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## On Galois representations defined by torsion points of modular elliptic curves

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**Abstract.** We verify Serre's conjecture (3.2.4?), stated in [Se4], for the special case of irreducible Galois representations defined by the $p$-torsion points of modular elliptic curves with potentially good, ordinary reduction at $p$. Also we treat the semi-stable case when the representation is not finite at $p$.

## Introduction

For a prime integer $p$, fix $\bar{\mathbb{F}}_p$ a closure of $\mathbb{F}_p$. Fix $\bar{\mathbb{Q}}$ a closure of $\mathbb{Q}$ and put $G_{\mathbb{Q}} = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. For every continuous Galois representation $\rho: G_{\mathbb{Q}} \to \mathrm{GL}_2(\bar{\mathbb{F}}_p)$, Serre defines two integers: the conductor $N_\rho$ and the weight $k_\rho$, as well as a Dirichlet character $\varepsilon_\rho: (\mathbb{Z}/N_\rho\mathbb{Z})^* \to \bar{\mathbb{F}}_p^*$ by means of a recipe. When $\rho$ is odd and irreducible, he conjectures that $\rho$ arises from a cusp form of type $(N_\rho, k_\rho, \varepsilon_\rho)$ with coefficients in $\bar{\mathbb{F}}_p$ which is an eigenfunction of all the Hecke operators (cf. [Se4], (3.2.4?)).

Suppose that $E$ is an elliptic curve over $\mathbb{Q}$. For each prime $p$, the action of the absolute Galois group $G_{\mathbb{Q}}$ on the $p$-torsion points $E_p$ of $E(\bar{\mathbb{Q}})$ defines a continuous representation

$$\rho: G_{\mathbb{Q}} \to \mathrm{Aut}(E_p) \simeq \mathrm{GL}_2(\mathbb{F}_p), \tag{1}$$

which is always odd and absolutely irreducible if $p > 163$ (cf. [Ma], [Se4]).

In this case, the character $\varepsilon_\rho$ is trivial. Depending on the arithmetic of $E$, we compute the conductor and the weight of $\rho$ in Section 1. If the representation $\rho$ is finite at $p$, $p \geqslant 5$, and the elliptic curve is modular and has semi-stable reduction at $p$, we know that $\rho$ verifies (3.2.4?) (cf. [Ri], [Ca]). In Sections 2 and 3 we show that $\rho$ verifies (3.2.4?) when $E$ is modular and has potentially good ordinary or semi-stable (non-finite) reduction at $p$. In these cases the weight $k_\rho$ is greater than 2 and the cusp form $f$ predicted by Serre's conjecture can be obtained by a manipulation of the weight-2 newform $F$ attached to $E$ by Eichler–Shimura theory.

Section 1 reproduces the text of a talk given by the second author in the Frey–Lamprecht–Zimmer Seminar of the Universität des Saarlandes, in September '89. The results in this section are in accordance with the ones given in [Kr].

## 1. Conductor and weight formulae

The action of $G_\mathbb{Q}$ on the Tate module $T_p(E)$ of $E$ gives rise to a $p$-adic representation $\rho_0: G_\mathbb{Q} \to \text{Aut}(T_p(E) \otimes \mathbb{Q}_p) \simeq \text{GL}_2(\mathbb{Q}_p)$ which is a lifting of the representation $\rho$. Let $N$ be the conductor of $\rho_0$; i.e., $N$ is the product of the local Artin conductors of $\rho_0$ at the primes $l \neq p$ (cf. [Se1]). We know that the conductor $N_\rho$ of $\rho$ must be a divisor of $N$. Since the wild ramification groups at $l$ are pro-$l$-groups and $l \neq p$, the wild exponents of $N$ and $N_\rho$ coincide. We have

$$v_l(N/N_\rho) = \text{codim}_{\mathbb{Q}_p}(T_p(E) \otimes \mathbb{Q}_p)^{I_l} - \text{codim}_{\mathbb{F}_p} E_p^{I_l},$$

for every prime $l \neq p$. Here, $I_l$ denotes some inertia group of $G_\mathbb{Q}$ for $l$.

Let now $\mathscr{E}_{\mathbb{F}_l}$ denote the special fibre of the Néron model of $E$ at $l$ and consider the exact sequence

$$0 \to \mathscr{E}_{\mathbb{F}_l}^0 \to \mathscr{E}_{\mathbb{F}_l} \to \Phi_l \to 0,$$

where $\mathscr{E}_{\mathbb{F}_l}^0$, $\Phi_l$ are the identity component and the group of components of $\mathscr{E}_{\mathbb{F}_l}$, respectively. We recall that

$$\mathscr{E}_{\mathbb{F}_l}^0 \simeq \begin{cases} \mathscr{E}_{\mathbb{F}_l} & \text{if } E \text{ has good reduction at } l, \\ \mathbb{G}_{m,\bar{\mathbb{F}}_l} \text{ (over } \bar{\mathbb{F}}_l) & \text{if } E \text{ has multiplicative reduction at } l, \\ \mathbb{G}_{a,\mathbb{F}_l} & \text{if } E \text{ has additive reduction at } l. \end{cases}$$

To compute $\text{codim}_{\mathbb{F}_p} E_p^{I_l}$, we use the isomorphism $E(\bar{\mathbb{Q}}_l)[p]^{I_l} \simeq \mathscr{E}_{\mathbb{F}_l}(\bar{\mathbb{F}}_l)[p]$ obtained by reduction (cf. [Se-Ta]); one easily checks that

$$\dim \mathscr{E}_{\mathbb{F}_l}(\bar{\mathbb{F}}_l)[p] = \begin{cases} 2 & \text{if } E \text{ has good reduction at } l, \\ 1 + \dim \Phi_l(\bar{\mathbb{F}}_l)[p] & \text{if } E \text{ has multiplicative reduction at } l, \\ \dim \Phi_l(\bar{\mathbb{F}}_l)[p] & \text{if } E \text{ has additive reduction at } l. \end{cases}$$

On the other hand,

$$\dim(T_p(E) \otimes \mathbb{Q}_p)^{I_l} = \begin{cases} 2 & \text{if } E \text{ has good reduction at } l, \\ 1 & \text{if } E \text{ has multiplicative reduction at } l, \\ 0 & \text{if } E \text{ has additive reduction at } l. \end{cases}$$

From what precedes it is clear that

$$v_l(N) - v_l(N_\rho) = p\text{-rank } \Phi_l,$$

for every prime $l \neq p$.

The following tables yield the exact value of the 'abaissement' $v_l(N/N_\rho)$ in terms of the reduction type of the Néron model at $l$.

For $p = 2$,

| $v_l(N/N_\rho)$ | Kodaira symbol of $\mathscr{E}_{\bar{\mathbb{F}}_l}$ |
| --- | --- |
| 2 | $I_{2v}^*$ |
| 1 | $I_{2v}, I_{2v+1}^*, III, III^*$ |
| 0 | $I_0, I_{2v+1}, II, II^*, IV, IV^*$ |

For $p = 3$,

| $v_l(N/N_\rho)$ | Kodaira symbol of $\mathscr{E}_{\bar{\mathbb{F}}_l}$ |
| --- | --- |
| 1 | $I_v$ if $3 \mid v, IV, IV^*$ |
| 0 | otherwise |

For $p \geqslant 5$,

| $v_l(N/N_\rho)$ | Kodaira symbol of $\mathscr{E}_{\bar{\mathbb{F}}_l}$ |
| --- | --- |
| 1 | $I_v$ if $p \mid v$ |
| 0 | otherwise |

The recipe to compute the weight $k_\rho$ is more subtle; it only involves the local representation at $p$. Let $\rho_p : G_p \to \mathrm{GL}_2(\bar{\mathbb{F}}_p)$ be the representation obtained from $\rho$ by restriction to some decomposition group $G_p = \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ of $G_\mathbb{Q}$ for $p$. Let $\rho_p^{ss}$ denote its semisimplification. The action under $\rho_p^{ss}$ of the tame inertia group $I_t$ of $G_p$ is given by two characters $\phi, \phi'$ of level 1 or 2; by writing them in a suitable way as powers of the fundamental characters one gets the value of $k_\rho$ (cf. [Se4]). Recall that each character of $I_t$ is determined by its invariant $\alpha \mod \mathbb{Z}[1/p]$, where $\alpha$ is a rational number (cf. [Se2]).

Assume for simplicity that $p > 7$ (cf. [Kr] for the cases $p \leqslant 7$). Let $c_4, c_6$ be the usual invariants of the elliptic curve $E$, determined up to $p$-units from a $p$-minimal Weierstrass equation for $E$.

PROPOSITION. *The weight* $k_\rho = 2 + \lambda(p-1)$ *and the invariants* $\alpha(\phi)$, $\alpha(\phi') = p/(p-1) - \alpha(\phi)$ *of the characters attached to the representation*

$$\rho : G_{\mathbb{Q}} \to \mathrm{Aut}(E_p)$$

*can be read from the following table:*

| $\mathscr{E}^e_{\overline{\mathbb{F}}_p}$ | Further conditions | $\alpha(\phi)$ | $\lambda$ |
|---|---|---|---|
| $I_v$ | if $p \mid v$ | $1/(p-1)$ | $0$ |
| | if $p \nmid v$ | $1/(p-1)$ | $1$ |
| $II$ | $p \equiv 1(3)$ | $(5p+1)/6(p-1)$ | $(p+5)/6$ |
| | $p \equiv 2(3),\ v_p(c_4) = 1$ | $(5p-1)/6(p-1)$ | $(p+7)/6$ |
| | $p \equiv 2(3),\ v_p(c_4) > 1$ | $(5p^2+1)/6(p^2-1)$ | $(p+7)/6$ |
| $III$ | $p \equiv 1(4)$ | $(3p+1)/4(p-1)$ | $(p+3)/4$ |
| | $p \equiv 3(4),\ v_p(c_6) = 2$ | $(3p-1)/4(p-1)$ | $(p+5)/4$ |
| | $p \equiv 3(4),\ v_p(c_6) > 2$ | $(3p^2+1)/4(p^2-1)$ | $(p+5)/4$ |
| $IV$ | $p \equiv 1(3)$ | $(2p+1)/3(p-1)$ | $(p+2)/3$ |
| | $p \equiv 2(3),\ v_p(c_4) = 2$ | $(2p-1)/3(p-1)$ | $(p+4)/3$ |
| | $p \equiv 2(3),\ v_p(c_4) > 2$ | $(2p^2+1)/3(p^2-1)$ | $(p+4)/3$ |
| $I_v^*$ | if $p \mid v$ | $(p+1)/2(p-1)$ | $(p+1)/2$ |
| | if $p \nmid v$ | $(p+1)/2(p-1)$ | $(p+3)/2$ |
| $II^*$ | $p \equiv 1(3)$ | $(p+5)/6(p-1)$ | $(p+11)/6$ |
| | $p \equiv 2(3),\ v_p(c_4) = 4$ | $(p+1)/6(p-1)$ | $(p+7)/6$ |
| | $p \equiv 2(3),\ v_p(c_4) > 4$ | $(p^2+5)/6(p^2-1)$ | $(p+1)/6$ |
| $III^*$ | $p \equiv 1(4)$ | $(p+3)/4(p-1)$ | $(p+7)/4$ |
| | $p \equiv 3(4),\ v_p(c_6) = 5$ | $(p+1)/4(p-1)$ | $(p+5)/4$ |
| | $p \equiv 3(4),\ v_p(c_6) > 5$ | $(p^2+3)/4(p^2-1)$ | $(p+1)/4$ |
| $IV^*$ | $p \equiv 1(3)$ | $(p+2)/3(p-1)$ | $(p+5)/3$ |
| | $p \equiv 2(3),\ v_p(c_4) = 3$ | $(p+1)/3(p-1)$ | $(p+4)/3$ |
| | $p \equiv 2(3),\ v_p(c_4) > 3$ | $(p^2+2)/3(p^2-1)$ | $(p+1)/3$ |

*Proof.* The results in the second table can be obtained from the results in the first table by twisting with the quadratic character ramified only at $p$. If $E$ has semi-stable reduction at $p$, the values of $\alpha(\phi)$ and $\lambda$ are already given in [Se4]. Therefore, we only need to consider the cases when $E$ has reduction type $II$, $III$, or $IV$ at $p$.

Let

$$e = \begin{cases} 6 & \text{if } E \text{ is of type } II \text{ at } p, \\ 4 & \text{if } E \text{ is of type } III \text{ at } p, \\ 3 & \text{if } E \text{ is of type } IV \text{ at } p. \end{cases}$$

The elliptic curve $E$ acquires good reduction over the field $L = \mathbb{Q}_p^{\mathrm{nr}}(\pi)$, where $\pi^e = p$. Let $E_L = E \otimes_{\mathbb{Q}_p^{\mathrm{nr}}} L$ and let $m : E_L(\overline{\mathbb{Q}}_p) \to E(\overline{\mathbb{Q}}_p)$ be the $\mathrm{Gal}(\overline{\mathbb{Q}}_p/L)$-

isomorphism obtained by base change. We search for some 1-dimensional subspace $V$ of the $\bar{\mathbb{F}}_p[I_t]$-module $(\bar{\mathbb{F}}_p \otimes_{\mathbb{F}_p} E_p)^{ss}$.

We first recall the basic setup of [Se2]. Let $\hat{E}$ be the formal group of $E_L$ and let $h$ be its height. The kernel $\hat{E}_p$ of the multiplication by $[p]$ contains always a non-zero $\mathbb{F}_p$-vector space $V_L$ on which the tame inertia group $I_{t,L}$ of $\mathrm{Gal}(\bar{\mathbb{Q}}_p/L)$ acts. The $\dim_{\mathbb{F}_p} V_L$ and the action of $I_{t,L}$ on $\mathbb{F}_{p^h} \otimes V_L$ can both be read from the Newton polygon of the formal series of the multiplication by $[p]$ on $\hat{E}$. A case-by-case verification shows:

| $h$ | Further conditions | $\dim_{\mathbb{F}_p} V_L$ | $I_{t,L}$ on $\bar{\mathbb{F}}_p \otimes V_L$ |
|---|---|---|---|
| 1 | $II, p \equiv 1(3)$ | 1 | $\theta_{p-1}^e$ |
|   | $III, p \equiv 1(4)$ | | |
|   | $IV, p \equiv 1(3)$ | | |
| 2 | $II, p \equiv 2(3), v_p(c_4) = 1$ | 1 | $\theta_{p-1}^4$ |
|   | $III, p \equiv 3(4), v_p(c_6) = 2$ | | $\theta_{p-1}^2$ |
|   | $IV, p \equiv 2(3), v_p(c_4) = 2$ | | $\theta_{p-1}$ |
| 2 | $II, p \equiv 2(3), v_p(c_4) > 1$ | 2 | $\theta_{p^2-1}^e, \theta_{p^2-1}^{pe}$ |
|   | $III, p \equiv 3(4), v_p(c_6) > 2$ | | |
|   | $IV, p \equiv 2(3), v_p(c_4) > 2$ | | |

where $\theta_{p-1}, \theta_{p^2-1} : I_{t,L} \to \bar{\mathbb{F}}_p^*$ are fundamental characters on $I_{t,L}$ of level one and two, respectively.

Let us take $V := \mathbb{F}_{p^h} \otimes m(V_{E_L})$, where $V_{E_L}$ is the image of $V_L$ under the embedding of $\hat{E}_p$ into the group $E_{L,p}$ of $p$-torsion points of $E_L(\bar{\mathbb{Q}}_p)$. Now $I_t$ acts on $V$ and its action can be computed by means of $m$. For that, we look at the action of $s_m := m^{-1} \circ s \circ m$ on $\mathbb{F}_{p^h} \otimes V_L$, for $s \in \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p^{nr})$. We carry out the whole computations only when $E$ has reduction type $IV$ at $p$, $p \equiv 2 \pmod 3$, and $v_p(c_4) > 2$:

$$E: Y^2 = X^3 - p^2 AX + p^2 B \quad \text{with } A, B \in \mathbb{Z}, \quad (p, B) = 1,$$

$$E_L: Y^2 = X^3 - \pi^2 AX + B, \quad \text{height}(\hat{E}) = 2, \quad \dim_{\mathbb{F}_p} V_L = 2,$$

$$m^{-1}: E(\bar{\mathbb{Q}}_p) \to E_L(\bar{\mathbb{Q}}_p), \quad m^{-1}(x, y) = \left( \frac{x}{\pi^2}, \frac{y}{\pi^3} \right),$$

and

$$V_L = \left\{ z = -\pi x/y \,\Big|\, \left( \frac{x}{\pi^2}, \frac{y}{\pi^3} \right) \in E_{L,p}, \, v_\pi(-\pi x/y) = \frac{3}{p^2-1} \right\}$$

$$= \left\{ z = -\pi x/y \,\Big|\, \left( \frac{x}{\pi^2}, \frac{y}{\pi^3} \right) \in E_{L,p}, \, v_p(-x/y) = \frac{-p^2+4}{3(p^2-1)} \right\},$$

$$V = \mathbb{F}_{p^2} \otimes \left\{ (x, y) \in E_p \,\Big|\, \frac{-\pi x}{y} \in V_L \right\}.$$

If $s \in \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p^{nr})$ and $z = -\pi x/y \in V_L$, then

$$s\left(\frac{x}{y}\right) = \theta_{p^2-1}^{(-p^2+4)/3}(s)\frac{x}{y} = \theta_{p^2-1}^{(2p^2+1)/3}(s)\frac{x}{y}.$$

Finally, for $1 \otimes (x, y) \in V$ we get

$$
\begin{aligned}
s(1 \otimes (x, y)) &= 1 \otimes (m \circ s_m \circ m^{-1})(x, y) \\
&= 1 \otimes (m \circ s_m)(x/\pi^2, y/\pi^3) \\
&= 1 \otimes m(sx/\pi^2, sy/\pi^3) \\
&= \theta_{p^2-1}^{(2p^2+1)/3}(s) \otimes m(x/\pi^2, y/\pi^3) \\
&= \theta_{p^2-1}^{(2p^2+1)/3}(s)(1 \otimes m(x/\pi^2, y/\pi^3)) \\
&= \theta_{p^2-1}^{(2p^2+1)/3}(s)(1 \otimes (x, y)).
\end{aligned}
$$

Now, case-by-case, we fill up the table by taking into account the recipe for the weight.

## 2. Serre's conjecture in the ordinary case

Our goal is to prove the following

THEOREM. *Serre's conjecture (3.2.4$_?$) is true for an irreducible representation*

$$\rho: G_{\mathbb{Q}} \to \mathrm{Aut}(E_p)$$

*provided that $E$ over $\mathbb{Q}$ is a modular elliptic curve with potentially good, ordinary reduction at $p > 7$.*

We look for a modular form $f = \Sigma a_n q^n$ of type $(N_\rho, k_\rho)$ with coefficients in $\bar{\mathbb{F}}_p$, eigenfunction of all the Hecke operators, and satisfying

$$\mathrm{trace}(\rho(\mathrm{Frob}_l)) = a_l \quad \text{and} \quad \det(\rho(\mathrm{Frob}_l)) = l^{k_\rho - 1},$$

for all $l \nmid pN_\rho$. We should like to remark that numerical evidence, collected by computer calculations, led us to guess how the normalized newform $F$ attached to our elliptic curve $E$ should be handled to produce $f$. First of all we fix some terminology:

Set $C_n$ for a cyclic group of order $n$. Fix a primitive root mod $p$ or, equivalently, fix an isomorphism $(\mathbb{Z}/p\mathbb{Z})^* \simeq C_{p-1}$. Let $\zeta_p$ in $\bar{\mathbb{Q}}$ be a primitive $p$th root of unity. Realize the group $C_{p-1}$ as Galois group as follows:

$G_{\mathbb{Q}} \to \text{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q}) \simeq (\mathbb{Z}/p\mathbb{Z})^*$, by sending a conjugation class $\text{Frob}_l$ to $l \mod p$ for all $l \neq p$.

Fix embeddings

$$\bar{\mathbb{Q}} \to \bar{\mathbb{Q}}_l$$

for each prime $l$ and define a Dirichlet character

$$\psi : (\mathbb{Z}/p\mathbb{Z})^* \to \mu_{p-1}$$

so that under the mapping $\mu_{p-1} \to \mathbb{Q}_p^*$ we have

$$\psi(n)n \equiv 1 \ (\text{mod } p), \quad \text{for all } n \in \mathbb{Z}, \quad (n, p) = 1.$$

We denote by

$$\psi_l : G_{\mathbb{Q}} \to \text{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q}) \to \mu_{p-1} \to \bar{\mathbb{Q}}_l^*,$$

the Galois $l$-adic character deduced from $\psi$. Note that by reduction mod $p$, $\psi_p$ yields the inverse of the cyclotomic character mod $p$

$$\chi : G_{\mathbb{Q}} \to \mathbb{F}_p^*,$$

which is given by the action of $G_{\mathbb{Q}}$ on the $p$th roots of unity.

Finally, for future use we state the following notations:

| Type | $I_0^*$ | $II$ | $III$ | $IV$ | $II^*$ | $III^*$ | $IV^*$ |
|------|---------|------|-------|------|--------|---------|--------|
| $e$  | 2       | 6    | 4     | 3    | 6      | 4       | 3      |
| $a$  | 0       | 1    | 1     | 1    | 1      | 1       | 1      |

From now on we put $m = \dfrac{p-1}{e}$.

LEMMA 1. *Let $F$ be the normalized newform associated to an elliptic curve $E$ satisfying the same hypothesis as in the theorem and let $N$ be as in Section 1. Then the level of $F$ is $p^2 N$ and there exists a newform*

$$g = \sum b_n q^n \in S_2(p^a N, \psi^{2m}), \qquad q = e^{2\pi i z},$$

*having the same eigenvalues system as the twisted form $F \otimes \psi^m$.*

*Proof.* The first claim follows from the assumptions and [Ca]. Observe that $F \otimes \psi^m$ is an eigenfunction for the Hecke operators with eigenvalues system $\{a_l \psi^m(l), \psi^{2m}\}_{l \nmid pN}$. Therefore there exists a divisor $M$ of $p^2 N$ and a newform $g \in S_2(M, \psi^{2m})$ having the same eigenvalues system as $F \otimes \psi^m$. In fact,

$$F \otimes \psi^m = g - g | U_p | B_p$$

being $U_p$, resp. $B_p$, the Atkin, resp. the degeneracy, operators as in [At-Li].

We must prove that $M = N$ if $E$ is of type $I_0^*$ at $p$, and $M = pN$ otherwise. Let $l$ be a prime such that $(l, pN) = 1$. Using Eichler–Shimura, if

$$\rho_l : G_\mathbb{Q} \to \mathrm{Aut}(T_l(E) \otimes \bar{\mathbb{Q}}_l) \simeq \mathrm{GL}_2(\bar{\mathbb{Q}}_l)$$

denotes the $l$-adic representation attached to $E$ then

$$\rho_l \otimes \psi_l^m : G_\mathbb{Q} \to \mathrm{GL}_2(\bar{\mathbb{Q}}_l)$$

is the $l$-adic representation arising from $F \otimes \psi^m$. Notice that $\rho_l \otimes \psi_l^m$ is also the $l$-adic representation for the newform $g$.

Since after Carayol's work it is known that the level of a newform agrees (outside $l$) with the Artin conductor of its $l$-adic representation, we have reduced the problem to compute the conductor of $\rho_l \otimes \psi_l^m$ attached to $g$. Moreover, since $\psi_l$ is unramified outside $p$ we only need to compute the action of $I_p$, the inertia subgroup of $\mathrm{Gal}\,(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$.

Since $E$ has potentially good ordinary reduction at $p$, $E$ acquires good reduction over the ring of integers of $\mathbb{Q}_p(\pi)$, with $\pi^e = p$ and $p \equiv 1 \pmod{e}$; therefore $\rho_l$ restricted at $I_p$ factorizes through a cyclic group $C_e$ of order $e$. In particular, this forces the restriction $\rho_{l/I_p}$ to be diagonalizable. Moreover $\det \rho_l(I_p) = 1$ and hence

$$\rho_{l/I_p} = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix}$$

with a character $\varepsilon : C_{p-1} \to \mu_e(\bar{\mathbb{Q}}_l)$. We may view $\varepsilon$ as a Galois character unramified outside $p$, still denoted by $\varepsilon$,

$$\varepsilon : G_\mathbb{Q} \to \mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q}) \simeq C_{p-1} \to C_e \to \mu_e(\bar{\mathbb{Q}}_l).$$

But, $\varepsilon$ should be $\psi_l^{\pm m}$. Since, being $\bar{\mathbb{Q}}_l$ fixed, only two characters of $C_{p-1} \to \mu_{p-1}(\bar{\mathbb{Q}}_l)$ of order $e$ exist if $e = 3, 4$ or $6$, and only one if $e = 2$. So, now it is

clear that $\rho_l \otimes \psi_l^m$ has Artin conductor with exponent 1 at $p$ if $e = 3, 4, 6$ or 0 if $e = 2$.

We also need some knowledge on the behavior of the 'hidden' coefficient $b_p$ of the newform $g$ attached to the modular form $F$. In the next lemma, we reject the easy case when $E$ is of type $I_0^*$ at $p$.

LEMMA 2. *Let $g = \Sigma b_n q^n$ be the normalized newform of Lemma 1 and assume $a = 1$. Let $K$ be the number field generated by the Fourier coefficients $b_n$ for all $n \geqslant 1$. Then we have:*

(i) *The field $K$ is equal to*

$$\mathbb{Q}(\psi^m) = \begin{cases} \mathbb{Q}(\sqrt{-3} & \text{if } e = 3, 6, \\ \mathbb{Q}(i) & \text{if } e = 4. \end{cases}$$

(ii) $|b_p| = \sqrt{p}$.

(iii) *If $v_p$ denotes the normalized $p$-adic valuation of $\bar{\mathbb{Q}}$ fixed by $\bar{\mathbb{Q}}_p$, then*

$$v_p(b_p) \in \{0, 1\}.$$

*Proof.* It is clear that the coefficients $b_l$ with $l \nmid pN$ belong to the field $\mathbb{Q}(\psi^m)$. Since $g$ is a newform, it then follows from a theorem of Miyake (cf. Theorem B in [Mi]), that all the coefficients of $g$ lie in $\mathbb{Q}(\psi^m)$.

Since $g$ is a normalized newform and it has primitive $p$-Nebentypus (in the sense that its level and the conductor of its character have the same valuation at $p$), a result of Ogg–Li–Asai (cf. [Og], [Li], [As]) yields $|b_p| = \sqrt{p}$, and therefore $v_p(b_p) + v_p(\overline{b_p}) = 1$, where the bar denotes complex conjugation. Since $b_p$ is an integer of $K$ and $p$ splits completely in $K$, we have $v_p(b_p) \in \{0, 1\}$.

The next lemma makes use of the theory of deforming residual Galois representations. Recent work of Mazur and Tilouine suggest how to make explicit links between ordinary deformations and ordinary (potentially) elliptic curves (cf. [Ma-Ti], [Hi]).

We let $\bar{g} = \Sigma \overline{b_n} q^n$ denote the 'complex conjugate' form of $g$.

LEMMA 3. *Either $g$ or $\bar{g}$ is ordinary at $p$, depending on the $p$-Kodaira symbol of $E$: II, III, IV or II\*, III\*, IV\*.*

*Proof.* Since $E$ has potentially good ordinary reduction at $p$, the restriction of $\rho$ to $I_p$ is reducible. It is given by

$$\begin{pmatrix} \chi^{p-m} & * \\ 0 & \chi^m \end{pmatrix} \quad \text{when } E \text{ is of type } II, III \text{ or } IV \text{ at } p,$$

and

$$\begin{pmatrix} \chi^{m+1} & * \\ 0 & \chi^{m(e-1)} \end{pmatrix} \quad \text{when } E \text{ is of type } II^*, III^*, IV^* \text{ at } p.$$

We obtain

| Form | Representation | $p$-type$(E) = II, III, IV$ | $p$-type$(E) = II^*, III^*, IV^*$ |
|------|----------------|------------------------------|-------------------------------------|
| $g$ | $\rho \otimes \chi_{|I_p}^{-m}$ | $\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}$ | $\begin{pmatrix} * & * \\ 0 & \chi^{(e-2)m} \end{pmatrix}$ |
| $\bar{g}$ | $\rho \otimes \chi_{|I_p}^{m}$ | $\begin{pmatrix} * & * \\ 0 & \chi^{2m} \end{pmatrix}$ | $\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}$ |

Therefore, by a theorem of Wiles (cf. [Wi]), $g$ cannot be ordinary in the cases when $E$ is of type $II^*$, $III^*$ or $IV^*$ at $p$, resp. $\bar{g}$ cannot be ordinary in the cases when $E$ is of type $II$, $III$ or $IV$ at $p$. Since we know a priori that one of the two forms is $p$-ordinary, the claim follows.

REMARK. From now on let $\mathfrak{P}$ be the place of $\bar{\mathbb{Q}}$ fixed by our choice of $\bar{\mathbb{Q}}_p$. Let $\tilde{g}$ denote the $p$-ordinary cusp form in Lemma 3. In the ring of integers $\mathcal{O}$ of $\mathbb{Q}(\psi^m)$ we write $(p) = \mathfrak{p}\bar{\mathfrak{p}}$ where $\mathfrak{p} = \mathfrak{P} \cap \mathcal{O}$. Denote by $A/_{\mathbb{Q}}$ the abelian subvariety of $J_1(pN)$ attached to $\tilde{g}$ by Shimura's theory and consider the $\mathbb{Q}$-rational multiplication

$$\iota \colon \mathcal{O} \to \text{End}(A/_{\mathbb{Q}})$$

under which the action of $\iota(\tilde{b}_n)$ is given by the Hecke operator $T_{n,\text{Alb}}$ acting on $A$.

By using a theorem of Langlands [La], $A$ acquires good reduction over $\mathbb{Q}_p(\zeta_p)$ (or even more precisely, over the splitting field of the character $\psi^{2m}$). Denote by $B$ the Néron model of $A$ over $\text{Spec } \mathbb{Z}_p[\zeta_p]$. Lemma 3 simply tells us that the $p$-divisible group $B[\mathfrak{p}^\infty]$ is ordinary at $\mathfrak{p}$; i.e. in the exact sequence of $p$-divisible groups

$$0 \to B[\mathfrak{p}^\infty]^0 \to B[\mathfrak{p}^\infty] \to B[\mathfrak{p}^\infty]^{\text{et}} \to 0$$

all the terms are non-zero.

The last lemma we need makes use of the Eisenstein series $E_{1,\psi}$. Briefly, we summarize its definition and properties:

$$E_{1,\psi} = 1 + c_\psi \sum_{m,m_1 > 0} \psi(m) e^{2\pi i m m_1 z},$$

where

$$c_\psi = -\frac{2\pi i C(\bar\psi)}{pL(1,\bar\psi)}.$$

Here, $L(s,\psi)$ is the Dirichlet's $L$-function for the character $\psi$, and

$$C(\psi) = \sum_{1\leqslant n\leqslant p-1} \psi(n)e^{2\pi in/p}$$

is the Gauss sum. The Eisenstein series $E_{1,\psi}$ is a modular form of weight 1 and character $\psi$ on $\Gamma_0(p)$ which satisfies

$$E_{1,\psi} \equiv 1 \pmod{\mathfrak{P}}$$

and

$$E_{1,\psi}\left|\begin{pmatrix} 0 & -1 \\ p & 0 \end{pmatrix}_1 = \frac{\sqrt{p}}{C(\bar\psi)}\frac{c_\psi}{c_{\bar\psi}} E_{1,\bar\psi} \quad \text{(cf. [Ko]).} \right. \tag{2}$$

Also we are going to recall the trace operator between modular forms. For sake of simplicity we only take care of the case

$$\text{Tr}: S_\kappa(pN,\phi) \to S_\kappa(N,\phi), \qquad \text{Tr}(g) = \sum_{j=1}^{p+1} g\,|\,[\sigma_j]_\kappa,$$

where $\phi$ is a Dirichlet character mod $N$, and $\sigma_j$ denotes a set of representatives of the right cosets $\Gamma_0(pN)\backslash\Gamma_0(N)$.

Let $W_p = \begin{pmatrix} px & 1 \\ pNy & p \end{pmatrix}$, with some integers $x$, $y$ satisfying $\det W_p = p$. It is the well-known Fricke involution. Let $U_p$ be the $p$th Hecke operator acting on $S_\kappa(pN,\phi)$. In our case, the trace operator has the simple expression

$$\text{Tr}(g) = g + \phi(p)^{-1}p^{1-\kappa/2}g\,|\,[W_p]_\kappa\,|\,U_p.$$

LEMMA 4. *Let $h = \Sigma c_n q^n \in S_\kappa(pN,\psi^{-t})$ be a newform with $1 \leqslant t < p-1$. Let $d$ be a positive integer with $d \equiv t \pmod{p-1}$. Then*

$$\text{Tr}(hE_{1,\psi}^d) \equiv h \pmod{\mathfrak{P}} \Leftrightarrow 2 - \kappa + \frac{d-t}{p-1} + v_p(\bar c_p) > 0.$$

*Proof.* For the proof we borrow some ideas of Koike and Shimura (cf. [Ko]).

We write for a moment

$$\text{Tr}(hE_{1,\psi}^d) = hE_{1,\psi}^d + p^{1-(\kappa+d)/2}(hE_{1,\psi}^d)|[W_p]_{k+d}|U_p.$$

Since $E_{1,\psi} \equiv 1 \pmod{\mathfrak{P}}$, we look for an integer $d$ such that

$$p^{1-(\kappa+d)/2}(hE_{1,\psi}^d)|[W_p]_{\kappa+d}|U_p \equiv 0 \pmod{\mathfrak{P}}.$$

Following Asai [As], let us define $c_n^{(p)}$ by

$$\begin{cases} c_n^{(p)} = \bar{\psi}^{-t}(n)c_n & \text{if } (n, p) = 1, \\ c_n^{(p)} = \bar{c}_n & \text{if } (n, N) = 1, \\ c_{nm}^{(p)} = c_n^{(p)}c_m^{(p)} & \text{if } (n, m) = 1, \end{cases}$$

and put

$$h^{(p)} = \sum c_n^{(p)}q^n.$$

Then $h|[W_p]_\kappa = \delta_p h^{(p)}$ being $\delta_p = C(\psi^{-t})p^{-\kappa/2}\bar{c}_p$ the pseudo-eigenvalue of $W_p$ at $h$.

By using (2) we rewrite

$$(hE_{1,\psi}^d)|[W_p]_{\kappa+d} = h|[W_p]_\kappa \cdot (E_{1,\psi}|[W_p]_1)^d = \delta_p h^{(p)} \cdot \left(\frac{\sqrt{p}}{C(\bar{\psi})}\frac{c_\psi}{c_{\bar{\psi}}}\right)^d E_{1,\bar{\psi}}^d.$$

Finally, by taking into account Stickelberger's theorem we get

$$v_p\left(p^{1-(\kappa+d)/2}\cdot\delta_p\cdot\left(\frac{\sqrt{p}}{C(\bar{\psi})}\frac{c_\psi}{c_{\bar{\psi}}}\right)^d\right) = 2-\kappa+\frac{d-t}{p-1}+v_p(\bar{c}_p),$$

and this proves the lemma.

*Proof of the theorem.* We give the proof when $E$ is of type *II, III,* or *IV* at $p$. For the other cases, except when $E$ is of type $I_0^*$ which is even easier, we proceed in the same way, and therefore the proof is left to the reader.

By Lemma 1 we take $g = \sum b_n q^n \in$ in $S_2(pN, \psi^{2m})$. From Lemmas 2 and 3 we know that in this case $g$ is a $p$-ordinary newform satisfying $v_p(\bar{b}_p) = 1$. By normalizing the exponent in the character of $g$, we can take $d = m(e-2)$ in accordance with Lemma 4.

Let $\lambda$ be as in Section 1. If $\theta$ denotes the Ramanujan operator (cf. [Se3]) it is immediate that

$$\theta^m \operatorname{Tr}(gE_{1,\psi}^{m(e-2)}) \in S_{2+\lambda(p-1)}(N)_{/\bar{\mathbb{F}}_p},$$

and its coefficients are congruent to the traces of $\rho(\operatorname{Frob}_l)$ for all $l \nmid pN$. The weight of this form is now the right one and its level is prime to $p$.

If $N$ is the conductor predicted by $(3.2.4_?)$ (cf. table of conductors in Section 1), we can take the form $f$ predicted by Serre's conjecture as $\theta^m \operatorname{Tr}(gE_{1,\psi}^d)$. Otherwise, since $N$ is prime to $p$, Theorem 1 in [Jo-Li] allows to change the form $\operatorname{Tr}(gE_{1,\psi}^d)$ by a form $g'$ such that $f = \theta^m g'$.

REMARK. Notice that the newform $\bar{g}$ can be used to produce the modular form predicted by the conjecture of Serre for the representation arising from the $p$-torsion points of $E^*$, the twisted elliptic curve of $E$ by the quadratic character ramified only at $p$.

## 3. Serre's conjecture in the semi-stable non-finite case

Using the same techniques as in the preceeding section we also give the following

THEOREM. *Serre's conjecture $(3.2.4_?)$ is true for an irreducible representation*

$$\rho: G_{\mathbb{Q}} \to \operatorname{Aut}(E_p)$$

*provided that $E$ over $\mathbb{Q}$ is a modular elliptic curve with semi-stable reduction at a prime $p \geqslant 5$.*

*Proof.* When the representation $\rho$ is finite, which is equivalent to say that the weight predicted by Serre's conjecture equals 2, it is a consequence of results given in [Ri] and [Ca].

Thus, we suppose $\rho$ is non-finite; then the weight attached to $\rho$ is $k_\rho = p+1$. Applying the same kind of arguments as in the previous section, now we have that $FE_{1,\psi}^{p-1}$ belongs to $S_{p+1}(pN)$, being $F = \Sigma a_n q^n$ the newform attached to the elliptic curve $E$. It is easy to check that $g = \operatorname{Tr}(FE_{1,\psi}^{p-1})$ will produce the cusp form predicted by $(3.2.4_?)$ because it has the correct weight, $p$ does not divide its level and since $a_p = \pm 1$ it satisfies

$$\operatorname{Tr}(FE_{1,\psi}^{p-1}) \equiv F \pmod{\mathfrak{P}}.$$

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## References

[As]     Asai, T.: On the Fourier coefficients of automorphic forms at various cusps and some applications to Rankin's convolution, *J. Math. Soc. Japan* 28 (1976), 48–61.

[At-Le]  Atkin, A.O.L. and Lehner, J.: Hecke operators on $\Gamma_0(N)$, *Math. Ann.* 185 (1970), 134–160.

[At-Li]  Atkin, A.O.L. and Li, W.-C.: Twists of newforms and pseudo-eigenvalues of *W*-operators, *Invent. Math.* 48 (1978), 221–243.

[Ca]     Carayol, H.: Sur les représentations *l*-adiques attachées aux formes modulaires de Hilbert, *Ann. scient. E.N.S.* 19 (1986), 409–468.

[Hi]     Hida, H.: Galois representations into $GL_2[[X]]$ attached to ordinary cusp forms, *Inv. Math.* 85 (1986), 545–577.

[Jo-Li]  Jordan, B.W. and Livné, R.: Conjecture 'epsilon' for weight $k > 2$, *Bull. Amer. Math. Soc.* 21 (1989), 51–69.

[Ko]     Koike, M.: Congruences between cusp forms and linear representations of the Galois group, *Nagoya Math. J.* 64 (1976), 63–85.

[Kr]     Kraus, A.: 'Thèse: Sur l'arithmétique des courbes elliptiques', 1990.

[La]     Langlands, R.P.: Automorphic forms and *l*-adic representations, *Modular Forms of One Variable II*, 361–500; *Springer Lect. Notes* 349, 1973.

[Ma]     Mazur, B.: Rational isogenies of prime degree, *Invent. Math.* 44 (1978), 129–162.

[Ma-Ti]  Mazur, B. and Tilouine, J.: Représentations galoisiennes, différentielles de Kähler, et 'conjectures principales',*Publ. Math. IHES* 71 (1990), 65–103.

[Mi]     Miyake, T.: On automorphic forms on $GL_2$ and Hecke operators, *Ann. of Math.* 94 (1971), 174–189.

[Og]     Ogg, A.: On the eigenvalues of the Hecke operators, *Math. Ann.* 79 (1969), 101–108.

[Ri]     Ribet, K.: On modular representations of $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ arising from modular forms, *Invent. Math.* 100 (1990), 431–476.

[Se1]    Serre, J.-P.: Facteurs locaux des fonctions zêta des variétés algébriques (définitions et conjectures); Séminaire Delange–Pisot–Poitou 19 (1969/70); Œ 87. Springer, 1986.

[Se2]    Serre, J.P.: Propriétés galoisiennes des points d'ordre fini des courbes elliptiques, *Invent. Math.* 15 (1972), 259–331; Œ 94, 1986.

[Se3]    Serre, J.-P.: Valeurs propres des opérateurs de Hecke modulo *l*, *Astérisque* 24–25 (1975), 109–117; Œ 104, Springer, 1986.

[Se4]    Serre, J.-P.: Sur les représentations modulaires de degré 2 de $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$, *Duke Math. J.* 54 (1987), 179–230.

[Se-Ta]  Serre, J.-P. and Tate, J.: Good reduction of abelian varieties, *Ann. of Math.* 88 (1968), 492–517; Œ 79, Springer, 1986.

[Sh]     Shimura, G.: On the factors of the Jacobian variety of a modular function field, *J. Math. Soc. Japan* 25 (1976), 523–544.

[Wi]     Wiles, A.: On *p*-adic representations for totally real fields, *Ann. of Math.* 123 (1986), 407–456.