{a}^*$  where  $U(0)$  satisfies the conditions of (4.6). Define  $F_0, H_F, \lambda_F, \chi_F, F \subseteq F_0$  as in (3.3) and fix  $\tau_1, \tau_2 \in \hat{K}(\chi)$ . Then the functions

$$\{\beta(H_F : \lambda_F : \chi_F : \tau_1 : \tau_2 : h : v : x)\}_{F \subseteq F_0}$$

satisfy the matching conditions of (4.1e) in  $U(0)$ . Fourth, there is  $m \geq 0$  so that for all  $H, (\lambda, \chi) \in X(T)$ ,  $\tau_j \in \hat{K}(\chi)$ ,

$$\text{supp } \beta(H : \lambda : \chi : \tau_1 : \tau_2) \subseteq \{h \in iv^* : \|\lambda(h)\| < m \text{ and } \|\tau_{j,h}\| < m, j = 1, 2\} \quad (4.15d)$$

where  $\text{supp } \beta(H : \lambda : \chi : \tau_1 : \tau_2)$  is the closure of

$$\{h \in iv^* : \beta(H : \lambda : \chi : \tau_1 : \tau_2 : h : v : x) \neq 0 \text{ for some } v \in \mathfrak{a}^*, x \in G\}.$$

Finally, for each connected component  $\mathcal{D}$  of  $\{h \in iv^* : \langle \lambda(h), \alpha \rangle \neq 0 \text{ for all } \alpha \in \Phi_M^+\}$

there are finitely many functions  $\Psi_i \in \mathcal{S}(M^\dagger: \lambda: \chi: \mathcal{D}: W)$ ,  $\alpha_i \in \mathcal{C}(\mathcal{D} \times \mathfrak{a}^*)_0$  such that for all  $h \in \mathcal{D}$ ,  $v \in \mathfrak{a}^*$ ,  $x \in G$ ,

$$\beta(H: \lambda: \chi: \tau_1: \tau_2: h: v: x) = \sum_i \alpha_i(h: v) E(P: \Psi_i: h: v: x). \quad (4.15e)$$

LEMMA 4.16. *Let  $f \in \mathcal{C}(G)_K$ . Then the functions  $\{\hat{F}(f: H: \lambda: \chi: \tau_1: \tau_2)\}$  defined as in (2.14) satisfy the conditions of (4.15).*

*Proof.* It is clear from (2.14) that the functions  $\hat{F}(f)$  satisfy the shift conditions of (4.15a). Suppose for  $i=1, 2$  we have  $\theta$ -stable Cartan subgroups  $H_i = T_i A_i$ ,  $(\lambda_i, \chi_i) \in X(T_i)$ , so that there is  $k \in K$  with  $H_1 = H_2^k$ ,  $\lambda_1 = \lambda_2^k$ ,  $\chi_1 = \chi_2^k$ . Then since  $k$  acts trivially on  $iv^*$ ,  $\lambda_1(h) = \lambda_2(h)^k$ ,  $\chi_1(h) = \chi_2(h)^k$  for all  $h \in iv^*$ . Thus

$$\Theta(H_1: \lambda_1: \chi_1: h: v_2^k: x) = \Theta(H_2: \lambda_2: \chi_2: h: v_2: x)$$

for all  $v_2 \in \mathfrak{a}_2^*$ ,  $h \in iv^*$ ,  $x \in G$ . Now it is clear from the definition that the  $\hat{F}(f)$  satisfy condition (4.15b). Suppose we have  $H_F, \lambda_F, \chi_F, F \subseteq F_0$  as in (4.15c). We know from (3.20) that the characters  $\{\Theta(H_F: \lambda_F: \chi_F: h: v_F: x): F \subseteq F_0\}$  satisfy the matching conditions of (4.1e). Using the same argument as that of [H1, 10.22] we see that the functions  $\{\hat{F}(f: H_F: \lambda_F: \chi_F: \tau_1: \tau_2: h: v: x)\}_{F \subseteq F_0}$  also satisfy these matching conditions. Finally, (2.13) and (2.20) show that the functions  $\hat{F}(f)$  satisfy (4.15d) and (4.15e) respectively.  $\square$

Let  $H' = T'A'$  be a  $\theta$ -stable Cartan subgroup,  $(\lambda', \chi') \in X(T')$ ,  $\tau'_1, \tau'_2 \in \hat{K}(\chi')$ . Define  $U'(0)$  as in (4.6) and  $F'_0, H'_F, \lambda'_F, \chi'_F, F \subseteq F'_0$ , as in (3.3) and assume that  $m(H': \lambda': \chi': h: v)$  is jointly smooth in  $U'(0) \times (\mathfrak{a}')^*$ . Suppose for each  $F \subseteq F'_0$  we have

$$\Phi(F): iv^* \times (\mathfrak{a}')^* \times G \rightarrow W = W(\tau'_1: \tau'_2) \quad (4.17a)$$

satisfying the conditions of (4.1). Now let  $H = TA$  be any  $\theta$ -stable Cartan subgroup of  $G$  and let  $(\lambda, \chi) \in X(T)$ ,  $\tau_1, \tau_2 \in \hat{K}(\chi)$ . Suppose there are  $F \subseteq F'_0$ ,  $h_0 \in iv^*$ , and  $k \in K$  so that  $H^k = H'_F$ ,  $\lambda^k = \lambda'_F(h_0)$ ,  $\chi^k = \chi'_F(h_0)$ ,  $\tau_1 = \tau'_{1, h_0}$ ,  $\tau_2 = \tau'_{2, h_0}$ . Then  $W(\tau_1: \tau_2) = W(\tau'_1: \tau'_2)$  and for all  $h \in iv^*$ ,  $v \in \mathfrak{a}^*$ ,  $x \in G$  we define

$$\Phi(H: \lambda: \chi: \tau_1: \tau_2: h: v: x) = \Phi(F: h + h_0: v^k: x). \quad (4.17b)$$

Otherwise, we set  $\Phi(H: \lambda: \chi: \tau_1: \tau_2: h: v: x) = 0$  for all  $h \in iv^*$ ,  $v \in \mathfrak{a}^*$ ,  $x \in G$ . Note that in (4.17b),  $h_0 \in iv^*$  is uniquely determined by  $\tau_1 = \tau'_{1, h_0}$ . We will show in (4.18) that  $F \subseteq F'_0$  is also unique and that definition (4.17b) is independent of the choice of  $k \in K$ .

LEMMA 4.18. *The collection of functions  $\{\Phi(H: \lambda: \chi: \tau_1: \tau_2)\}$  of (4.17) is well-*

defined and satisfies the conditions of (4.15). Further, let  $\Phi(x)$  be the elementary mixed wave packet defined as in (4.1a) corresponding to the functions  $\Phi(F)$ . Then

$$\begin{aligned} \Phi(x) &= \sum_{H \in \text{Cart}(G)} \sum_{(\lambda, \chi) \in X_0(T)/W_H} \sum_{\tau_1 \in \hat{K}(\chi)} \sum_{\tau_2 \in [\hat{K}(\chi)/S_\tau]} \\ &\quad \times \int_{iv^*} \int_{a^*} \Phi(H : \lambda : \chi : \tau_1 : \tau_2 : h : v : x) m(H : \lambda : \chi : h : v) \, dv \, dh. \end{aligned}$$

*Proof.* Suppose there are  $k_1, k_2 \in K$  and  $F_1, F_2 \in F_0$  such that  $H^{k_1} = H'_{F_1}$ ,  $H^{k_2} = H'_{F_2}$ ,  $\lambda^{k_1} = \lambda'_{F_1}(h_0)$ ,  $\lambda^{k_2} = \lambda'_{F_2}(h_0)$ ,  $\chi^{k_1} = \chi'_{F_1}(h_0)$ ,  $\chi^{k_2} = \chi'_{F_2}(h_0)$ . Let  $k = k_1(k_2^{-1})$ . Then  $H'_{F_1} = (H'_{F_2})^k$ ,  $\lambda'_{F_1}(h_0) = (\lambda'_{F_2}(h_0))^k$ , and  $\chi'_{F_1}(h_0) = (\chi'_{F_2}(h_0))^k$ . But since  $k$  acts trivially on  $h_0$ , we also have  $\lambda'_{F_1} = (\lambda'_{F_2})^k$  and  $\chi'_{F_1} = (\chi'_{F_2})^k$ .

Since  $H'_{F_1}$  is conjugate to  $H'_{F_2}$  there is  $w \in W(G, H')$  so that  $wF_1 = F_2$ . In fact it is not hard to check cases as in (5.12) to see that we can take  $w \in W(M', T')$  and  $w^2 = 1$ . Now  $v = w \text{Ad } k \in W(G, H'_{F_2})$  and  $\lambda'_{F_1} = w^{-1}v(\lambda'_{F_2})$ . Suppose  $F_1 \neq F_2$ . Then there is  $\alpha \in F_2$  such that  $w\alpha \notin F_2$ . Then  $w\alpha \in F_1$  so that  $w\alpha \in \Phi_{M'_{F_2}}$ . Now since  $v \in W(G, H'_{F_2})$ ,  $v^{-1}w\alpha \in \Phi_{M'_{F_2}}$ . Now

$$\langle v^{-1}w\alpha, \lambda' \rangle = \langle v^{-1}w\alpha, \lambda'_{F_2} \rangle = \langle w\alpha, w\lambda'_{F_1} \rangle = \langle \alpha, \lambda' \rangle = 0.$$

Thus  $v^{-1}w\alpha \in F'_0 \setminus F_2$ . But now if we identify  $(\mathfrak{h}')^{\natural}$ ,  $(\mathfrak{h}'_{F_1})^{\natural}$ ,  $(\mathfrak{h}'_{F_2})^{\natural}$  and the corresponding roots systems and Weyl groups using the Cayley transforms  $c_{F_1}$ ,  $c_{F_2}$ , we have  $s = w^{-1}v \in W$  and  $\beta = s^{-1}\alpha \neq \alpha \in F'_0$  such that  $s\lambda' = \lambda'$  and  $s\beta = \alpha$ . Now  $\alpha$  and  $\beta$  must be in the same simple factor of  $\Phi(\lambda')$  and are also in the same simple factor of  $\Phi_{M'}$ . But  $\Phi(\lambda') \cap \Phi_{M'}$  is of type  $A_1^{\natural}$  by (3.1). Again, by checking cases, this cannot happen. Thus  $F_1 = F_2$  so that the  $F$  in (4.17b) is unique. Further,  $\text{Ad } k$  represents an element  $w \in W_{F_1}(\lambda, \chi)$ . Now using (4.1c),

$$\Phi(F_1 : h + h_0 : wv : x) = \Phi(F_1 : h + h_0 : v : x)$$

so (4.17b) is well-defined.

Since (4.17b) is well-defined, the functions  $\Phi(H : \lambda : \chi : \tau_1 : \tau_2)$  satisfy (4.15a) and (4.15b). The fact that there is a compact subset  $\omega$  of  $iv^*$  so that each  $\Phi(F : h : v_F : x)$  is zero unless  $h \in \omega$  implies that the functions  $\Phi(H : \lambda : \chi : \tau_1 : \tau_2)$  satisfy (4.15d) and condition (4.1d) implies they satisfy (4.15e). Thus we need only show that they satisfy all matching conditions.

Fix  $H = TA$ ,  $(\lambda, \chi) \in X(T)$ , and  $U(0)$  as in (4.6) so that  $m(H : \lambda : \chi : h : v)$  is jointly smooth in  $U(0) \times \alpha^*$ . Define  $F_0, H_F, \lambda_F, \chi_F, F \subseteq F_0$ , as usual. We want to show that for any  $\tau_1, \tau_2 \in \hat{K}(\chi)$  the functions  $\Phi(H_F : \lambda_F : \chi_F : \tau_1 : \tau_2)$  satisfy matching conditions in  $U(0)$ . We first note that for any  $h \in U(0)$ ,  $\Phi_M^+(h) \subseteq F_0$  by (4.6), and in fact  $\Phi_M^+(h)$  must be the union of the  $F_0^i$  such that  $h \in \mathcal{H}_i$ . Thus the matching conditions corresponding to  $(\lambda(h), \chi(h))$  are a subset of the matching conditions

corresponding to  $(\lambda, \chi)$ . Conversely, suppose for each  $h \in U(0)$  there is a neighborhood  $U(h)$  of  $h$  such that in  $U(h)$  the  $\Phi(H_F: \lambda_F: \chi_F: \tau_1: \tau_2)$  satisfy the matching conditions corresponding to hyperplanes  $\mathcal{H}_i$  such that  $h \in \mathcal{H}_i$ . Then they satisfy the matching conditions corresponding to  $(\lambda, \chi)$  in  $U(0)$ . Thus we need only show that matching conditions are satisfied in some neighborhood of 0.

Now fix  $F \subseteq F_0$ . Then  $\Phi(H_F: \lambda_F: \chi_F: \tau_1: \tau_2)$  is identically zero unless there are  $F' \subseteq F'_0, h_0 \in i\nu^*, k \in K$  so that  $H_F^k = H_{F'}, \lambda_F^k = \lambda'_{F'}(h_0), \chi_F^k = \chi'_{F'}(h_0), \tau_1 = \tau'_{1,h_0}, \tau_2 = \tau'_{2,h_0}$ . Assume this is the case. Then we may as well assume that  $k = 1$ . Now

$$\Phi(H_F: \lambda_F: \chi_F: \tau_1: \tau_2: h: \nu_F: x) = \Phi(F': h + h_0: \nu: x) = 0$$

unless  $h + h_0 \in \omega \subset U'(0)$ . Thus  $\Phi(H_F: \lambda_F: \chi_F: \tau_1: \tau_2)$  is zero in a neighborhood of  $h = 0$  unless  $h_0 \in U'(0)$ . Thus by the above we may as well assume that  $h_0 \in U'(0)$ . Now consider  $H_{F'}, (\lambda'_{F'}(h_0), \chi'_{F'}(h_0))$ . They correspond by (4.10) to  $H, (\lambda, \chi)$ . Thus there is  $\Psi$  as in the proof of (4.10) so that  $SOS(H) = SOS(H_{F'}) \setminus c^{-1}\Psi$ . Now  $\Psi$  is a subset of  $\Phi_{F',R}^+(h_0)$ , the set of real roots of  $H_{F'}$  such that  $m_\alpha^*(H_{F'}: \lambda'_{F'}: \chi'_{F'}: h: \nu)$  is not jointly smooth at  $h = h_0$ . Using (4.7) we see that  $\Phi_{F',R}^+(h_0) \subseteq \Phi_{F',R}^+(0)$ . Fix  $\alpha \in \Psi$ . We can assume that  $\alpha' = c_F^{-1}\alpha \in F'$ . Now suppose  $\beta \in \Psi$  with  $\alpha \sim_0 \beta$  where  $\sim_0$  is the equivalence relation on  $\Phi_{F',R}^+(0)$  defined as in (4.12) using  $H_{F'}, (\lambda'_{F'}(0), \chi'_{F'}(0))$ . Then  $m_\beta^*(H_{F'}: \lambda'_{F'}: \chi'_{F'})$  is jointly smooth at  $h = 0$  where  $F'' = F' \setminus \{\alpha'\}$ . Now by (4.7), since  $h_0 \in U'(0)$ ,  $m_\beta^*(H_{F'}: \lambda'_{F'}: \chi'_{F'})$  is jointly smooth at  $h = h_0$ . Thus  $\alpha \sim_{h_0} \beta$  so that  $\alpha = \beta$  since  $\Psi$  contains a unique representative of each conjugacy class with respect to  $\sim_{h_0}$ . Thus we may as well assume that  $\Psi \subseteq c_{F'}F'$ . Write  $F_1 = F' \setminus c_F^{-1}\Psi$ . Thus  $SOS(H) = SOS(H') \cup (F_1)$  so that  $H = H_{F_1}, \lambda = \lambda'_{F_1}(h_0), \chi = \chi'_{F_1}(h_0)$ . Now the matching conditions corresponding to  $H, (\lambda, \chi)$  are a subset of the matching conditions corresponding to  $H', (\lambda', \chi')$ .

Finally, suppose that we have  $H = TA \in \text{Car}(G), (\lambda, \chi) \in X_0(T)/W_H, \tau_1 \in \widehat{K}(\chi), \tau_2 \in [\widehat{K}(\chi)/S_T]$ , such that there are  $F \subseteq F'_0, h_0 \in i\nu^*$ , and  $k \in K$  so that  $H^k = H'_F, \lambda^k = \lambda'_F(h_0), \chi^k = \chi'_F(h_0), \tau_1 = \tau'_{1,h_0}, \tau_2 = \tau'_{2,h_0}$ . Then

$$\begin{aligned} & \int_{i\nu^*} \int_{\mathfrak{a}^*} \Phi(H: \lambda: \chi: h: \nu: x) m(H: \lambda: \chi: h: \nu) \, d\nu \, dh \\ &= \int_{i\nu^*} \int_{\mathfrak{a}^*} \Phi(F: h + h_0: \nu^k: x) m(H: \lambda: \chi: h: \nu) \, d\nu \, dh \\ &= \int_{i\nu^*} \int_{\mathfrak{a}_F^*} \Phi(F: h: \nu_F: x) m(H'_F: \lambda'_F: \chi'_F: h: \nu_F) \, d\nu_F \, dh. \end{aligned}$$

Further, given  $H = TA \in \text{Car}(G), (\lambda, \chi) \in X_0(T)/W_H, \tau_1 \in \widehat{K}(\chi), \tau_2 \in [\widehat{K}(\chi)/S_T]$  such that  $\Phi(H: \lambda: \chi: \tau_1: \tau_2)$  is not identically zero, there are a unique  $F \subseteq F'_0, h_0 \in i\nu^*$  such that there is  $k \in K$  with  $H^k = H'_F, \lambda^k = \lambda'_F(h_0), \chi^k = \chi'_F(h_0), \tau_1 = \tau'_{1,h_0},$

$\tau_2 = \tau'_{2,h_0}$ . Conversely, given  $F \subseteq F'_0$ , there are unique  $H = TA \in \text{Car}(G)$ ,  $(\lambda, \chi) \in X_0(T)/W_H$ ,  $\tau_1 \in \widehat{K}(\chi)$ ,  $\tau_2 \in [\widehat{K}(\chi)/S_T]$ ,  $h_0 \in iv^*$  such that for some  $k \in K$ ,  $H^k = H'_F$ ,  $\lambda^k = \lambda'_F(h_0)$ ,  $\chi^k = \chi'_F(h_0)$ ,  $\tau_1 = \tau'_{1,h_0}$ ,  $\tau_2 = \tau'_{2,h_0}$ . Thus each term

$$\int_{iv^*} \int_{\mathfrak{a}^*} \Phi(F : h : v_F : x) m(H'_F : \lambda'_F : \chi'_F : h : v_F) dv_F dh$$

occurs exactly once in

$$\begin{aligned} & \sum_{H \in \text{Car}(G)} \sum_{(\lambda, \chi) \in X_0(T)/W_H} \sum_{\tau_1 \in \widehat{K}(\chi)} \sum_{\tau_2 \in [\widehat{K}(\chi)/S_T]} \\ & \times \int_{iv^*} \int_{\mathfrak{a}^*} \Phi(H : \lambda : \chi : \tau_1 : \tau_2 : h : v : x) m(H : \lambda : \chi : h : v) dv dh \end{aligned}$$

and all non-zero terms are of this type. □

Write  $\text{Car}(G) = \{H_1, H_2, \dots, H_k\}$  where the Cartan subgroups  $H_i = T_i A_i$  are ordered so that  $\dim A_1 \leq \dim A_2 \leq \dots \leq \dim A_k$ . Now suppose  $H = TA$  is a  $\theta$ -stable Cartan subgroup of  $G$ . For any  $(\lambda, \chi) \in X(T)$ , we say  $(\lambda, \chi)$  is of level  $d$ ,  $1 \leq d \leq k$ , if the Cartan subgroup  $H'$  associated to  $(\lambda, \chi)$  by (4.10) is conjugate to  $H_d$ .

Suppose for every  $0 \leq d \leq k+1$ ,  $\theta$ -stable Cartan subgroup  $H$  of  $G$ ,  $(\lambda, \chi) \in X(T)$ ,  $\tau_1, \tau_2 \in \widehat{K}(\chi)$  we have a function

$$\beta_d(H : \lambda : \chi : \tau_1 : \tau_2) : iv^* \times \mathfrak{a}^* \times G \rightarrow W = W(\tau_1 : \tau_2) \tag{4.19a}$$

so that for each  $d$  the collection  $\{\beta_d\}$  satisfies the conditions of (4.15). Suppose in addition that if  $(\lambda, \chi) \in X(T)$  has level  $d' < d$ , then

$$0 \notin \text{supp } \beta_d(H : \lambda : \chi : \tau_1 : \tau_2). \tag{4.19b}$$

**THEOREM 4.20.** *Let  $f \in \mathcal{C}(G)_K$ . Then there is a collection of functions  $\{\beta_d(H : \lambda : \chi : \tau_1 : \tau_2)\}$  satisfying the conditions of (4.19) so that*

$$\beta_0(H : \lambda : \chi : \tau_1 : \tau_2) = \widehat{F}(f : H : \lambda : \chi : \tau_1 : \tau_2)$$

for all  $H, \lambda, \chi, \tau_1, \tau_2$  and such that if

$$\begin{aligned} f_d(x) &= \sum_{H \in \text{Car}(G)} \sum_{(\lambda, \chi) \in X_0(T)/W_H} \sum_{\tau_1 \in \widehat{K}(\chi)} \sum_{\tau_2 \in [\widehat{K}(\chi)/S_T]} \\ & \times \int_{iv^*} \int_{\mathfrak{a}^*} \beta_d(H : \lambda : \chi : \tau_1 : \tau_2 : h : v : x) (1 : 1) m(H : \lambda : \chi : h : v) dv dh, \end{aligned}$$

then  $f(x) - f_d(x)$  is a finite sum of scalar-valued elementary mixed wave packets for all  $d \geq 0$ .

**COROLLARY 4.21.** *Every  $f \in \mathcal{C}(G)_K$  is a finite sum of scalar-valued elementary mixed wave packets.*

*Proof.* Define the functions  $\beta_d$  corresponding to  $f$  as in (4.20). Let  $k = |\text{Car}(G)|$ . Then  $\beta_{k+1}$  is identically zero using property (4.19b) since for any  $\theta$ -stable Cartan subgroup  $H$ , every  $(\lambda, \chi) \in X(T)$  has level  $d \leq k < k + 1$ . Thus  $f_{k+1} = 0$ . Now  $f(x) = f(x) - f_{k+1}(x)$  is a finite sum of scalar-valued elementary mixed wave packets. □

**PROOF OF THEOREM 4.20.** Let  $f \in \mathcal{C}(G)_K$ . Define

$$\beta_0(H : \lambda : \chi : \tau_1 : \tau_2 : h : v : x) = \widehat{F}(f : H : \lambda : \chi : \tau_1 : \tau_2 : h : v : x)$$

for all  $H, \lambda, \chi, \tau_1, \tau_2, h, v, x$ . Then  $\beta_0$  satisfies the conditions of (4.15) by (4.16). Further, condition (4.19b) is vacuous when  $d = 0$ . Finally,  $f(x) - f_0(x) = 0$ .

Now let  $0 \leq d \leq k$  and assume for  $d' \leq d$  that we have constructed functions  $\beta_{d'}$  satisfying the conditions of (4.19) such that  $f - f_{d'}$  is a finite sum of elementary mixed wave packets. We will show how to construct  $\beta_{d+1}$  satisfying (4.19) so that  $f - f_{d+1}$  is a finite sum of elementary mixed wave packets.

Fix  $H' = H_d \in \text{Car}(G)$ ,  $(\lambda, \chi) \in X_0(T')/W_{H'}$ ,  $\tau_1 \in \widehat{K}(\chi)$ ,  $\tau_2 \in [\widehat{K}(\chi)/S_{T'}]$ . For every  $h \in iv^*$ , let  $\Phi_{M'}^+(h) = \{\alpha \in \Phi_{M'}^+ : \langle \alpha, \lambda(h) \rangle = 0\}$ . For any  $F \subseteq \Phi_{M'}^+(h)$ , define  $H'_F, \lambda(h)_F, \chi(h)_F$  as in (3.3). Let  $\lambda_F = \lambda|_{i_F}$ ,  $\chi_F = \chi \otimes e^{\lambda - \rho_M}$ . Then  $(\lambda_F, \chi_F) \in X(T'_F)$  and  $\lambda(h)_F = \lambda_F(h)$ ,  $\chi(h)_F = \chi_F(h)$ . Then define

$$S = \left\{ h \in iv^* : h \in \bigcup_{F \subseteq \Phi_{M'}^+(h)} \text{supp } \beta_d(H'_F : \lambda_F : \chi_F : \tau_1 : \tau_2) \right\}.$$

We claim that  $cl(S) \subseteq T = \{h_0 \in iv^* : m(H' : \lambda : \chi : h : v)\}$  is jointly smooth at  $(h_0, v)$  for all  $v \in (\mathfrak{a}')^*$ . Fix  $h_0 \notin T$  and define  $U(h_0)$  as in (4.6). Then since  $m(H' : \lambda : \chi : h : v)$  is not jointly smooth at  $(h_0, v)$  for some  $v \in (\mathfrak{a}')^*$ ,  $H', \lambda(h_0), \chi(h_0)$  correspond via (4.10) to some  $H'' = T''A'', \lambda'', \chi''$  with  $\dim A'' < \dim A'$ . Thus the level  $d'$  of  $(\lambda(h_0), \chi(h_0))$  is strictly less than  $d$ . Suppose  $F \subseteq \Phi_{M'}^+(h_0)$ . Then  $H'_F, \lambda(h_0)_F, \chi(h_0)_F$  correspond via (4.10) to the same  $H'', \lambda'', \chi''$  as  $H', \lambda(h_0), \chi(h_0)$ . Thus the level of  $(\lambda(h_0)_F, \chi(h_0)_F)$  is also  $d' < d$  so that

$$0 \notin \text{supp } \beta_d(H'_F : \lambda_F(h_0) : \chi_F(h_0) : \tau_1(h_0) : \tau_2(h_0))$$

which implies that

$$h_0 \notin \text{supp } \beta_d(H'_F : \lambda_F : \chi_F : \tau_1 : \tau_2).$$



Let  $V_F(h_0)$  be a neighborhood of  $h_0 \in iv^*$  so that

$$V_F(h_0) \cap \text{supp } \beta_d(H'_F: \lambda_F: \chi_F: \tau_1: \tau_2) = \emptyset.$$

Let  $V = U(h_0) \cap \bigcap_{F \subseteq \Phi_{M'}^+(h_0)} V_F(h_0)$ . Now if  $h \in V$ ,  $h \in U(h_0)$  so that  $\Phi_{M'}^+(h) \subseteq \Phi_{M'}^+(h_0)$ . Thus for

$$F \subseteq \Phi_{M'}^+(h) \subseteq \Phi_{M'}^+(h_0), \quad h \in V_F(h_0)$$

so that

$$h \notin \text{supp } \beta_d(H'_F: \lambda_F: \chi_F: \tau_1: \tau_2).$$

Thus  $h \notin \text{supp } \beta_d(H'_F: \lambda_F: \chi_F: \tau_1: \tau_2)$  for every  $F \subseteq \Phi_{M'}^+(h)$ , so that  $h \notin S$ . Thus there is a neighborhood  $V$  of  $h_0$  so that  $V \cap S = \emptyset$  and so  $h_0 \notin cl(S)$ . Now for every  $h \in S$ ,  $h \in \text{supp } \beta_d(H'_F: \lambda_F: \chi_F: \tau_1: \tau_2)$  for some  $F \subseteq \Phi_{M'}^+(h)$  implies by (4.15d) that there is  $m$  such that  $\|\tau_1(h)\| < m$ . But there is  $c_1 = \|\tau_1\|$  so that  $|h| \leq \|\tau_1(h)\| + c_1$  for all  $h \in iv^*$ . Thus  $|h| \leq c_1 + m$  for all  $h \in cl(S)$  so that  $cl(S)$  is compact. Now choose a relatively compact open subset  $U \subseteq iv^*$  so that  $cl(S) \subset U \subset \omega = cl(U) \subset T$ .

We are now ready to define elementary mixed wave packets corresponding to  $H'$ ,  $\lambda$ ,  $\chi$ ,  $\tau_1$ ,  $\tau_2$ . Define  $\omega$  as above, and for each  $h \in \omega$ , let  $U(h)$  be a neighborhood of  $h$  defined as in (4.6) with radius  $\varepsilon < 1$ . Then there are finitely many  $h_1, h_2, \dots, h_k \in \omega$  such that  $\omega \subseteq \bigcup_{i=1}^k U(h_i)$ . Further, since each  $h_i \in \omega \subset T$ ,  $m(H': \lambda: \chi: h: \nu)$  is jointly smooth at  $h_i$  so that by (4.7),  $m(H': \lambda: \chi: h: \nu)$  is jointly smooth in  $U(h_i) \times (\mathfrak{a}')^*$ . Choose  $\alpha_i \in C_c^\infty(iv^*)$  such that  $\text{supp } \alpha_i \subset U(h_i)$  and  $\sum_{i=1}^k \alpha_i(h) = 1$  for all  $h \in \omega$ .

For  $1 \leq i \leq k$ , let  $F_0(i) = \Phi_{M'}^+(h_i)$  and for  $F \subseteq F_0(i)$  define  $H'_F$ ,  $\lambda(h_i)_F$ ,  $\chi(h_i)_F$  as in (3.3) and  $\lambda_F = \lambda|_{\mathfrak{t}_F}$ ,  $\chi_F = \chi \otimes e^{\lambda - \rho_M}$  as above. Now define

$$\begin{aligned} \Phi_i(H': \lambda: \chi: \tau_1: \tau_2: x) &= \Phi_i(x) \\ &= \sum_{F \subseteq F_0(i)} \int_{iv^*} \int_{(\mathfrak{a}')^*} \Phi_i(F: h: \nu_F: x) m(H'_F: \lambda_F(h_i): \chi_F(h_i): h: \nu_F) d\nu_F dh \end{aligned}$$

where

$$\Phi_i(F: h: \nu_F: x) = \beta_d(H'_F: \lambda_F(h_i): \chi_F(h_i): \tau_1(h_i): \tau_2(h_i): h: \nu_F: x) \alpha_i(h + h_i).$$

We claim that  $\Phi_i$  is an elementary mixed wave packet. Now  $\Phi_i(F: h: \nu_F: x) = 0$  unless  $h + h_i \in \text{supp } \alpha_i \subset U(h_i)$  so that  $h \in \omega_i = \text{supp } \alpha_i - h_i \subset U_i = U(h_i) - h_i$ . Conditions (4.15b) and (4.15e) imply that  $\Phi_i(F)$  satisfies (4.1c) and (4.1d). Finally,

since the  $\beta_d$  satisfy all matching conditions, and  $\alpha_i(h+h_i)$  is a smooth function of  $h$  which is independent of  $F$ , the  $\Phi(F)$  satisfy the matching conditions of (4.1e).

Suppose  $\Phi_i(H' : \lambda : \chi : \tau_1 : \tau_2)$  is not identically zero. Then there is  $F \subseteq F_0(i)$  such that  $\beta_d(H_F : \lambda_F(h_i) : \chi_F(h_i) : \tau_1(h_i) : \tau_2(h_i) : h : \nu_F : x) \alpha_i(h+h_i)$  is not zero for some  $h$ . Now  $\alpha_i(h+h_i) \neq 0$  implies that  $|h| < 1$ . Now

$$\beta_d(H_F : \lambda_F(h_i) : \chi_F(h_i) : \tau_1(h_i) : \tau_2(h_i) : h : \nu_F : x) \neq 0$$

implies that

$$\|\lambda_F(h_i+h)\| < m \quad \text{and} \quad \|\tau_j(h_i+h)\| < m, j=1,2.$$

But

$$\|\lambda\| \leq \|\lambda(h_i)\| = \|\lambda_F(h_i)\| \leq \|\lambda_F(h_i+h)\| + |h| \leq m+1.$$

Thus  $(\lambda, \chi) \in X_0^{m+1}(T')$ . Further  $\|\tau_j\| \leq \|\tau_j(h_i+h)\| + |h_i| + |h| < m+c_1+m+1$  so that  $\tau_j \in \widehat{K}^{2m+c_1+1}(\chi)$ . Now by (2.13),  $X_0^{m+1}(T')$  and  $K^{2m+c_1+1}(\chi)$  are finite sets. Thus only finitely many of the  $\Phi_i(H' : \lambda : \chi : \tau_1 : \tau_2)$  are non-zero.

Now for each

$$(\lambda', \chi') \in X_0(T')/W_H, \quad \tau'_1 \in \widehat{K}(\chi), \quad \tau'_2 \in [\widehat{K}(\chi)/S_T], \quad 1 \leq i \leq k$$

we can use (4.17b) to define the collection of functions

$$\{\Phi_i(H' : \lambda' : \chi' : \tau'_1 : \tau'_2)(H : \lambda : \chi : \tau_1 : \tau_2)\}$$

corresponding to the elementary mixed wave packet  $\Phi_i(H' : \lambda' : \chi' : \tau'_1 : \tau'_2)$ . By (4.18) this collection satisfies the conditions of (4.15). Now for all  $H, (\lambda, \chi) \in X(T), \tau_1, \tau_2 \in \widehat{K}(\chi)$ , we define

$$\begin{aligned} \beta_{d+1}(H : \lambda : \chi : \tau_1 : \tau_2) &= \beta_d(H : \lambda : \chi : \tau_1 : \tau_2) \\ &- \sum_{(\lambda', \chi') \in X_0(T')/W_H} \sum_{\tau'_1 \in \widehat{K}(\chi)} \sum_{\tau'_2 \in [\widehat{K}(\chi)/S_T]} \sum_i \Phi_i(H' : \lambda' : \chi' : \tau'_1 : \tau'_2)(H : \lambda : \chi : \tau_1 : \tau_2). \end{aligned}$$

Since  $\beta_d$  and the  $\Phi_i(H' : \lambda' : \chi' : \tau'_1 : \tau'_2)$  satisfy the conditions of (4.15), so does  $\beta_{d+1}$  since the sum is finite.

Define

$$\begin{aligned} f_{d+1}(x) &= \sum_{H \in \text{Car}(G)} \sum_{(\lambda, \chi) \in X_0(T)/W_H} \sum_{\tau_1 \in \widehat{K}(\chi)} \sum_{\tau_2 \in \widehat{K}(\chi)/S_T} \\ &\times \int_{i\nu^*} \int_{\mathfrak{a}^*} \beta_{d+1}(H : \lambda : \chi : \tau_1 : \tau_2 : h : \nu : x)(1 : 1) m(H : \lambda : \chi : h : \nu) \, d\nu \, dh. \end{aligned}$$

Then it is clear from (4.18) that

$$\begin{aligned}
 & f_d(x) - f_{d+1}(x) \\
 &= \sum_{(\lambda', \chi') \in X_0(T')/W_{H'}} \sum_{\tau'_1 \in \widehat{K}(\chi')} \sum_{\tau'_2 \in [\widehat{K}(\chi')/S_{T'}]} \sum_i \Phi_i(H' : \lambda' : \chi' : \tau'_1 : \tau'_2 : x)(1 : 1).
 \end{aligned}$$

Now, as above, there are only finitely many non-zero terms in the sum so that  $f_d - f_{d+1}$  is a finite sum of elementary mixed wave packets. By the induction hypothesis,  $f - f_d$  is a finite sum of elementary mixed wave packets. Thus  $f - f_{d+1}$  is a finite sum of elementary mixed wave packets.

Finally, we must show that  $\beta_{d+1}$  satisfies the additional conditions of (4.19). Fix  $(\lambda', \chi') \in X_0(T')/W_{H'}$ ,  $\tau_1 \in \widehat{K}(\chi')$ ,  $\tau_2 \in [\widehat{K}(\chi')/S_{T'}]$ . Then, for each  $1 \leq i \leq k$  and  $F \subseteq F_0(i)$ , using the change of variables  $h \rightarrow h - h_i$ ,

$$\begin{aligned}
 & \int_{iv^*} \int_{(\alpha')^\#} \Phi_i(F : h : v_F : x) m(H'_F : \lambda'_F(h_i) : \chi'_F(h_i) : h : v_F) dv_F dh \\
 &= \int_{iv^*} \int_{(\alpha')^\#} \beta_d(H'_F : \lambda'_F : \chi'_F : \tau'_1 : \tau'_2 : h : v_F : x) \alpha_i(h) m(H'_F : \lambda'_F : \chi'_F : h : v_F) dv_F dh.
 \end{aligned}$$

Thus

$$\begin{aligned}
 & \sum_i \Phi_i(H' : \lambda' : \chi' : \tau'_1 : \tau'_2 : x) \\
 &= \sum_i \sum_{F \subseteq F_0(i)} \int_{iv^*} \int_{(\alpha')^\#} \beta_d(H'_F : \lambda'_F : \chi'_F : \tau'_1 : \tau'_2 : h : v_F : x) \\
 & \quad \times \alpha_i(h) m(H'_F : \lambda'_F : \chi'_F : h : v_F) dv_F dh \\
 &= \sum_F \int_{iv^*} \int_{(\alpha')^\#} \beta_d(H'_F : \lambda'_F : \chi'_F : \tau'_1 : \tau'_2 : h : v_F : x) \\
 & \quad \times m(H'_F : \lambda'_F : \chi'_F : h : v_F) \alpha(F : h) dv_F dh
 \end{aligned}$$

where we sum over all  $F$  such that  $F \subseteq F_0(i)$  for some  $1 \leq i \leq k$  and for such an  $F$ ,

$$I(F) = \{1 \leq i \leq k : F \subseteq F_0(i)\}, \quad \alpha(F : h) = \sum_{i \in I(F)} \alpha_i(h).$$

Note that for all  $F \subseteq F_0(i)$ ,  $(\lambda'_F(h_i), \chi'_F(h_i))$  has level  $d$ .

Let  $H = TA$  be a  $\theta$ -stable Cartan subgroup,  $(\lambda, \chi) \in X(T)$ , and suppose the level of  $(\lambda(h), \chi(h))$  is not equal to  $d$  for any  $h \in iv^*$ . Then  $H$  and the family  $\{(\lambda(h), \chi(h)) : h \in iv^*\}$  cannot occur as  $H'_F$ ,  $\{(\lambda'_F(h), \chi'_F(h)) : h \in iv^*\}$  for any

$(\lambda', \chi') \in X_0(T')$ ,  $F \subseteq F_0(i)$ . Thus in this case we have

$$\beta_{d+1}(H: \lambda: \chi: \tau_1: \tau_2) = \beta_d(H: \lambda: \chi: \tau_1: \tau_2)$$

for all  $\tau_1, \tau_2 \in \widehat{K}(\chi)$ . Now suppose the level of  $(\lambda(h_0), \chi(h_0))$  is equal to  $d$  for some  $h_0 \in \mathfrak{iv}^*$ . Then we may as well assume  $H = H'_F$  and the family  $\{(\lambda(h), \chi(h)): h \in \mathfrak{iv}^*\} = \{(\lambda'_F(h), \chi'_F(h)): h \in \mathfrak{iv}^*\}$  for some  $(\lambda', \chi') \in X_0(T')/W_H$  and  $F$  such that  $F \subseteq \Phi_M^+(h)$  for some  $h \in \mathfrak{iv}^*$ . Define  $h_1, \dots, h_k$  and  $\alpha_i$ ,  $1 \leq i \leq k$  as above for  $(\lambda', \chi')$ . In this case we have

$$\beta_{d+1}(H: \lambda'_F: \chi'_F: \tau_1: \tau_2: h: v: x) = \beta_d(H: \lambda'_F: \chi'_F: \tau_1: \tau_2: h: v: x)(1 - \alpha(F: h)).$$

Note that in either case,

$$\beta_{d+1}(H: \lambda: \chi: \tau_1: \tau_2: h: v: x) = \beta_d(H: \lambda: \chi: \tau_1: \tau_2: h: v: x)\beta(h)$$

where  $\beta$  is a smooth function of  $h$ . Thus

$$\text{supp } \beta_{d+1}(H: \lambda: \chi: \tau_1: \tau_2) \subseteq \text{supp } \beta_d(H: \lambda: \chi: \tau_1: \tau_2).$$

Let  $H \in \text{Car}(G)$ ,  $(\lambda, \chi) \in X(T)$  with level  $d' < d + 1$ . If  $d' < d$ , then by (4.19b) applied to  $\beta_d$ ,

$$0 \notin \text{supp } \beta_d(H: \lambda: \chi: \tau_1: \tau_2) \Rightarrow 0 \notin \text{supp } \beta_{d+1}(H: \lambda: \chi: \tau_1: \tau_2)$$

for all  $\tau_j$ . Thus we need only check the case that  $d' = d$ . Now we can assume as above that  $H = H'_F$  and the family

$$\{(\lambda(h), \chi(h)): h \in \mathfrak{iv}^*\} = \{(\lambda'_F(h), \chi'_F(h)): h \in \mathfrak{iv}^*\}$$

and

$$\beta_{d+1}(H: \lambda'_F: \chi'_F: \tau_1: \tau_2: h: v: x) = \beta_d(H: \lambda'_F: \chi'_F: \tau_1: \tau_2: h: v: x)(1 - \alpha(F: h)).$$

Let  $h_0 \in \mathfrak{iv}^*$  such that  $(\lambda = \lambda'_F(h_0), \chi = \chi'_F(h_0))$  has level  $d$ . We must show that  $h_0 \notin \text{supp } \beta_{d+1}(H: \lambda'_F: \chi'_F: \tau_1: \tau_2)$ . This is true if  $h_0 \notin \text{supp } \beta_d(H: \lambda'_F: \chi'_F: \tau_1: \tau_2)$ . Thus we may as well assume that

$$h_0 \in \text{supp } \beta_d(H: \lambda'_F: \chi'_F: \tau_1: \tau_2).$$

Since

$$F \subseteq \Phi_{M'}^+(h_0), \text{ this implies that } h_0 \in \mathcal{S}.$$

Pick a neighborhood  $V_0(h_0)$  of  $h_0$  so that  $V_0(h_0) \subset \omega$ . Let

$$I(h_0) = \{1 \leq i \leq k : h_0 \in U(h_i)\}.$$

Now for any  $i \in I(h_0)$ ,  $F \subseteq \Phi_{M'}^+(h_0) \subseteq F_0(i)$ . Thus  $I(h_0) \subseteq I(F)$ . If  $i \notin I(h_0)$ , there is a neighborhood  $V_i(h_0)$  so that  $\alpha_i(h) = 0$  for all  $h \in V_i(h_0)$ . Define  $V = V_0(h_0) \cap \bigcap_{i \notin I(h_0)} V_i(h_0)$ . Then for any  $h \in V$ ,  $\alpha(F : h) = \sum_{i \in I(F)} \alpha_i(h) = \sum_{i=1}^k \alpha_i(h) = 1$ . Thus for all  $h \in V$ ,

$$\beta_{d+1}(H : \lambda'_F : \chi'_F : \tau_1 : \tau_2 : h : v : x) = \beta_d(H : \lambda'_F : \chi'_F : \tau_1 : \tau_2 : h : v : x)(1 - \alpha(F : h)) = 0. \quad \square$$

### 5. Plancherel factors

Let  $H = TA$  be a  $\theta$ -stable Cartan subgroup of  $G$ ,  $(\lambda, \chi) \in X(T)$ . Let  $U(0)$  be defined as in (4.6) and assume that the Plancherel function  $m^*(H : \lambda : \chi : h : v)$  is jointly smooth on  $U(0) \times \mathfrak{a}^*$ . Define  $H_F, \lambda_F, \chi_F, c_F : \mathfrak{h}_{\mathbb{C}} \rightarrow \mathfrak{h}_{F, \mathbb{C}}$ ,  $F \subseteq F_0$  as in (3.3). Define  $\Phi^+, \Phi_R^+$  as in (4.5) and for  $F \subseteq F_0$ , set  $\Phi_F^+ = c_F \Phi^+$  and  $\Phi_{F,R}^+ = \{\alpha \in \Phi_F^+ : \alpha \text{ takes real values on } \mathfrak{h}_F\}$ . We will identify  $\Phi_F^+$  with  $\Phi^+$  via  $c_F$ . For each  $\alpha \in \Phi_{F,R}^+$ , define  $m_\alpha^*(H_F : \lambda_F : \chi_F : h : v_F) = m_\alpha^*(F : h : v_F)$  as in (4.5b). Write  $\Phi_{F,R}^+(0) = \{\alpha \in \Phi_{F,R}^+ : m_\alpha^*(F : h : v_F) \text{ is not jointly smooth at } (0, 0)\}$ .

Define the equivalence relation on  $\Phi_{F_0,R}^+(0)$  as in (4.12), and for each  $\alpha \in \Phi_{F_0,R}^+(0)$ , write  $[\alpha]$  for the equivalence class containing  $\alpha$ . For each  $\alpha \in \Phi_{F_0,R}^+(0)$ , let  $\Phi_{F_0}(\alpha)$  be the simple factor of  $\Phi_{F_0,R}$  containing  $\alpha$  and write

$$\Phi_{F_0}[\alpha] = \bigcup_{\beta \in [\alpha]} \Phi_{F_0}(\beta).$$

Finally, for any  $F \subseteq F_0$ ,  $\alpha \in F_0$ , define  $\Phi_F[\alpha] = \Phi_{F_0}[\alpha] \cap \Phi_{F,R}$  and set  $\Phi_{F,R}' = \bigcup_{\alpha \in F_0} \Phi_F^+[\alpha]$ ,  $\Phi_{F,R}'' = \Phi_{F,R}^+ \setminus \Phi_{F,R}' = \Phi_{F_0,R}'$ . Then if  $\alpha \in F$ ,  $\Phi_F[\alpha] = \Phi_{F_0}[\alpha]$  and  $\Phi_F[\alpha] \cap \Phi_{F,R}^+(0) = [\alpha]$ . If  $\alpha \in F_0 \setminus F$ , then  $\Phi_F[\alpha] = \{\beta \in \Phi_{F_0}[\alpha] : \beta \perp \alpha\}$  and  $\Phi_F[\alpha] \cap \Phi_{F,R}^+(0) = \emptyset$ .

For each  $F \subseteq F_0$ , we will use the notation of (4.5) and write for  $(h, v) \in iv^* \times \mathfrak{a}_F^*$ ,

$$m^*(H_F : \lambda_F : \chi_F : h : v) = m^*(F : h : v) = \pi(F : h : v) \prod_{\alpha \in \Phi_{F,R}^+} m_\alpha(F : h : v) \tag{5.1a}$$

where

$$\pi(F : h : v) = \prod_{\alpha \in \Phi_F^+} \langle \lambda_F(h) + iv, \alpha \rangle \tag{5.1b}$$

and for  $\alpha \in \Phi_{F,R}^+$ ,

$$m_\alpha(F : h : v) = \frac{\sinh \pi v_\alpha}{\cosh \pi v_\alpha - \varepsilon_\alpha(F : h)}. \tag{5.1c}$$

For  $\alpha \in \Phi_{F,R}''$ , define  $h_\alpha = 2 \langle h_M(h), c_F^{-1} \alpha \rangle / \langle \alpha, \alpha \rangle$  and set

$$p_F(h : v) = \prod_{\alpha \in c_F F} (v_\alpha + ih_\alpha), \tag{5.1e}$$

$$q(F : h : v) = \prod_{\alpha \in \Phi_{F,R}^+(0)} (v_\alpha + ih_\alpha) \prod_{\alpha \in F} \prod_{\beta \in [\alpha], \beta \neq \alpha} (v_\beta - v_\alpha)^{-1}. \tag{5.1f}$$

For each  $F \subseteq F_0$ , let  $\mathcal{S}_F = \mathcal{S}(\Phi_{F,R}'')$  denote the set of all two-structures for  $\Phi_{F,R}''$ . Thus if  $\Phi_{F,R}''$  is of type  $C_1 = A_1$ , the only two-structure for  $\Phi_{F,R}''$  is  $\psi = \Phi_{F,R}''$  and  $\varepsilon(\psi) = 1$ . If  $\Phi_{F,R}''$  is of type  $C_s, s \geq 2$ , then the two-structures  $\psi$  for  $\Phi_{F,R}''$  and associated signs  $\varepsilon(\psi) = \pm 1$  are described in Section 6. For  $\Phi_{F,R}''$  not simple, two-structures are the union of two-structures for the simple factors and the signs are multiplied. Thus each  $\psi \in \mathcal{S}_F$  is of the form  $\psi = \psi_1 \cup \dots \cup \psi_r$  where each  $\psi_i$  is of type  $A_1$  or  $C_2$ . For each  $\alpha \in \Phi_{F,R}''$  write

$$t_\alpha(F : h : v) = \frac{\sinh \pi(v_\alpha + ih_\alpha)}{\cosh \pi(v_\alpha + ih_\alpha) - \varepsilon_\alpha(F : 0)}. \tag{5.2a}$$

If  $\psi_i^+ = \{\alpha\}$  is of type  $A_1$ , write

$$t(F : \psi_i : h : v) = t_\alpha(F : h : v). \tag{5.2b}$$

If  $\psi_i$  is of type  $C_2$ , let  $\psi_{i,s}^+$  denote the short roots in  $\psi_i^+$  and write

$$t(F : \psi_i : h : v) = \prod_{\alpha \in \psi_i^+} t_\alpha(F : h : v) + \prod_{\alpha \in \psi_{i,s}^+} t_\alpha(F : h : v). \tag{5.2c}$$

Finally, write

$$t(F : \psi : h : v) = \prod_{i=1}^r t(F : \psi_i : h : v). \tag{5.2d}$$

**THEOREM 5.3.** Let  $\tau_1, \tau_2 \in \widehat{K}(\chi)$  and let  $W = W(\tau_1 : \tau_2)$ . Suppose for each

$F \subseteq F_0$  we have functions  $\Phi(F): \mathfrak{iv}^* \times \mathfrak{a}_F^* \times G \rightarrow W$  satisfying (4.1b–e). Then there is a constant  $c \neq 0$  independent of  $F$  so that

$$\int_{\mathfrak{a}_F^*} \Phi(F: h: v: x) m(F: h: v) \, dv = c(\pi i)^{-|F|} \int_{\mathfrak{a}_F^*} \frac{g(F: h: v: x)}{p_F(h: v)} \, dv$$

where for any  $\varepsilon \in \Sigma_0$ ,  $h \in \mathcal{D}_F(\varepsilon)$ ,

$$g(F: h: v: x) = \sigma_F(\varepsilon)(\pi/2)^{|F|} \Phi(F: h: v: x) \pi(F: h: v) q(F: h: v) \\ \times \prod_{\alpha \in \Phi_{F,R}} m_\alpha(F: h: v) \sum_{\psi \in \mathcal{F}_F} \varepsilon(\psi) t(F: \psi: h: v).$$

Further, the functions  $g(F)$  have the following properties. For any  $\varepsilon \in \Sigma_0$ ,

$$(h, v, x) \rightarrow g(F: h: v: x) \text{ is jointly smooth on } \text{cl}(\mathcal{D}_F(\varepsilon)) \times \mathfrak{a}_F^* \times G.$$

For any  $D \in D(\mathfrak{iv}^* \times \mathfrak{a}_F^*)$ ,  $r \geq 0$ ,  $g_1, g_2 \in \mathcal{U}(\mathfrak{g}_C)$ , there are constants  $C, s \geq 0$  so that

$$\|g(F: h: v; D: g_1; x; g_2)\| (1 + |v|)^r \leq C \Xi(x) (1 + \tilde{\sigma}(x))^s$$

for all  $x \in G$ ,  $h \in \mathcal{D}_F(\varepsilon)$ ,  $v \in \mathfrak{a}_F^*$ . Finally, the functions  $\{g(F): F \subseteq F_0\}$  satisfy the matching conditions of (5.18).

The theorem will be proven by a series of lemmas. Some technical results on two-structures and Plancherel functions for roots systems of type  $C_n$ ,  $n \geq 2$ , will be deferred to the next section. Define  $W_F(\lambda, \chi) = \{w \in W(G, H_F): w\lambda_F = \lambda_F, w\chi_F = \chi_F\}$ .

LEMMA 5.4. Let  $\alpha \in F$ . Then for any  $\beta \in [\alpha]$  there is  $w \in W_F(\lambda, \chi)$  such that  $w\alpha = \beta$ ,  $w\beta = \alpha$ , and  $w\delta = \delta$  for every  $\delta \in \Phi_{F,R}$  such that  $\langle \delta, \alpha \rangle = \langle \delta, \beta \rangle = 0$ .

Proof. Suppose  $\beta \in [\alpha]$ . Then if  $\Phi(\alpha) = \Phi(\beta)$  is of type  $C_n$ , let  $w$  be the reflection in the Weyl group of  $\Phi(\alpha)$  which interchanges  $\alpha$  and  $\beta$  and fixes every other long root. If  $\Phi(\alpha) \neq \Phi(\beta)$ , choose  $\gamma, \bar{\gamma}$  as in (4.12) and let  $w = s_\gamma s_{\bar{\gamma}}$ . In either case  $w$  represents an element of  $W(G, H_F)$  with  $w\lambda_F = \lambda_F$ ,  $w\chi_F = \chi_F$ .  $\square$

For any  $F \subseteq F_0$  we can identify  $\Phi_{F,R}^+ = \{\alpha \in \Phi_{F_0,R}^+ : \langle \alpha, \beta \rangle = 0 \text{ for all } \beta \in F_0 \setminus F\}$  and  $\mathfrak{a}_F^* = \{v \in \mathfrak{a}_{F_0}^* : \langle v, \beta \rangle = 0 \text{ for all } \beta \in F_0 \setminus F\}$ .

LEMMA 5.5. Suppose  $\alpha \in \Phi_{F,R}^+$  is a long root in a simple factor of  $\Phi_{F,R}''$  of type  $C_n$  which has empty intersection with  $\Phi_{F,R}^+(0)$ . Then  $\varepsilon_\alpha(F: h) = -\varepsilon_\alpha(F_0: h)$  for all  $h \in \mathfrak{iv}^*$ . For any other  $\alpha \in \Phi_{F,R}^+$  we have  $\varepsilon_\alpha(F: h) = \varepsilon_\alpha(F_0: h)$ . Let  $\alpha \in \Phi_{F,R}''$ . Then if  $\alpha \in \text{SOS}(H_F)$ ,  $\varepsilon_\alpha(F: h) = \varepsilon_\alpha(F: 0) \cos \pi h_\alpha$  for all  $h \in \mathfrak{iv}^*$ . If  $\alpha \notin \text{SOS}(H_F)$ , then  $\varepsilon_\alpha(F: h) = \varepsilon_\alpha(F: 0)$  is independent of  $h \in \mathfrak{iv}^*$ . In either case,  $\varepsilon_\alpha(F: 0) = \pm 1$ .

*Proof.* Since

$$SOS(H_{F_0}) = SOS(H_F) \cup (F_0 \setminus F), \quad \gamma_\alpha \in Z_{M_F}(M_F^0) \subseteq Z_{M_{F_0}}(M_{F_0}^0)$$

is independent of  $F$  and  $\chi_F(h)$  is the restriction of  $\chi_{F_0}(h)$  to  $Z_{M_F}(M_F^0)$ . Thus  $\varepsilon_\alpha(F : h) = \varepsilon_\alpha(F_0 : h)$  if and only if  $(-1)^{\rho_{F,\alpha}} = (-1)^{\rho_{F_0,\alpha}}$ . But using [HW1, 4.13],  $(-1)^{\rho_{F,\alpha}} = -e^{\rho_F - \rho_{F,R}}(\gamma_\alpha)$  where  $\rho_F = 1/2 \sum_{\beta \in \Phi_F^+} \beta$  and  $\rho_{F,R} = 1/2 \sum_{\beta \in \Phi_{F,R}^+} \beta$ . But  $e^{\rho_F}(\gamma_\alpha)$  is independent of  $F$ , and  $e^{\rho_{F,R}}(\gamma_\alpha)$  is independent of  $F$  as long as the simple factor of  $\Phi_{F,R}$  containing  $\alpha$  is the same as the simple factor of  $\Phi_{F_0,R}$  containing  $\alpha$ . This happens unless  $\alpha \in \Phi_{F,R}^+$  is in a simple factor of  $\Phi_{F,R}''$  of type  $C_n$  which has empty intersection with  $\Phi_{F,R}^+(0)$  so that the simple factor of  $\Phi_{F_0,R}$  containing  $\alpha$  is of type  $C_{n+1}$ . This can happen only if  $\alpha$  is a root in a simple factor of  $G$  which is isomorphic to the universal covering group of  $Sp(m, \mathbf{R})$  for some  $m \geq n + 1$ . In this case the result is proven in (6.1).

It is enough to prove the second part of the lemma when  $F = F_0$ . Now if  $\alpha \in \Phi_{F_0,R}''$ ,  $\alpha$  must come from a simple factor of  $G$  which is non-compact, simply connected, and of hermitian type. Thus as in (4.9),  $\gamma_\alpha$  is central in  $Z_0 = Z_{M_{F_0}}(M_{F_0}^0)$ . Let  $\Gamma$  be the central subgroup of  $Z_0$  generated by the  $\gamma_\alpha, \alpha \in \Phi_{F_0,R}''$  and let  $\zeta$  be the  $\Gamma$ -character of  $\chi_{F_0}(0)$ . Then

$$\varepsilon_\alpha(F_0 : h) = (-1)^{\rho_\alpha} \frac{e^{h(\gamma_\alpha)\zeta(\gamma_\alpha)} + e^{h(\gamma_\alpha^{-1})\zeta(\gamma_\alpha^{-1})}}{2}.$$

Thus  $\varepsilon_\alpha(F_0 : 0) = \pm 1$  just in case  $\zeta(\gamma_\alpha) = \zeta(\gamma_\alpha^{-1}) = \pm (-1)^{\rho_\alpha}$ .

Every simple factor of  $\Phi_{F_0,R}''$  is of type  $A_1$  or  $C_n, n \geq 2$ . If  $\alpha$  is in a simple factor of type  $A_1$ , then  $\alpha \sim \beta$  for some  $\beta \in c_{F_0}F_0$ , so that  $\alpha \in \Phi_{F_0,R}^+(0)$  and  $\varepsilon_\alpha(F_0 : 0) = 1$ . If  $\alpha$  is in a simple factor of type  $C_n$  and is long, then  $\varepsilon_\alpha(F_0 : 0) = \pm 1$  by (6.1). Now in either case  $\alpha \in SOS(H_{F_0})$ , and  $h_\alpha = h(c_{F_0}^{-1}H_\alpha^*) = -ih(Z_\alpha)$ . Thus

$$\varepsilon_\alpha(F_0 : h) = \varepsilon_\alpha(F_0 : 0) \frac{e^{h(\gamma_\alpha)} + e^{h(\gamma_\alpha^{-1})}}{2} = \varepsilon_\alpha(F_0 : 0) \cos \pi h_\alpha$$

since  $e^{h(\gamma_\alpha)} = e^{\pi h(Z_\alpha)} = e^{\pi i h_\alpha}$ . Finally, if  $\alpha \notin SOS(H_{F_0})$ , then  $\alpha$  is in a simple factor of type  $C_n$  and is short, so that  $\varepsilon_\alpha(F_0 : h) = \varepsilon_\alpha(F_0 : 0) = \pm 1$  by (6.1).  $\square$

LEMMA 5.6. For each  $F \subseteq F_0$ ,

$$\begin{aligned} & \int_{\mathfrak{a}_F^*} \Phi(F : h : v : x) \pi(F : h : v) \prod_{\alpha \in \Phi_{F,R}^+} m_\alpha(F : h : v) \, dv \\ &= \int_{\mathfrak{a}_F^*} \Phi(F : h : v : x) \pi(F : h : v) \prod_{\alpha \in \Phi_{F,R}^+} m_\alpha(F : h : v) \sum_{\psi \in \mathcal{F}_F} \varepsilon(\psi) t(F : \psi : h : v) \, dv. \end{aligned}$$



*Proof.* Write  $\Phi_{F,R}^+ = \Phi'_{F,R} \cup \Phi''_{F,R}$ . Now by [HW1, 4.17],

$$\prod_{\alpha \in \Phi_{F,R}^+} m_\alpha(F:h:v) = \sum_{\psi \in \mathcal{T}_F} \varepsilon(\psi) \prod_{\alpha \in \psi^+} m_\alpha(F:h:v).$$

Thus we can write

$$\begin{aligned} & \int_{\mathfrak{a}_F^*} \Phi(F:h:v:x) \pi(F:h:v) \prod_{\alpha \in \Phi_{F,R}^+} m_\alpha(F:h:v) dv \\ &= \sum_{\psi \in \mathcal{T}_F} \varepsilon(\psi) \int_{\mathfrak{a}_F^*} f(h:v:x) \prod_{\alpha \in \psi^+} m_\alpha(F:h:v) dv \end{aligned}$$

where

$$f(h:v:x) = \Phi(F:h:v:x) \pi(F:h:v) \prod_{\alpha \in \Phi'_{F,R}} m_\alpha(F:h:v).$$

Suppose  $w \in W(\Phi''_{F,R})$  such that  $w\chi_F = \chi_F$ . Then  $\pi(F:h:vw) = \det w \pi(F:h:v)$ ,  $\Phi(F:h:vw:x) = \Phi(F:h:v:x)$  by assumption (4.1c), and  $m_\alpha(F:h:vw) = m_\alpha(F:h:v)$  for all  $\alpha \in \Phi'_{F,R}$  since every root in  $\Phi'_{F,R}$  is orthogonal to every root in  $\Phi''_{F,R}$ . Thus  $f(h:vw:x) = \det w f(h:v:x)$ .

Fix  $\psi \in \mathcal{T}_F$  and write  $\psi = \psi_1 \cup \dots \cup \psi_r$  as before. Then if  $\psi_i^+ = \{\alpha\}$  is of type  $A_1$ , then using (5.5),

$$m_\alpha(F:h:v) = \frac{\sinh \pi v_\alpha}{\cosh \pi v_\alpha - \varepsilon_\alpha(F:0) \cos \pi h_\alpha}.$$

But for all  $x, y \in \mathbf{R}$ ,  $\varepsilon = \pm 1$ ,

$$\frac{\sinh(x+iy)}{\cosh(x+iy) - \varepsilon} = \frac{\sinh x - i\varepsilon \sin y}{\cosh x - \varepsilon \cos y}.$$

Thus

$$\begin{aligned} m_\alpha(F:h:v) &= \frac{\sinh \pi(v_\alpha + ih_\alpha)}{\cosh \pi(v_\alpha + ih_\alpha) - \varepsilon_\alpha(F:0)} + \frac{i\varepsilon_\alpha(F:0) \sin \pi h_\alpha}{\cosh \pi v_\alpha - \varepsilon_\alpha(F:0) \cos \pi h_\alpha} \\ &= t_\alpha(F:h:v) + \frac{i\varepsilon_\alpha(F:0) \sin \pi h_\alpha}{\cosh \pi v_\alpha - \varepsilon_\alpha(F:0) \cos \pi h_\alpha}. \end{aligned}$$

But for  $\alpha$  as above,  $\alpha \in \text{SOS}(H_F)$  and the reflection  $s_\alpha$  in  $\alpha$  centralizes  $Z_{M_F}(M_F^0)$ .

Thus  $f(h:s_\alpha v:x) = -f(h:v:x)$  as above and for

$\beta \in \psi^+ \setminus \{\alpha\}$ ,  $m_\beta(F:h:s_\alpha v) = m_\beta(F:h:v)$ ,  $t_\beta(F:h:s_\alpha v) = t_\beta(F:h:v)$ . Thus if

$$f'(h:v:x) = f(h:v:x) \prod_{\beta \in \psi^+ \setminus \{\alpha\}} m_\beta(F:h:v) \frac{\sin \pi h_\alpha}{\cosh \pi v_\alpha - \varepsilon_\alpha(F:0) \cos \pi h_\alpha},$$

$f'(h:s_\alpha v:x) = -f'(h:v:x)$  for all  $v \in \mathfrak{a}_F^*$ , so that  $\int_{\mathfrak{a}_F^*} f'(h:v:x) dv = 0$ . Thus if  $\psi_i^+ = \{\alpha\}$  is of type  $A_1$ , we can replace  $m_\alpha(F:h:v)$  by  $t_\alpha(F:h:v) = t(\psi_i:h:v)$  under the integral.

Now suppose  $\psi_i$  is of type  $C_2$ . Then as in the proof of [HW1, 5.6],

$$\int_{\mathfrak{a}_F^*} f''(h:v:x) \prod_{\alpha \in \psi_i^+} m_\alpha(F:h:v) dv = \int_{\mathfrak{a}_F^*} f''(h:v:x) t(\psi_i:h:v) dv$$

if  $f''(h:vw:x) = \det w f''(h:v:x)$  for  $w \in W_i(\psi_i)$ , the subgroup of  $W(\psi_i)$  generated by reflections in the long roots. But again, the long roots of  $\psi_i$  are in  $SOS(H_F)$  and act trivially on  $Z_{M_F}(M_F^0)$ . Thus for  $w \in W_i(\psi_i)$ ,  $\Phi(F:h:vw:x) = \Phi(F:h:v:x)$ ,  $\pi(F:h:vw:x) = \det w \pi(F:h:v:x)$ , and  $m_\alpha(F:h:vw) = m_\alpha(F:h:v)$ ,  $\alpha \in \Phi'_{F,R} \cup \psi_j$ ,  $j \neq i$ , and  $t_\alpha(F:h:vw) = t_\alpha(F:h:v)$  for  $\alpha \in \psi_j$ ,  $j \neq i$ . Thus

$$\int_{\mathfrak{a}_F^*} f(h:v:x) \prod_{\alpha \in \psi^+} m_\alpha(F:h:v) dv = \int_{\mathfrak{a}_F^*} f(h:v:x) t(F:\psi:h:v) dv. \quad \square$$

LEMMA 5.7. Let  $U(0)$  be a neighborhood of 0 in  $iv^*$  as in (4.6). Then for any  $\alpha \in \Phi''_{F,R}$ ,

$$(h, v) \rightarrow t_\alpha(F:h:v) \text{ is jointly smooth on } U(0) \times \mathfrak{a}_F^* \text{ if } \varepsilon_\alpha(F:0) = -1$$

and

$$(h, v) \rightarrow (v_\alpha + ih_\alpha)t_\alpha(F:h:v)$$

is jointly smooth on

$$U(0) \times \mathfrak{a}_F^* \text{ if } \varepsilon_\alpha(F:0) = 1.$$

*Proof.* It is clear that  $t_\alpha(F:h:v)$  is jointly smooth except at points  $(h, v)$  such that  $\cosh \pi(v_\alpha + ih_\alpha) = \varepsilon_\alpha(F:0)$ , thus  $v_\alpha = 0$  and  $h_\alpha \in \pi\mathbf{Z}$ . But as in the proof of (4.7), for  $h \in U(0)$ ,  $h_\alpha \in \pi\mathbf{Z}$  implies that  $h_\alpha = 0$ . Thus if  $\varepsilon_\alpha(F:0) = -1$ ,  $\cosh \pi(v_\alpha + ih_\alpha) \neq \varepsilon_\alpha(F:0)$  for any  $(h, v) \in U(0) \times \mathfrak{a}_F^*$ . If  $\varepsilon_\alpha(F:0) = 1$ ,  $(h, v) \rightarrow (v_\alpha + ih_\alpha)t_\alpha(F:h:v)$  is jointly smooth on  $U(0) \times \mathfrak{a}_F^*$ .  $\square$

LEMMA 5.8. *The function*

$$(h \ v) \rightarrow \pi(F : h : v) \prod_{\alpha \in \Phi_{F,R}^+(0)} (v_\alpha + ih_\alpha) \prod_{\alpha \in F} \prod_{\beta \in [\alpha], \beta \neq \alpha} (v_\beta - v_\alpha)^{-2} \sum_{\psi \in \mathcal{F}_F} \varepsilon(\psi) t(F : \psi : h : v)$$

is jointly smooth on  $U(0) \times \mathfrak{a}_F^*$ .

*Proof.* First, let

$$\Phi_{F,R}''(1) = \{ \alpha \in \Phi_{F,R}'' : \varepsilon_\alpha(F : 0) = 1 \text{ and } c^{-1}\alpha \in \text{SOS}(H_F) \text{ or } c^{-1}\alpha \text{ is compact} \}.$$

Then by (5.7) and (6.5),

$$(h, v) \rightarrow \prod_{\alpha \in \Phi_{F,R}''(1)} (v_\alpha + ih_\alpha) \sum_{\psi \in \mathcal{F}_F} \varepsilon(\psi) t(F : \psi : h : v)$$

is jointly smooth on  $U(0) \times \mathfrak{a}_F^*$ . Suppose  $\alpha \in \Phi_{F,R}''(1)$ . If  $\alpha \notin \Phi_{F,R}^+(0)$ , then  $v_\alpha + ih_\alpha$  divides  $\pi(F : h : v)$ . Thus

$$(h, v) \rightarrow \pi(F : h : v) \prod_{\alpha \in \Phi_{F,R}^+(0)} (v_\alpha + ih_\alpha) \sum_{\psi \in \mathcal{F}_F} \varepsilon(\psi) t(F : \psi : h : v)$$

is jointly smooth on  $U(0) \times \mathfrak{a}_F^*$ . Finally, let  $\alpha \in F, \beta \in [\alpha], \beta \neq \alpha$ . If  $\Phi(\alpha) = \Phi(\beta)$  it is proven in (6.5) that  $(v_\beta - v_\alpha)$  divides  $\sum_{\psi \in \mathcal{F}_F} \varepsilon(\psi) t(F : \psi : h : v)$ . Since  $\frac{\alpha - \beta}{2}$  is a real root,  $(v_\beta - v_\alpha)$  also divides  $\pi(F : h : v)$ . If  $\Phi(\alpha) \neq \Phi(\beta)$ , then let  $\gamma, \bar{\gamma}$  be defined as in (4.12) and define  $w = s_\gamma s_{\bar{\gamma}}$ . Let  $v_0 \in \mathfrak{a}_F^*$  such that  $v_\beta = v_\alpha$ . Then  $w(\lambda_F(h) + iv_0) = \lambda_F(h) + iv_0$  for all  $h \in i\mathfrak{v}^*$  and  $w\gamma = -\gamma, w\bar{\gamma} = -\bar{\gamma}$ . Thus

$$\langle \lambda_F(h) + iv_0, \gamma \rangle = \langle \lambda_F(h) + iv_0, \bar{\gamma} \rangle = 0 \text{ for all } h \in i\mathfrak{v}^*.$$

Thus  $(v_\beta - v_\alpha)$  divides both  $\langle \lambda_F(h) + iv, \gamma \rangle$  and  $\langle \lambda_F(h) + iv, \bar{\gamma} \rangle$  so that  $(v_\beta - v_\alpha)^2$  divides  $\pi(F : h : v)$ . □

LEMMA 5.9. *Define  $g(F)$  as in (5.3). Then for  $\varepsilon \in \Sigma_0, h \in \mathcal{D}_F(\varepsilon)$ ,*

$$\begin{aligned} & \int_{\mathfrak{a}_F^*} \Phi(F : h : v : x) \pi(F : h : v) \prod_{\alpha \in \Phi_{F,R}^+} m_\alpha(F : h : v) \, dv \\ &= (2/\pi)^{|F|} \sigma_F(\varepsilon) \prod_{\alpha \in F} |[\alpha]| \int_{\mathfrak{a}_F^*} \frac{g(F : h : v : x)}{p_F(h : v)} \, dv. \end{aligned}$$

Further, for any  $\varepsilon \in \Sigma_0, (h, v, x) \rightarrow g(F : h : v : x)$  is jointly smooth on  $cl(\mathcal{D}_F(\varepsilon)) \times \mathfrak{a}_F^* \times G$ .

*Proof.* Suppose  $\alpha_0 \in F$ . Then  $\Phi_F[\alpha_0]$  is of type  $C_n, n \geq 2$ , or  $A_1^n, n \geq 1$ , and

$\Phi_F[\alpha_0] \cap \Phi_{F,R}^+(0) = [\alpha_0]$ . Further,  $\Phi_{F,R}^+(0) = \bigcup_{\alpha \in F} [\alpha]$ . Write

$$\prod_{\alpha \in [\alpha_0]} (v_\alpha + ih_\alpha)^{-1} = \sum_{\alpha \in [\alpha_0]} \left[ (v_\alpha + ih_\alpha)^{-1} \prod_{\beta \in [\alpha_0], \beta \neq \alpha} (v_\beta - v_\alpha)^{-1} \right].$$

Now, using (5.4), for any  $\alpha \in [\alpha_0]$  there is  $w \in W_F(\lambda, \chi)$  which interchanges  $\alpha$  and  $\alpha_0$ . Thus we have  $\Phi(F : h : wv : x) = \Phi(F : h : v : x)$ . Further,

$$\pi(F : h : wv) = \det w \pi(F : h : v), \quad \prod_{\alpha \in [\alpha_0]} ((wv)_\alpha + ih_\alpha) = \prod_{\alpha \in [\alpha_0]} (v_\alpha + ih_\alpha).$$

Further,

$$\sum_{\psi \in \mathcal{F}_F} \varepsilon(\psi) t(F : \psi : h : wv) = \det w \sum_{\psi \in \mathcal{F}_F} \varepsilon(\psi) t(F : \psi : h : v)$$

by (6.2) in the case that  $\Phi_F[\alpha_0]$  is of type  $C_n$ . When  $\Phi_F[\alpha_0]$  is of type  $A_1^n$  this is also true since  $\det w = 1$  and  $t(F : \psi : h : wv) = t(F : \psi : h : v)$  for all  $\psi \in \mathcal{F}_F$ . Thus

$$\begin{aligned} & \int_{\mathfrak{a}_F^*} \Phi(F : h : v : x) \pi(F : h : v) \prod_{\alpha \in \Phi_{F,R}^+} m_\alpha(F : h : v) \sum_{\psi \in \mathcal{F}_F} \varepsilon(\psi) t(F : \psi : h : v) dv \\ &= \int_{\mathfrak{a}_F^*} \Phi(F : h : v : x) \pi(F : h : v) \prod_{\alpha \in \Phi_{F,R}^+} m_\alpha(F : h : v) \sum_{\psi \in \mathcal{F}_F} \varepsilon(\psi) t(F : \psi : h : v) \\ & \quad \times \prod_{\alpha_0 \in F} \left[ \prod_{\alpha \in [\alpha_0]} (v_\alpha + ih_\alpha) \sum_{\alpha \in [\alpha_0]} \left( (v_\alpha + ih_\alpha)^{-1} \prod_{\beta \in [\alpha_0], \beta \neq \alpha} (v_\beta - v_\alpha)^{-1} \right) \right] dv \\ &= \int_{\mathfrak{a}_F^*} \Phi(F : h : v : x) \pi(F : h : v) \prod_{\alpha \in \Phi_{F,R}^+} m_\alpha(F : h : v) \sum_{\psi \in \mathcal{F}_F} \varepsilon(\psi) t(F : \psi : h : v) \\ & \quad \times \prod_{\alpha \in \Phi_{F,R}^+(0)} (v_\alpha + ih_\alpha) \prod_{\alpha_0 \in F} |[\alpha_0]| (v_{\alpha_0} + ih_{\alpha_0})^{-1} \prod_{\beta \in [\alpha_0], \beta \neq \alpha_0} (v_\beta - v_{\alpha_0})^{-1} dv \\ &= \prod_{\alpha_0 \in F} |[\alpha_0]| \int_{\mathfrak{a}_F^*} \Phi(F : h : v : x) \pi(F : h : v) q(F : h : v) p_F(h : v)^{-1} \\ & \quad \times \prod_{\alpha \in \Phi_{F,R}^+} m_\alpha(F : h : v) \sum_{\psi \in \mathcal{F}_F} \varepsilon(\psi) t(F : \psi : h : v) dv. \end{aligned}$$

By (4.1d),  $\Phi(F : h : v : x)$  is jointly smooth on  $cl(\mathcal{D}_F(\varepsilon)) \times \mathfrak{a}_F^* \times G$ . Suppose that  $\alpha \in \Phi_{F,R}^+$ . Then  $m_\alpha^*(h : v) = \prod_{\beta \in \Phi_\alpha^+} \langle \lambda_F(h) + iv, \beta \rangle m_\alpha(h : v)$  is jointly smooth.

Thus to prove the second part of the lemma we must prove that

$$\begin{aligned} & \pi(F : h : v) q(F : h : v) \sum_{\psi \in \mathcal{F}_F} \varepsilon(\psi) t(F : \psi : h : v) \\ &= \pi(F : h : v) \prod_{\alpha \in \Phi_{F,R}^+(0)} (v_\alpha + ih_\alpha) \prod_{\alpha_0 \in F} \prod_{\beta \in [\alpha_0], \beta \neq \alpha_0} (v_\beta - v_{\alpha_0})^{-1} \sum_{\psi \in \mathcal{F}_F} \varepsilon(\psi) t(F : \psi : h : v) \end{aligned}$$

is jointly smooth. This follows from (5.8). □

Recall from (4.5) that for any  $F \subseteq F_0$ ,  $\varepsilon \in \Sigma_0$ , there is a constant  $c(F:\varepsilon) = c(H_F:\lambda_F:\chi_F:\mathcal{D}_F(\varepsilon))$  so that

$$m(F:h:v) = c(F:\varepsilon)m^*(F:h:v) \tag{5.10a}$$

for all  $h \in \mathcal{D}_F(\varepsilon)$ ,  $v \in \mathfrak{a}_F^*$ . What was actually proven in [HW1, 6.17], as modified in (2.12), is that there is a constant  $c_{H_F}$  defined as in [HW1, 6.17] so that

$$m(F:h:v) = [\lambda_F:\chi_F]c_{H_F}|m^*(F:h:v)| \tag{5.10b}$$

for all  $h \in i\nu^*$ ,  $v \in \mathfrak{a}_F^*$ , where  $[\lambda_F:\chi_F]$  is the number of distinct elements in  $\{(w\lambda_F, w\chi_F): w \in W(G, H_F)\}$ .

LEMMA 5.11. *Normalize the Haar measures  $dv_F$  on  $\mathfrak{a}_F^*$  as in (7.8). Then*

$$c_{H_F}|W(G, H_F)| = c_{H_\emptyset}|W(G, H_\emptyset)|$$

for all  $F \subseteq F_0$ .

*Proof.* In the notation on [HW1, 6.17] we have a constant  $c_G$  so that for any Cartan subgroup  $H = TA$  of  $G$ ,

$$c_H^{-1} = c_G|W(G, H)||T/(T \cap M_B^+)|[L(\psi):L(\Phi_R)] \prod_{\alpha \in \text{SOS}(H)} \|\alpha\|$$

where  $B$  is a fundamental Cartan subgroup of  $G$ . Now the Cartan subgroups  $H_F$  differ only in simple factors  $G_i$  of  $G$  which are simply connected, non-compact simple groups of hermitian type. Thus  $|T/(T \cap M_B^+)|$  is independent of  $F$  since for  $G_i$  as above, the fundamental Cartan subgroup  $B_i$  is relatively compact and  $M_{B_i}^+ = G_i$ . Further,  $\Phi_{F,R}$  differs from  $\Phi_{\emptyset,R}$  by simple factors of type  $A_1$  or  $C_n$  and so, as in [HW1, 4.18],  $[L(\psi):L(\Phi_{F,R})]$  is also independent of  $F$ . Finally, the normalization of Haar measures  $dv_F$  on  $\mathfrak{a}_F^*$  used in (7.8) differs from that of [HW1] by the factor  $\prod_{\alpha \in \text{SOS}(H)} \|\alpha\|$  so that this factor does not occur in the constants  $c_H$  for the normalization used in this paper. □

LEMMA 5.12. *For any  $F \subseteq F_0$ ,  $c_{H_F}[\lambda_F, \chi_F]2^{|F|} \prod_{\alpha \in F} \|\alpha\| = c_{H_\emptyset}[\lambda_\emptyset, \chi_\emptyset]$ .*

*Proof.* By (5.11) we have a constant  $c$  independent of  $F$  so that

$$c_{H_F} = c|W(G, H_F)|^{-1} \text{ for all } F \subseteq F_0.$$

Thus  $c_{H_F}[\lambda_F, \chi_F] = c|W_F(\lambda, \chi)|^{-1}$  for all  $F \subseteq F_0$

where  $W_F(\lambda, \chi) = \{w \in W(G, H_F): w\lambda_F = \lambda_F, w\chi_F = \chi_F\}$ . Thus it suffices to prove

that  $|W_F(\lambda, \chi)| = 2^{|\mathcal{F}|} \prod_{\alpha \in \mathcal{F}} |[\alpha]| |W_{\mathcal{O}}(\lambda, \chi)|$ . We may as well assume that  $G$  is simply connected, non-compact, simple, and of hermitian type.

Every  $w \in W(G, H_F)$  is of the form  $w = \text{ad } k$  where  $k \in K$  such that  $\text{Ad } k$  stabilizes both  $\mathfrak{t}_F$  and  $\mathfrak{a}_F$ . Let  $w = \text{ad } k \in W_F(\lambda, \chi)$ . Since  $k \in K$ ,  $\text{Ad } k(h) = h$  for all  $h \in \mathfrak{iv}^*$ . Now since  $\text{Ad } k$  stabilizes  $\mathfrak{t}_F$ ,  $\text{Ad } kh_{M_F}(h) = h_{M_F}(h)$  for all  $h \in \mathfrak{iv}^*$ . Thus  $w\lambda_F(h) = \lambda_F(h)$  for all  $h \in \mathfrak{iv}^*$ . But  $\lambda_F(h)$  is generically regular with respect to  $\Phi_{M_F}^+$ . Thus  $W_F(\lambda, \chi) \cap W(\Phi_{M_F}) = \{1\}$ .

Assume  $\Phi^+$  has been chosen so that  $\langle \lambda, \beta \rangle \geq 0$  for all  $\beta \in \Phi^+$ . Recall  $W(G, H_F) = W(\Phi_{F,R})W_0(G, H_F)W(M_F, T_F)$  where

$$W_0(G, H_F) = \{w \in W(G, H_F) : w\Phi_{F,R}^+ = \Phi_{F,R}^+, w\Phi_{M_F}^+ = \Phi_{M_F}^+\}.$$

Write  $w \in W_F(\lambda, \chi)$  as  $w = w_R w_0 w_I$  where  $w_R \in W(\Phi_{F,R})$ ,  $w_0 \in W_0(G, H_F)$ , and  $w_I \in W(M_F, T_F)$ . Then  $\lambda_F = w_R w_0 w_I \lambda_F = w_0 w_I \lambda_F$  since  $w_R$  acts trivially on  $\mathfrak{iv}^*$ . But

$$\langle \lambda_F, \beta \rangle = \langle w_0 w_I \lambda_F, \beta \rangle = \langle w_I \lambda_F, w_0^{-1} \beta \rangle \geq 0 \quad \text{for all } \beta \in \Phi_{M_F}^+.$$

But  $w_0 \Phi_{M_F}^+ = \Phi_{M_F}^+$  so that  $\langle w_I \lambda_F, \beta \rangle \geq 0$  for all  $\beta \in \Phi_{M_F}^+$ . Thus  $w_I \lambda_F = \lambda_F$ . But  $w_I$  acts trivially on  $Z_{M_F}(M_F^0)$  so that  $w_I \in W_F(\lambda, \chi) \cap W(M_F, T_F) = \{1\}$ . Thus  $w = w_R w_0$  where  $w_R \lambda_F = \lambda_F$  and  $w_0 \lambda_F = \lambda_F$ .

Since we assume that  $G$  is simple, non-compact, and of hermitian type,  $\Phi_{F,R}$  is of type  $A_1^k \times C_s$ . Now  $w\chi_F = \chi_F$  if and only if  $\varepsilon_{w\alpha}(F:0) = \varepsilon_\alpha(F:0)$  for all  $\alpha$  in the  $A_1^k$  factor and all long roots  $\alpha$  in the  $C_s$  factor. But  $w_R$  acts trivially on  $\gamma_\alpha$  if  $\alpha$  is in the  $A_1^k$  factor and  $w_0$  acts trivially on  $\gamma_\alpha$  if  $\alpha$  is a long root in the  $C_s$  factor. Thus  $w\chi_F = \chi_F$  if and only if  $w_R \chi_F = \chi_F$  and  $w_0 \chi_F = \chi_F$ . Thus for all  $w = w_R w_0 \in W_F(\lambda, \chi)$ ,  $w_R, w_0 \in W_F(\lambda, \chi)$ .

Now  $W(\Phi_{F,R})$  is the semidirect product of subgroups  $W_1(F)$  and  $W_2(F)$  where  $W_1(F)$  is the group of order  $2^{\dim \mathfrak{a}_F}$  generated by reflections in the  $A_1^k$  roots and the long  $C_s$  roots and  $W_2(F)$  is the group of permutations of the long  $C_s$  roots. Every element of  $W_1(F)$  centralizes  $Z_{M_F}(M_F^0)$  and hence belongs to  $W_F(\lambda, \chi)$ . An element  $w \in W_2(F)$  belongs to  $W_F(\lambda, \chi)$  just in case  $\varepsilon_\alpha(F:0) = \varepsilon_{w\alpha}(F:0)$  for all long  $C_s$  roots. Thus if there are  $r$  long roots with  $\varepsilon_\alpha(F:0) = 1$  and  $s-r$  long roots with  $\varepsilon_\alpha(F:0) = -1$ , then  $|W_F(\lambda, \chi) \cap W_2(F)| = r!(s-r)!$ . Now if there is  $\alpha \in F$  in the  $C_s$  factor,  $|[\alpha]| = r$  and we see using (6.1) that  $|W_{\mathcal{O}}(\lambda, \chi) \cap W_2(\mathcal{O})| = (r-1)!(s-r)!$ . If there is no  $\alpha \in F$  in the  $C_s$  factor, then  $|W_F(\lambda, \chi) \cap W_2(F)| = |W_{\mathcal{O}}(\lambda, \chi) \cap W_2(\mathcal{O})|$ . Thus in all cases we have

$$|W_F(\lambda, \chi) \cap W(\Phi_{F,R})| = 2^{|\mathcal{F}|} \prod_{\alpha \in \mathcal{F}'} |[\alpha]| |W_{\mathcal{O}}(\lambda, \chi) \cap W(\Phi_{\mathcal{O},R})|$$

where we define  $F' = \{\alpha \in F : \Phi_F(\alpha) \text{ is of type } C_s\}$  and  $F'' = \{\alpha \in F : \Phi_F(\alpha) \text{ is of type } A_1\}$ .

To prove the lemma it now suffices to show that

$$|W_0(G, H_F) \cap W_F(\lambda, \chi)| = \prod_{\alpha \in F''} |[\alpha]| |W_0(G, H_\emptyset) \cap W_\emptyset(\lambda, \chi)|.$$

Let  $W = W(\Phi_\emptyset)$ . For any  $F \subseteq F_0$  we can identify  $W$  with  $W(\Phi_F)$  via the Cayley transform  $c_F$ . Then

$$W_0(G, H_F) = \{w \in W : wt_F = t_F, w\alpha_F = \alpha_F, w\Phi_{F,R}^+ = \Phi_{F,R}^+, w\Phi_{M_F}^+ = \Phi_{M_F}^+\}.$$

Suppose  $w \in W_0(G, H_F) \cap W_F(\lambda, \chi)$ . Every element of  $W_0(G, H_F)$  acts on  $\Phi_{F,R}$  by permuting  $\Phi^+(F : A_1)$ , the positive  $A_1^k$  roots, and fixing the  $C_s$  factor pointwise. Now fix  $\alpha \in \Phi^+(F : A_1)$  with  $\varepsilon_\alpha(F : 0) = 1$  and let  $w \in W_F(\lambda, \chi) \cap W_0(G, H_F)$ . Now  $\alpha \notin \Phi_{F,R}^+(0)$  if and only if  $m_\alpha^*(F : h : v)$  is jointly smooth at  $(0, 0)$  if and only if there is a root  $\gamma \in \Phi_\alpha^+$  with  $\langle \lambda_F(h) + iv, \gamma \rangle = c(h_\alpha + iv_\alpha)$ . But then  $\pm w\gamma \in \Phi_{w\alpha}^+$  and  $\langle \lambda_F(h) + iv, \pm w\gamma \rangle = \pm c(h_{w\alpha} + iv_{w\alpha})$  since  $w^{-1}\lambda_F = \lambda_F$ . Thus  $\alpha \in \Phi_{F,R}^+(0)$  if and only if  $w\alpha \in \Phi_{F,R}^+(0)$ . Now by (5.4), if  $\alpha, \beta \in \Phi_{F,R}^+(0) \cap \Phi^+(F : A_1)$  and  $\beta \in [\alpha]$ , then there is  $w \in W_0(G, H_F) \cap W_F(\lambda, \chi)$  such that  $w$  interchanges  $\alpha$  and  $\beta$  and fixes all other roots in  $\Phi_{F,R}^+$ . Conversely, suppose  $\alpha \in F''$  and there is  $w \in W_0(G, H_F) \cap W_F(\lambda, \chi)$  such that  $w\alpha = \beta$ . Then there is  $w_0 \in W_0(G, H_F) \cap W_F(\lambda, \chi)$  which interchanges  $\alpha$  and  $\beta$  and fixes all other roots in  $\Phi_{F,R}^+$ . As in the proof of (4.14) we define  $H_2, \lambda_2$  such that  $SOS(H_2) = SOS(H_F) \setminus c^{-1}\{\alpha, \beta\}, \lambda_2|_{t_F} = \lambda_F, \langle \lambda_2, \alpha \rangle = \langle \lambda_2, \beta \rangle = 0$ . Then  $\lambda_2 \in \Lambda_{M_2,1}$  if and only if  $\beta \notin [\alpha]$ . Suppose  $\lambda_2 \in \Lambda_{M_2,1}$ . Then

$$\Phi(\lambda_2) = \{\delta \in \Phi_{M_2} : \langle \delta, \lambda_2 \rangle = 0\}$$

is of type  $A_1^k$  by (3.1), and so if

$$W(\lambda_2) = \{w \in W(\Phi_{M_2}) : w\lambda_2 = \lambda_2, w\delta = \pm\delta \text{ for all } \delta \in \Phi(\lambda_2), w \in W(\lambda_2)\}.$$

But  $w_0 \in W(\lambda_2), \alpha, \beta \in \Phi(\lambda_2)$ , and  $w\alpha = \beta$ . Thus  $\lambda_2 \notin \Lambda_{M_2,1}$  so that  $\beta \in [\alpha]$ . Thus

$$|W_0(G, H_F) \cap W_F(\lambda, \chi)| = \prod_{\alpha \in F''} |[\alpha]| |W_{0,F}(G, H_F) \cap W_F(\lambda, \chi)|$$

where  $W_{0,F}(G, H_F) = \{w \in W_0(G, H_F) : w\alpha = \alpha \text{ for all } \alpha \in F\}$ . Thus to complete the proof of the lemma it suffices to show that

$$|W_{0,F}(G, H_F) \cap W_F(\lambda, \chi)| = |W_0(G, H_\emptyset) \cap W_\emptyset(\lambda, \chi)|.$$

Let  $w \in W_{0,F}(G, H_F) \cap W_F(\lambda, \chi)$ . Since  $t_\emptyset = t_F \oplus \sum_{\alpha \in F} i\mathbf{R}H_\alpha$  and  $\alpha_F = \alpha_\emptyset \oplus \sum_{\alpha \in F} \mathbf{R}H_{c_F\alpha}, wt_\emptyset = t_\emptyset$  and  $w\alpha_\emptyset = \alpha_\emptyset$ . Further, via the Cayley transform identifications,  $\lambda_F = \lambda_\emptyset$  and  $\Phi_{\emptyset,R} \subseteq \Phi_{F,R}$ . Thus  $w\lambda_\emptyset = \lambda_\emptyset$  and  $w\chi_\emptyset = \chi_\emptyset$ . Finally,

recall that  $\langle \lambda_\emptyset, \beta \rangle \geq 0$  for all  $\beta \in \Phi_{M_\emptyset}^+$ . Now  $\langle \lambda_\emptyset, \beta \rangle = 0$  just in case  $\beta \in F$  and we know that  $wF = F$ . Now for any  $\beta \in \Lambda_{H_\emptyset}$  with  $\langle \beta, \lambda_\emptyset \rangle > 0$ ,  $\langle w\beta, \lambda_\emptyset \rangle = \langle w\beta, w\lambda_\emptyset \rangle = \langle \beta, \lambda_\emptyset \rangle > 0$  so that  $w\beta \in \Phi_{M_\emptyset}^+$ . Thus

$$w \in W_0(G, H_\emptyset) \cap W_\emptyset(\lambda, \chi).$$

Conversely, suppose  $w \in W_0(G, H_\emptyset) \cap W_\emptyset(\lambda, \chi)$ . If we can show that  $w\alpha = \alpha$  for all  $\alpha \in F$  it will follow as above that  $w \in W_{0,F}(G, H_F) \cap W_F(\lambda, \chi)$ . Thus it is enough to show that  $w\alpha = \alpha$  for all  $\alpha \in F_0$ . Since  $w\lambda = \lambda$  and  $w\Phi_M^+ = \Phi_M^+$ ,  $wF_0 = F_0$ . Suppose there are  $\beta_1 \neq \beta_2 \in F_0$  such that  $w\beta_1 = \beta_2$ . Since  $w \neq 1$  there are also  $\alpha_1 \neq \alpha_2 \in \Phi(\emptyset : A_1)$  such that  $w\alpha_1 = \alpha_2$ . Thus  $\dim \mathfrak{a}_{F_0} = \dim \mathfrak{a}_\emptyset + |F_0| \geq 4$  so that  $G$  has real rank at least four. Now we see from the list in [HW1, 1.4] that  $G$  must be the universal covering group of  $SU(p, q)$ ,  $Sp(n, \mathbf{R})$  or  $SO^*(2n)$ . In the first case,  $W_0(G, H_\emptyset)$  acts trivially on  $\Phi_M^+$ . In the other two cases,  $\Phi_M$  is of type  $A_1^k \times C_r$  or  $A_1^k \times D_r$ , for some  $k, r \geq 0$  and  $W_0(G, H_\emptyset)$  permutes the positive  $A_1^k$  roots and acts trivially on the  $C_r$  or  $D_r$  factor. However, all of the  $A_1^k$  roots are in  $\Phi_{M,1}$ , hence cannot be in  $F_0$ . Thus  $W_0(G, H_\emptyset)$  acts trivially on  $F_0$ .  $\square$

For any  $F \subseteq F_0$ , let  $F^c = F_0 \setminus F$ . Then for all  $\varepsilon \in \Sigma_0$ ,  $\sigma_F(\varepsilon) = \prod_{\alpha \in F^c} \varepsilon_\alpha$ . We also define  $\sigma_F(\lambda) = \text{sign} \prod_{\alpha \in \Phi_{M_F}^+ \setminus F^c} \langle \alpha, \lambda_F \rangle$ .

LEMMA 5.13. *There is a complex number  $c \neq 0$  so that for any  $F \subseteq F_0$ ,  $\varepsilon \in \Sigma_0$ ,  $h \in \mathcal{D}_F(\varepsilon)$ ,*

$$|m^*(F : h : v)| = c(-i)^{|F|} \sigma_F(\varepsilon) m^*(F : h : v).$$

*Proof.* We will identify  $\Phi_F^+$  with  $\Phi_\emptyset^+$  via the Cayley transform  $c_F$ . Write  $\Phi_F^+ = \Phi_{F,R}^+ \cup \Phi_{M_F}^+ \cup \Phi_{F,CPX}^+$ . For every  $\alpha \in \Phi_{F,R}^+$ ,

$$\langle \lambda_F(h) + iv, \alpha \rangle m_\alpha(F : h : v) = \frac{i \langle \alpha, \alpha \rangle}{2} \frac{v_\alpha \sinh \pi v_\alpha}{\cosh \pi v_\alpha - \varepsilon_\alpha(F : h)}$$

so that

$$|\langle \lambda_F(h) + iv, \alpha \rangle m_\alpha(F : h : v)| = -i \langle \lambda_F(h) + iv, \alpha \rangle m_\alpha(F : h : v).$$

For any

$$\alpha \in \Phi_{M_F}^+, \langle \lambda_F(h) + iv, \alpha \rangle = \langle \lambda_F(h), \alpha \rangle.$$

Now if

$$\alpha \in F^c, h \in \mathcal{D}_F(\varepsilon), \text{sign} \langle \lambda_F(h), \alpha \rangle = \text{sign} \langle h_{M_F}(h), \alpha \rangle = \varepsilon_\alpha.$$



If

$$\alpha \in \Phi_{M_F}^+ \setminus F^c, \langle \lambda_F, \alpha \rangle \neq 0,$$

so that

$$\text{sign} \langle \lambda_F(h), \alpha \rangle = \text{sign} \langle \lambda_F, \alpha \rangle$$

is independent of  $\varepsilon \in \Sigma_0$ ,  $h \in \mathcal{D}_F(\varepsilon)$ . Finally, for each  $\gamma \in \Phi_{F,CPX}^+$  there is  $\bar{\gamma} \in \Phi_{F,CPX}^\pm$  such that  $\langle \lambda_F(h) + iv, \bar{\gamma} \rangle = \langle -\lambda_F(h) + iv, \gamma \rangle$ . Now

$$\langle \lambda_F(h) + iv, \pm \bar{\gamma} \rangle \langle \lambda_F(h) + iv, \gamma \rangle = \mp (\langle \lambda_F(h), \gamma \rangle^2 + \langle v, \gamma \rangle^2).$$

Let  $n_F$  denote the number of roots  $\gamma \in \Phi_{F,CPX}^+$  such that  $\bar{\gamma} \in \Phi_{F,CPX}^+$ . Since  $\bar{\bar{\gamma}} = \gamma$ ,  $n_F$  is even. Then for  $h \in \mathcal{D}_F(\varepsilon)$ ,

$$|m^*(F : h : v)| = (-i)^{|\Phi_{F,R}^+|} \sigma_F(\varepsilon) \sigma_F(\lambda) (-1)^{n_F/2} \pi(F : h : v) \prod_{\alpha \in \Phi_{F,R}^+} m_\alpha(F : h : v).$$

Thus to finish the proof it suffices to show that

$$(-i)^{|\Phi_{F,R}^+|} \sigma_F(\lambda) (-1)^{n_F/2} = (-i)^{|F|} (-i)^{|\Phi_{\emptyset,R}^+|} \sigma_{\emptyset}(\lambda) (-1)^{n_{\emptyset}/2}.$$

We may as well assume  $\Phi^+ = \Phi_{\emptyset}^+ = \Phi_F^+$  is chosen so that every  $\alpha \in F_0$  is simple. Recall for any  $F \subseteq F_0$ ,  $\Phi_{F,R}^+ = \{\beta \in \Phi^+ : \beta|_{t_F} = 0\}$ ,  $\Phi_{M_F}^+ = \{\beta \in \Phi^+ : \beta|_{a_F} = 0\}$ , and  $\Phi_{F,CPX}^+ = \{\beta \in \Phi^+ : \beta|_{a_F} \neq 0 \text{ and } \beta|_{t_F} \neq 0\}$ . First let  $\beta \in \Phi_{F,R}^+ \setminus F$ . Then the restriction of  $\beta$  to  $a_{\emptyset}$  is non-zero. If  $\langle \alpha, \beta \rangle = 0$  for all  $\alpha \in F$ , then  $\beta \in \Phi_{\emptyset,R}^+$ . If  $\langle \alpha, \beta \rangle \neq 0$  for some  $\alpha \in F$ , then  $\beta \in \Phi_1 = \{\gamma \in \Phi_{\emptyset,CPX}^+ : \gamma|_{t_F} = 0\}$ . Thus

$$(-i)^{|\Phi_{F,R}^+|} = (-i)^{|F|} (-i)^{|\Phi_{\emptyset,R}^+|} (-i)^{|\Phi_1|}.$$

Now let  $\Phi_2 = \{\beta \in \Phi_M^+ \setminus F_0 : \beta \notin \Phi_{M_F}^+ \setminus F^c\}$ . Then  $\Phi_2 = \{\beta \in \Phi_M^+ \setminus F_0 : \langle \beta, \alpha \rangle \neq 0 \text{ for some } \alpha \in F\}$ . For every  $\alpha \in F_0$ , let  $s_\alpha$  denote the reflection in  $\alpha$ . Then since  $\alpha$  is simple,  $s_\alpha \Phi_2 = \Phi_2$ . Further, since  $\langle \lambda, \alpha \rangle = 0$ ,  $\langle \lambda, \beta \rangle = \langle \lambda, s_\alpha \beta \rangle$  for all  $\beta$ . But for all  $\beta \in \Phi_2$  there is  $\alpha \in F$  such that  $s_\alpha \beta \neq \beta$  and  $\text{sign} \langle \lambda, \beta \rangle \langle \lambda, s_\alpha \beta \rangle = 1$ . Thus

$$\sigma_{\emptyset}(\lambda) = \sigma_F(\lambda).$$

Now write

$$\Phi_{F,CPX}^+ = \Phi_3 \cup \Phi_4 \quad \text{where } \Phi_3 = \{\gamma \in \Phi_{F,CPX}^+ : \gamma|_{a_{\emptyset}} \neq 0\}$$

and

$$\Phi_4 = \{\gamma \in \Phi_{F,CPX}^+ : \gamma|_{\mathfrak{a}_\emptyset} = 0\}.$$

Let  $s_F = \prod_{\alpha \in F} s_\alpha$ . Then  $s_F \Phi_4 = \Phi_4$ . Thus for each  $\gamma \in \Phi_4$ ,  $\bar{\gamma} = -s_F \gamma \notin \Phi_{F,CPX}^+$ . Now  $\Phi_3 = \{\gamma \in \Phi_{\emptyset,CPX}^+ : \gamma|_{\mathfrak{t}_F} \neq 0\}$ . For  $\gamma \in \Phi_3$ , let  $\bar{\gamma}_F$  denote the conjugate of  $\gamma$  considered as an element of  $\Phi_{F,CPX}$  and let  $\bar{\gamma}_\emptyset$  denote the conjugate of  $\gamma$  considered as an element of  $\Phi_{\emptyset,CPX}$ . Then  $\bar{\gamma}_\emptyset = s_F \bar{\gamma}_F$  so that  $\bar{\gamma}_\emptyset \in \Phi^+$  if and only if  $\bar{\gamma}_F \in \Phi^+$ . Finally, for any  $\gamma \in \Phi_1$ ,  $\bar{\gamma}_\emptyset = s_F \gamma \in \Phi^+$ . Thus since  $\Phi_{\emptyset,CPX}^+ = \Phi_1 \cup \Phi_3$  we have

$$(-1)^{n_{\emptyset/2}} = (-i)^{|\Phi_1|} (-1)^{n_F/2}. \quad \square$$

LEMMA 5.14. *There is a constant  $c \neq 0$  so that for any  $F \subseteq F_0$ ,*

$$\int_{\mathfrak{a}_F^*} \Phi(F : h : v : x) m(F : h : v) dv = \frac{c}{(\pi i)^{|F|}} \int_{\mathfrak{a}_F^*} \frac{g(F : h : v : x)}{p_F(h : v)} dv.$$

*Proof.* The result follows from combining (5.9), (5.10), (5.12) and (5.13).  $\square$

LEMMA 5.15. *Define  $g(F)$  as in (5.3). Then for all  $\varepsilon \in \Sigma_0$ ,  $D \in D(iv^* \times \mathfrak{a}_F^*)$ ,  $r \geq 0$ ,  $g_1, g_2 \in \mathcal{U}(\mathfrak{g}_\mathbb{C})$ , there are constants  $C, s \geq 0$  so that*

$$\|g(F : h : v; D : g_1; x; g_2)\| (1 + |v|)^r \leq C \Xi(x) (1 + \bar{\sigma}(x))^s$$

for all  $x \in G$ ,  $h \in \mathcal{D}_F(\varepsilon)$ ,  $v \in \mathfrak{a}_F^*$ .

*Proof.* This follows from (2.22) since

$$\pi(F : h : v) \prod_{\alpha \in \Phi_{F,R}} m_\alpha(F : h : v) q(F : h : v) \sum_{\psi \in \mathcal{F}_F} \varepsilon(\psi) t(F : \psi : h : v)$$

and all its derivatives have polynomial growth in  $(h, v) \in iv^* \times \mathfrak{a}_F^*$ .  $\square$

We now want to show that the functions  $g(F)$ ,  $F \subseteq F_0$  satisfy the matching conditions of (4.1e). Thus as in Section 3 we fix  $E \subseteq F_0$ ,  $1 \leq i \leq m$ .

LEMMA 5.16. *For any  $E \subseteq F \subseteq E(i)$ ,  $v_E \in \mathfrak{a}_E^*$ ,  $h_0 \in \mathcal{H}_i$ ,  $k \geq 0$ ,*

$$\left( \partial/\partial h_i - i \sum_{\alpha \in F \setminus E} \partial/\partial \mu_\alpha \right)^k \pi(F : h_0 : (v_E, 0)) = (\partial/\partial h_i)^k \pi(E : h_0 : v_E).$$

*Proof.* For any  $F \subseteq F_0$ ,  $\alpha \in \Phi_F$ , define  $H_\alpha \in \mathfrak{h}_{F,C}$  to be dual to  $\alpha$  under the Killing form. Then we can decompose

$$\mathfrak{h}_F = \mathfrak{t}_{F_0} \oplus \sum_{\alpha \in F_0 \setminus F} \mathbf{R}iH_{c_F \alpha} \oplus \sum_{\alpha \in F} \mathbf{R}H_{c_F \alpha} \oplus \mathfrak{a}_\emptyset.$$

Thus we can consider  $\lambda_{F_0}(h) \in \mathfrak{h}_{F,C}^*$  by extending trivially on

$$\oplus \sum_{\alpha \in F_0 \setminus F} \mathbf{R}iH_{c_F \alpha} \oplus \sum_{\alpha \in F} \mathbf{R}H_{c_F \alpha} \oplus \mathfrak{a}_{\emptyset}.$$

Further, for  $v \in \mathfrak{a}_F^*$  we let  $v_{\emptyset}$  denote the element of  $\mathfrak{h}_{F,C}^*$  which is equal to  $v$  on  $\mathfrak{a}_{\emptyset}$  and is trivial on  $\mathfrak{t}_{F_0} \oplus \sum_{\alpha \in F_0 \setminus F} \mathbf{R}iH_{c_F \alpha} \oplus \sum_{\alpha \in F} \mathbf{R}H_{c_F \alpha}$ . Now we see that

$$\lambda_F(h) + iv_F = \lambda_{F_0}(h) + \sum_{\alpha \in F_0 \setminus F} \frac{\langle h, \alpha \rangle}{\langle \alpha, \alpha \rangle} c_F \alpha + iv_{\emptyset} + \sum_{\alpha \in F} \frac{\langle iv_F, c_F \alpha \rangle}{\langle \alpha, \alpha \rangle} c_F \alpha.$$

Write

$$\pi(F : h : v_F) = \prod_{\beta \in \Phi_F^+} \langle \lambda_F(h) + iv_F, \beta \rangle = \prod_{\beta \in \Phi_{\emptyset}^+} \langle c_F^{-1}(\lambda_F(h) + iv_F), \beta \rangle.$$

For any  $v \in \mathfrak{a}_{F_0}^*$ ,  $F \subseteq F_0$ , let  $v_F$  be the restriction of  $v$  to  $\mathfrak{a}_F$ . Now for every

$$\begin{aligned} & \beta \in \Phi_{\emptyset}^+, (h, v) \in i\mathfrak{v}^* \times \mathfrak{a}_{F_0}^*, \langle c_F^{-1}(\lambda_F(h) + iv_F), \beta \rangle - \langle c_E^{-1}(\lambda_E(h) + iv_E), \beta \rangle \\ &= - \sum_{\alpha \in F \setminus E} \frac{\langle h, \alpha \rangle}{\langle \alpha, \alpha \rangle} \langle \alpha, \beta \rangle + \sum_{\alpha \in F \setminus E} \frac{\langle iv_F, c_F \alpha \rangle}{\langle \alpha, \alpha \rangle} \langle \alpha, \beta \rangle. \end{aligned}$$

Thus if  $h_0 \in \mathcal{H}_i$  and  $v_E \in \mathfrak{a}_E^*$ ,

$$\langle c_F^{-1}(\lambda_F(h_0) + i(v_E, 0)), \beta \rangle - \langle c_E^{-1}(\lambda_E(h_0) + iv_E), \beta \rangle = 0$$

for all  $E \subseteq F \subseteq E(i)$  since  $\langle h_0, \alpha \rangle = 0$  for all  $\alpha \in F \setminus E \subseteq F_0^i$  and  $\langle (v_E, 0), c_F \alpha \rangle = 0$  for all  $\alpha \in F \setminus E$ .

Further, in the notation of (3.7),

$$\partial/\partial h_i (\langle c_F^{-1}(\lambda_F(h) + iv_F), \beta \rangle - \langle c_E^{-1}(\lambda_E(h) + iv_E), \beta \rangle) = \sum_{\alpha \in F \setminus E} \frac{\langle h_i, \alpha \rangle \langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle}$$

while for each  $\alpha \in F \setminus E$ ,

$$\partial/\partial \mu_{\alpha} (\langle c_F^{-1}(\lambda_F(h) + iv_F), \beta \rangle - \langle c_E^{-1}(\lambda_E(h) + iv_E), \beta \rangle) = i \frac{\langle \mu_{\alpha}, c_{F_0} \alpha \rangle \langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle}.$$

But for all  $\alpha \in F \setminus E \subseteq F_0^i$ ,  $\langle \mu_{\alpha}, c_{F_0} \alpha \rangle = \langle h_i, \alpha \rangle$ . Thus

$$\left( \partial/\partial h_i - i \sum_{\alpha \in F \setminus E} \partial/\partial \mu_{\alpha} \right) (\langle c_F^{-1}(\lambda_F(h) + iv_F), \beta \rangle - \langle c_E^{-1}(\lambda_E(h) + iv_E), \beta \rangle) = 0$$

for all  $(h, v) \in i\mathfrak{v}^* \times \mathfrak{a}_{F_0}^*$ . Thus for every  $k \geq 1$ ,

$$\left( \partial/\partial h_i - i \sum_{\alpha \in F \setminus E} \partial/\partial \mu_\alpha \right)^k \left( \langle c_F^{-1}(\lambda_F(h) + iv_F), \beta \rangle - \langle c_E^{-1}(\lambda_E(h) + iv_E), \beta \rangle \right) = 0.$$

Now, since

$$\partial/\partial \mu_\alpha \langle c_E^{-1}(\lambda_E(h) + iv_E), \beta \rangle = 0$$

for all  $\alpha \in F \setminus E$ , we have shown that for all  $k \geq 0$ ,  $\beta \in \Phi_{\emptyset}^+$ ,

$$\begin{aligned} & \left( \partial/\partial h_i - i \sum_{\alpha \in F \setminus E} \partial/\partial \mu_\alpha \right)^k \left( \langle c_F^{-1}(\lambda_F(h_0) + i(v_E, 0)), \beta \rangle \right) \\ &= (\partial/\partial h_i)^k \langle c_E^{-1}(\lambda_E(h_0) + iv_E), \beta \rangle \end{aligned}$$

for all  $E \subseteq F \subseteq E(i)$ . Thus for all  $k \geq 0$ ,

$$\left( \partial/\partial h_i - i \sum_{\alpha \in F \setminus E} \partial/\partial \mu_\alpha \right)^k \pi(F: h_0: (v_E, 0)) = (\partial/\partial h_i)^k \pi(E: h_0: v_E)$$

for all  $E \subseteq F \subseteq E(i)$ . □

**LEMMA 5.17.** *Suppose  $h_0 \in \mathcal{H}_i$  and  $v_E \in \mathfrak{a}_E^*$  such that  $\langle v_E, \alpha \rangle \neq 0$  for all  $\alpha \in \Phi_{E,R}^+$ . Then for all  $E \subseteq F \subseteq E(i)$ ,*

$$\begin{aligned} & \lim_{(h, v_F) \rightarrow (h_0, (v_E, 0))} q(F: h: v_F) \sum_{\psi \in \mathcal{T}_F} \varepsilon(\psi) t(F: \psi: h: v_F) \\ &= (2/\pi)^{|F \setminus E|} q(E: h_0: v_E) \sum_{\psi \in \mathcal{T}_E} \varepsilon(\psi) t(E: \psi: h_0: v_E). \end{aligned}$$

*Proof.* For all  $h \in i\mathfrak{v}^*$ ,  $v \in \mathfrak{a}_F^*$ , we can write

$$q(F: h: v) = \prod_{\alpha \in F} (v_\alpha + ih_\alpha) \prod_{\beta \in [\alpha], \beta \neq \alpha} (v_\beta + ih_\beta)(v_\beta - v_\alpha)^{-1}.$$

Thus if for  $v \in \mathfrak{a}_F^*$ , we let  $v_E$  denote the restriction of  $v$  to  $\mathfrak{a}_E$ ,

$$q(F: h: v) = q(E: h: v_E) \prod_{\alpha \in F \setminus E} (v_\alpha + ih_\alpha) \prod_{\beta \in [\alpha], \beta \neq \alpha} (v_\beta + ih_\beta)(v_\beta - v_\alpha)^{-1}.$$

For every  $\alpha \in F_0$ ,  $\Phi_F[\alpha]$  is a union of some of the simple factors of  $\Phi_{F,R}'$ . Now

since  $\Phi''_{F,R} = \bigcup_{\alpha \in F_0} \Phi_F[\alpha]$ , we can write

$$\sum_{\psi \in \mathcal{T}_F} \varepsilon(\psi) t(F: \psi: h: v) = \prod_{\alpha \in F_0} \left( \sum_{\psi_\alpha \in \mathcal{T}_F(\alpha)} \varepsilon(\psi_\alpha) t(F: \psi_\alpha: h: v) \right)$$

where  $\mathcal{T}_F(\alpha)$  denotes the set of two-structures of  $\Phi_F[\alpha]$  and if  $\psi_\alpha = \psi_1 \cup \dots \cup \psi_k \in \mathcal{T}_F(\alpha)$ ,

$$t(F: \psi_\alpha: h: v) = \prod_{i=1}^k t(F: \psi_i: h: v).$$

Suppose  $\alpha \in F \setminus E$ . Then if  $\Phi_{F_0}(\alpha)$  is of type  $C_n$  for some  $n \geq 2$  it follows from (6.6) that

$$\begin{aligned} & \lim_{(h, v_F) \rightarrow (h_0, (v_E, 0))} (v_\alpha + ih_\alpha) \prod_{\beta \in [\alpha], \beta \neq \alpha} (v_\beta + ih_\beta)(v_\beta - v_\alpha)^{-1} \sum_{\psi_\alpha \in \mathcal{T}_F(\alpha)} \varepsilon(\psi_\alpha) t(F: \psi_\alpha: h: v_F) \\ &= (2/\pi) \sum_{\psi_\alpha \in \mathcal{T}_E(\alpha)} \varepsilon(\psi_\alpha) t(E: \psi_\alpha: h: v_E). \end{aligned}$$

If  $\Phi_{F_0}(\alpha)$  is of type  $A_1$ , then when  $v_F = (v_E, 0)$ ,  $h = h_0 \in \mathcal{H}_i$ ,  $v_\alpha = h_\alpha = h_\beta = 0$  for all  $\beta \in [\alpha]$ . Thus

$$\lim_{(h, v_F) \rightarrow (h_0, (v_E, 0))} \prod_{\beta \in [\alpha], \beta \neq \alpha} (v_\beta + ih_\beta)(v_\beta - v_\alpha)^{-1} = 1.$$

Further  $\mathcal{T}_F(\alpha) = \{\Phi_F[\alpha]\}$ . Now

$$\lim_{(h, v_F) \rightarrow (h_0, (v_E, 0))} \frac{(h_\alpha + iv_\alpha) \sinh \pi(v_\alpha + ih_\alpha)}{\cosh \pi(v_\alpha + ih_\alpha) - 1} = 2/\pi$$

while for all  $\beta \in \Phi_F[\alpha]$ ,  $\beta \neq \alpha$ ,

$$\lim_{(h, v_F) \rightarrow (h_0, (v_E, 0))} t_\beta(F: h: v_F) = t_\beta(E: h_0: v_E).$$

Further,  $\Phi_E^+[\alpha] = \Phi_F^+[\alpha] \setminus \{\alpha\}$ . Thus in this case as well we have

$$\begin{aligned} & \lim_{(h, v_F) \rightarrow (h_0, (v_E, 0))} (v_\alpha + ih_\alpha) \prod_{\beta \in [\alpha], \beta \neq \alpha} (v_\beta + ih_\beta)(v_\beta - v_\alpha)^{-1} \sum_{\psi_\alpha \in \mathcal{T}_F(\alpha)} \varepsilon(\psi_\alpha) t(\psi_\alpha: h: v_F) \\ &= (2/\pi) \sum_{\psi_\alpha \in \mathcal{T}_E(\alpha)} \varepsilon(\psi_\alpha) t(E: \psi_\alpha: h: v_E). \end{aligned}$$

Finally, when  $\alpha \in F_0 \setminus F$ ,  $\Phi_F[\alpha] = \Phi_E[\alpha]$  and

$$\lim_{(h, v_F) \rightarrow (h_0, (v_E, 0))} \sum_{\psi_\alpha \in \mathcal{F}_F(\alpha)} \varepsilon(\psi_\alpha) t(F: \psi_\alpha: h: v_F) = \sum_{\psi_\alpha \in \mathcal{F}_E(\alpha)} \varepsilon(\psi_\alpha) t(E: \psi_\alpha: h: v_E) \quad \square$$

For any  $F \subseteq F_0$ ,  $v \in \mathfrak{a}_{F_0}^*$ , let  $v_F$  denote the restriction of  $v$  to  $\mathfrak{a}_F \subseteq \mathfrak{a}_{F_0}$ . Then for each  $F \subseteq F_0$  we can extend  $g(F)$  to a function on  $iv^* \times \mathfrak{a}_{F_0}^* \times G$  by

$$g(F: h: v: x) = g(F: h: v_F: x), (h, v, x) \in iv^* \times \mathfrak{a}_{F_0}^* \times G.$$

Then for any  $\alpha \in F_0 \setminus F$ ,  $\partial/\partial\mu_\alpha g(F: h: v: x) = 0$  for all  $(h, v, x)$  since  $(v + t\mu_\alpha)_F = v_F$  for all  $v \in \mathfrak{a}_{F_0}^*$ ,  $t \in \mathbf{R}$ . For any  $\varepsilon \in \Sigma_0$ , the restriction of  $g(F: h: v: x)$  to  $\mathcal{D}_F(\varepsilon)$  extends to a  $C^\infty$  function on  $cl(\mathcal{D}_F(\varepsilon)) \times \mathfrak{a}_{F_0}^* \times G$ . Let  $g(F: \varepsilon: h: v: x)$  denote a  $C^\infty$  function on  $iv^* \times \mathfrak{a}_{F_0}^* \times G$  which agrees with  $g(F)$  on  $cl(\mathcal{D}_F(\varepsilon)) \times \mathfrak{a}_{F_0}^* \times G$ . Now fix  $E \subseteq F_0$ ,  $1 \leq i \leq m$ , and  $\varepsilon \in \Sigma_i$ . For  $v_E \in \mathfrak{a}_E^*$ , let  $(v_E, 0) \in \mathfrak{a}_{F_0}^*$  be defined by  $(v_E, 0)|_{\mathfrak{a}_E} = v_E$ ,  $\langle (v_E, 0), c_{F_0}\alpha \rangle = 0$  for all  $\alpha \in F_0 \setminus E$ . For  $E \subseteq F \subseteq E(i)$ ,  $v_E \in \mathfrak{a}_E^*$ ,  $h_0 \in \mathcal{H}_i \cap cl(\mathcal{D}_E(\varepsilon))$ ,  $x \in G$ , write

$$g^\pm(F: k: h_0: v_E: x) = \left( \partial/\partial h_i - i \sum_{\alpha \in F \setminus E} \partial/\partial \mu_\alpha \right)^k g(F: \varepsilon^\pm(i): h_0: (v_E, 0): x). \quad (5.18a)$$

To complete the proof of Theorem 5.3 we must show that for all  $E \subseteq F \subseteq E(i)$ ,  $k \geq 0$ ,  $v_E \in \mathfrak{a}_E^*$ ,  $h_0 \in \mathcal{H}_i \cap cl(\mathcal{D}_E(\varepsilon))$ ,  $x \in G$ , we have

$$\begin{aligned} & g^+(F: k: h_0: v_E: x) - g^-(F: k: h_0: v_E: x) \\ &= \sum_{F \subseteq F' \subseteq E(i)} c_{|F' \setminus F|} (g^+(F': k: h_0: v_E: x) + g^-(F': k: h_0: v_E: x)). \end{aligned} \quad (5.18b)$$

For  $(h, v, x) \in iv^* \times \mathfrak{a}_{F_0}^* \times G$ , define

$$\begin{aligned} g(E: F: i: h: v: x) &= g(F: \varepsilon^+(i): h: v: x) - g(F: \varepsilon^-(i): h: v: x) \\ &- \sum_{F \subset F' \subseteq E(i)} c_{|F' \setminus F|} (g(F': \varepsilon^+(i): h: v: x) + g(F': \varepsilon^-(i): h: v: x)). \end{aligned} \quad (5.18c)$$

Then since

$$\begin{aligned} & \left( \partial/\partial h_i - i \sum_{\alpha \in F \setminus E} \partial/\partial \mu_\alpha \right)^k g(F: \varepsilon^\pm(i): h: v: x) \\ &= \left( \partial/\partial h_i - i \sum_{\alpha \in E(i) \setminus E} \partial/\partial \mu_\alpha \right)^k g(F: \varepsilon^\pm(i): h: v: x) \end{aligned}$$

for all  $k \geq 0$ ,  $E \subseteq F \subseteq E(i)$ ,  $(h, v, x) \in iv^* \times \mathfrak{a}_{F_0}^* \times G$ , we see that proving (5.18b) is

equivalent to proving that for all  $E \subseteq F \subseteq E(i)$ ,  $k \geq 0$ ,  $v_E \in \alpha_E^*$ ,  $h_0 \in \mathcal{H}_i \cap cl(\mathcal{D}_E(\varepsilon))$ ,  $x \in G$ ,

$$\left( \partial/\partial h_i - i \sum_{\alpha \in E(i) \setminus E} \partial/\partial \mu_\alpha \right)^k g(E: F: i: h_0: (v_E, 0): x) = 0. \quad (5.18d)$$

To do this we first need a simple calculus lemma. For  $p, q \geq 0$ , write coordinates in  $\mathbf{R}^{1+p+q}$  as  $(t, x, y)$ ,  $t \in \mathbf{R}$ ,  $x \in \mathbf{R}^p$ ,  $y \in \mathbf{R}^q$ .

LEMMA 5.19. *A function  $f \in C^\infty(\mathbf{R}^{1+p+q})$  satisfies*

$$\left( \partial/\partial t - i \sum_{i=1}^p \partial/\partial x_i - i \sum_{j=1}^q \partial/\partial y_j \right)^k f(0, 0, y) = 0 \quad \text{for all } y \in \mathbf{R}^q, \quad k \geq 0$$

if and only if

$$\left( \partial/\partial t - i \sum_{i=1}^p \partial/\partial x_i \right)^k f(0, 0, y) = 0 \quad \text{for all } y \in \mathbf{R}^q, \quad k \geq 0.$$

*Proof.* For any  $k \geq 0$  write

$$\begin{aligned} & \left( \partial/\partial t - i \sum_{i=1}^p \partial/\partial x_i - i \sum_{j=1}^q \partial/\partial y_j \right)^k f(t, x, y) \\ &= \sum_{r=0}^k \binom{k}{r} \left( -i \sum_{j=1}^q \partial/\partial y_j \right)^{k-r} \left( \partial/\partial t - i \sum_{i=1}^p \partial/\partial x_i \right)^r f(t, x, y). \end{aligned}$$

Now since  $t, x$ , and  $y$  are independent variables, we can evaluate  $(\partial/\partial t - i \sum_{i=1}^p \partial/\partial x_i) f(t, x, y)$  at  $(t, x) = (0, 0)$  before differentiating with respect to  $(-i \sum_{j=1}^q \partial/\partial y_j)^{k-r}$ . Thus the second condition in the lemma clearly implies the first. Conversely, assume the first condition is satisfied. We will prove that the second holds by induction on  $k$ . If  $k=0$  the result is the same as the  $k=0$  case of the first condition. Now for  $k \geq 1$ ,

$$0 = \left( \partial/\partial t - i \sum_{i=1}^p \partial/\partial x_i - i \sum_{j=1}^q \partial/\partial y_j \right)^k f(0, 0, y) = \left( \partial/\partial t - i \sum_{i=1}^p \partial/\partial x_i \right)^k f(0, 0, y)$$

since using the induction hypothesis, for all  $0 \leq r \leq k-1$ ,

$$\left( -i \sum_{j=1}^q \partial/\partial y_j \right)^{k-r} \left( \partial/\partial t - i \sum_{i=1}^p \partial/\partial x_i \right)^r f(0, 0, y) = 0. \quad \square$$

We will apply this lemma as follows. For any  $F \subseteq F_0$ , let  $\Phi_F(i, l) = \bigcup_{\alpha \in F_0} \Phi_F[\alpha, l]$  where  $\Phi_F[\alpha, l]$  denotes the long roots in  $\Phi_F[\alpha]$ .

LEMMA 5.20. Let  $E \subseteq F \subseteq E(i)$ ,  $h_0 \in \mathcal{H}_i \cap cl(\mathcal{D}_E(\varepsilon))$ ,  $x \in G$ . Then

$$\left( \partial/\partial h_i - i \sum_{\alpha \in E(i) \setminus E} \partial/\partial \mu_\alpha \right)^k g(E: F: i: h_0: (v_E, 0), x) = 0$$

for all  $v_E \in \alpha_E^*$ ,  $k \geq 0$  if and only if

$$\left( \partial/\partial h_i - i \sum_{\alpha \in \Phi_{F_0}(i, l)} \partial/\partial \mu_\alpha \right)^k g(E: F: i: h_0: (v_E, 0), x) = 0$$

for all  $v_E \in \alpha_E^*$ ,  $k \geq 0$ .

*Proof.*  $\{\mu_\alpha: \alpha \in \Phi_{F_0}(i, l)\}$  is a linearly independent set in  $\alpha_{F_0}^*$  and  $\Phi_{F_0}(i, l)$  is the disjoint union of  $E(i) \setminus E$  and  $\Phi_E(i, l)$ . Let  $p = |E(i) \setminus E|$ ,  $q = |\Phi_E(i, l)|$ . Then for any

$$\begin{aligned} v_E \in \alpha_E^*, (t, x, y) &\rightarrow (h_0(t), v(v_E, x, y)) \\ &= \left( h_0 + th_i, (v_E, 0) + \sum_{\alpha \in E(i) \setminus E} x_\alpha \mu_\alpha + \sum_{\beta \in \Phi_E(i, l)} y_\beta \mu_\beta \right) \end{aligned}$$

gives an embedding of  $\mathbf{R}^{1+p+q}$  into  $iv^* \times \alpha_{F_0}^*$ . Further, if we consider  $\alpha_E^* \subseteq \alpha_{F_0}^*$  by  $\alpha_E^* = \{(v_E, 0): v_E \in \alpha_E^*\}$ , then for  $\alpha \in E(i) \setminus E$ ,  $\mu_\alpha$  is orthogonal to  $\alpha_E^*$ , while for  $\alpha \in \Phi_E(i, l)$ ,  $\mu_\alpha \in \alpha_E^*$ . For  $v_E \in \alpha_E^*$  define

$$f(t, x, y) = g(E: F: i: h_0(t): v(v_E, x, y)).$$

Then for any  $k \geq 0$ ,

$$\begin{aligned} &\left( \partial/\partial t - i \sum_{\alpha \in E(i) \setminus E} \partial/\partial x_\alpha \right)^k f(0, 0, y) \\ &= \left( \partial/\partial h_i - i \sum_{\alpha \in E(i) \setminus E} \partial/\partial \mu_\alpha \right)^k g\left(E: F: i: h_0: \left(v_E + \sum_{\beta \in \Phi_E(i, l)} y_\beta v_\beta, 0\right)\right) \end{aligned}$$

while

$$\begin{aligned} &\left( \partial/\partial t - i \sum_{\alpha \in E(i) \setminus E} \partial/\partial x_\alpha - \sum_{\beta \in \Phi_E(i, l)} \partial/\partial y_\beta \right)^k f(0, 0, y) \\ &= \left( \partial/\partial h_i - i \sum_{\alpha \in \Phi_{F_0}(i, l)} \partial/\partial \mu_\alpha \right)^k g\left(E: F: i: h_0: \left(v_E + \sum_{\beta \in \Phi_E(i, l)} y_\beta v_\beta, 0\right)\right). \end{aligned}$$



Now the result follows from Lemma 5.19. □

LEMMA 5.21. For all  $E \subseteq F \subseteq E(i)$ ,  $k \geq 0$ ,  $v_E \in \mathfrak{a}_E^*$ ,  $h_0 \in \mathcal{H}_i \cap cl(\mathcal{D}_E(\varepsilon))$ ,  $x \in G$ , we have

$$\left( \partial/\partial h_i - i \sum_{\alpha \in \Phi_{F_0}(i,l)} \partial/\partial \mu_\alpha \right)^k g(E:F:i:h_0:(v_E, 0):x) = 0.$$

*Proof.* Write  $D_i = (\partial/\partial h_i - i \sum_{\alpha \in \Phi_{F_0}(i,l)} \partial/\partial \mu_\alpha)$ . Since  $g(E:F:i:h_0:(v_E, 0):x)$  is a smooth function of  $v_E \in \mathfrak{a}_E^*$ , it suffices to prove the result for  $v_E$  regular as in (5.17). First, using the formulas in the proof of (5.17) we see that for all  $E \subseteq F \subseteq E(i)$

$$q(F:h:v) = q(E:h:v) \prod_{\alpha \in F \setminus E} (v_\alpha + ih_\alpha) \prod_{\beta \in [\alpha], \beta \neq \alpha} (v_\beta + ih_\beta)(v_\beta - v_\alpha)^{-1}$$

and

$$\sum_{\psi \in \mathcal{F}_F} \varepsilon(\psi)t(F:\psi:h:v) = \prod_{\alpha \in F_0} \left[ \sum_{\psi_\alpha \in \mathcal{F}_F(\alpha)} \varepsilon(\psi_\alpha)t(F:\psi_\alpha:h:v) \right].$$

Now since  $F \setminus E \subseteq F_0^i$ , for  $\alpha \in F_0^j$ ,  $j \neq i$ , we will have

$$\sum_{\psi_\alpha \in \mathcal{F}_F(\alpha)} \varepsilon(\psi_\alpha)t(F:\psi_\alpha:h:v) = \sum_{\psi_\alpha \in \mathcal{F}_E(\alpha)} \varepsilon(\psi_\alpha)t(E:\psi_\alpha:h:v).$$

For each  $E \subseteq F \subseteq E(i)$ , write

$$\begin{aligned} q(F:h:v) &= \sum_{\psi \in \mathcal{F}_F} \varepsilon(\psi)t(F:\psi:h:v) \\ &= p(E:F:h:v)q(E:h:v) \prod_{\alpha \in F_0 \setminus F_0^i} \left[ \sum_{\psi_\alpha \in \mathcal{F}_E(\alpha)} \varepsilon(\psi_\alpha)t(E:\psi_\alpha:h:v) \right] \end{aligned}$$

where

$$\begin{aligned} p(E:F:h:v) &= \prod_{\alpha \in F \setminus E} (v_\alpha + ih_\alpha) \prod_{\beta \in [\alpha], \beta \neq \alpha} (v_\beta + ih_\beta)(v_\beta - v_\alpha)^{-1} \prod_{\alpha \in F_0^i} \left[ \sum_{\psi_\alpha \in \mathcal{F}_F(\alpha)} \varepsilon(\psi_\alpha)t(F:\psi_\alpha:h:v) \right]. \end{aligned}$$

Now, by (5.2) and (6.3), for all  $E \subseteq F \subseteq E(i)$ ,  $p(E:F:h:v)$  depends on  $(h, v) \in iv^* \times \mathfrak{a}_{F_0}^*$  only as a function of  $v_\alpha + ih_\alpha$ ,  $\alpha \in \Phi_{F_0}(i, l)$ . Now if  $\alpha, \beta \in \Phi_{F_0}(i, l)$ , then

$$(v + t\mu_\alpha)_\beta = \begin{cases} v_\beta, & \text{if } \alpha \neq \beta; \\ v_\beta + \frac{2t\langle h_i, \beta \rangle}{\langle \beta, \beta \rangle}, & \text{if } \alpha = \beta; \end{cases}$$

and

$$(h + th_i)_\beta = h_\beta + \frac{2t\langle h_i, \beta \rangle}{\langle \beta, \beta \rangle} \text{ for all } t \in \mathbf{R}.$$

Thus

$$D_i(h_\beta) = \frac{2t\langle h_i, \beta \rangle}{\langle \beta, \beta \rangle} \text{ and } D_i(v_\beta) = -i \frac{2t\langle h_i, \beta \rangle}{\langle \beta, \beta \rangle}$$

so that  $D_i(v_\beta + ih_\beta) = 0$ . Thus for all  $k \geq 1$ ,  $D_i^k p(E:F:h:v) = 0$  for all  $(h, v) \in i\nu^* \times \mathfrak{a}_{F_0}^*$ . Further, using (5.17),

$$\lim_{(h,v) \rightarrow (h_0, (v_E, 0))} p(E:F:h:v) = (2/\pi)^{|F \setminus E|} p(E:E:h_0:v_E).$$

Thus for all  $k \geq 0$ ,  $E \subseteq F \subseteq E(i)$  we have

$$\begin{aligned} D_i^k g(F : \varepsilon^\pm(i) : h_0 : (v_E, 0) : x) \\ = (\pi/2)^{|E|} D_i^k g_1(F : \varepsilon^\pm(i) : h_0 : (v_E, 0) : x) p(E:E:h_0:v_E) \end{aligned}$$

where for any  $\varepsilon \in \Sigma_0$ ,  $h \in \mathcal{D}_F(\varepsilon)$ ,

$$\begin{aligned} g_1(F : \varepsilon : h : v : x) &= \sigma_F(\varepsilon) \Phi(F : h : v : x) \pi(F : h : v) \prod_{\alpha \in \Phi_{F,R}} m_\alpha(F : h : v) \\ &\times q(E : h : v) \prod_{\alpha \in F_0 \setminus F_0^i} \left[ \sum_{\psi_\alpha \in \mathcal{F}_E(\alpha)} \varepsilon(\psi_\alpha) t(E : \psi_\alpha : h : v) \right]. \end{aligned}$$

Thus it is enough to prove that  $D_i^k g_1(E:F:i:h_0:(v_E, 0):x) = 0$  where  $g_1(E:F:i:h:v:x)$  is defined as in (5.18c) with  $g_1(F)$  replacing  $g(F)$ .

But

$$q(E : h : v) \prod_{\alpha \in F_0 \setminus F_0^i} \left[ \sum_{\psi_\alpha \in \mathcal{F}_E(\alpha)} \varepsilon(\psi_\alpha) t(E : \psi_\alpha : h : v) \right]$$

is independent of  $E \subseteq F \subseteq E(i)$  and by (5.5) we know that  $\prod_{\alpha \in \Phi_{F,R}} m_\alpha(F : h : v)$  is independent of  $F \subseteq F_0$  for all  $(h, v) \in i\nu^* \times \mathfrak{a}_{F_0}^*$ . Thus using (5.16) and assumption (4.1e) we can conclude that

$$\left( \partial/\partial h_i - \sum_{\alpha \in E(i) \setminus E} \partial/\partial \mu_\alpha \right)^k g_1(E:F:i:h_0:(v_E, 0):x) = 0$$

for all  $v_E \in \mathfrak{a}_E^*$ ,  $k \geq 0$ . Now as in (5.20) this implies that

$$D_i^k g_1(E: F: i: h_0: (v_E, 0): x) = 0$$

for all  $v_E \in \mathfrak{a}_E^*$ ,  $k \geq 0$ . □

### 6. $Sp(n, \mathbf{R})$ calculations

In this section we will assume that  $G$  is the universal covering group of  $Sp(n, \mathbf{R})$  for some  $n \geq 2$ . Let  $B$  be a relatively compact Cartan subgroup of  $G$ . Then we will write

$$\Phi_B^+ = \Phi^+(\mathfrak{g}_C, \mathfrak{b}_C) = \{e_i \pm e_j, 2e_k: 1 \leq i < j \leq n, 1 \leq k \leq n\}$$

and

$$\Phi^+(\mathfrak{k}_C, \mathfrak{b}_C) = \{e_i - e_j: 1 \leq i < j \leq n\}.$$

In this case  $\mathfrak{iv}^*$  is one-dimensional and for  $h \in \mathfrak{iv}^*$ ,  $2\langle h, \alpha \rangle / \langle \alpha, \alpha \rangle$  is independent of  $\alpha \in \{2e_1, \dots, 2e_n\}$ . By abuse of notation we will use  $h$  to denote both an element of  $\mathfrak{iv}^*$  and the real number  $2\langle h, \alpha \rangle / \langle \alpha, \alpha \rangle$ ,  $\alpha \in \{2e_1, \dots, 2e_n\}$ .

Fix  $0 \leq r \leq n - 1$  and let  $H = TA$  be the Cartan subgroup of  $G$  corresponding to the set of strongly orthogonal non-compact roots

$$SOS(H) = \{2e_1, \dots, 2e_r\}, c: \mathfrak{b}_C \rightarrow \mathfrak{h}_C$$

the corresponding Cayley transform, and  $P = MAN$  a cuspidal parabolic subgroup corresponding to  $H$ . Use  $c$  to identify  $\Phi_B^+$  and  $\Phi_H^+ = c\Phi_B^+$ . Now

$$\Phi_M^+ = \Phi^+(\mathfrak{m}_C, \mathfrak{t}_C) = \{e_i \pm e_j, 2e_k: r+1 \leq i < j \leq n, r+1 \leq k \leq n\}.$$

Fix  $(\lambda, \chi) \in X(T)$  such that  $F_0 = \{2e_{r+1}\}$  and for  $F \subseteq F_0$  define  $H_F, \lambda_F, \chi_F, c_F: \mathfrak{h}_C \rightarrow \mathfrak{h}_{F_C}$  as in (3.3). Define  $\Phi_F, \Phi_{F,R}$  as in (5.1) and identify  $\Phi_F^+$  with  $\Phi_B^+$  via the Cayley transform  $cc_F$ . Thus

$$\Phi_{\mathcal{O},R}^+ = \{e_i \pm e_j, 2e_k: 1 \leq i < j \leq r, 1 \leq k \leq r\}$$

and

$$\Phi_{F_0,R}^+ = \{e_i \pm e_j, 2e_k: 1 \leq i < j \leq r+1, 1 \leq k \leq r+1\}.$$

Further,  $Z_{M_F}(M_F^0)$  is abelian and generated by  $Z_{M_F^0} = Z_{M_F}(M_F^0) \cap T_F^0$  and the subgroup  $\Gamma_F$  generated by  $\{\gamma_\alpha : \alpha \in \Phi_{F,R}^+\}$  where  $\gamma_\alpha$  is defined as in (5.1d). For  $1 \leq i \leq r+1$ , let  $\gamma_i = \gamma_{2e_i}$ . Then as in [HW1, 1.5], for  $1 \leq i \neq j \leq r+1$ ,  $\gamma_{e_i \pm e_j} = \gamma_i \gamma_j^{-1}$  has order two. Thus  $\Gamma_\emptyset$  is generated by  $\gamma_1, \dots, \gamma_r$  and  $\Gamma_{F_0}$  is generated by  $\gamma_1, \dots, \gamma_{r+1}$ . For  $\alpha \in \Phi_{F,R}^+$ , define  $m_\alpha(F : h : v)$ ,  $\rho_{F,\alpha}$ ,  $\varepsilon_\alpha(F : h)$  as in (5.1). For  $1 \leq i \leq r+1$ , define  $\varepsilon_i = \varepsilon_{2e_i}(F_0 : 0)$ .

LEMMA 6.1. *For all  $1 \leq i < j \leq r+1$ ,  $1 \leq k \leq r+1$ ,  $h \in iv^*$ ,  $\varepsilon_{2e_k}(F_0 : h) = \varepsilon_k \cos \pi h$  and  $\varepsilon_{e_i \pm e_j}(F_0 : h) = -\varepsilon_i \varepsilon_j$ . For all  $1 \leq i < j \leq r$ ,  $1 \leq k \leq r$ ,  $h \in iv^*$ ,  $\varepsilon_{2e_k}(\emptyset : h) = -\varepsilon_k \cos \pi h$  and  $\varepsilon_{e_i \pm e_j}(\emptyset : h) = -\varepsilon_i \varepsilon_j$ . Further  $\varepsilon_i = \pm 1$  for  $1 \leq i \leq r$  and  $\varepsilon_{r+1} = 1$ . Finally, for all  $1 \leq i \leq r+1$ ,  $m_{2e_i}^*(F_0 : h : v)$  is jointly smooth at  $(0, 0)$  if and only if  $\varepsilon_i = -1$  and  $m_{2e_i}^*(\emptyset : h : v)$  is jointly smooth on  $iv^* \times \alpha_{\emptyset}^*$  for all  $1 \leq i \leq r$ .*

*Proof.* Because  $2e_{r+1} \in F_0$  we have  $\varepsilon_{r+1} = 1$  by (4.9). Now as in (5.5), for  $1 \leq i \leq r+1$ ,  $\varepsilon_i = \pm 1$  just in case  $\chi_{F_0}(0 : \gamma_i) = \pm (-1)^{\rho_{F_0} 2e_i}$ . But now  $\varepsilon_{r+1} = 1$  implies that  $\chi_{F_0}(0 : \gamma_{r+1}) = \pm 1$  so that for

$$1 \leq i \leq r, \quad \chi_{F_0}(0 : \gamma_i) = \chi_{F_0}(0 : \gamma_{r+1}) \chi_{F_0}(0 : \gamma_i \gamma_{r+1}^{-1}) = \pm 1$$

since  $\gamma_i \gamma_{r+1}^{-1}$  has order two.

Now as in (5.5), for  $F \subseteq F_0$  we have  $\varepsilon_\alpha(F : h) = \varepsilon_\alpha(F : 0) \cos \pi h$  for any  $h \in iv^*$  and any long root  $\alpha \in \Phi_{F,R}^+$ . If  $\alpha \in \Phi_{F,R}$  is a short root,  $\gamma_\alpha$  has order two and  $e^h(\gamma_\alpha) = 1$  for all  $h \in iv^*$  so that  $\varepsilon_\alpha(F : h) = \varepsilon_\alpha(0 : h) = \pm 1$ .

To finish the proof of the first part we must compute  $(-1)^{\rho_{F,\alpha}}$  for  $F \subseteq F_0$ ,  $\alpha \in \Phi_{F,R}^+$ . Suppose  $F = \emptyset$ . Then for  $1 \leq i \leq r$ ,  $\Phi_{\emptyset, 2e_i}^+ = \{2e_i, e_i \pm e_j : r+1 \leq j \leq n\}$  so that  $\rho_{\emptyset, 2e_i} = n - r + 1$ . For  $1 \leq i < j \leq r$ ,  $\Phi_{\emptyset, e_i \pm e_j}^+ = \{e_i \pm e_j\}$  so that  $\rho_{\emptyset, e_i \pm e_j} = 1$ . Similarly for  $F = F_0$ , for  $1 \leq i \leq r+1$ ,  $\Phi_{F_0, 2e_i}^+ = \{2e_i, e_i \pm e_j : r+2 \leq j \leq n\}$  so that  $\rho_{F_0, 2e_i} = n - r$  and for  $1 \leq i < j \leq r+1$ ,  $\Phi_{F_0, e_i \pm e_j}^+ = \{e_i \pm e_j\}$  so that  $\rho_{F_0, e_i \pm e_j} = 1$ .

Now  $\chi_\emptyset(0)$  is the restriction to  $Z_M(M^0)$  of  $\chi_{F_0}(0)$  so that for  $1 \leq i \leq r$ ,  $\chi_\emptyset(0 : \gamma_i) = \chi_{F_0}(0 : \gamma_i)$ . Thus  $\varepsilon_{2e_i}(\emptyset, 0) = (-1)^{n-r+1} \chi_\emptyset(0 : \gamma_i) = -\varepsilon_{2e_i}(F_0 : 0) = -\varepsilon_i$ . Further, for  $1 \leq i < j \leq r$ ,  $\varepsilon_{e_i \pm e_j}(\emptyset : 0) = -\chi_\emptyset(0 : \gamma_i \gamma_j^{-1}) = -\varepsilon_i \varepsilon_j$ . Similarly, for  $1 \leq i < j \leq r+1$ ,  $\varepsilon_{e_i \pm e_j}(F_0 : 0) = -\chi_{F_0}(0 : \gamma_i \gamma_j^{-1}) = -\varepsilon_i \varepsilon_j$ .

Now for  $1 \leq i \leq r+1$ ,  $m_{2e_i}(F_0 : h : v)$  is jointly smooth at  $(0, 0)$  if and only if  $\varepsilon_i = -1$ . But  $\Phi_{F_0, 2e_i}^+ = \{2e_i, e_i \pm e_j : r+2 \leq j \leq n\}$ , and for  $\gamma = e_i \pm e_j$ ,  $\langle \gamma, \lambda_{F_0} \rangle = \langle \pm e_j, \lambda \rangle \neq 0$  since  $2e_j \notin F_0$  for  $r+2 \leq j \leq n$ . Thus  $m_{2e_i}^*(F_0 : h : v)$  is jointly smooth if and only if  $m_{2e_i}(F_0 : h : v)$  is jointly smooth. However, for  $1 \leq i \leq r$ ,  $e_i + e_{r+1} \in \Phi_{\emptyset, 2e_i}^+$  and  $\langle \lambda(h) + iv, e_i + e_{r+1} \rangle = c(h + v_{2e_i})$  since  $\langle 2e_{r+1}, \lambda \rangle = 0$ . Thus  $m_{2e_i}^*(\emptyset : h : v)$  is jointly smooth even when  $\varepsilon_{2e_i}(\emptyset : 0) = 1$ .  $\square$

Fix  $F \subseteq F_0$ , and define  $\mathcal{T} = \mathcal{T}_F$  as in (5.2). Since  $\Phi_{F,R}$  is of type  $C_s$  where  $s = r$  or  $r+1$ ,  $\mathcal{T}$  can be described as follows. For any  $1 \leq i < j \leq s$ , let

$$\psi(i, j) = \{\pm 2e_i, \pm 2e_j, \pm e_i \pm e_j\}.$$

For  $1 \leq i \leq s$ , let  $\psi(i) = \{\pm 2e_i\}$  and define

$$\psi_0 = \begin{cases} \bigcup_{i=1}^{\lfloor s/2 \rfloor} \psi(2i-1, 2i) \cup \psi(s), & \text{if } s \text{ is odd;} \\ \bigcup_{i=1}^{\lfloor s/2 \rfloor} \psi(2i-1, 2i), & \text{if } s \text{ is even.} \end{cases}$$

Let  $S_s$  denote the set of all permutations  $\sigma$  of  $\{1, 2, \dots, s\}$  considered as a subgroup of  $W(\Phi_{F,R})$ . Then  $\mathcal{T} = \{\sigma\psi_0 : \sigma \in S_s\}$ . Further, if  $\sigma \in S_s$  such that  $\sigma(\psi_0^+) = (\sigma\psi_0)^+$  where  $\psi^+ = \psi \cap \Phi_{F,R}^+$  for any  $\psi \in \mathcal{T}$ , then  $\varepsilon(\sigma\psi_0) = \det \sigma$ .

LEMMA 6.2. For  $\psi \in \mathcal{T}$ , define  $t(F : \psi : h : v)$  as in (5.2). Then for any  $\sigma \in S_s$  such that  $\varepsilon_{\sigma\alpha}(F : 0) = \varepsilon_\alpha(F : 0)$  for all  $\alpha \in \Phi_{F,R}^+$ ,

$$\varepsilon(\psi)t(F : \psi : h : \sigma v) = \det \sigma \varepsilon(\sigma\psi)t(F : \sigma\psi : h : v).$$

*Proof.* Since  $\sigma \in S_s$  permutes  $\{2e_1, \dots, 2e_s\}$ ,  $\{e_i + e_j : 1 \leq i \neq j \leq s\}$  and  $\{e_i - e_j : 1 \leq i \neq j \leq s\}$ , we have  $h_{\sigma\alpha} = h_\alpha$  for all  $\alpha \in \Phi_{F,R}^+$ . Thus if  $\varepsilon_{\sigma\alpha}(F : 0) = \varepsilon_\alpha(F : 0)$  for all  $\alpha \in \Phi_{F,R}^+$ ,  $t_\alpha(F : h : \sigma v) = t_{\sigma\alpha}(F : h : v)$  for all  $\alpha \in \Phi_{F,R}^+$ . Further, the only case in which it is possible to have  $\alpha \in \Phi_{F,R}^+$  and  $\sigma\alpha \in \Phi_{F,R}^-$  is when  $\alpha = e_i - e_j$  so that  $h_\alpha = h_{\sigma\alpha} = 0$ . Thus for such  $\alpha$  we have  $t_{-\alpha}(F : h : v) = -t_\alpha(F : h : v)$ . Thus

$$\varepsilon(\psi)t(F : \psi : h : \sigma v) = \varepsilon(\psi) \prod_{\alpha \in \sigma(\psi^+)} t_\alpha(F : h : v) = (-1)^p \varepsilon(\psi) \prod_{\alpha \in (\sigma\psi)^+} t_\alpha(F : h : v)$$

where  $p$  is the number of roots  $\alpha \in \psi^+$  such that  $\sigma\alpha \in (\sigma\psi)^-$ . Now  $\varepsilon(\sigma\psi) = \det \sigma (-1)^p \varepsilon(\psi)$ . □

For  $1 \leq i \leq s$ , write  $v_i = v_{2e_i}$ ,  $v \in \mathfrak{a}_F^*$ , and set  $\delta_i = \varepsilon_{2e_i}(F : 0)$ . Thus for  $1 \leq i < j \leq s$  we can write

$$\begin{aligned} t(F : \psi(i, j) : h : v) &= \left( \frac{\sinh \pi(v_i + ih)}{(\cosh \pi(v_i + ih) - \delta_i)} \frac{\sinh \pi(v_j + ih)}{(\cosh \pi(v_j + ih) - \delta_j)} + 1 \right) \\ &\times \frac{\sinh \pi(v_i + v_j + 2ih)}{(\cosh \pi(v_i + v_j + 2ih) + \delta_i \delta_j)} \frac{\sinh \pi(v_i - v_j)}{(\cosh \pi(v_i - v_j) + \delta_i \delta_j)}. \end{aligned} \tag{6.3}$$

LEMMA 6.4. If  $\delta_i = \delta_j = -1$ , then

$$(h, v) \rightarrow (v_i - v_j)^{-1} t(F : \psi(i, j) : h : v)$$

is jointly smooth on  $U(0) \times \mathfrak{a}_F^*$ . If  $\delta_i = \delta_j = 1$ , then

$$(h, v) \rightarrow \frac{(v_i + ih)(v_j + ih)}{(v_i - v_j)} t(F : \psi(i, j) : h : v)$$

is jointly smooth on  $U(0) \times \mathfrak{a}_F^*$ . If  $\delta_i = 1$  and  $\delta_j = -1$ , then

$$(h, v) \rightarrow (v_i + ih)(v_i - v_j)t(F : \psi(i, j) : h : v)$$

is jointly smooth on  $U(0) \times \mathfrak{a}_F^*$ .

*Proof.* The first two statements follow directly from (5.7) and formula (6.3). To prove the third statement we first note, again using (5.7) and (6.3), that

$$(h, v) \rightarrow t(v_i, v_j, h) = (v_i + ih)(v_i + v_j + 2ih)(v_i - v_j)t(F : \psi(i, j) : h : v)$$

is jointly smooth on  $U(0) \times \mathfrak{a}_F^*$ . Write  $D = \partial/\partial h - i\partial/\partial v_i - i\partial/\partial v_j$ . Then to show

that  $\frac{t(v_i, v_j, h)}{(v_i + v_j + 2ih)}$  is jointly smooth on  $U(0) \times \mathfrak{a}_F^*$  it suffices to show that

$D^k t(v_i, v_j, h) = 0$  for all  $k \geq 0$  when  $h = 0$  and  $v_i = -v_j$ . But since  $t(v_i, v_j, h)$  is a function of  $v_i + ih$  and  $v_j + ih$ ,  $D^k t(v_i, v_j, h) = 0$  for all  $v_i, v_j, h$  if  $k \geq 1$ . Finally, for any

$$\begin{aligned} x \in \mathbf{R}, t(x, -x, 0) &= \frac{x \sinh \pi x}{(\cosh \pi x - 1)} \frac{\sinh \pi(-x)}{(\cosh \pi(-x) + 1)} \frac{2}{\pi} \frac{2x \sinh 2\pi x}{(\cosh 2\pi x - 1)} \\ &+ \frac{2x}{\pi} \frac{2x \sinh 2\pi x}{(\cosh 2\pi x - 1)} = 0. \quad \square \end{aligned}$$

Let  $\Phi_{F,R}^+(1) = \{\alpha \in \Phi_{F,R}^+ : \varepsilon_\alpha(F : 0) = 1 \text{ and } \alpha \text{ is either a long root or is a short root of the form } e_i - e_j, 1 \leq i < j \leq s\}$ . Let  $\Phi_{F,R}^+(0) = \{\alpha \in \Phi_{F,R}^+ : m_\alpha(F : h : v) \text{ is not jointly smooth at } (0, 0) \in i\nu^* \times \mathfrak{a}_F^*\}$ . Thus by (6.1),  $\Phi_{F_0,R}^+(0) = \{2e_i : 1 \leq i \leq r + 1, \varepsilon_i = 1\}$  and  $\Phi_{\emptyset,R}^+(0) = \emptyset$ .

LEMMA 6.5. *Suppose  $F = \emptyset$ . Then*

$$(h, v) \rightarrow \prod_{\alpha \in \Phi_{F,R}^+(1)} (v_\alpha + ih_\alpha) \sum_{\psi \in \mathcal{F}_F} \varepsilon(\psi)t(F : \psi : h : v)$$

is jointly smooth on  $i\nu^* \times \mathfrak{a}_F^*$ . Suppose  $F = F_0$ . Then

$$(h, v) \rightarrow \prod_{\alpha \in \Phi_{F,R}^+(1)} (v_\alpha + ih_\alpha) \prod_{\alpha \in \Phi_{F,R}^+(0) \setminus \{2e_{r+1}\}} (v_\alpha - v_{r+1})^{-1} \sum_{\psi \in \mathcal{F}_F} \varepsilon(\psi)t(F : \psi : h : v)$$

is jointly smooth on  $i\nu^* \times \mathfrak{a}_F^*$ .

*Proof.* For any  $F \subseteq F_0$  it follows from (6.4) that

$$(h, v) \rightarrow \prod_{\alpha \in \Phi_{F,R}^+(1)} (v_\alpha + ih_\alpha) \sum_{\psi \in \mathcal{F}_F} \varepsilon(\psi)t(F : \psi : h : v)$$

is jointly smooth. Now to prove the lemma it suffices to show that if  $1 \leq i \leq r$  with  $\varepsilon_{2e_i}(F : 0) = 1$  and if  $v_0 \in \mathfrak{a}_{F_0}^*$  with  $(v_0)_i = (v_0)_{r+1}$ , then

$\sum_{\psi \in \mathcal{T}_F} \varepsilon(\psi)t(F:\psi:h:v_0)=0$ . But let  $\sigma$  be the reflection interchanging  $2e_i$  and  $2e_{r+1}$ . Then  $\sigma v_0 = v_0$ . Note that  $\varepsilon_{\sigma\alpha}(F:0)=\varepsilon_\alpha(F:0)$  for all  $\alpha \in \Phi_{F,R}^+$  since  $\varepsilon_{2e_i}(F:0)=\varepsilon_{2e_{r+1}}(F:0)$ . Thus, by (6.2), for any  $\psi \in \mathcal{T}$ ,

$$\varepsilon(\psi)t(F:\psi:h:v_0) = \varepsilon(\psi)t(F:\psi:h:\sigma v_0) = \det \sigma \varepsilon(\sigma\psi)t(F:\sigma\psi:h:v_0).$$

Thus if  $\sigma\psi = \psi$ ,  $t(F:\psi:h:v_0) = 0$ , while if  $\sigma\psi \neq \psi$ , then

$$\varepsilon(\psi)t(F:\psi:h:v) + \varepsilon(\sigma\psi)t(F:\sigma\psi:h:v) = 0.$$

Thus  $\sum_{\psi \in \mathcal{T}_F} \varepsilon(\psi)t(F:\psi:h:v_0) = 0$ . □

Define  $q(F:h:v)$  as in (5.1f) so that  $q(\emptyset:h:v) = 1$  and

$$q(F_0:h:v) = \prod_{\alpha \in \Phi_{F_0,R}^+(0)} (v_\alpha + ih_\alpha) \prod_{\beta \in \Phi_{F_0,R}^+(0) \setminus \{2e_{r+1}\}} (v_\beta - v_{2e_{r+1}})^{-1}.$$

**THEOREM 6.6.** *Suppose  $v_0 \in \mathfrak{a}_{F_0}^*$  such that  $(v_0)_{2e_{r+1}} = 0$  and  $(v_0)_\alpha \neq 0$  for all  $\alpha \in \Phi_{F_0,R}^+ \setminus \alpha \neq 2e_{r+1}$ . Then*

$$\lim_{(h,v) \rightarrow (0,v_0)} q(F_0:h:v) \sum_{\psi \in \mathcal{T}_{F_0}} \varepsilon(\psi)t(F_0:\psi:h:v) = \frac{2}{\pi} \sum_{\psi \in \mathcal{T} \setminus \emptyset} \varepsilon(\psi)t(\emptyset:\psi:0:v_0).$$

Before we prove Theorem 6.6 we will need some lemmas. Let  $F \subseteq F_0$  and as above write

$$s = s(F) = \begin{cases} r, & \text{if } F = \emptyset; \\ r+1, & \text{if } F = F_0. \end{cases}$$

Then each  $\psi \in \mathcal{T}_F$  is determined by

$$P(\psi) = \{(i, j): 1 \leq i < j \leq s(F) \text{ and } \psi(i, j) \subseteq \psi\}.$$

Let

$$P_0(F) = \{(i, j): 1 \leq i < j \leq s(F)\},$$

and for each  $P \subseteq P_0(F)$ , let

$$\mathcal{T}_{T,P} = \{\psi \in \mathcal{T}_F: P \subseteq P(\psi)\}.$$

For  $\psi \in \mathcal{T}_{F,P}$ , define

$$\psi^+(P) = \psi^+ \bigcup_{(i,j) \in P} \psi^+(i, j), \quad \psi_P^+ = \{e_i \pm e_j: (i, j) \in P\}$$

and

$$\Psi_F^+(P) = \{\alpha \in \Phi_{F,R}^+ : \alpha \perp \psi(i, j) \text{ for all } (i, j) \in P\}.$$

We can consider  $P_0(\emptyset) \subset P_0(F_0)$ .

LEMMA 6.7. For all  $(h, v) \in \mathfrak{iv}^* \times \mathfrak{a}_F^*$ ,

$$\sum_{\psi \in \mathcal{T}_F} \varepsilon(\psi) t(F : \psi : h : v) = \sum_{P \subseteq P_0(F)} \varepsilon(P) \prod_{\alpha \in \psi_P^+} t_\alpha(F : h : v) \prod_{\alpha \in \Psi_F^+(P)} t_\alpha(F : h : v).$$

Here  $\varepsilon(P) = \pm 1$  for all  $P \subseteq P_0(F)$  and if  $P \subseteq P_0(\emptyset) \subseteq P_0(F_0)$ ,  $\varepsilon(P)$  is independent of  $F$ .

Proof. Let  $\psi \in \mathcal{T}_F$  and write  $\psi = \psi_1 \cup \dots \cup \psi_k \cup \psi_{k+1}$  where  $k = [s(F)/2]$ , each  $\psi_i$ ,  $1 \leq i \leq k$ , is of type  $C_2$ , and  $\psi_{k+1}$  is of type  $A_1$  if  $s(F) = 2k + 1$  and is the empty set if  $s(F) = 2k$ . Recall from (5.2) that

$$\begin{aligned} t(F : \psi : h : v) &= \prod_{i=1}^k \left( \prod_{\alpha \in \psi_i^+} t_\alpha(F : h : v) + \prod_{\alpha \in \psi_i^{+,s}} t_\alpha(F : h : v) \right) \prod_{\alpha \in \psi_{k+1}} t_\alpha(F : h : v) \\ &= \sum_{P \subseteq P(\psi)} \prod_{\alpha \in \psi_P^+} t_\alpha(F : h : v) \prod_{\alpha \in \psi^+(P)} t_\alpha(F : h : v). \end{aligned}$$

Thus

$$\sum_{\psi \in \mathcal{T}_F} \varepsilon(\psi) t(F : \psi : h : v) = \sum_{P \subseteq P_0(F)} \prod_{\alpha \in \psi_P^+} t_\alpha(F : h : v) \sum_{\psi \in \mathcal{T}_{F,P}} \varepsilon(\psi) \prod_{\alpha \in \psi^+(P)} t_\alpha(F : h : v).$$

Fix  $P \subseteq P_0(F_0)$  and write  $P = \{(p_{2i-1}, p_{2i}) : 1 \leq i \leq |P|\}$ . Let

$$T(P) = \{1 \leq j \leq r + 1 : j \notin \{p_i : 1 \leq i \leq 2|P|\}\}$$

and write  $T(P) = \{t_1, t_2, \dots, t_{r+1-2|P|}\}$  where  $t_1 < t_2 < \dots < t_{r+1-2|P|}$ . Now define the permutation  $\sigma_P \in S_{r+1}$  by

$$\sigma_P(i) = p_i, 1 \leq i \leq 2|P|, \sigma_P(i + 2|P|) = t_i, 1 \leq i \leq r + 1 - 2|P|.$$

Define  $\varepsilon(P) = \det \sigma_P$ . Suppose  $P \subseteq P_0(\emptyset) \subset P_0(F_0)$ . Then  $r + 1 = t_{r+1-2|P|} \in T(P)$  and  $\sigma_P(r + 1) = r + 1$ . Thus  $\sigma_P$  permutes  $\{1, \dots, r\}$ .

Now for any  $F \subseteq F_0$  and  $P \subseteq P_0(F)$ ,  $\Psi_F^+(P)$  is a root system of type  $C_{s(F)-2|P|}$  and  $\psi \rightarrow \psi(P)$  gives a bijection between  $\mathcal{T}_{F,P}$  and  $\mathcal{T}(\Psi_F(P))$ , the set of all two-structures for  $\Psi_F(P)$ . Let

$$\psi(P)_0 = \begin{cases} \psi(t_1, t_2) \cup \dots \cup \psi(t_{s(F)-2|P|-1}, t_{s(F)-2|P|}), & \text{if } s(F) \text{ is even;} \\ \psi(t_1, t_2) \cup \dots \cup \psi(t_{s(F)-2|P|-2}, t_{s(F)-2|P|-1}) \cup \psi(t_{s(F)-2|P|}), & \text{if } s(F) \text{ is odd} \end{cases}$$



be the standard two-structure for  $\Psi_F^+(P)$ . Now if  $\psi \in \mathcal{F}_{F,P}$ , let  $\sigma \in S_{S(F)}$  so that  $\sigma(\psi_0^+) = \psi^+$ . We can assume that  $\sigma(i) = p_i$  for  $1 \leq i \leq 2|P|$  and  $\sigma\psi_0(P)^+ = \psi(P)^+$  where  $P = \{(p_{2i-1}, p_{2i})\}$  as above and  $P' = \{(2i-1, 2i) : 1 \leq i \leq |P|\}$ . Now  $\sigma\sigma_p^{-1}(p_i) = p_i$  for all  $1 \leq i \leq 2|P|$  so that  $\sigma' = \sigma\sigma_p^{-1}$  is a permutation  $T(P)$  and  $\sigma'\psi(P)_0^+ = \sigma\psi_0(P)^+ = \psi(P)^+$ . Thus  $\varepsilon(\psi(P)) = \det \sigma' = \varepsilon(\psi)\varepsilon(P)$ .

Now for any  $v \in \mathfrak{a}_F^*$ ,

$$\sum_{\psi(P) \in \mathcal{F}(\Psi_F(P))} \varepsilon(\psi(P)) \prod_{\alpha \in \psi^+(P)} m_\alpha(F : h : v) = \prod_{\alpha \in \Psi_F^+(P)} m_\alpha(F : h : v)$$

by [HW1, 4.17]. Since both sides are meromorphic functions of  $v \in \mathfrak{a}_{F,C}^*$ , the equality persists for  $v \in \mathfrak{a}_{F,C}^*$ . Now for  $h \in iv^*$  we can define  $\nu(h) \in \mathfrak{a}_{F,C}^*$  so that  $\nu(h)_\alpha = ih_\alpha$  for all  $\alpha \in \Phi_{F,R}^+$ . Then  $t_\alpha(F : h : v) = m_\alpha(F : 0 : v + \nu(h))$  for all  $(h, v) \in iv^* \times \mathfrak{a}_F^*$ . Thus we also have the equality

$$\sum_{\psi(P) \in \mathcal{F}(\Psi_F(P))} \varepsilon(\psi(P)) \prod_{\alpha \in \psi^+(P)} t_\alpha(F : h : v) = \prod_{\alpha \in \Psi_F^+(P)} t_\alpha(F : h : v)$$

for all  $(h, v) \in iv^* \times \mathfrak{a}_F^*$ . Thus

$$\begin{aligned} \sum_{\psi \in \mathcal{F}_{F,P}} \varepsilon(\psi) \prod_{\alpha \in \psi_F^+(P)} t_\alpha(F : h : v) &= \varepsilon(P) \sum_{\psi(P) \in \mathcal{F}_F(\Psi(P))} \varepsilon(\psi(P)) \prod_{\alpha \in \psi^+(P)} t_\alpha(F : h : v) \\ &= \varepsilon(P) \prod_{\alpha \in \Psi_F^+(P)} t_\alpha(F : h : v). \end{aligned} \quad \square$$

LEMMA 6.8. *Let  $v_0 \in \mathfrak{a}_{F_0}^*$  be as in (6.6). Let  $P \subseteq P_0(\emptyset)$ . Then*

$$\lim_{(h,v) \rightarrow (0,v_0)} (v_{r+1} + ih) \prod_{\alpha \in \Psi_{F_0}^+(P)} t_\alpha(F_0 : h : v) = (2/\pi) \prod_{\alpha \in \Psi_{\emptyset}^+(P)} t_\alpha(\emptyset : 0 : v_0).$$

*Proof.* For simplicity of notation we will assume that

$$P = \{(2i - 1, 2i) : 1 \leq i \leq q\}, \quad 0 \leq 2q \leq r$$

so that  $\Psi_{F_0}^+(P)$  is the root system of type  $C_{r+1-2q}$  generated by  $2e_{2q+1}, \dots, 2e_r, 2e_{r+1}$  and  $\Psi_{\emptyset}^+(P)$  is the root system of type  $C_{r-2q}$  generated by  $2e_{2q+1}, \dots, 2e_r$ . Then

$$\begin{aligned} \prod_{\alpha \in \Psi_{F_0}^+(P)} t_\alpha(F_0 : h : v) &= \prod_{2q+1 \leq i \leq r+1} \frac{\sinh \pi(v_i + ih)}{\cosh \pi(v_i + ih) - \varepsilon_i} \\ &\times \prod_{2q+1 \leq i < j \leq r+1} \frac{\sinh \pi(v_i + v_j + 2ih)}{(\cosh \pi(v_i + v_j + 2ih) + \varepsilon_i \varepsilon_j)} \frac{\sinh \pi(v_i - v_j)}{(\cosh \pi(v_i - v_j) + \varepsilon_i \varepsilon_j)}. \end{aligned}$$

Now

$$\lim_{(h,v) \rightarrow (0,v_0)} (v_{r+1} + ih) \frac{\sinh \pi(v_{r+1} + ih)}{\cosh \pi(v_{r+1} + ih) - 1} = 2/\pi.$$

For  $q + 1 \leq i \leq r$ ,

$$\begin{aligned} \lim_{(h,v) \rightarrow (0,v_0)} & \frac{\sinh \pi(v_i + ih)}{(\cosh \pi(v_i + ih) - \varepsilon_i)} \frac{\sinh \pi(v_i + v_{r+1} + 2ih)}{(\cosh \pi(v_i + v_{r+1} + 2ih) + \varepsilon_i)} \\ & \times \frac{\sinh \pi(v_i - v_{r+1})}{(\cosh \pi(v_i - v_{r+1}) + \varepsilon_i)} = \frac{\sinh \pi(v_0)_i}{(\cosh \pi(v_0)_i - \varepsilon_i)} \\ & \times \frac{\sinh \pi(v_0)_i}{(\cosh \pi(v_0)_i + \varepsilon_i)} \frac{\sinh \pi(v_0)_i}{(\cosh \pi(v_0)_i + \varepsilon_i)} \\ & = \frac{\sinh \pi(v_0)_i}{(\cosh \pi(v_0)_i + \varepsilon_i)} = t_{2e_i}(\emptyset : 0 : v_0). \end{aligned}$$

Finally, for  $q + 1 \leq i < j \leq r$ ,

$$\begin{aligned} \lim_{(h,v) \rightarrow (0,v_0)} & \frac{\sinh \pi(v_i + v_j + 2ih)}{(\cosh \pi(v_i + v_j + 2ih) + \varepsilon_i \varepsilon_j)} \frac{\sinh \pi(v_i - v_j)}{(\cosh \pi(v_i - v_j) + \varepsilon_i \varepsilon_j)} \\ & = \frac{\sinh \pi((v_0)_i + (v_0)_j)}{(\cosh \pi((v_0)_i + (v_0)_j) + \varepsilon_i \varepsilon_j)} \frac{\sinh \pi((v_0)_i - (v_0)_j)}{(\cosh \pi((v_0)_i - (v_0)_j) + \varepsilon_i \varepsilon_j)} \\ & = t_{e_i + e_j}(\emptyset : 0 : v_0) t_{e_i - e_j}(\emptyset : 0 : v_0). \end{aligned}$$

□

**PROOF OF THEOREM 6.6.** Write

$$q(F_0 : h : v) = (v_{r+1} + ih) \prod_{\alpha \in \Phi_{F,R}^+(0) \setminus \{2e_{r+1}\}} (v_\alpha + ih)(v_\alpha - v_{r+1})^{-1}.$$

Now when  $h=0$  and  $v=v_0$ ,  $v_{r+1}=0$  so that

$$\prod_{\alpha \in \Phi_{F,R}^+(0) \setminus \{2e_{r+1}\}} (v_\alpha + ih)(v_\alpha - v_{r+1})^{-1} = 1.$$

Now by (6.7),

$$\begin{aligned} & (v_{r+1} + ih) \sum_{\psi \in \mathcal{F}_{F_0}} \varepsilon(\psi) t(F_0 : \psi : h : v) \\ & = (v_{r+1} + ih) \sum_{P \subseteq P_0(F_0)} \varepsilon(P) \prod_{\alpha \in \psi_P^+} t_\alpha(F_0 : h : v) \prod_{\alpha \in \Psi_{F_0}^+(P)} t_\alpha(F_0 : h : v). \end{aligned}$$

Now fix  $P \subseteq P_0(F_0)$ . Now all roots in  $\psi_P^+$  are short, so not equal to  $2e_{r+1}$ . Suppose  $P \not\subseteq P_0(\emptyset)$ . Then  $2e_{r+1} \notin \Psi_{F_0}^+(P)$ , so that  $t_\alpha(F_0 : h : v)$  is jointly smooth at  $(0, v_0)$  for all  $\alpha \in \psi_P^+ \cup \Psi_{F_0}^+(P)$ , so that

$$\lim_{(h,v) \rightarrow (0,v_0)} (v_{r+1} + ih) \prod_{\alpha \in \psi_P^+} t_\alpha(F_0 : h : v) \prod_{\alpha \in \Psi_{F_0}^+(P)} t_\alpha(F_0 : h : v) = 0.$$

Now if  $P \subseteq P_0(\emptyset)$ , then by (6.8),

$$\lim_{(h,v) \rightarrow (0,v_0)} (v_{r+1} + ih) \prod_{\alpha \in \Psi_{F_0}^+(P)} t_\alpha(F_0 : h : v) = (2/\pi) \prod_{\alpha \in \Psi_{\emptyset}^+(P)} t_\alpha(\emptyset : 0 : v_0).$$

Further, for  $\alpha \in \psi_P^+$ ,  $t_\alpha(F_0 : 0 : v_0) = t_\alpha(\emptyset : 0 : v_0)$ . Thus

$$\begin{aligned} & \lim_{(h,v) \rightarrow (0,v_0)} \sum_{P \subseteq P_0(F_0)} \varepsilon(P) (v_{r+1} + ih) \prod_{\alpha \in \psi_P^+} t_\alpha(F_0 : h : v) \prod_{\alpha \in \Psi_{F_0}^+(P)} t_\alpha(F_0 : h : v) \\ &= (2/\pi) \sum_{P \subseteq P_0(\emptyset)} \varepsilon(P) \prod_{\alpha \in \psi_P^+} t_\alpha(\emptyset : 0 : v_0) \prod_{\alpha \in \Psi_{\emptyset}^+(P)} t_\alpha(\emptyset : 0 : v_0) \\ &= (2/\pi) \sum_{\psi \in \mathcal{F}_{\emptyset}} \varepsilon(\psi) t(\emptyset : \psi : 0 : v_0). \end{aligned} \quad \square$$

### 7. Matching families

Let  $H = TA$  be a  $\theta$ -stable Cartan subgroup of  $G$  and fix  $(\lambda, \chi) \in X(T)$ ,  $\tau_1, \tau_2 \in \widehat{K}(\chi)$ ,  $W = W(\tau_1 : \tau_2)$ , and  $U(0)$  as in (4.6). For  $F \subseteq F_0$  and  $\varepsilon \in \Sigma_0$ , set  $U_F(\varepsilon) = \{h \in U(0) : \varepsilon_\alpha(h) = \varepsilon_\alpha \text{ for all } \alpha \in F_0 \setminus F\}$ . Define  $\mathcal{C}(U_F(\varepsilon) \times \alpha_F^* : W)$  to be the set of all  $g \in C^\infty(U_F(\varepsilon) \times \alpha_F^* : W)$  such that  $g$  has a  $C^\infty$  extension to the closure of

$$U_F(\varepsilon) \times \alpha_F^* \quad \text{and for all} \quad D \in D(i\nu^* \times \alpha_F^*), \quad r \geq 0,$$

$$\|g\|_{D,r} = \sup_{(h,v) \in U_F(\varepsilon) \times \alpha_F^*} \|Dg(h : v)\| (1 + |v|)^r < \infty.$$

Now suppose for each  $F \subseteq F_0$  we have a  $W$ -valued function  $g(F : h : v_F) \in (h, v_F) \in U(0) \times \alpha_F^*$  such that

$$g(F : \varepsilon) = g(F)|_{U_F(\varepsilon) \times \alpha_F^*} \in \mathcal{C}(U_F(\varepsilon) \times \alpha_F^* : W) \quad \text{for all } \varepsilon \in \Sigma_0. \quad (7.1a)$$

We will say  $\{g(F)\}_{F \subseteq F_0}$  is a matching family if in addition the functions  $g(F)$  satisfy the following matching conditions. Fix  $E \subseteq F_0$ ,  $k \geq 0$ ,  $1 \leq i \leq m$ ,  $\varepsilon \in \Sigma_i$ . For  $E \subseteq F \subseteq E(i) = E \cup F_0^i$ ,  $v_E \in \alpha_E^*$ ,  $h_0 \in \mathcal{H}_i \cap cl(U_E(\varepsilon))$ , write

$$a^\pm(F : h_0 : v_E) = \left( \partial/\partial h_i - i \sum_{\alpha \in F \setminus E} \partial/\partial \mu_\alpha \right)^k g(F : \varepsilon^\pm(i) : h_0 : (v_E, 0)).$$

Then we require that for all  $E \subseteq F \subseteq E(i)$ ,

$$\begin{aligned}
 & a^+(F : h_0 : v_E) - a^-(F : h_0 : v_E) \\
 &= \sum_{F \subset F' \subseteq E(i)} c_{|F' \setminus F|} (a^+(F' : h_0 : v_E) + a^-(F' : h_0 : v_E))
 \end{aligned} \tag{7.1b}$$

where  $c_k, k \geq 0$  are defined as in (3.11).

For  $F \subseteq F_0, h \in i\mathfrak{v}^*, v_F \in \mathfrak{a}_F^*$ , define  $p_F(h : v_F) = \prod_{\alpha \in \mathfrak{a}_F} (v_\alpha + ih_\alpha)$  as in (5.1e).

**THEOREM 7.2.** *Suppose  $\{g(F)\}_{F \subseteq F_0}$  is a matching family. For  $h \in U(0)$ , define*

$$f(h) = \sum_{F \subseteq F_0} \frac{1}{(\pi i)^{|F|}} \int_{\mathfrak{a}_F^*} \frac{g(F : h : v_F)}{p_F(h : v_F)} dv_F$$

where the measures  $dv_F$  are normalized as in (7.8). Then  $f \in C^\infty(U(0))$ .

**COROLLARY 7.3.** *Let  $\Phi$  be an elementary mixed wave packet as in (4.1) and define  $\Phi(h : x)$  as in (4.3). Then  $(h, x) \rightarrow \Phi(h : x)$  is a jointly smooth function on  $i\mathfrak{v}^* \times G$ .*

*Proof.* Combining definition (4.1) with (5.3) we see that

$$\Phi(h : x) = c \sum_{F \subseteq F_0} \frac{1}{(\pi i)^{|F|}} \int_{\mathfrak{a}_F^*} \frac{g(F : h : v_F : x)}{p_F(h : v_F)} dv_F.$$

Further, by Theorem 5.3, for each  $x \in G, \{g(F : x) : F \subseteq F_0\}$  is a matching family as defined in (7.1). Now the theorem says that  $h \rightarrow \Phi(h : x)$  is a smooth function on  $U(0)$  for all  $x \in G$ . Further, since  $\Phi(h : x)$  is supported in a compact subset of  $U(0)$ , it is smooth for all  $h \in i\mathfrak{v}^*$ . However, it will be clear in the proof of Theorem 7.2 that since for each  $\varepsilon \in \Sigma_0$ , the functions  $g(F : \varepsilon)$  are jointly smooth on  $cl(U_F(\varepsilon)) \times \mathfrak{a}_F^* \times G$  and the Schwartz norms  $\|g(F : \varepsilon : x)\|_{D,r}$  are uniformly bounded on compact subsets of  $G, \Phi$  is in fact jointly smooth on  $i\mathfrak{v}^* \times G$ .  $\square$

The remainder of this section will be devoted to the proof of Theorem 7.2. Fix  $\varepsilon \in \Sigma_0, F \subseteq F_0$ . Then  $g(F : h : v_F)$  is jointly smooth on  $U_F(\varepsilon) \times \mathfrak{a}_F^*$  and satisfies Schwartz estimates as a function of  $v_F$  uniformly in  $h$ . Further,  $p_F(h : v_F)$  is a polynomial in  $(h, v_F)$  which has no zeros in  $U_F(\varepsilon) \times \mathfrak{a}_F^*$ . Thus  $f_F(h) = \int_{\mathfrak{a}_F^*} \frac{g(F : h : v_F)}{p_F(h : v_F)} dv_F$  is a smooth function on  $U_F(\varepsilon)$ . We must show that  $f_F(h)$  extends to be  $C^\infty$  on  $cl(U_F(\varepsilon))$ , the closure of  $U_F(\varepsilon)$  in  $U(0)$ , and compute  $Df_F(\varepsilon : h_0) = \lim_{h \rightarrow h_0} Df_F(h)$  for any differential operator  $D \in D(i\mathfrak{v}^*)$  and  $h_0 \in cl(U_F(\varepsilon))$  where the limit is taken through  $h \in U_F(\varepsilon)$ .

We start with some elementary calculus lemmas. For any  $g \in \mathcal{C}(\mathbf{R})$ , define

$$\text{P.V.} \int_{\mathbf{R}} \frac{g(x)}{x} dx = \lim_{M \rightarrow +\infty, \epsilon \downarrow 0} \int_{\epsilon \leq |x| \leq M} \frac{g(x)}{x} dx.$$

The limit exists since

$$\int_{\epsilon \leq |x| \leq M} \frac{g(x)}{x} dx = \int_{\epsilon}^M \frac{g(x) - g(-x)}{x} dx \quad \text{and} \quad \frac{g(x) - g(-x)}{x}$$

is bounded as  $x \rightarrow 0$  and rapidly decreasing at infinity. Thus we can rewrite

$$\begin{aligned} \text{P.V.} \int_{\mathbf{R}} \frac{g(x)}{x} dx &= \int_{x \geq 0} \frac{g(x) - g(-x)}{x} dx \\ &= \int_{|x| \leq 1} \frac{g(x) - g(0)}{x} dx + \int_{|x| \geq 1} \frac{g(x)}{x} dx. \end{aligned}$$

Now let  $a > 0$  and write  $I^+(0, a) = (0, a)$ ,  $I^-(0, a) = (-a, 0)$ ,  $I^\pm[0, a] = I^\pm(0, a) \cup \{0\}$ . For any integer  $p \geq 0$  define

$$\begin{aligned} \mathcal{C}(\mathbf{R}^p \times I^\pm[0, a]: W) \\ \{g \in C^\infty(\mathbf{R}^p \times I^\pm[0, a]: W) : \|g\|_{D,r} < \infty \text{ for all } D \in D(\mathbf{R}^{p+1}), r \geq 0\} \end{aligned}$$

where

$$\|g\|_{D,r} = \sup_{(x,y) \in \mathbf{R}^p \times I^\pm[0,a]} \|Dg(x,y)\| (1 + |x|)^r.$$

Write coordinates in  $\mathbf{R}^p \times I^\pm[0, a]$  as  $(x, y, t)$ ,  $x \in \mathbf{R}$ ,  $y \in \mathbf{R}^{p-1}$ ,  $t \in I^\pm[0, a]$ . For  $g \in \mathcal{C}(\mathbf{R}^p \times I^\pm[0, a]: W)$ ,  $(y, t) \in \mathbf{R}^{p-1} \times I^\pm(0, a)$ , define

$$I(g)(y, t) = \int_{\mathbf{R}} \frac{g(x, y, t)}{x + it} dx.$$

LEMMA 7.4. Let  $g \in \mathcal{C}(\mathbf{R}^p \times I^\pm[0, a]: W)$ . Then

$$I(g) \in C^\infty(\mathbf{R}^{p-1} \times I^\pm(0, a): W)$$

and extends continuously to  $\mathbf{R}^{p-1} \times I^\pm[0, a]$  where for  $y \in \mathbf{R}^{p-1}$ , we define

$$I(g)(y, 0) = \text{P.V.} \int_{\mathbf{R}} \frac{g(x, y, 0)}{x} dx - \epsilon \pi i g(0, y, 0)$$

where

$$\varepsilon = \begin{cases} 1, & \text{if } I^\pm[0, a) = I^+[0, a); \\ -1, & \text{if } I^\pm[0, a) = I^-[0, a). \end{cases}$$

*Proof.* Clearly  $I(g) \in C^\infty(\mathbf{R}^{p-1} \times I^\pm(0, a))$  since  $g(x, y, t)/(x + it)$  is a Schwartz function of  $x$ , uniformly on compact subsets of  $\mathbf{R}^{p-1} \times I^\pm(0, a)$ . Now for all  $t \neq 0$ ,

$$\int_{|x| \leq 1} \frac{1}{x + it} dx = -\pi i \operatorname{sign}(t) + 2i \arctan t.$$

Thus we can write  $I(g)(y, t) = \sum_{j=1}^3 g_j(y, t)$  where

$$g_1(y, t) = \int_{|x| \geq 1} \frac{g(x, y, t)}{x + it} dx, \quad g_2(y, t) = \int_{|x| \leq 1} \frac{g(x, y, t) - g(0, y, t)}{x + it} dx,$$

and

$$g_3(y, t) = g(0, y, t)(-\pi i \varepsilon + 2i \arctan t).$$

Now  $\|g(x, y, t)/(x + it)\| \leq \|g(x, y, t)\|$  for all  $|x| \geq 1, y \in \mathbf{R}^{p-1}, t \in I^\pm[0, a)$ , so  $g_1$  extends continuously to  $t=0$  with

$$g_1(y, 0) = \int_{|x| \geq 1} \frac{g(x, y, 0)}{x} dx.$$

Further, for  $|x| \leq 1, y \in \mathbf{R}^{p-1}, t \in I^\pm[0, a)$ ,

$$\left\| \frac{g(x, y, t) - g(0, y, t)}{x + it} \right\| \leq \left\| \frac{g(x, y, t) - g(0, y, t)}{x} \right\| \leq \sup_{|x| \leq 1} \|(\partial/\partial x)g(x, y, t)\|.$$

Thus  $g_2$  extends continuously to  $t=0$  with

$$g_2(y, 0) = \int_{|x| \leq 1} \frac{g(x, y, 0) - g(0, y, 0)}{x} dx.$$

Note

$$g_1(y, 0) + g_2(y, 0) = \text{P.V.} \int_{\mathbf{R}} \frac{g(x, y, 0)}{x} dx.$$

Finally, for  $y \in \mathbf{R}^{p-1}$ ,  $t \in I^\pm(0, a)$ ,  $g_3(y, t) = g(0, y, t)(-\pi i \varepsilon + 2i \arctan t)$  extends continuously to  $t=0$  with  $g_3(y, 0) = -\varepsilon \pi i g(0, y, 0)$ .  $\square$

LEMMA 7.5. Suppose  $g \in \mathcal{C}(\mathbf{R}^p \times I^\pm[0, a): W)$ . Then for any  $k \geq 0$ ,  $D_y \in D(\mathbf{R}^{p-1})$ ,  $(y, t) \in \mathbf{R}^{p-1} \times I^\pm(0, a)$ ,

$$(\partial/\partial t)^k D_y \int_{\mathbf{R}} \frac{g(x, y, t)}{x + it} dx = \int_{\mathbf{R}} \frac{(\partial/\partial t - i\partial/\partial x)^k D_y g(x, y, t)}{x + it} dx.$$

*Proof.* Write

$$\begin{aligned} & (\partial/\partial t)^k D_y \int_{\mathbf{R}} \frac{g(x, y, t)}{x + it} dx \\ &= \int_{\mathbf{R}} \sum_{q=0}^k \binom{k}{q} ((\partial/\partial t)^{k-q} D_y g(x, y, t)) (\partial/\partial t)^q (x + it)^{-1} dx \\ &= \int_{\mathbf{R}} \sum_{q=0}^k \binom{k}{q} ((\partial/\partial t)^{k-q} D_y g(x, y, t)) (i\partial/\partial x)^q (x + it)^{-1} dx. \end{aligned}$$

But integrating by parts, this is equal to

$$\begin{aligned} & \int_{\mathbf{R}} \sum_{q=0}^k \binom{k}{q} ((\partial/\partial t)^{k-q} (-i\partial/\partial x)^q D_y g(x, y, t)) (x + it)^{-1} dx \\ &= \int_{\mathbf{R}} \frac{(\partial/\partial t - i\partial/\partial x)^k D_y g(x, y, t)}{x + it} dx. \end{aligned} \quad \square$$

LEMMA 7.6. Let  $g \in \mathcal{C}(\mathbf{R}^p \times I^\pm[0, a): W)$ . Then  $I(g) \in \mathcal{C}(\mathbf{R}^{p-1} \times I^\pm[0, a): W)$  and for all  $D \in D(\mathbf{R}^p)$ , there is a finite subset  $F \subset D(\mathbf{R}^{p+1})$  such that for all  $r \geq 0$ ,

$$\|I(g)\|_{D,r} \leq \sum_{D' \in F} \|g\|_{D',r+2}.$$

*Proof.* Combining Lemmas 7.4 and 7.5 we see that for all  $D_y \in D(\mathbf{R}^{p-1})$ ,  $k \geq 0$ ,  $(\partial/\partial t)^k D_y I(g)(y, t)$  extends continuously to  $\mathbf{R}^{p-1} \times I^\pm[0, a)$ . Thus to show that  $I(g) \in C^\infty(\mathbf{R}^{p-1} \times I^\pm[0, a): W)$  we need only check that each of the one-sided derivatives  $(\partial/\partial t)^k D_y I(g)(y, 0)$  exists and is equal to

$$\text{P.V.} \int_{\mathbf{R}} \frac{(\partial/\partial t - i\partial/\partial x)^k D_y g(x, y, 0)}{x} dx - \varepsilon \pi i (\partial/\partial t - i\partial/\partial x)^k D_y g(0, y, 0).$$

The proof will be by induction on  $k$ . Suppose  $k = 0$ . Then clearly

$$\text{P.V.} \int_{\mathbf{R}} \frac{g(x, y, 0)}{x} dx - \varepsilon \pi i g(0, y, 0)$$

is a smooth function of  $y$  and the derivative  $D_y$  can be brought inside the P.V. integral. Now suppose  $k > 0$ . By the induction hypothesis, for all  $(y, t) \in \mathbf{R}^{p-1} \times I^\pm[0, a)$  we can write

$$(\partial/\partial t)^{k-1} D_y I(g)(y, t) = \sum_{j=1}^3 \tilde{g}_j(y, t)$$

where

$$\tilde{g}(x, y, t) = (\partial/\partial t - i\partial/\partial x)^{k-1} D_y g(x, y, t)$$

and the  $\tilde{g}_j(y, t)$  are defined as in the proof of (7.4). Now, assuming that  $t$  is restricted to lie in  $I^\pm(0, a)$ ,

$$\begin{aligned} \lim_{t \rightarrow 0} \frac{\tilde{g}_1(y, t) - \tilde{g}_1(y, 0)}{t} &= \lim_{t \rightarrow 0} \frac{1}{t} \int_{|x| \geq 1} \frac{\tilde{g}(x, y, t)}{x + it} - \frac{\tilde{g}(x, y, 0)}{x} dx \\ &= \lim_{t \rightarrow 0} \int_{|x| \geq 1} \frac{\tilde{g}(x, y, t) - \tilde{g}(x, y, 0)}{t(x + it)} - \frac{i\tilde{g}(x, y, 0)}{x(x + it)} dx \\ &= \int_{|x| \geq 1} \frac{(\partial/\partial t)\tilde{g}(x, y, 0)}{x} - \frac{i\tilde{g}(x, y, 0)}{x^2} dx. \end{aligned}$$

Now, integrating

$$\int_{|x| \geq 1} \frac{\tilde{g}(x, y, 0)}{x^2} dx$$

by parts, this is equal to

$$\begin{aligned} &\int_{|x| \geq 1} \frac{(\partial/\partial t - i\partial/\partial x)\tilde{g}(x, y, 0)}{x} dx - i(\tilde{g}(1, y, 0) + \tilde{g}(-1, y, 0)) \\ &= \int_{|x| \geq 1} \frac{(\partial/\partial t - i\partial/\partial x)^k D_y g(x, y, 0)}{x} dx - i(\tilde{g}(1, y, 0) + \tilde{g}(-1, y, 0)). \end{aligned}$$



Similarly,

$$\begin{aligned}
 \lim_{t \rightarrow 0} \frac{\tilde{g}_2(y, t) - \tilde{g}_2(y, 0)}{t} &= \lim_{t \rightarrow 0} \frac{1}{t} \int_{|x| \leq 1} \frac{\tilde{g}(x, y, t) - \tilde{g}(0, y, t)}{x + it} \\
 &\quad - \frac{\tilde{g}(x, y, 0) - \tilde{g}(0, y, 0)}{x} dx \\
 &= \lim_{t \rightarrow 0} \int_{|x| \leq 1} \frac{\tilde{g}(x, y, t) - \tilde{g}(x, y, 0)}{t(x + it)} - \frac{\tilde{g}(0, y, t) - \tilde{g}(0, y, 0)}{t(x + it)} dx \\
 &\quad - i \int_{|x| \leq 1} \frac{\tilde{g}(x, y, 0) - \tilde{g}(0, y, 0) - x(\partial/\partial x)\tilde{g}(0, y, 0)}{x(x + it)} dx \\
 &\quad - i(-\pi i \varepsilon + 2i \arctan t)(\partial/\partial x)\tilde{g}(0, y, 0) \\
 &= \int_{|x| \leq 1} \frac{(\partial/\partial t)\tilde{g}(x, y, 0) - (\partial/\partial t)\tilde{g}(0, y, 0)}{x} \\
 &\quad - i \frac{\tilde{g}(x, y, 0) - \tilde{g}(0, y, 0) - x(\partial/\partial x)\tilde{g}(0, y, 0)}{x^2} dx - \varepsilon \pi (\partial/\partial x)\tilde{g}(0, y, 0) \\
 &= \int_{|x| \leq 1} \frac{(\partial/\partial t - i\partial/\partial x)^k D_y g(x, y, 0) - (\partial/\partial t - i\partial/\partial x)^k D_y g(0, y, 0)}{x} dx \\
 &\quad + i(\tilde{g}(1, y, 0) + \tilde{g}(-1, y, 0)) - 2i\tilde{g}(0, y, 0) - \varepsilon \pi (\partial/\partial x)\tilde{g}(0, y, 0).
 \end{aligned}$$

Finally,

$$\lim_{t \rightarrow 0} \frac{\tilde{g}_3(y, t) - \tilde{g}_3(y, 0)}{t} = -\varepsilon \pi i (\partial/\partial t)\tilde{g}(0, y, 0) + 2i\tilde{g}(0, y, 0).$$

Thus  $(\partial/\partial t)^k D_y I(g)(y, 0)$  exists and is equal to

$$\text{P.V.} \int_{\mathbf{R}} \frac{(\partial/\partial t - \partial/\partial x)^k D_y g(x, y, 0)}{x} dx - \varepsilon \pi i (\partial/\partial t - \partial/\partial x)^k D_y g(0, y, 0).$$

Finally, for any  $r \geq 0$ ,  $D \in D(\mathbf{R}^p)$ , we must compute

$$\|I(g)\|_{D,r} = \sup_{(y,t) \in \mathbf{R}^{p-1} \times I^\pm(0,a)} (1 + |y|)^r \|DI(g)(y, t)\|.$$

But, from (7.5) we see that there is  $D' \in D(\mathbf{R}^{p+1})$  so that  $DI(g) = I(D'g)$ . Thus  $\|I(g)\|_{D,r} = \|I(D'g)\|_{1,r}$ . Write  $D'g = \tilde{g}$ . As above, write  $I(\tilde{g})(y, t) = \sum_{j=1}^3 \tilde{g}_j(y, t)$ . But

there is a constant  $C > 0$  so that

$$\left\| \int_{|x| \geq 1} \frac{\tilde{g}(x, y, t)}{x + it} dx \right\| \leq \int_{\mathbf{R}} \|\tilde{g}(x, y, t)\| dx \leq C \sup_x (1 + |x|)^2 \|\tilde{g}(x, y, t)\|.$$

Further,

$$\begin{aligned} \left\| \int_{|x| \leq 1} \frac{\tilde{g}(x, y, t) - \tilde{g}(0, y, t)}{x + it} dx \right\| &\leq \int_{|x| \leq 1} \frac{\|\tilde{g}(x, y, t) - \tilde{g}(0, y, t)\|}{|x|} dx \\ &\leq 2 \sup_{|x| \leq 1} \|(\partial/\partial x)\tilde{g}(x, y, t)\|. \end{aligned}$$

Finally,

$$\|\tilde{g}(0, y, t)\| |(\pi\varepsilon - 2 \arctan t)| \leq 2\pi \|\tilde{g}(0, y, t)\|.$$

Thus for any  $r \geq 0$ ,

$$\|I(g)\|_{D,r} = \|I(\tilde{g})\|_{1,r} \leq C \|g\|_{D',r+2} + 2 \|g\|_{\partial/\partial x D',r} + 2\pi \|g\|_{D',r}. \quad \square$$

**COROLLARY 7.7.** *Suppose  $g(x_1, \dots, x_p, y, t) \in \mathcal{C}(\mathbf{R}^p \times \mathbf{R}^q \times I^\pm[0, a): W)$ . Then*

$$\begin{aligned} &\lim_{t \rightarrow 0} \int_{\mathbf{R}} \dots \int_{\mathbf{R}} \prod_{j=1}^p \frac{1}{x_j + it} g(x_1, \dots, x_p, y, t) dx_1 \dots dx_p \\ &= \sum_{I \subseteq \{1,2,\dots,p\}} (-\varepsilon\pi i)^{p-|I|} \prod_{j \in I} \text{P.V.} \int_{\mathbf{R}} \frac{dx_j}{x_j} g(x_1, \dots, x_p, y, 0)|_{x_k=0, k \notin I} \end{aligned}$$

where the limit is taken through  $t \in I^\pm(0, a)$ . Further, for any  $D_y \in D(\mathbf{R}^q)$ ,  $k \geq 0$ ,

$$\begin{aligned} &(\partial/\partial t)^k D_y \int_{\mathbf{R}} \dots \int_{\mathbf{R}} \frac{g(x_1, \dots, x_p, y, t)}{\prod_{j=1}^p (x_j + it)} dx_1 \dots dx_p \\ &= \int_{\mathbf{R}} \dots \int_{\mathbf{R}} \frac{(\partial/\partial t - i \sum_{j=1}^p \partial/\partial x_j)^k D_y g(x_1, \dots, x_p, y, t)}{\prod_{j=1}^p (x_j + it)} dx_1 \dots dx_p. \end{aligned}$$

*Proof.* This follows from (7.4), (7.5), and (7.6) by an easy induction argument. □

Now let  $F \subseteq F_0$ ,  $1 \leq i \leq m$ ,  $\varepsilon \in \Sigma_i$ . Recall  $\varepsilon_\alpha = \pm 1$  is independent of  $\alpha$  for  $\alpha \in F_0^i$ . Write  $\varepsilon_i$  for this common value and write  $F^i = F \cap F_0^i$ . Let  $h_0 \in \mathcal{H}_i \cap cl(U_F(\varepsilon))$ . We will assume that  $h_0$  is “semi-regular”, that is  $h_0 \notin \mathcal{H}_j$ ,  $j \neq i$ . Then there is

$a > 0$  so that if we define  $h_i$  as in (3.7),

$$h_0(t) = h_0 + th_i \in U_F(\varepsilon) \quad \text{for } t \in I_{\varepsilon_i}(0, a) = \begin{cases} I^+(0, a), & \text{if } \varepsilon_i = 1; \\ I^-(0, a), & \text{if } \varepsilon_i = -1. \end{cases}$$

We will identify  $\mathfrak{a}_F^*$  with  $\mathfrak{a}_{F \setminus F^i}^* \times \mathbf{R}^{|F^i|}$  by  $v = v_0 + \sum_{\alpha \in F^i} x_\alpha \mu_\alpha \leftrightarrow (v_0, (x_\alpha)_{\alpha \in F^i})$  where  $v_0 = v|_{\mathfrak{a}_{F \setminus F^i}}$  and for  $\alpha \in F^i$ ,  $\mu_\alpha$  is defined as in (3.7). Suppose  $E \subseteq F^i$ . Then for any  $v_0 \in \mathfrak{a}_{F \setminus F^i}^*$ , write  $(v_0, (x_\alpha)_{\alpha \in E}, 0)$  for the element  $(v_0, (x_\alpha)_{\alpha \in F^i}) \in \mathfrak{a}_F^*$  with  $x_\alpha = 0$  for all  $\alpha \in F^i \setminus E$ . For  $\alpha \in F$ , write  $p_\alpha(h : v) = v_\alpha + ih_\alpha$ ,  $h \in iv^*$ ,  $v \in \mathfrak{a}_F^*$ . Then for

$$\alpha \in F^i, p_\alpha(h_0(t) : (v_0, (x_\beta)_{\beta \in F^i})) = \frac{2 \langle h_i, \alpha \rangle}{\langle \alpha, \alpha \rangle} (x_\alpha + it).$$

For  $\alpha \in F \setminus F^i$ ,  $p_\alpha(h : v) \neq 0$  in a neighborhood of  $h_0$  for all  $v \in \mathfrak{a}_F^*$ . Assume that  $a > 0$  is chosen small enough so that  $p_\alpha(h_0(t) : v) \neq 0$  for all  $t \in I_{\varepsilon_i}[0, a]$ ,  $v \in \mathfrak{a}_F^*$ . For  $g \in \mathcal{C}(U_F(\varepsilon) \times \mathfrak{a}_F^* : W)$ , write

$$g_i(h : v_F) = g(h : v_F) p_{F \setminus F^i}(h : v_F)^{-1}, \quad (h, v_F) \in U_F(\varepsilon) \times \mathfrak{a}_F^*.$$

LEMMA 7.8. *Suppose  $g \in \mathcal{C}(U_F(\varepsilon) \times \mathfrak{a}_F^* : W)$ . Then*

$$\begin{aligned} \lim_{h \rightarrow h_0} \int_{\mathfrak{a}_F^*} \frac{g(h : v_F)}{p_F(h : v_F)} dv_F &= 2^{-|F^i|} \sum_{E \subseteq F^i} (-\varepsilon_i \pi i)^{|F^i \setminus E|} \\ &\times \int_{\mathfrak{a}_{F \setminus F^i}^*} dv_0 \prod_{\alpha \in E} \text{P.V.} \int_{\mathbf{R}} \frac{dx_\alpha}{x_\alpha} g_i(h_0 : (v_0, (x_\alpha)_{\alpha \in E}, 0)). \end{aligned}$$

Here the limit is taken through  $h \in U_F(\varepsilon)$ .

*Proof.* Write  $F^i = \{\alpha_1, \dots, \alpha_p\}$ ,  $x_j = x_{\alpha_j}$ ,  $1 \leq j \leq p$ . Then we assume that the Haar measure  $dv_F$  on  $\mathfrak{a}_F^*$  is normalized so that

$$dv_F = dv_0 \prod_{j=1}^p \frac{\langle h_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle} dx_j,$$

so we can write

$$\begin{aligned} &\int_{\mathfrak{a}_F^*} \frac{g(h_0(t) : v_F)}{p_F(h_0(t) : v_F)} dv_F \\ &= 2^{-p} \int_{\mathbf{R}} \dots \int_{\mathbf{R}} \prod_{j=1}^p \frac{1}{x_j + it} \int_{\mathfrak{a}_{F \setminus F^i}^*} \frac{g(h_0(t) : (v_0, x_1, \dots, x_p))}{p_{F \setminus F^i}(h_0(t) : v_0)} dv_0 dx_1 \dots dx_p. \end{aligned}$$

Now since  $g \in \mathcal{C}(U_F(\mathfrak{e}) \times \mathfrak{a}_F^* : W)$  and  $p_{F \setminus F^i}(h_0(t) : v_0)$  is a polynomial in  $t$  and  $v_0$  with no zeros on  $I_{\mathfrak{e}_i}[0, a) \times \mathfrak{a}_{F \setminus F^i}^*$ ,

$$g'(x_1, \dots, x_p, t) = \int_{\mathfrak{a}_{F \setminus F^i}^*} g_i(h_0(t) : (v_0, x_1, \dots, x_p)) dv_0$$

is an element of  $\mathcal{C}(\mathbf{R}^p \times I_{\mathfrak{e}_i}[0, a) : W)$ . Thus we can write

$$\int_{\mathfrak{a}_F^*} \frac{g(h_0(t) : v_F)}{p_F(h_0(t) : v_F)} dv_F = 2^{-p} \int_{\mathbf{R}} \dots \int_{\mathbf{R}} \prod_{j=1}^p \frac{1}{x_j + it} g'(x_1, \dots, x_p, t) dx_1 \dots dx_p.$$

Now the result follows from (7.7). □

LEMMA 7.9. Suppose  $g \in \mathcal{C}(U_F(\mathfrak{e}) \times \mathfrak{a}_F^* : W)$ . Then for any  $k \geq 0$ ,  $h \in U_F(\mathfrak{e})$ ,

$$(\partial/\partial h_i)^k \int_{\mathfrak{a}_F^*} \frac{g(h : v_F)}{p_F(h : v_F)} dv_F = \int_{\mathfrak{a}_F^*} \frac{D_{F^i}^k g_i(h : v_F)}{p_{F^i}(h : v_F)} dv_F$$

where  $D_{F^i} = \partial/\partial h_i - i \sum_{\alpha \in F^i} \partial/\partial \mu_\alpha$ .

*Proof.* Using the notation of (7.8) we can write

$$(\partial/\partial h_i)^k \int_{\mathfrak{a}_F^*} \frac{g(h : v_F)}{p_F(h : v_F)} dv_F = 2^{-p} (\partial/\partial t)^k \int_{\mathbf{R}} \dots \int_{\mathbf{R}} \frac{g'(x_1, \dots, x_p, t)}{\prod_{j=1}^p x_j + it} dx_1 \dots dx_p.$$

Now using (7.7) this is equal to

$$\begin{aligned} & 2^{-p} \int_{\mathbf{R}} \dots \int_{\mathbf{R}} \frac{(\partial/\partial t - i \sum_{j=1}^p \partial/\partial x_j)^k g'(x_1, \dots, x_p, t)}{\prod_{j=1}^p x_j + it} dx_1 \dots dx_p \\ & = \int_{\mathfrak{a}_F^*} \frac{D_{F^i}^k g_i(h : v_F)}{p_{F^i}(h : v_F)} dv_F. \end{aligned} \quad \square$$

Write  $iv^* = \mathcal{H}_i \oplus \mathbf{R}h_i$ . Then we can write any differential operator  $D \in D(iv^*)$  as a sum of terms of the form  $D_0(\partial/\partial h_i)^k$  where  $D_0 \in D(\mathcal{H}_i)$ ,  $k \geq 0$ . Now combining Lemmas 7.8 and 7.9 we obtain the following.

PROPOSITION 7.10. Let  $g \in \mathcal{C}(U_F(\mathfrak{e}) \times \mathfrak{a}_F^* : W)$ . Write

$$f_F(h) = \int_{\mathfrak{a}_F^*} \frac{g(h : v_F)}{p_F(h : v_F)} dv_F.$$

Then  $f_F(h) \in \mathcal{C}(U_F(\varepsilon): W)$ . Further, for any  $D_0 \in D(\mathcal{H}_i)$ ,  $k \geq 0$ ,  $h \in U_F(\varepsilon)$ ,

$$D_0(\partial/\partial h_i)^k \int_{\alpha_F^*} \frac{g(h: v_F)}{p_F(h: v_F)} dv_F = \int_{\alpha_F^*} \frac{D_0 D_{F^i}^k g_i(h: v_F)}{p_{F^i}(h: v_F)} dv_F$$

Finally, if  $h_0 \in \mathcal{H}_i$  is semi-regular, then

$$\begin{aligned} \lim_{h \rightarrow h_0} D_0(\partial/\partial h_i)^k \int_{\alpha_F^*} \frac{g(h: v_F)}{p_F(h: v_F)} dv_F &= 2^{-|F^i|} \sum_{E \subseteq F^i} (-\varepsilon_i \pi i)^{|F^i \setminus E|} \\ &\times \int_{\alpha_{F \setminus F^i}^*} dv_0 \prod_{\alpha \in E} \text{P.V.} \int_{\mathbf{R}} \frac{dx_\alpha}{x_\alpha} D_0 D_{F^i}^k g_i(h_0: (v_0, (x_\alpha)_{\alpha \in E}, 0)). \end{aligned}$$

Here the limit is taken through  $h \in U_F(\varepsilon)$ .

Now suppose  $\{g(F)\}_{F \subseteq F_0}$  is a matching family and define

$$f(h) = \sum_{F \subseteq F_0} \frac{1}{(\pi i)^{|F|}} \int_{\alpha_F^*} \frac{g(F: h: v_F)}{p_F(h: v_F)} dv_F$$

as in (7.2). Using Proposition 7.10 we know that for all  $\varepsilon \in \Sigma_0$ ,  $f(h)$  extends to a smooth function  $f(h: \varepsilon)$  on  $cl(U_\varnothing(\varepsilon))$  since  $cl(U_\varnothing(\varepsilon)) \subseteq cl(U_F(\varepsilon))$  for all  $F \subseteq F_0$ . Thus to prove Theorem 7.2 it suffices to show that for any  $1 \leq i \leq m$ ,  $\varepsilon \in \Sigma_i$ ,  $h_0 \in \mathcal{H}_i \cap cl(U_\varnothing(\varepsilon))$  semi-regular, and  $D \in D(i\sigma^*)$ ,  $Df(h_0: \varepsilon^+(i)) = Df(h_0: \varepsilon^-(i))$ .

Fix  $i, \varepsilon, h_0$ , as above and let  $D = D_0(\partial/\partial h_i)^k$ ,  $D_0 \in D(\mathcal{H}_i)$ ,  $k \geq 0$ . Write

$$F_0^i = F'_0, \quad F_0 \setminus F'_0 = F''_0 \quad \text{and} \quad D'_F = D_0 D_{F^i}^k.$$

LEMMA 7.11.

$$Df(h_0: \varepsilon^+(i)) - Df(h_0: \varepsilon^-(i))$$

$$\begin{aligned} &= \sum_{F'' \subseteq F''_0} \frac{1}{(\pi i)^{|F''|}} \int_{\alpha_{F''}^*} dv_{F''} \sum_{E' \subseteq F'_0} \frac{1}{(\pi i)^{|E'|}} \prod_{\alpha \in E'} \text{P.V.} \int_{\mathbf{R}} \frac{dx_\alpha}{x_\alpha} \sum_{E' \subseteq F' \subseteq F'_0} 2^{-|F'|} \\ &\quad \times [(-1)^{|F' \setminus E'|} D'_{F'} g_i(F' \cup F'' : \varepsilon^+(i): h_0: (v_{F''}, (x_\alpha)_{\alpha \in E'}, 0)) \\ &\quad - D'_{F'} g_i(F' \cup F'' : \varepsilon^-(i): h_0: (v_{F''}, (x_\alpha)_{\alpha \in E'}, 0))]. \end{aligned}$$

*Proof.* Using (7.10) we can write

$$Df(h_0: \varepsilon^+(i)) - Df(h_0: \varepsilon^-(i))$$

$$\begin{aligned} &= \sum_{F \subseteq F_0} \frac{1}{(\pi i)^{|F|}} 2^{-|F^i|} \sum_{E' \subseteq F^i} (\pi i)^{|F^i \setminus E'|} \int_{\alpha_{F \setminus F^i}^*} dv_0 \prod_{\alpha \in E'} \text{P.V.} \int_{\mathbf{R}} \frac{dx_\alpha}{x_\alpha} \\ &\quad \times [(-1)^{|F^i \setminus E'|} D'_F g_i(F: \varepsilon^+(i): h_0: (v_0, (x_\alpha)_{\alpha \in E'}, 0)) \\ &\quad - D'_F g_i(F: \varepsilon^-(i): h_0: (v_0, (x_\alpha)_{\alpha \in E'}, 0))]. \end{aligned}$$

But for any  $F \subseteq F_0$ , write  $F' = F \cap F'_0 = F^i$ ,  $F'' = F \cap F''_0$ . Thus  $\mathfrak{a}_{F \setminus F'}^* = \mathfrak{a}_{F''}^*$ . Finally, we can rewrite the double summation

$$\sum_{F \subseteq F_0} \sum_{E' \subseteq F'}$$

as

$$\sum_{F'' \subseteq F''_0} \sum_{E' \subseteq F'_0} \sum_{E' \subseteq F' \subseteq F_0}$$

and simplify the constants to obtain the formula in the lemma. □

**PROOF OF THEOREM 7.2.** Fix  $E' \subseteq F'_0$  and  $F'' \subseteq F''_0$  and write  $E = E' \cup F''$ . Then

$$\{F' \cup F'' : E' \subseteq F' \subseteq F'_0\} = \{F : E \subseteq F \subseteq E(i) = E \cup F'_0\}.$$

For  $E \subseteq F \subseteq E(i)$  and  $v_E \in \mathfrak{a}_E^*$ ,  $l \geq 0$ , write as in (7.1),

$$a^\pm(F : l : h_0 : v_E) = \left( \partial / \partial h_i - i \sum_{\alpha \in F \setminus E} \partial / \partial \mu_\alpha \right)^l g(F : \varepsilon^\pm(i) : h_0 : (v_E, 0));$$

$$a_i^\pm(F : l : h_0 : v_E) = \left( \partial / \partial h_i - i \sum_{\alpha \in F \setminus E} \partial / \partial \mu_\alpha \right)^l g_i(F : \varepsilon^\pm(i) : h_0 : (v_E, 0));$$

$$b^\pm(F : h_0 : v_E) = D_{F^i}^k g_i(F : \varepsilon^\pm(i) : (v_E, 0));$$

$$c^\pm(F : h_0 : v_E) = D_F^k g_i(F : \varepsilon^\pm(i) : (v_E, 0)).$$

Then using Lemma 7.11, to show that  $Df(h_0 : \varepsilon^+(i)) = Df(h_0 : \varepsilon^-(i))$  it suffices to show for all  $v_E \in \mathfrak{a}_E^*$  that

$$\sum_{E \subseteq F \subseteq E(i)} 2^{-|F|} (-1)^{|F \setminus E|} c^+(F : h_0 : v_E) - c^-(F : h_0 : v_E) = 0.$$

Now for each  $E \subseteq F \subseteq E(i)$ ,

$$g_i(F : \varepsilon^\pm(i) : h : v_F) = g(F : \varepsilon^\pm(i) : h : v_F) p_{F''}(h : v_F)^{-1}$$

where  $p_{F''}(h : v_F)^{-1}$  is smooth near  $h_0$ . For  $l \geq 0$  write

$$p(l : h_0 : v_E) = \left( \partial / \partial h_i - i \sum_{\alpha \in F \setminus E} \partial / \partial \mu_\alpha \right)^l p_{F''}(h_0 : (v_E, 0))^{-1}.$$

Now since  $\{g(F)\}$  is a matching family, using (7.1b) we have for all  $r \geq 0$ ,

$$\begin{aligned} & a_i^+(F:r:h_0:v_E) - a_i^-(F:r:h_0:v_E) \\ &= \sum_{l=0}^r \binom{r}{l} p(r-l:h_0:v_E)(a^+(F:l:h_0:v_E) - a^-(F:l:h_0:v_E)) \\ &= \sum_{l=0}^r \binom{r}{l} p(r-l:h_0:v_E) \\ &\quad \times \sum_{F \subset F_1 \subseteq E(i)} c_{|F_1 \setminus F|} (a^+(F_1:l:h_0:v_E) + a^-(F_1:l:h_0:v_E)) \\ &= \sum_{F \subset F_1 \subseteq E(i)} c_{|F_1 \setminus F|} (a_i^+(F_1:r:h_0:v_E) + a_i^-(F_1:r:h_0:v_E)). \end{aligned}$$

Now for all  $E \subseteq F \subset F_1 \subseteq E(i)$ ,  $v_E \in \alpha_E^*$ , since

$$F_1^i = F_1' = (F_1' \setminus E') \cup E' \quad \text{and} \quad F_1' \setminus E' = F_1 \setminus E,$$

$$b^\pm(F_1:h_0:v_E) = \sum_{l=0}^k \binom{k}{l} \left( -i \sum_{\alpha \in E'} \partial / \partial \mu_\alpha \right)^{k-l} a_i^\pm(F_1:l:h_0:v_E).$$

Now using the identities proved above for  $a_i$  we have

$$\begin{aligned} & b^+(F:h_0:v_E) - b^-(F:h_0:v_E) \\ &= \sum_{l=0}^k \binom{k}{l} \left( -i \sum_{\alpha \in E'} \partial / \partial \mu_\alpha \right)^{k-l} (a_i^+(F:l:h_0:v_E) - a_i^-(F:l:h_0:v_E)) \\ &= \sum_{l=0}^k \binom{k}{l} \left( -i \sum_{\alpha \in E'} \partial / \partial \mu_\alpha \right)^{k-l} \\ &\quad \times \sum_{F \subset F_1 \subseteq E(i)} c_{|F_1 \setminus F|} (a_i^+(F_1:l:h_0:v_E) + a_i^-(F_1:l:h_0:v_E)) \\ &= \sum_{F \subset F_1 \subseteq E(i)} c_{|F_1 \setminus F|} (b^+(F_1:h_0:v_E) + b^-(F_1:h_0:v_E)). \end{aligned}$$

Finally, since  $D_0 \in D(\mathcal{H}_i)$  we have  $c^\pm(F:h_0:v_E) = D_0 b^\pm(F:h_0:v_E)$  for all  $E \subseteq F \subseteq E(i)$ . Now as above, for all  $E \subseteq F \subseteq E(i)$ ,  $v_E \in \alpha_E^*$ ,

$$\begin{aligned} & b^+(F:h_0:v_E) - b^-(F:h_0:v_E) \\ &= \sum_{F \subset F_1 \subseteq E(i)} c_{|F_1 \setminus F|} (b^+(F_1:h_0:v_E) + b^-(F_1:h_0:v_E)). \end{aligned}$$

But differentiating both sides of the equation using the differential operator  $D_0$  we obtain

$$c^+(F:h_0:v_E) - c^-(F:h_0:v_E) = \sum_{F \subset F_1 \subseteq E(i)} c_{|F_1 \setminus F|} (c^+(F_1:h_0:v_E) + c^-(F_1:h_0:v_E)).$$

Now applying (3.23) we see that

$$\sum_{E \subseteq F \subseteq E(i)} 2^{-|F|} ((-1)^{|F \setminus E|} c^+(F : h_0 : v_E) - c^-(F : h_0 : v_E)) = 0. \quad \square$$

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