COMPOSITIO MATHEMATICA

KUNRUI YU

Linear forms in p-adic logarithms. II

Compositio Mathematica, tome 74, nº 1 (1990), p. 15-113
<http://www.numdam.org/item?id=CM_1990__74_1_15_0>

Compositio Mathematica **74**: 15–113, 1990.

Linear forms in p-adic logarithms II

Dedicated to the memory of Professor Loo-keng Hua

KUNRUI YU*

Institute of Mathematics, Academia Sinica, Beijing 100080, China

Received 20 March 1989; accepted 17 August 1989

0. Introduction and results

0.1 The present paper is a continuation of the study in Yu [20] and [21], where a brief history of the theory of linear forms in p-adic logarithms was given, and precise results subject to a Kummer condition were proved. In this paper we shall remove the Kummer condition, thereby establishing the p-adic analogue of a celebrated theorem of Baker on linear forms in logarithms of algebraic numbers (i.e. Theorem 2 of Baker [2]) and the p-adic analogue of Baker's well-known Sharpening II (i.e. Baker [1]).

Let $\alpha_1, \ldots, \alpha_n$ be $n(\geqslant 2)$ non-zero algebraic numbers and let K be the field of degree d generated by $\alpha_1, \ldots, \alpha_n$ over the rationals $\mathbb{Q}$. We denote by p a prime number and by $\mathfrak{p}$ any prime ideal of the ring of integers in K, lying above p. We shall establish estimates for

$$\Xi = \operatorname{ord}_{\mathfrak{p}}(\alpha_1^{b_1} \ldots \alpha_n^{b_n} - 1),$$

where $b_1, \ldots, b_n$ are non-zero rational integers and $\operatorname{ord}_{\mathfrak{p}}$ denotes the exponent to which $\mathfrak{p}$ divides the principal fractional ideal generated by the expression (assumed non-zero) in parentheses. Our result will be in terms of real numbers $h_1, \ldots, h_n$ satisfying $h_1 \leqslant \cdots \leqslant h_n$ and

$$h_j \geqslant \max(h(\alpha_j), |\log \alpha_j|/(2\pi d), \log p) \quad (1 \leqslant j \leqslant n),$$

where $\log \alpha_j$ has its imaginary part in the interval $(-\pi, \pi]$ and $h(\alpha)$ denotes the

*Forschungsinstitut für Mathematik, ETH Zentrum, CH-8092 Zürich, Switzerland.

logarithmic absolute height of α. This is defined by

$$h(\alpha) = \frac{1}{m}\log\left(|a|\prod_{j=1}^{m}\max(1,|\alpha^{(j)}|)\right),$$

where m is the degree of α, a is the leading coefficient of the minimal polynomial of α over the rational integers $\mathbb{Z}$, and $\alpha^{(1)},\ldots,\alpha^{(m)}$ are the conjugates of α. Then as a simple consequence of our main result (see Section 0.2), we have

$$\Xi < \Phi\log(2dB),$$

where B is the maximum of the $|b_j|$ $(1 \leqslant j \leqslant n)$ and

$$\Phi = 7\cdot 10^5(10nd/\sqrt{\log p})^{2(n+1)}p^{d'}h_1\ldots h_n\log(24ndh')$$

with $d' = \max(d,2)$ and $h' = \max(h_n,1)$. When $\mathrm{ord}_p b_n = \min \mathrm{ord}_p b_j$, h' can be replaced by $\max(h_{n-1},1)$. This is the p-adic analogue of Baker's [2] Theorem 2. As a second corollary, analogous with Baker's [1] Sharpening II, we suppose that the above condition on $\mathrm{ord}_p b_n$ is satisfied and h' is modified accordingly; then for any δ with $0 < \delta \leqslant 1$, we have

$$\Xi < \max(\Phi\log(\delta^{-1}\Phi|b_n|/h_n), \delta B/|b_n|).$$

Thus we have overcome all the difficulties associated with the work of [14] – see the discussion in our earlier papers [20], [21] – and except for the minor replacement of p by p^2 in the case $d = 1$, we have established and strengthened all the main assertions (Theorems 1, 3 and 4) given there.

In order to overcome the essential probem in applying the Kummer theory to the final descent in the p-adic case, we introduce a new ingredient into the analytic part of our proof. It is an irreducibility criterion for the polynomial $x^{r^k} - a$, where r is a prime number (see Lemma 1.8), and it is obtained as a consequence of the Vahlen-Capelli Theorem (see Capelli [6] and Rédei [15]). This enables us to construct a new auxiliary function (see the proof of Lemma 2.1), and both the extrapolation and the passage from the Jth step to the $(J+1)$th step in the proof of the main inductive argument depend strongly on this criterion (see the proof of Lemmas 2.3, 2.4 and 2.5).

The research of this paper was partly done at the Max-Planck-Institut für Mathematik Bonn and at the ETH Zürich. The author would like to express his gratitude to the Alexander von Humboldt-Stiftung, the Max-Planck-Institut für Mathematik Bonn, the Forschungsinstitut für Mathematik and the Mathematikdepartement of the ETH Zürich for their support. The author is also very grateful to Professors A. Baker, R. Tijdeman, M. Waldschmidt and G. Wüstholz for valuable discussions. Finally, the author expresses his gratitude to his wife, Dehua Liu, for her assistance in many respects during the course of the work.

0.2 Detailed statements of the main results. Let $\alpha_1, \ldots, \alpha_n$ be $n(\geqslant 2)$ non-zero algebraic numbers and

$$K_0 = \mathbb{Q}(\alpha_1, \ldots, \alpha_n), \quad D_0 = [K_0 : \mathbb{Q}]. \tag{0.1}$$

Let p be a prime number. Set

$$q = \begin{cases} 2, & \text{if } p > 2, \\ 3, & \text{if } p = 2. \end{cases} \tag{0.2}$$

Let K be an algebraic number field of degree D over $\mathbb{Q}$ such that

$$K \supseteq \begin{cases} K_0(\zeta_4), & \text{if } p > 2, \\ K_0(\zeta_3), & \text{if } p = 2 \end{cases} \quad \text{with} \quad \zeta_m = e^{2\pi i/m} \quad (m = 1, 2, \ldots). \tag{0.3}$$

Denote by $\mathfrak{p}$ a prime ideal of the ring of integers in K, lying above p. For $\alpha \in K \backslash \{0\}$, write $\operatorname{ord}_{\mathfrak{p}} \alpha$ for the exponent of $\mathfrak{p}$ in the prime factorization of the fractional ideal (α); define $\operatorname{ord}_{\mathfrak{p}} 0 = \infty$. Denote by $e_{\mathfrak{p}}$ the ramification index of $\mathfrak{p}$ and by $f_{\mathfrak{p}}$ the residue class degree of $\mathfrak{p}$. Write $K_{\mathfrak{p}}$ for the completion of K with respect to the (additive) valuation $\operatorname{ord}_{\mathfrak{p}}$; and the completion of $\operatorname{ord}_{\mathfrak{p}}$ will be denoted again by $\operatorname{ord}_{\mathfrak{p}}$. Now let Σ be an algebraic closure of $\mathbb{Q}_p$. Write $\mathbb{C}_p$ for the completion of Σ with respect to the valuation of Σ, which is the unique extension of the valuation $| \ |_p$ of $\mathbb{Q}_p$. Denote by ord_p the additive form of the valuation on $\mathbb{C}_p$. According to Hasse [9], pp. 298–302, we can embed $K_{\mathfrak{p}}$ into $\mathbb{C}_p$: there exists a $\mathbb{Q}$-isomorphism ψ from K into Σ such that $K_{\mathfrak{p}}$ is value-isomorphic to $\mathbb{Q}_p(\psi(K))$, whence we can identify $K_{\mathfrak{p}}$ with $\mathbb{Q}_p(\psi(K))$. Obviously

$$\operatorname{ord}_{\mathfrak{p}} \beta = e_{\mathfrak{p}} \operatorname{ord}_p \beta \quad \text{for all} \quad \beta \in K_{\mathfrak{p}}.$$

Let $\mathbb{N}$ be the set of non-negative rational integers and define

$$u := \max\{t \in \mathbb{N} \mid \zeta_{q^t} \in K\}, \tag{0.4}$$

$$v := \max\{t \in \mathbb{N} \mid \zeta_{p^t} \in K\}, \tag{0.5}$$

$$\alpha_0 := e^{2\pi i/(p^v q^u)}. \tag{0.6}$$

Set $\mathscr{L}_K := \{l \in \mathbb{C} | e^l \in K\}$. For $l \in \mathscr{L}_K$ define

$$V(l) := \max\left\{h(e^l), \frac{|l|}{2\pi D}, \frac{f_{\mathfrak{p}} \log p}{D}\right\}, \tag{0.7}$$

where $h(\alpha)$ denotes the logarithmic absolute height of an algebraic number α (see, for example, Lang [10], Chapter IV). Let $V_1, \ldots, V_n$ be real numbers satisfying

$$V_1 \leqslant \ldots \leqslant V_n \tag{0.8}$$

and

$$V_j \geqslant V(\log \alpha_j) \quad (1 \leqslant j \leqslant n), \tag{0.9}$$

where and in the sequel $\log \alpha_j = \log|\alpha_j| + i \arg \alpha_j$ with $-\pi < \arg \alpha_j \leqslant \pi$ $(1 \leqslant j \leqslant n)$. Let $b_1, \ldots, b_n \in \mathbb{Z}$, not all zero, and let $B, B_1, \ldots, B_n$ be positive numbers such that

$$B \geqslant \max_{1 \leqslant j \leqslant n} |b_j|, \qquad \max(1, |b_j|) \leqslant B_j \leqslant B \quad (1 \leqslant j \leqslant n). \tag{0.10}$$

Set

$$V = \begin{cases} V_{n-1}, & \text{if } \operatorname{ord}_p b_n = \min_{1 \leqslant j \leqslant n} \operatorname{ord}_p b_j \text{ or } \log \alpha_n \text{ is} \\ & \text{linearly dependent on } \pi i, \log \alpha_1, \ldots, \log \alpha_{n-1} \text{ over } \mathbb{Q}, \\ V_n, & \text{otherwise.} \end{cases} \tag{0.11}$$

Define

$$\sigma = 1/(2q f_{\mathfrak{p}} \log p). \tag{0.12}$$

THEOREM 1. *Suppose that*

$$\operatorname{ord}_{\mathfrak{p}} \alpha_j = 0 \quad (1 \leqslant j \leqslant n) \tag{0.13}$$

and

$$\Theta := (\alpha_1^{b_1} \ldots \alpha_n^{b_n} - 1) \neq 0. \tag{0.14}$$

Then we have

$$\operatorname{ord}_{\wp}\Theta < C_0(n+1)^{n+2}n^{n+\sigma}\cdot\frac{p^{f_\wp}-1}{q^u}\cdot\left(\frac{2+1/(p-1)}{f_\wp\log p}\right)^{n+2}\cdot$$

$$\cdot D^{n+2}V_1\ldots V_n\log(D^2B)\max(n\log(2^{10}qn(n+\sigma)D^2V), f_\wp\log p),$$

where

$$C_0=\begin{cases}404746\cdot 10^n, & \text{if } p>2,\\ 848625\cdot 12^n, & \text{if } p=2.\end{cases}$$

COROLLARY 1. *Suppose that* (0.13) *and* (0.14) *hold. Then*

$$\operatorname{ord}_{\wp}\Theta < C_1(n+1)^{2n+4}\cdot\frac{p^{f_\wp}}{(f_\wp\log p)^{n+2}}\cdot D^{n+2}V_1\ldots V_n\log(D^2B)$$

$$\cdot\max(\log(2^{10}q(n+1)^2D^2V), (f_\wp\log p)/n),$$

where

$$C_1=\begin{cases}56345\cdot(\tfrac{45}{2})^n, & \textit{if } p\equiv 1 \pmod 4,\\ 67587\cdot 25^n, & \textit{if } p\equiv 3 \pmod 4,\\ 273297\cdot 36^n, & \textit{if } p=2.\end{cases}$$

THEOREM 2. *Suppose that* (0.13), (0.14) *hold and*

$$\operatorname{ord}_p b_n = \min_{1\leqslant j\leqslant n}\operatorname{ord}_p b_j. \qquad (0.15)$$

Let

$$\Phi = C_2(n+1)^{2n+3}\frac{p^{f_\wp}}{(f_\wp\log p)^{n+2}}\cdot$$

$$\cdot D^{n+2}V_1\ldots V_n\max(\log(2^{10}qn^2D^2V_{n-1}), (f_\wp\log p)/n) \qquad (0.16)$$

with

$$C_2=\rho' C_1,\qquad \rho'=\begin{cases}1.0752, & \textit{if } p>2,\\ 1.1114, & \textit{if } p=2.\end{cases}$$

Let $Z = \omega\Phi/V_j$ *with*

$$\omega = \begin{cases} \frac{15}{7}, & \text{if } j < n \text{ and } \pi i, \log\alpha_1, \ldots, \log\alpha_n \text{ are linearly independent over } \mathbb{Q}, \\ 1, & \text{otherwise,} \end{cases} \tag{0.17}$$

$$Q = p(10nD)^{2(n+1)}(DV_{n-1})^n. \tag{0.18}$$

Then for any j *with* $1 \leqslant j \leqslant n$ *and any* δ *with* $0 < \delta \leqslant Zf_{\not p}(\log p)/D$, *we have*

$$\operatorname{ord}_{\not p}\Theta < \max(ZV_j \log(\delta^{-1}ZB_jQ), \delta B/B_j). \tag{0.19}$$

When $\alpha_1, \ldots, \alpha_n$ are n ($\geqslant 2$) non-zero rational numbers, the hypothesis (0.13) in Theorems 1, 2 and Corollary 1 may be omitted. For example, Theorem 1 has the following

COROLLARY 2. *Suppose that* (0.14) *holds and*

$$\alpha_j = p_j/q_j \quad \text{with} \quad p_j, q_j \in \mathbb{Z}\setminus\{0\} \quad \text{and} \quad \text{g.c.d.}(p_j, q_j) = 1 \quad (1 \leqslant j \leqslant n).$$

Let $A_1, \ldots, A_n$ *be real numbers such that* $A_1 \leqslant \cdots \leqslant A_n$ *and*

$$A_j \geqslant \max(|p_j|, |q_j|, p) \quad (1 \leqslant j \leqslant n).$$

Set $A = A_{n-1}$ *if* $\operatorname{ord}_p b_n = \min_{1 \leqslant j \leqslant n} \operatorname{ord}_{\not p} b_j$ *or* $\log\alpha_n$ *is linearly dependent on* $\pi i, \log\alpha_1, \ldots, \log\alpha_{n-1}$, *and set* $A = A_n$ *otherwise. Let*

$$C_1^* = \begin{cases} 225380 \cdot 45^n, & \text{if } p \equiv 1 \pmod 4, \\ 67587 \cdot 25^n, & \text{if } p \equiv 3 \pmod 4, \\ 273297 \cdot 36^n, & \text{if } p = 2, \end{cases} \qquad f = \begin{cases} 1, & \text{if } p \equiv 1 \pmod 4, \\ 2, & \text{otherwise.} \end{cases}$$

Then we have

$$\operatorname{ord}_p \Theta < C_1^*(n+1)^{2n+4} \frac{p^f}{(\log p)^{n+2}} \log A_1 \ldots \log A_n \log(4B) \cdot$$
$$\cdot \max(\log(2^{12}q(n+1)^2 \log A), f(\log p)/n).$$

In the general case, the hypothesis (0.13) can also be removed. The following Theorems 1′ and 2′ are the version in terms of the additive valuation on $K_0 = \mathbb{Q}(\alpha_1, \ldots, \alpha_n)$ and without assuming (0.13). Denote by $\not p_0$ any prime ideal of the ring of integers in K_0, lying above p. Let $\operatorname{ord}_{\not p_0}, e_{\not p_0}, f_{\not p_0}$ be defined with respect to the field K_0. Set

$$f_0 = \begin{cases} f_{\mathfrak{p}_0}, & \text{if } p \equiv 1 \pmod 4, \\ \max(f_{\mathfrak{p}_0}, 2), & \text{otherwise.} \end{cases} \tag{0.20}$$

Let $V_1, \ldots, V_n$ be real numbers satisfying $V_1 \leqslant \cdots \leqslant V_n$ and

$$V_j \geqslant \max(h(\alpha_j), |\log \alpha_j|/(12D_0), \tfrac{1}{2}(f_{\mathfrak{p}_0}/D_0)^2 \log p) \quad (1 \leqslant j \leqslant n), \tag{0.21}$$

and let $B, B_1, \ldots, B_n$ and V be defined by (0.10) and (0.11).

THEOREM 1′. *Suppose that* (0.14) *holds. Then we have*

$$\operatorname{ord}_{\mathfrak{p}_0} \Theta < C_1'(n+1)^{2n+4} \frac{p^{f_0}}{(\log p)^{n+2}} (D_0/f_0)^{2n+2} V_1 \ldots V_n \log(4D_0^2 B)$$
$$\cdot \max(\log(2^{13} q(n+1)^2 D_0^3 V), f_0(\log p)/n),$$

where

$$C_1' = \begin{cases} 225380 \cdot 90^n, & \text{if } p \equiv 1 \pmod 4, \\ 270348 \cdot 100^n, & \text{if } p \equiv 3 \pmod 4,\ f_{\mathfrak{p}_0} \geqslant 2, \\ 1093188 \cdot 144^n, & \text{if } p = 2,\ f_{\mathfrak{p}_0} \geqslant 2 \end{cases}$$

and

$$C_1' = \begin{cases} 270348 \cdot 200^n, & \text{if } p \equiv 3 \pmod 4, f_{\mathfrak{p}_0} = 1, \\ 1093188 \cdot 288^n, & \text{if } p = 2, f_{\mathfrak{p}_0} = 1. \end{cases}$$

THEOREM 2′. *Suppose that* (0.14) *and* (0.15) *hold. Let*

$$\Phi = \rho' C_1'(n+1)^{2n+3} \frac{p^{f_0}}{(\log p)^{n+2}} (D_0/f_0)^{2n+2} V_1 \ldots V_n \cdot$$
$$\cdot \max(\log(2^{13} q n^2 D_0^3 V_{n-1}), f_0(\log p)/n)$$

with $\rho' = 1.0752$ *if* $p > 2$ *and* $\rho' = 1.1114$ *if* $p = 2$. *Let*

$$Q = p(20nD_0)^{2(n+1)}(4D_0^2 V_{n-1})^n.$$

Then for any j *with* $1 \leqslant j \leqslant n$ *and any* δ *with* $0 < \delta \leqslant \frac{1}{4}\omega\Phi/(D_0 V_j)$, *we have*

$$\operatorname{ord}_{\mathfrak{p}_0} \Theta < \max(\omega\Phi \log((2\delta)^{-1} \omega\Phi B_j Q/V_j), \delta B/B_j),$$

where ω *is given by* (0.17).

1. Preliminaries

For the basic facts about p-adic exponential and logarithmic functions in $\mathbb{C}_p$, we refer to Hasse [9], pp. 262–274, or Section 1.1 of Yu [21]. We assume that the variable z takes values from $\mathbb{C}_p$. If $\mathrm{ord}_p z \geqslant 0$, we say that z is integral. The following concepts of normal series and functions are due to Mahler [13]. A p-adic power series

$$f(z) = \sum_{h=0}^{\infty} f_h(z - z_0)^h, \qquad f_h \in \mathbb{C}_p \quad (h = 0, 1, \ldots),$$

where $z_0 \in \mathbb{C}_p$ is integral, is called a normal series, if

$$\mathrm{ord}_p f_h \geqslant 0 \ (h = 0, 1, \ldots) \quad \text{and} \quad \mathrm{ord}_p f_h \to \infty \quad (h \to \infty).$$

A p-adic function, which is definable by a normal series in a neighborhood of an integral point in $\mathbb{C}_p$, is called a normal function. For the fundamental properties of normal functions, we refer to Mahler [13].

LEMMA 1.1. *Let $\kappa \in \mathbb{Z}$ be defined by*

$$p^{\kappa-1}(p-1) \leqslant (1 + (p-1)/p)e_{\mathfrak{p}} < p^{\kappa}(p-1) \tag{1.1}$$

and set

$$\theta = \begin{cases} 1, & \text{if } \kappa \geqslant 1 \text{ and } p^{\kappa-1}(p-1) > e_{\mathfrak{p}}, \\ \dfrac{p^{\kappa}}{(2 + 1/(p-1))e_{\mathfrak{p}}}, & \text{otherwise.} \end{cases} \tag{1.2}$$

If $\beta \in \mathbb{C}_p$ satisfies

$$\mathrm{ord}_p(\beta - 1) \geqslant 1/e_{\mathfrak{p}},$$

then

$$\mathrm{ord}_p(\beta^{p^{\kappa}} - 1) > \theta + \frac{1}{p-1}.$$

Proof. This is Lemma 1.2 of Yu [21].

For later references, note that by (1.1) and (1.2) we have

$$\frac{1}{p} < \theta \leqslant 1 \tag{1.3}$$

and

$$\frac{p^\kappa}{e_\wp} \leqslant \frac{p^\kappa}{e_\wp \theta} \leqslant 2 + 1/(p-1). \tag{1.4}$$

LEMMA 1.2. *Suppose that $\theta > 0$ is a rational number, q is a prime number with $q \neq p$, and $M > 0$, $R > 0$ are rational integers with $q|R$. Suppose further that $F(z)$ is a p-adic normal function and*

$$\min_{\substack{1 \leqslant s \leqslant R, (s,q)=1 \\ t=0,\dots,M-1}} \left(\operatorname{ord}_p \frac{F^{(t)}(sp^\theta)}{t!} + t\theta \right)$$

$$\geqslant \left(1 - \frac{1}{q}\right) RM\theta + M \operatorname{ord}_p(R!) + (M-1) \frac{\log R}{\log p}. \tag{1.5}$$

Then for all rational integers k, we have

$$\operatorname{ord}_p F\left(\frac{k}{q} p^\theta\right) \geqslant \left(1 - \frac{1}{q}\right) RM\theta.$$

REMARK. Here $\log R$ and $\log p$ denote the usual logarithms for positive numbers.

Proof. This is Lemma 1.4 of Yu [21].

Let E be an algebraic number field, $\wp'$ be a prime ideal of the ring of integers in E, lying above the prime number p. Let $\operatorname{ord}_{\wp'}, e_{\wp'}, f_{\wp'}$ be defined in the same way as in Section 0.2. For a polynomial P, denote by $L(P)$ its length, i.e. the sum of the absolute values of its coefficients.

LEMMA 1.3. *Suppose that $P(x_1, \dots, x_m) \in \mathbb{Z}[x_1, \dots, x_m]$ satisfies*

$$\deg_{x_j} P \leqslant N_j, \quad 1 \leqslant j \leqslant m.$$

If $\beta_1, \dots, \beta_m \in E$ and $P(\beta_1, \dots, \beta_m) \neq 0$, then

$$\operatorname{ord}_{\wp'} P(\beta_1, \dots, \beta_m) \leqslant \frac{[E:\mathbb{Q}]}{f_{\wp'} \log p} \left(\log L(P) + \sum_{j=1}^{n} N_j h(\beta_j) \right).$$

Proof. This is Lemma 2.1 of Yu [21].

LEMMA 1.4. *Suppose that $\alpha \neq 0$ is an algebraic number in K and $b \in \mathbb{Z}\backslash\{0\}$. If $\alpha^b \neq 1$, then*

$$\operatorname{ord}_\wp(\alpha^b - 1) \leqslant \frac{D}{f_\wp \log p} \{\log(2|b|) + (p^{f_\wp} - 1)(1 + 1/(p-1)) e_\wp h(\alpha)\}.$$

REMARK. Note that here K may be chosen to be any algebraic number field containing α.

Proof. If $\mathrm{ord}_{\mathfrak{p}}\alpha \neq 0$, then it is easily seen that $\mathrm{ord}_{\mathfrak{p}}(\alpha^b - 1) \leqslant 0$; and when α is a root of unity, we have, by Lemma 1.3,

$$\mathrm{ord}_{\mathfrak{p}}(\alpha^b - 1) \leqslant \frac{D}{f_{\mathfrak{p}} \log p} \cdot \log 2.$$

Thus we may assume that $\mathrm{ord}_{\mathfrak{p}}\,\alpha = 0$ and α is not a root of unity. Let s be the least positive integer such that

$$\alpha^s \equiv 1 \pmod{\mathfrak{p}}.$$

Then

$$1 \leqslant s \leqslant p^{f_{\mathfrak{p}}} - 1 \quad \text{and} \quad \mathrm{ord}_p(\alpha^s - 1) \geqslant 1/e_{\mathfrak{p}}. \tag{1.6}$$

By an argument similar to that in the proof of Lemma 1.1 (see Yu [20], p. 418) we see that if $\beta \in \mathbb{C}_p$ satisfies $\mathrm{ord}_p(\beta - 1) \geqslant 1/e_{\mathfrak{p}}$, then

$$\mathrm{ord}_p(\beta^{p^\kappa} - 1) > \frac{1}{p-1}, \tag{1.7}$$

where $\kappa \in \mathbb{Z}$ is defined by the inequality $p^{\kappa-1}(p-1) \leqslant e_{\mathfrak{p}} < p^\kappa(p-1)$, whence

$$p^\kappa \leqslant (1 + 1/(p-1))e_{\mathfrak{p}}. \tag{1.8}$$

On applying (1.7) to α^s, we get

$$\mathrm{ord}_p(\alpha^{sp^\kappa} - 1) > \frac{1}{p-1}.$$

Note that $\alpha^{sp^\kappa} \neq 1$, since α is not a root of unity. By the basic properties of the p-adic exponential and logarithmic functions (see, for example, Yu[21], §1.1) and by Lemma 1.3, (1.6), (1.8), we obtain

$$\begin{aligned}
\mathrm{ord}_p(\alpha^b - 1) &\leqslant \mathrm{ord}_p(\alpha^{bsp^\kappa} - 1) \\
&= \mathrm{ord}_p\{\exp(b \log(\alpha^{sp^\kappa})) - 1\} \\
&= \mathrm{ord}_p(b \log(\alpha^{sp^\kappa})) = \mathrm{ord}_p\, b + \mathrm{ord}_p(\alpha^{sp^\kappa} - 1) \\
&\leqslant \frac{\log|b|}{\log p} + \frac{D}{e_{\mathfrak{p}} f_{\mathfrak{p}} \log p}\{\log 2 + (p^{f_{\mathfrak{p}}} - 1)(1 + 1/(p-1))e_{\mathfrak{p}} h(\alpha)\}.
\end{aligned}$$

On noting the inequality $e_{\not{p}}f_{\not{p}} \leqslant D$, the lemma follows at once.

LEMMA 1.5. *Let* $\beta_1, \ldots, \beta_r \in K$. *Suppose that*

$$P_{ij} \in \mathbb{Z}[x_1, \ldots, x_r] \quad (1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m)$$

(not all zero) satisfy

$$\deg_{x_k} P_{ij} \leqslant N_{jk} \quad (1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m, 1 \leqslant k \leqslant r).$$

Write

$$X = \max_{1 \leqslant j \leqslant m} \left\{ \left(\sum_{i=1}^{n} L(P_{ij}) \right) \exp\left(\sum_{k=1}^{r} N_{jk} h(\beta_k) \right) \right\}$$

and

$$\gamma_{ij} = P_{ij}(\beta_1, \ldots, \beta_r) \quad (1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m).$$

If $n > mD$, *then there exist* $y_1, \ldots, y_n \in \mathbb{Z}$ *with*

$$0 < \max_{1 \leqslant i \leqslant n} |y_i| \leqslant X^{mD/(n-mD)}$$

such that

$$\sum_{i=1}^{n} \gamma_{ij} y_i = 0 \quad (1 \leqslant j \leqslant m).$$

Proof. This is Lemma 2.2 of Yu [21].

Define for $z \in \mathbb{C}$

$$\Delta(z;k) = (z+1)\ldots(z+k)/k! \ (k \in \mathbb{Z}, k \geqslant 1) \quad \text{and} \quad \Delta(z;0) = 1, \tag{1.9}$$

and for $l, m \in \mathbb{N}$

$$\Delta(z;k,l,m) = \frac{1}{m!} \left\{ \frac{d^m}{dy^m} (\Delta(y;k))^l \right\}_{y=z}. \tag{1.10}$$

For every positive integer k, let $v(k)$ be the least common multiple of $1, 2, \ldots, k$.

LEMMA 1.6. *For any $z \in \mathbb{C}$ and any integers $k \geqslant 1, l \geqslant 1, m \geqslant 0$, we have*

$$|\Delta(z;k,l,m)| \leqslant \left(e\cdot\frac{|z|+k}{k}\right)^{kl}. \tag{1.11}$$

Let q be a positive integer, and let x be a rational number such that qx is a positive integer. Then

$$q^{2kl}(v(k))^m \Delta(x;k,l,m) \in \mathbb{Z}, \tag{1.12}$$

and we have

$$v(k) \leqslant 3^k. \tag{1.13}$$

Finally, for any positive integers k, R and L with $k \geqslant R$, the polynomials $(\Delta(z+r;k))^l (r = 0, 1, \ldots, R-1; l = 1, \ldots, L)$ are linearly independent.

Proof. (1.11) is a slight improvement of Lemma 2.4 of Waldschmidt [18] and Lemma 2.3 of Yu [21], and will be proved below. (1.12) is just Lemma T1 of Tijdeman [17]. For a proof of (1.13), see the proof of Lemma 2.3 of Yu [21]. The last assertion of the lemma is just Lemma 4 of Cijsouw and Waldschmidt [7].

To prove (1.11), we may assume $m \leqslant kl$. Thus

$$\Delta(y;k,l,m) = (\Delta(y;k))^l \sum((y+j_1)\ldots(y+j_m))^{-1}, \tag{1.14}$$

where the summation is over all selections $j_1, \ldots, j_m$ of m integers from the set $1, \ldots, k$ repeated l times. Now (1.14) implies that

$$|\Delta(z;k,l,m)| \leqslant \Delta(|z|;k,l,m).$$

Hence it suffices to show that

$$\Delta(x;k,l,m) \leqslant \left(e\cdot\frac{x+k}{k}\right)^{kl}, \quad x \geqslant 0. \tag{1.15}$$

(1.15) is obviously true for $k = 1$, and we may assume $k \geqslant 2$. Write

$$f(x) = \frac{(x+k)^k}{k!}, \quad g(x) = \Delta(x;k).$$

It is easy to see that the polynomial $f(x) - g(x)$ has non-negative coefficients, whence so does the polynomial $(f(x))^l - (g(x))^l$, because of $f^l - g^l =$

$(f-g)(f^{l-1}+f^{l-2}g+\cdots+g^{l-1})$. By this observation we get

$$\frac{1}{m!}\frac{d^m}{dx^m}(f(x))^l-\Delta(x;k,l,m)\geqslant 0,\quad x\geqslant 0.$$

Thus to prove (1.15) it suffices to show that

$$\frac{1}{m!}\frac{d^m}{dx^m}(f(x))^l\leqslant\left(e\cdot\frac{x+k}{k}\right)^{kl},\quad x\geqslant 0,k\geqslant 2. \tag{1.16}$$

For $x\geqslant 0$, we have

$$\frac{d^m}{dx^m}(x+k)^{kl}=\frac{kl}{xl+kl}\cdots\frac{(kl-m+1)}{xl+kl}l^m(x+k)^{kl}\leqslant l^m(x+k)^{kl}.$$

From this and the inequality $k!>(2\pi k)^{1/2}k^ke^{-k}$ (see Yu [21], Lemma 2.7) we obtain, for $x\geqslant 0$ and $k\geqslant 2$,

$$\frac{1}{m!}\frac{d^m}{dx^m}(f(x))^l\leqslant\frac{l^m}{m!}\frac{(x+k)^{kl}}{(k!)^l}<e^l\left(\frac{e^k}{(2\pi k)^{1/2}k^k}\right)^l(x+k)^{kl}\leqslant\left(e\cdot\frac{x+k}{k}\right)^{kl}.$$

This is just (1.16), whence the proof of (1.11) is complete.

Let B', B_n be positive numbers, $T, L_1,\ldots,L_n$ $(n\geqslant 2)$ be positive integers. Set $L'=\max_{1\leqslant j<n}L_j$.

LEMMA 1.7. *Suppose that* $b_1,\ldots,b_n,\lambda_1,\ldots,\lambda_n,\tau_1,\ldots,\tau_{n-1}$ *are rational integers satisfying*

$$|b_j|\leqslant B'\quad(1\leqslant j<n),\qquad |b_n|\leqslant B_n,\qquad 0\leqslant\lambda_j\leqslant L_j\quad(1\leqslant j\leqslant n),$$
$$\tau_j\geqslant 0\quad(1\leqslant j<n),\quad \tau_1+\cdots+\tau_{n-1}\leqslant T.$$

Then

$$\prod_{j=1}^{n-1}|\Delta(b_n\lambda_j-b_j\lambda_n;\tau_j)|\leqslant e^T\cdot\left(1+\frac{(n-1)(B_nL'+B'L_n)}{T}\right)^T.$$

Proof. This is Lemma 2.4 of Yu [21], which is a slight improvement of a Lemma in Loxton, Mignotte, van der Poorten and Waldschmidt [12].

For a field E and a positive integer h, write $E^h=\{a^h\,|\,a\in E\}$.

LEMMA 1.8. *Let r be a prime number, k a positive integer, and E a field. When*

$r = 2$, we suppose further that $-1 \in E^2$. If $a \in E$ and $a \notin E^r$, then the polynomial

$$x^{r^k} - a$$

is irreducible in $E[x]$.

Proof. This is a simple consequence of the following

VAHLEN-CAPELLI THEOREM: *Over a field F a polynomial*

$$x^n - \alpha \quad (n \geqslant 2; \alpha \in F, \neq 0)$$

is reducible if, and only if,

$$\alpha = \beta^d \quad (d \mid n, > 1; \beta \in F)$$

or

$$4 \mid n, \qquad \alpha = -4\gamma^4 \quad (\gamma \in F).$$

(For a proof see Capelli [6] (when F is a number field) and Rédei [15], pp. 675–679 for the general case.)

LEMMA 1.9. *Let $\alpha_1, \ldots, \alpha_n$ be non-zero elements of an algebraic number field K and let $\alpha_1^{1/p}, \ldots, \alpha_n^{1/p}$ denote fixed pth roots for some prime p. Further let $K' = K(\alpha_1^{1/p}, \ldots, \alpha_{n-1}^{1/p})$. Then either $K'(\alpha_n^{1/p})$ is an extension of K' of degree p or we have*

$$\alpha_n = \alpha_1^{j_1} \ldots \alpha_{n-1}^{j_{n-1}} \gamma^p$$

for some γ in K and some integers $j_1, \ldots, j_{n-1}$ with $0 \leqslant j_r < p$.

Proof. This is a lemma of Baker and Stark [4].

LEMMA 1.10. *Let α be a non-zero algebraic integer of degree d with conjugates $\alpha_1 = \alpha, \alpha_2, \ldots, \alpha_d$. Set $\overline{|\alpha|} = \max_{1 \leqslant j \leqslant d} |\alpha_j|$. If α is not a root of unity, then*

$$\log \overline{|\alpha|} > \frac{1}{2d^2}.$$

Proof. The lemma holds for $d = 1$, since $\log \overline{|\alpha|} \geqslant \log 2 > \frac{1}{2}$. By a result of Dobrowolski [8], which states that

$$\log \overline{|\alpha|} > \frac{\log d}{6d^2},$$

the lemma is valid for $d \geqslant 21$, since $\log d \geqslant \log 21 > 3$. Thus we may assume that $2 \leqslant d \leqslant 20$ in the sequel. By Smyth [16], we see that if α is not reciprocal, then

$$\log \lceil \alpha \rceil \geqslant \frac{\log \theta_0}{d} > \frac{1}{2d^2},$$

where $\theta_0 = 1.324\ldots$ is the real root of $x^3 - x - 1 = 0$. For α reciprocal we see that the lemma holds for $d = 2, 4, \ldots, 16$, in virtue of a result of Boyd [5]. It remains to verify that the lemma holds for $d = 18, 20$. Obviously $p = 61$ is a prime satisfying

$$3d < p < 5d \quad \text{for } d = 18, 20.$$

On replacing 6 by 5 in the proof of Dobrowolski [8], we conclude that

$$\log \lceil \alpha \rceil > \frac{\log d}{5d^2} > \frac{1}{2d^2} \quad \text{for } d = 18,\ 20.$$

This completes the proof of the lemma.

LEMMA 1.11. *Let K be a number field of degree D over* $\mathbb{Q}$, *and* $l_1, \ldots, l_m$ *linearly dependent (over* $\mathbb{Q}$) *elements of* $\mathscr{L}_K$. *Then there exist* $t_1, \ldots, t_m \in \mathbb{Z}$, *not all zero, such that*

$$t_1 l_1 + \cdots + t_m l_m = 0$$

and

$$|t_k| \leqslant (2(m-1)D^3)^{m-1} V_1 \ldots V_m / V_k \quad (1 \leqslant k \leqslant m),$$

where $V_1, \ldots, V_m$ *are positive numbers satisfying*

$$V_j \geqslant \max\left(h(e^{l_j}), \frac{|l_j|}{2\pi D} \right) \quad (1 \leqslant j \leqslant m).$$

Proof. This is a slight improvement of Lemma 4.1 of Waldschmidt [18]. By virtue of Lemma 1.10, we may replace $C_0(D) = 9D^2$ in the proof of Lemma 4.1 in [18] by $C_0(D) = 2D^2$, and the lemma follows at once.

LEMMA 1.12. *Let K and* $f_{\wp}$ *be defined in Section* 0.2. *If* $p = 2$ *or* $p \equiv 3 \pmod 4$, *then* $f_{\wp} \geqslant 2$.

Proof. By (0.3) we may assume

$$K = \begin{cases} \mathbb{Q}(\zeta_4), & \text{if } p > 2, \\ \mathbb{Q}(\zeta_3), & \text{if } p = 2. \end{cases}$$

Now the conclusion of the lemma follows immediately from Lemma A in the Appendix, where we take $K_0 = \mathbb{Q}$.

We record two simple inequalities for later references. For any real number $\sigma > 0$ and integer $m \geqslant 2$, we have

$$\prod_{j=2}^{m} (j + \sigma) = m! \exp\left(\sum_{j=2}^{m} \log\left(1 + \frac{\sigma}{j}\right)\right) \leqslant m! \exp\left(\sigma \sum_{j=2}^{m} \frac{1}{j}\right) \leqslant m! m^{\sigma}. \tag{1.17}$$

Secondly, it is easy to verify that

$$\log\left(\frac{4(x-1)}{\log x}\right) \geqslant \tfrac{1}{6}\log x \quad \text{for } x \geqslant 3. \tag{1.18}$$

2. Results subject to a new Kummer condition.

Let p be a prime number, K be an algebraic number field of degree D over $\mathbb{Q}$ such that

$$\zeta_4 \in K \quad \text{if } p > 2 \qquad \text{and} \qquad \zeta_3 \in K \quad \text{if } p = 2. \tag{2.1}$$

Denote by $\mathfrak{p}$ a prime ideal of the ring of integers in K, lying above p. Let $\mathrm{ord}_{\mathfrak{p}}, e_{\mathfrak{p}}, f_{\mathfrak{p}}$ be defined as in Section 0.2. We have, by Lemma 1.12,

$$f_{\mathfrak{p}} \geqslant 2 \quad \text{if} \quad p = 2 \qquad \text{or} \qquad p \equiv 3 \pmod 4. \tag{2.2}$$

Let q, u, v, α_0 be defined as follows

$$q = \begin{cases} 2, & \text{if } p > 2, \\ 3, & \text{if } p = 2, \end{cases} \tag{2.3}$$

$$u = \max\{t \in \mathbb{N} \mid \zeta_{q^t} \in K\}, \tag{2.4}$$

$$v = \max\{t \in \mathbb{N} \mid \zeta_{p^t} \in K\}, \tag{2.5}$$

$$\alpha_0 = e^{2\pi i/(p^v q^u)}. \tag{2.6}$$

Thus $\alpha_0 \in K$ and

$$q^u \leqslant 2D \quad \text{if } p > 2; \qquad q^u \leqslant \tfrac{3}{2}D \quad \text{if } p = 2. \tag{2.7}$$

Suppose that $\alpha_1, \ldots, \alpha_n \in K$ ($n \geqslant 2$) and $V_1, \ldots, V_n, V_{n-1}^*$ are real numbers such that

$$V_j \geqslant \max\left(h(\alpha_j), \frac{f_{\mathfrak{p}} \log p}{D}\right) \quad (1 \leqslant j \leqslant n), \tag{2.8}$$

$$V_1 \leqslant \cdots \leqslant V_{n-1}, \tag{2.9}$$

$$V_{n-1}^* = \max(p^{f_{\mathfrak{p}}}, (2^{11} qnD^2 V_{n-1})^n). \tag{2.10}$$

Let $b_1, \ldots, b_n \in \mathbb{Z}$, not all zero, B, B', B_n, B_0, W, W^* be positive numbers such that

$$B \geqslant \max_{1 \leqslant j \leqslant n} |b_j|, \quad B' \geqslant \max_{1 \leqslant j < n} |b_j|, \quad B_n \geqslant |b_n|, \quad B_0 \geqslant \min_{1 \leqslant j \leqslant n, b_j \neq 0} |b_j|, \tag{2.11}$$

$$W \geqslant \max\left\{\log\left(1 + \frac{1}{\rho n} \cdot \frac{f_{\mathfrak{p}} \log p}{D}\left(\frac{B_n}{V_1} + \frac{B'}{V_n}\right)\right), \rho'' \log B_0, \frac{f_{\mathfrak{p}} \log p}{D}\right\}, \tag{2.12}$$

where

$$\rho = \begin{cases} \frac{8}{3}, & \text{if } p = 2, \\ 5, & \text{if } p > 2 \end{cases} \quad \text{and} \quad \rho'' = \begin{cases} 1, & \text{if } p | b_n, \\ 0, & \text{otherwise,} \end{cases}$$

$$W^* \geqslant \max(W, n \log(2^{11} qnD)). \tag{2.13}$$

In this section we shall prove the following Theorems and Corollaries.

THEOREM 2.1. *Suppose that*

$$\mathbb{Q}(\alpha_0, \alpha_1, \ldots, \alpha_n) = K, \tag{2.14}$$

$$[K(\alpha_0^{1/q}, \alpha_1^{1/q}, \ldots, \alpha_n^{1/q}) : K] = q^{n+1}, \tag{2.15}$$

$$\operatorname{ord}_{\mathfrak{p}} \alpha_j = 0 \quad (1 \leqslant j \leqslant n), \tag{2.16}$$

$$\operatorname{ord}_p b_n = \min_{1 \leqslant j \leqslant n} \operatorname{ord}_p b_j \tag{2.17}$$

and

$$\Theta := (\alpha_1^{b_1} \ldots \alpha_n^{b_n} - 1) \neq 0, \tag{2.18}$$

then

$$\operatorname{ord}_{\wp}\Theta < ca^n\cdot\frac{(n+1)^{n+1}n^n}{n!}\cdot q^{2n+1}(q-1)\cdot$$
$$\cdot\frac{p^{f_\wp}-1}{q^u}\cdot\left(\frac{2+1/(p-1)}{f_\wp\log p}\right)^{n+2}\cdot D^{n+2}V_1\dots V_nW^*\log V_{n-1}^*,$$

where a, c are constants given by the following tables

$p = 2$

n	$2 \leqslant n \leqslant 7$	$n \geqslant 8$
a	$\frac{18}{5}$	$\frac{8}{3}$
c	6803.1852	70718.74

$p > 2$

n	$2 \leqslant n \leqslant 5$	$n = 6, 7$	$n \geqslant 8$
a	7	$\frac{27}{4}$	5
c	14016.196	12314.974	101186.36

REMARK. Here $\alpha_0^{1/q},\dots,\alpha_n^{1/q}$ are fixed qth roots in $\mathbb{C}_p$. If (2.15) holds for a choice of qth roots in $\mathbb{C}_p$, then it holds for any choice of qth roots in $\mathbb{C}_p$, since K contains qth roots of unity by (2.1) and (2.3). In the proof of Theorem 2.1, the choice of qth roots will be fixed by (2.23) and (2.25).

THEOREM 2.2. *In Theorem* 2.1, (2.14) *may be omitted.*

COROLLARY 2.3. *Suppose that* (2.15)–(2.18) *hold. Then we have*

$$\operatorname{ord}_{\wp}\Theta < c'(a')^n\cdot\frac{(n+1)^{n+1}n^n}{n!}\cdot\frac{p^{f_\wp}-1}{q^u}\cdot\left(\frac{2+1/(p-1)}{f_\wp\log p}\right)^{n+2}\cdot$$
$$\cdot D^{n+2}V_1\dots V_n\max\left(\log B, n\log(2^{11}qnD), \frac{f_\wp\log p}{D}\right)\cdot$$
$$\cdot\max(n\log(2^{11}qnD^2V_{n-1}), f_\wp\log p),$$

where

$$a' = \begin{cases} 20, & \text{if } p > 2,\\ 24, & \text{if } p = 2,\end{cases}\qquad c' = \begin{cases} 202373, & \text{if } p > 2,\\ 424312.44, & \text{if } p = 2.\end{cases}$$

COROLLARY 2.4. *Let Z', Z, δ, W' be positive numbers satisfying*

$$Z' \geqslant c'(a')^n \cdot \frac{(n+1)^{n+1} n^n}{n!} \cdot \frac{p^{f_{\mathfrak{p}}} - 1}{q^u} \cdot \left(\frac{2 + 1/(p-1)}{f_{\mathfrak{p}} \log p} \right)^{n+2} \cdot$$

$$\cdot D^{n+2} V_1 \ldots V_{n-1} \max(n \log(2^{11} qnD^2 V_{n-1}), f_{\mathfrak{p}} \log p),$$

$$0 < \delta \leqslant \frac{f_{\mathfrak{p}} \log p}{D} Z,$$

$$W' \geqslant \max\left\{ \rho' \log\left(\delta^{-1} \frac{f_{\mathfrak{p}} \log p}{D} Z B_n \right), n \log(2^{11} qnD), \frac{f_{\mathfrak{p}} \log p}{D} \right\},$$

where a', c' are given in Corollary 2.3 *and*

$$\rho' = \begin{cases} 1.0752, & \text{if } p > 2, \\ 1.1114, & \text{if } p = 2. \end{cases}$$

Suppose that (2.15)–(2.18) *hold. Then*

$$\mathrm{ord}_{\mathfrak{p}} \Theta < \max\left(Z' W' V_n, \frac{\delta B}{B_n} \cdot \frac{Z'}{Z} \right).$$

Write

$$G = p^{f_{\mathfrak{p}}} - 1. \tag{2.19}$$

By Hasse [9], p. 220 and (2.3), (2.4), we see that

$$q^u | G.$$

Let μ be the order to which q divides G, and let G_0, G_1 be the integers such that

$$G = q^u G_0 = q^\mu G_1. \tag{2.20}$$

Denote by ζ a fixed Gth primitive root of unity in $K_{\mathfrak{p}}$ such that

$$\zeta^{G_0} = \zeta_{q^u} (= \alpha_0^{p^v}) \tag{2.21}$$

and by ξ a fixed qGth primitive root of unity in $\mathbb{C}_p$ such that

$$\xi^q = \zeta. \tag{2.22}$$

By (2.21), (2.22), we can fix a qth root $\alpha_0^{1/q}\in\mathbb{C}$ such that

$$\xi^{G_0} = (\alpha_0^{1/q})^{p^v}. \tag{2.23}$$

By (2.16) and Lemma 1.3 of Yu [21], there exist $r'_1,\ldots,r'_n\in\mathbb{Z}$ such that $\mathrm{ord}_p(\alpha_j\zeta^{r'_j}-1)\geqslant 1/e_{\mathfrak{p}}(1\leqslant j\leqslant n)$. Let $r_1,\ldots,r_n\in\mathbb{Z}$ be such that

$$r_j\equiv p^\kappa r'_j \ (\mathrm{mod}\ G), \qquad 0\leqslant r_j<G \quad (1\leqslant j\leqslant n),$$

then, by Lemma 1.1,

$$\mathrm{ord}_p(\alpha_j^{p^\kappa}\zeta^{r_j}-1)>\theta+\frac{1}{p-1} \quad (1\leqslant j\leqslant n), \tag{2.24}$$

where κ,θ are defined by (1.1) and (1.2). By (2.24) and (2.3) we see that

$$(\alpha_j^{p^\kappa}\zeta^{r_j})^{1/q}:=\exp\left(\frac{1}{q}\log(\alpha_j^{p^\kappa}\zeta^{r_j})\right) \quad (1\leqslant j\leqslant n),$$

where the logarithmic and exponential functions are p-adic functions, are well defined. Furthermore it is easy to verify that there exist qth roots $\alpha_1^{1/q},\ldots,\alpha_n^{1/q}\in\mathbb{C}_p$ such that

$$(\alpha_j^{p^\kappa}\zeta^{r_j})^{1/q}=(\alpha_j^{1/q})^{p^\kappa}\xi^{r_j} \quad (1\leqslant j\leqslant n). \tag{2.25}$$

2.1. The statement of a proposition towards the proof of Theorem 2.1

We define $h_0,\ldots,h_8,\varepsilon_1,\varepsilon_2,\eta$ by the following formulae, which will be referred as (2.26).

$$h_0=n\log(2^{11}qnD),$$

$$h_1=c_0c_4c_2^n\cdot\frac{(n+1)^{n+1}n^n}{n!}\cdot q^n(q-1)f_{\mathfrak{p}}\left(2+\frac{1}{p-1}\right)^n,$$

$$h_2=h_1\left(c_2n(n+1)q\left(2+\frac{1}{p-1}\right)\right)^{-1}, \quad 1+\varepsilon_1=\left(1-\frac{1}{h_2}\right)^{-n},$$

$$h_3=(h_1-1)/n^2, \quad 1+\varepsilon_2=e^{1/h_3},$$

$$h_4=c_0c_3c_2^n\cdot\frac{(n+1)^{n+1}n^n}{n!}\cdot q^n(q-1)\cdot\frac{D}{q^u}\cdot\left(2+\frac{1}{p-1}\right)^n\cdot\frac{h_0}{h_0+1},$$

$$h_5 = \frac{(1+\varepsilon_1)(1+\varepsilon_2)c_0c_4}{\sqrt{2\pi n}\left(1-\dfrac{1}{c_3(n+1)}\right)q^u},$$

$$h_6 = c_0c_1c_2^nc_3c_4\cdot\frac{(n+1)^{n+1}n^{n-1}}{n!}\cdot q^n(q-1)\cdot\frac{D}{q^u}\cdot\left(2+\frac{1}{p-1}\right)^n h_0,$$

$$h_7^{-1} = 6.17\times 10^{-12}\cdot\frac{D}{nh_6q^u}+(n+1)\frac{\log(nh_0h_6)}{nh_0h_6},$$

$$h_8 = c_2n(q-1)\left(1-\frac{1}{c_3(n+1)}\right)\left(1-\frac{1}{h_1}\right)\left(1+\frac{1}{p-1}\right),$$

$$\eta = \begin{cases}1/14, & \text{if } p>2,\\ 0.108672, & \text{if } p=2.\end{cases} \tag{2.26}$$

In Section 2.1–2.5, we suppose that c_0, c_1, c_2, c_3, c_4 are real numbers satisfying the following conditions (2.27)–(2.29):

If $p = 2$, $n \geqslant 8$, then

$$c_0 = 17,\quad 16/9 \leqslant c_1 \leqslant 7/2,\quad c_2 = 8/3,\quad 64 \leqslant c_3 \leqslant 128,\quad 128 \leqslant c_4 \leqslant 256;$$

if $p = 2$, $2 \leqslant n \leqslant 7$, then

$$c_0 = 9,\quad 16/9 \leqslant c_1 \leqslant 3,\quad c_2 = 18/5,\quad 32 \leqslant c_3 \leqslant 64,\quad 64 \leqslant c_4 \leqslant 120;$$

if $p > 2, n \geqslant 8$, then

$$c_0 = 9,\quad 16/9 \leqslant c_1 \leqslant 3,\quad c_2 = 18/5,\quad 32 \leqslant c_3 \leqslant 64,\quad 64 \leqslant c_4 \leqslant 120;$$

if $p > 2, 2 \leqslant n \leqslant 5$, then

$$c_0 = 9,\quad 4/5 \leqslant c_1 \leqslant 11/10,\quad c_2 = 7,\quad 60 \leqslant c_3 \leqslant 80,\quad 64 \leqslant c_4 \leqslant 120;$$

if $p > 2, n = 6, 7$, then

$$c_0 = 9,\quad 3/4 \leqslant c_1 \leqslant 11/10,\quad c_2 = 27/4,\quad 56 \leqslant c_3 \leqslant 80,\quad 64 \leqslant c_4 \leqslant 120. \tag{2.27}$$

$$\left(1-\frac{1}{q}\right)^2\left(1-\frac{1}{c_3(n+1)}\right)\left(1-\frac{1}{h_1}\right)$$
$$\geqslant \left(1+\frac{1}{c_0-1}\right)\left(\frac{1}{h_6}+\frac{1}{h_7}\right)c_1 + \left(1+\frac{1}{c_0-1}\right)\frac{1}{c_2}+$$
$$+\left(\frac{1}{q}+\frac{1}{c_0-1}\right)\left(\log 3\cdot\left(1+\frac{1}{h_0}\right)+1\right)\left(2+\frac{1}{p-1}\right)\frac{1}{c_3}+$$

$$+\left(1+\frac{1}{h_4}\right)\left\{\left(1+\frac{1}{c_0-1}\right)\frac{1}{n}+4+\frac{1}{2^{10}qnD}+\frac{2\log h_5}{h_0}+\right.$$

$$\left.+\left(1+\frac{1}{p-1}\right)\frac{1}{q^{n+1}f_{\mathfrak{p}}}\right\}\frac{1}{c_4}. \tag{2.28}$$

$$c_1 \geqslant \left(1+\frac{1}{h_8}\right)\left(1-\frac{1}{q}\right)\left(2+\frac{1}{p-1}-\frac{1}{q}\right)+$$

$$+\frac{2+1/(p-1)}{q^{n+1}}\cdot\frac{1}{c_3}\left\{\frac{\log(h_0+1)}{h_0}+\frac{1}{n+1}\cdot\right.$$

$$\left.\cdot\left(1-\frac{1}{q}\right)\left(1+\frac{1}{n}+\frac{1}{h_8}+\frac{\log h_0}{h_0}+\frac{\log q}{qh_0}+\eta\right)\right\}. \tag{2.29}$$

Set

$$U=(1+\varepsilon_1)(1+\varepsilon_2)c_0c_1c_2^nc_3c_4\cdot\frac{(n+1)^{n+1}n^n}{n!}q^{2n+1}(q-1)\frac{p^{f_{\mathfrak{p}}}-1}{q^u}\cdot$$

$$\cdot\frac{(2+1/(p-1))^n}{e_{\mathfrak{p}}(f_{\mathfrak{p}}\log p)^{n+2}}\cdot D^{n+2}V_1\ldots V_nW^*\log V_{n-1}^*. \tag{2.30}$$

PROPOSITION 2.1. *Suppose that* (2.14)–(2.18) *hold. Then*

$$\operatorname{ord}_p\Theta < U.$$

2.2. Notations

The following formulae will be referred as (2.31).

$$Y=\frac{e_{\mathfrak{p}}f_{\mathfrak{p}}\log p}{q^{n+1}D}\cdot U,$$

$$S=q\left[\frac{c_3(n+1)DW^*}{f_{\mathfrak{p}}\log p}\right],$$

$$T=\left[\frac{f_{\mathfrak{p}}\log p}{q^{n+1}D}\cdot\frac{U}{c_1c_3W^*\theta}\right],$$

$$L_{-1}=[W^*],$$

$$L_0=\left[\frac{Y}{c_1c_4(L_{-1}+1)\log V_{n-1}^*}\right],$$

$$L_j = \left[\frac{Y}{c_1 c_2 n p^\kappa S V_j}\right] \quad (1 \leqslant j \leqslant n),$$

$$X_0 = D \prod_{j=-1}^{n} (L_j + 1) \cdot 3^{T(L_{-1}+1)} \left(e\left(2 + \frac{S}{L_{-1}+1}\right)\right)^{(L_{-1}+1)(L_0+1)} \cdot$$

$$\cdot \left(1 + \frac{(n-1)(B_n L_1 + B' L_n)}{T}\right)^T \cdot$$

$$\cdot \exp\left(p^\kappa S \sum_{j=1}^{n} L_j V_j + nD \max_{1 \leqslant j \leqslant n} V_j\right). \tag{2.31}$$

For later covenience we need the following inequalities (2.32)–(2.47).

$$(L_{-1}+1)(L_0+1) \prod_{j=1}^{n} (L_j + 1 - G_0) \geqslant c_0 G_0 \left(1 - \frac{1}{q}\right) S \binom{T+n}{n}, \tag{2.32}$$

(note that, by (2.19), (2.20), $G_0 = (p^{f_\mathfrak{p}} - 1)/q^u$.)

$$\frac{1}{n+1} q^n S T \theta > \left(1 - \frac{1}{c_3(n+1)}\right)\left(1 - \frac{1}{h_1}\right) \frac{1}{c_1} U, \tag{2.33}$$

$$p^\kappa S \sum_{j=1}^{n} L_j V_j \leqslant \frac{1}{c_1 c_2} Y, \tag{2.34}$$

$$T(L_{-1}+1) \leqslant \left(1 + \frac{1}{h_0}\right)\left(2 + \frac{1}{p-1}\right) \frac{1}{c_1 c_3} Y, \tag{2.35}$$

$$T \log\left(1 + \frac{(n-1)q(B_n L_1 + B' L_n)}{T}\right) \leqslant \left(2 + \frac{1}{p-1}\right) \frac{1}{c_1 c_3} Y, \tag{2.36}$$

$$(L_{-1}+1)(L_0+1)\left(\theta + \frac{1}{p-1}\right)$$

$$\leqslant \left(1 + \frac{1}{h_4}\right)\left(1 + \frac{1}{p-1}\right) \frac{1}{q^{n+1} f_\mathfrak{p}} \cdot \frac{1}{c_1 c_4} U, \tag{2.37}$$

$$(L_{-1}+1)(L_0+1) \log\left(e\left(2 + \frac{S}{L_{-1}+1}\right)\right) \leqslant \left(1 + \frac{1}{h_4}\right) \frac{1}{n} \cdot \frac{1}{c_1 c_4} Y, \tag{2.38}$$

$$(L_{-1}+1)(L_0+1) \log(q L_n)$$

$$\leqslant \left(1 + \frac{1}{h_4}\right)\left(2 + \frac{1}{2^{11} q n D} + \frac{\log h_5}{h_0}\right) \frac{1}{c_1 c_4} Y, \tag{2.39}$$

$$nD \max_{1 \leqslant j \leqslant n} V_j \leqslant \frac{1}{h_6} Y, \tag{2.40}$$

$$\log\left(D \prod_{j=-1}^{n} (L_j + 1)\right) \leqslant \frac{1}{h_7} Y, \tag{2.41}$$

$$\frac{T \log(L_{-1} + 1)}{\log p} \leqslant \frac{\log(h_0 + 1)}{h_0} \cdot \frac{2 + 1/(p-1)}{q^{n+1}} \cdot \frac{1}{c_1 c_3} U. \tag{2.42}$$

In (2.43)–(2.45), J, k are rational integers with $0 \leqslant J \leqslant [\log L_n / \log q], 0 \leqslant k \leqslant n, \eta$ is given in (2.26).

$$\left(\left(1 - \frac{1}{q}\right) \frac{1}{n+1} q^{-J} T + 1\right) \operatorname{ord}_p b_n$$
$$< \left(1 - \frac{1}{q}\right)\left(1 + \frac{1}{h_8}\right) \frac{2 + 1/(p-1)}{(n+1)q^{n+1}} \cdot \frac{1}{c_1 c_3} U, \tag{2.43}$$

$$\left(\left(1 - \frac{1}{q}\right) \frac{1}{n+1} q^{-J} T + 1\right) q^{J+k} S \left(\left(1 - \frac{1}{q}\right)\theta + \frac{1}{p-1}\right)$$
$$< \left(1 - \frac{1}{q}\right)\left(1 + \frac{1}{h_8}\right)\left(2 + \frac{1}{p-1} - \frac{1}{q}\right) \frac{1}{c_1} U, \tag{2.44}$$

$$\left(1 - \frac{1}{q}\right) \frac{1}{n+1} q^{-J} T \cdot \frac{\log(q^{J+k} S)}{\log p}$$
$$\leqslant \left(1 - \frac{1}{q}\right)\left(\frac{1}{n} + \frac{\log h_0}{h_0} + \frac{\log q}{q h_0} + \eta\right) \frac{2 + 1/(p-1)}{(n+1)q^{n+1}} \cdot \frac{1}{c_1 c_3} U, \tag{2.45}$$

$$L_1 + \cdots + L_{n-1} < \tfrac{1}{2} T, \tag{2.46}$$

$$(L_{-1} + 1)(L_0 + 1) < \tfrac{1}{4} S T. \tag{2.47}$$

Proof of (2.32). Similar to the proof of (3.12) of Yu [21]. Note that we use (1.3), (1.4) and the fact $e_{\mathfrak{p}} f_{\mathfrak{p}} \leqslant D$ to show

$$\prod_{j=1}^{n} \left(1 - \frac{G_0 c_1 c_2 n p^{\kappa} S V_j}{Y}\right) \geqslant \frac{1}{1 + \varepsilon_1}$$

and

$$\left(1 + \frac{n}{T}\right)^n \leqslant 1 + \varepsilon_2.$$

Proof of (2.33)–(2.37). Similar to the proof of (3.13)–(3.17) of [21].

Proof of (2.38). Similar to the proof of (3.18) of [21]. We use the inequality $c_3 \leqslant 160$ (see (2.27)) to show

$$\log\left(e\left(2 + \frac{S}{L_{-1} + 1}\right)\right) \leqslant \log(2^{11}qnD) \leqslant \frac{1}{n}\log V_{n-1}^*.$$

Proof of (2.39). From (2.31) and the definition of h_4 (see (2.26)) we get

$$(L_{-1} + 1)(L_0 + 1) \leqslant \left(1 + \frac{1}{h_4}\right)\frac{1}{c_1 c_4} Y \cdot \frac{1}{\log V_{n-1}^*}.$$

Thus to prove (2.39) it suffices to show

$$qL_n \leqslant h_5(V_{n-1}^*)^{2 + 1/(2^{11}qnD)}, \tag{2.48}$$

since $\log V_{n-1}^* \geqslant h_0$. Now by (2.30), (2.31) we have

$$\begin{aligned} qL_n &\leqslant h_5\left(\frac{(n+1)e}{n}\right)^n q(q-1)\cdot \\ &\quad \cdot \frac{(2 + 1/(p-1))^n}{p^\kappa (f_\mathfrak{p} \log p)^n}(c_2 qn)^{n-1} D^n V_1 \dots V_{n-1} G \log V_{n-1}^* \\ &\leqslant h_5(\tfrac{3}{2}e)^{2(n-1)} q(q-1)\cdot \\ &\quad \cdot \left(\frac{2 + 1/(p-1)}{f_\mathfrak{p} \log p}\right)^n (c_2 qnD^2 V_{n-1})^{n-1} G \log V_{n-1}^*. \end{aligned} \tag{2.49}$$

By (2.3), (2.2) it is easy to verify that

$$q(q-1)\left(\frac{2 + 1/(p-1)}{f_\mathfrak{p} \log p}\right)^n \leqslant \begin{cases} (6 \times (\frac{3}{\log 4})^2)^{n-1}, & \text{if } p = 2, \\ (2 \times (\frac{5}{2})^2)^{n-1}, & \text{if } p > 2. \end{cases}$$

On combining this and (2.27), (2.10), (2.49), we obtain

$$qL_n \leqslant h_5(2^{11}qnD^2 V_{n-1})^{n-1} G \log V_{n-1}^* \leqslant h_5(V_{n-1}^*)^{2 - 1/n} \log V_{n-1}^*.$$

Now this inequality and the following inequality

$$\log V_{n-1}^* \leqslant (V_{n-1}^*)^{(1+\eta')/n} \quad \text{with} \quad \eta' = \frac{1}{2^{11}qD},$$

which can be verified similarly as in the proof of (3.19) of [21], yield (2.48) at once.

Proof of (2.40), (2.42)–(2.44), (2.46), (2.47). Similar to the proof of (3.20), (3.22)–(3.24), (3.26), (3.27) of [21].

Proof of (2.41). Similar to the proof of (3.21) in [21]. Here we need (2.7).

Proof of (2.45). Similar to the proof of (3.25) in [21]. We need to use the definition of η in (2.26), from which it follows that

$$n \log q \leqslant \eta h_0 \leqslant \eta W^*.$$

So far we have established the inequalities (2.32)–(2.47). Now we introduce some more notations. For $(J, \lambda_{-1}, \ldots, \lambda_n, \tau_0, \ldots, \tau_{n-1}) \in \mathbb{N}^{2n+3}$ set

$$\Lambda_J(z, \tau) = \Delta(q^{-J}z + \lambda_{-1}; L_{-1} + 1, \lambda_0 + 1, \tau_0) \prod_{j=1}^{n-1} \Delta(b_n\lambda_j - b_j\lambda_n; \tau_j), \qquad (2.50)$$

where $\Delta(z; k)$ and $\Delta(z; k, l, m)$ are defined by (1.9) and (1.10). In the sequel, we abbreviate $(\lambda_{-1}, \ldots, \lambda_n)$ as λ, $(\tau_0, \ldots, \tau_{n-1})$ as τ and write $|\tau| = \tau_0 + \cdots + \tau_{n-1}$. Let

$$D_0 = [\mathbb{Q}(\alpha_0) : \mathbb{Q}], \qquad D_1 = [K : \mathbb{Q}(\alpha_0)] \quad (= D/D_0). \qquad (2.51)$$

By (2.14) we can fix a basis of K over $\mathbb{Q}$ of the shape

$$\xi_{d_0,d} = \alpha_0^{d_0}\alpha_1^{k_{1d}} \ldots \alpha_n^{k_{nd}} \text{ with } (k_{1d}, \ldots, k_{nd}) \in \mathbb{N}^n \text{ and } \sum_{j=1}^{n} k_{jd} \leqslant D_1 - 1 < D,$$
$$d_0 = 0, \ldots, D_0 - 1, \quad d = 1, \ldots, D_1. \qquad (2.52)$$

2.3. Construction of the rational integers $p(\lambda, d_0, d)$

We recall that $r_1, \ldots, r_n$ are the rational integers in (2.24); G, G_0, G_1 are defined by (2.19), (2.20); X is given in (2.31); D_0, D_1 are given in (2.51).

LEMMA 2.1. *For*

$$d_0 = 0, \ldots, D_0 - 1, \quad d = 1, \ldots, D_1 \qquad (2.53)$$

and $\lambda = (\lambda_{-1}, \ldots, \lambda_n)$ *in the range*

$$0 \leqslant \lambda_j \leqslant L_j \quad (-1 \leqslant j \leqslant n), \qquad r_1\lambda_1 + \cdots + r_n\lambda_n \equiv 0 \pmod{G_1}, \qquad (2.54)$$

there exist $p(\lambda, d_0, d) \in \mathbb{Z}$ *with*

$$0 < \max_{\lambda, d_0, d} |p(\lambda, d_0, d)| \leqslant X_0^{1/(c_0-1)}$$

such that

$$\sum_{\lambda} \sum_{d_0, d} p(\lambda, d_0, d) \xi_{d_0, d} \Lambda_0(s, \tau) \prod_{j=1}^{n} (\alpha_j^{p^{\kappa}} \zeta^{r_j})^{\lambda_j s} = 0 \tag{2.55}$$

for all $(s, \tau_0, \ldots, \tau_{n-1}) \in \mathbb{N}^{n+1}$ *satisfying*

$$1 \leqslant s \leqslant S, \quad (s, q) = 1, \quad |\tau| \leqslant T,$$

where Σ_λ *ranges over* (2.54), $\Sigma_{d_0, d}$ *ranges over* (2.53).

REMARK. In the sequel s always denotes a rational integer and τ always denotes a point $(\tau_0, \ldots, \tau_{n-1}) \in \mathbb{N}^n$. The expression $(s, \tau_0, \ldots, \tau_{n-1}) \in \mathbb{N}^{n+1}$ will be omitted.

Proof. For $t \in \mathbb{Z}$, define

$$\mathscr{C}_t = \{\lambda = (\lambda_{-1}, \ldots, \lambda_n) \in \mathbb{N}^{n+2} \mid 0 \leqslant \lambda_j \leqslant L_j (-1 \leqslant j \leqslant n),$$
$$r_1 \lambda_1 + \cdots + r_n \lambda_n \equiv t G_1 (\mathrm{mod}\, G_0)\}. \tag{2.56}$$

Let

$$\mathscr{T} = \{t \in \mathbb{Z} \mid 0 \leqslant t < q^{\mu - u}, \mathscr{C}_t \neq \emptyset\}. \tag{2.57}$$

By (2.20) we have

$$\mathscr{C}_t \cap \mathscr{C}_{t'} = \emptyset \quad \text{for} \quad t, t' \in \mathscr{T} \quad \text{with } t \neq t'. \tag{2.58}$$

Denote by $\mathscr{C}$ the set of $\lambda = (\lambda_{-1}, \ldots, \lambda_n) \in \mathbb{N}^{n+2}$ satisfying (2.54). Then

$$\mathscr{C} = \bigcup_{t \in \mathscr{T}} \mathscr{C}_t. \tag{2.59}$$

By (2.59), (2.58), (2.56), we see that every $\lambda \in \mathscr{C}$ determines uniquely $t = t(\lambda_1, \ldots, \lambda_n) \in \mathscr{T}$ and $k = k(\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}$ such that

$$r_1 \lambda_1 + \cdots + r_n \lambda_n = t G_1 + k G_0. \tag{2.60}$$

Write $h = h(\lambda_1, \ldots, \lambda_n, d_0, s)$ for the rational integer satisfying

$$h \equiv p^v k(\lambda_1, \ldots, \lambda_n)s + d_0 \pmod{p^v q^u} \quad \text{and} \quad 0 \leqslant h < p^v q^u. \tag{2.61}$$

By (2.60), (2.21), (2.6), (2.61) we obtain

$$\begin{aligned} \alpha_0^{d_0} \zeta^{(r_1\lambda_1 + \cdots + r_n\lambda_n)s} &= \alpha_0^{d_0 + p^v k(\lambda_1, \ldots, \lambda_n)s} \zeta^{G_1 st(\lambda_1, \ldots, \lambda_n)} \\ &= \alpha_0^{h(\lambda_1, \ldots, \lambda_n, d_0, s)} \zeta^{G_1 st(\lambda_1, \ldots, \lambda_n)}. \end{aligned} \tag{2.62}$$

For λ, d_0, d, s, τ with $\lambda \in \mathscr{C}$, $0 \leqslant d_0 < D_0$, $1 \leqslant d \leqslant D_1$, $1 \leqslant s \leqslant S$ and $(s, q) = 1$, and $|\tau| \leqslant T$, set

$$P_{\lambda, d_0, d; s, \tau}(x_0, x_1, \ldots, x_n) = (v(L_{-1} + 1))^{\tau_0} \Lambda_0(s, \tau) x_0^{h(\lambda_1, \ldots, \lambda_n, d_0, s)} \prod_{j=1}^{n} x_j^{p^\kappa \lambda_j s + k_j d}.$$

By Lemmas 1.6 and 1.7 we see that each $P_{\lambda, d_0, d; s, \tau}$ is a monomial in $x_0, x_1, \ldots, x_n$ with rational integer coefficient, the absolute value of which is at most

$$\begin{aligned} &3^{(L_{-1}+1)\tau_0} e^{T - \tau_0} \left(1 + \frac{(n-1)(B_n L_1 + B' L_n)}{T - \tau_0}\right)^{T - \tau_0} \left(e\left(2 + \frac{S}{L_{-1} + 1}\right)\right)^{(L_{-1}+1)(L_0+1)} \\ &\leqslant 3^{(L_{-1}+1)T} \left(1 + \frac{(n-1)(B_n L_1 + B' L_n)}{T}\right)^{T} \left(e\left(2 + \frac{S}{L_{-1} + 1}\right)\right)^{(L_{-1}+1)(L_0+1)}. \end{aligned}$$

Further

$$\deg_{x_j} P_{\lambda, d_0, d; s, \tau} \leqslant p^\kappa S L_j + D \quad (1 \leqslant j \leqslant n).$$

Note that (2.1), (2.3), (2.4) and $(s, q) = 1$ imply that

$$-1 \in K^2 \quad \text{when} \quad q = 2; \qquad \text{and} \quad (\zeta_{q^u})^s \notin K^q.$$

By (2.20), (2.21) we see that $\zeta^{G_1 s}$ is a root of $x^{q^{\mu - u}} - (\zeta_{q^u})^s$. Thus, in virtue of Lemma 1.8, it follows that the $q^{\mu - u}$ elements

$$\zeta^{G_1 st}, \quad t = 0, 1, \ldots, q^{\mu - u} - 1$$

are linearly independent over K. On combining (2.58), (2.59), (2.62) and the above fact, we see that (2.55) is equivalent to that for each $t \in \mathscr{T}$

$$\begin{aligned} &\sum_{\lambda \in \mathscr{C}_t} \sum_{d_0, d} P_{\lambda, d_0, d; s, \tau}(\alpha_0, \alpha_1, \ldots, \alpha_n) p(\lambda, d_0, d) = 0 \\ &1 \leqslant s \leqslant S, \quad (s, q) = 1, \quad |\tau| \leqslant T. \end{aligned} \tag{2.63}$$

For each $t \in \mathscr{T}$, in (2.63) there are $(1 - 1/q)S\binom{T+n}{n}$ equations and at least

$$D_0 D_1 (L_{-1} + 1)(L_0 + 1) \prod_{j=1}^{n} \left[\frac{L_j + 1}{G_0}\right] \cdot G_0^{n-1} \text{g.c.d.}(r_1, \ldots, r_n, G_0)$$

$$\geqslant \frac{1}{G_0} D(L_{-1} + 1)(L_0 + 1) \prod_{j=1}^{n} (L_j + 1 - G_0)$$

unknowns $p(\lambda, d_0, d)$. By (2.32), we can apply Lemma 1.5 to (2.63) for each $t \in \mathscr{T}$ (note that $h(\alpha_0) = 0$), and the lemma follows at once.

2.4. The main inductive argument

For rational integers $r^{(J)}, L_j^{(J)} (-1 \leqslant j \leqslant n)$ and $p^{(J)}(\lambda, d_0, d) = p^{(J)}(\lambda_{-1}, \ldots, \lambda_n, d_0, d)$, which will be constructed in the following main inductive argument, set

$$\phi_J(z, \tau) = \sum_{\lambda} \sum_{d_0, d} p^{(J)}(\lambda, d_0, d) \xi_{d_0, d} \Lambda_J(z, \tau) \prod_{j=1}^{n} (\alpha_j^{p^\kappa} \zeta^{r_j})^{\lambda_j z}, \tag{2.64}$$

where Σ_λ is taken over the set $\mathscr{C}^{(J)}$ of $\lambda = (\lambda_{-1}, \ldots, \lambda_n)$ satisfying

$$0 \leqslant \lambda_j \leqslant L_j^{(J)} \quad (-1 \leqslant j \leqslant n), \qquad r_1 \lambda_1 + \cdots + r_n \lambda_n \equiv r^{(J)} \pmod{G_1}. \tag{2.65}$$

Note that by (2.24), the p-adic functions

$$(\alpha_j^{p^\kappa} \zeta^{r_j})^{\lambda_j z} = \exp(\lambda_j z \log(\alpha_j^{p^\kappa} \zeta^{r_j})) \quad (1 \leqslant j \leqslant n)$$

are normal.

The main inductive argument. Suppose that there are algebraic numbers $\alpha_1, \ldots, \alpha_n$ *in K and rational integers* $b_1, \ldots, b_n$ *satisfying* (2.14)–(2.18), *such that*

$$\operatorname{ord}_p \Theta \geqslant U. \tag{2.66}$$

Then for every $J \in \mathbb{Z}$ *with* $0 \leqslant J \leqslant [\log L_n / \log q] + 1$ *there exist* $r^{(J)} \in \mathbb{Z}$, $L_j^{(J)} \in \mathbb{Z}$ $(-1 \leqslant j \leqslant n)$ *with*

$$0 \leqslant r^{(J)} < G_1, \qquad L_j^{(J)} = L_j \quad (j = -1, 0),$$
$$0 \leqslant L_j^{(J)} \leqslant q^{-J} L_j \quad (1 \leqslant j \leqslant n), \qquad \mathscr{C}^{(J)} \neq \emptyset$$

and $p^{(J)}(\lambda, d_0, d) \in \mathbb{Z}$ *for* $\lambda \in \mathscr{C}^{(J)}, 0 \leqslant d_0 < D_0, 1 \leqslant d \leqslant D_1$ *with*

$$0 < \max_{\lambda, d_0, d} |p^{(J)}(\lambda, d_0, d)| \leqslant X_0^{1/(c_0 - 1)},$$

such that

$$\phi_J(s, \tau) = 0 \quad \textit{for} \quad 1 \leqslant s \leqslant q^J S, \qquad (s, q) = 1, \qquad |\tau| \leqslant q^{-J} T.$$

The main inductive argument will be proved by an induction on J. On taking $r^{(0)} = 0$, $L_j^{(0)} = L_j (-1 \leqslant j \leqslant n)$, $p^{(0)}(\lambda, d_0, d) = p(\lambda, d_0, d)$, which are constructed in Lemma 2.1, we see, by Lemma 2.1, that the case $J = 0$ is true. In the rest of Section 2.4, we suppose the main inductive argument is valid for some J with $0 \leqslant J \leqslant [\log L_n/\log q]$, and we are going to prove it for $J + 1$. We always keep the hypothesis (2.66). We first prove the following Lemmas 2.2, 2.3, 2.4, then deduce from Lemma 2.4 the main inductive argument for $J + 1$.

Set

$$\gamma_j = \lambda_j - \frac{b_j}{b_n} \lambda_n \quad (1 \leqslant j < n), \qquad p^{(J)}(\lambda) = \sum_{d_0, d} p^{(J)}(\lambda, d_0, d) \xi_{d_0, d}.$$

Write $\mathscr{C}_t^{(J)} (t \in \mathbb{Z})$ for the set of $\lambda = (\lambda_{-1}, \ldots, \lambda_n)$ satisfying

$$\begin{aligned} &0 \leqslant \lambda_j \leqslant L_j^{(J)} \quad (-1 \leqslant j \leqslant n), \\ &\quad r_1 \lambda_1 + \cdots + r_n \lambda_n \equiv r^{(J)} + t G_1 \pmod{G_0} \end{aligned} \tag{2.67}$$

and define

$$\mathscr{T}^{(J)} = \{t \in \mathbb{Z} \mid 0 \leqslant t < q^{\mu - u}, \mathscr{C}_t^{(J)} \neq \emptyset\}. \tag{2.68}$$

By (2.20),

$$\mathscr{C}_t^{(J)} \cap \mathscr{C}_{t'}^{(J)} = \emptyset \quad \text{for} \quad t, t' \in \mathscr{T}^{(J)} \quad \text{with} \quad t \neq t'. \tag{2.69}$$

By (2.65), (2.67), (2.68),

$$\mathscr{C}^{(J)} = \bigcup_{t \in \mathscr{T}^{(J)}} \mathscr{C}_t^{(J)}. \tag{2.70}$$

Define

$$f_J(z, \tau) = \sum_{\lambda \in \mathscr{C}^{(J)}} p^{(J)}(\lambda) \Lambda_J(z, \tau) \prod_{j=1}^{n-1} (\alpha_j^{p^\kappa} \zeta^{r_j})^{\gamma_j z}, \tag{2.71}$$

and for $t \in \mathcal{T}^{(J)}$ define

$$f_{J,t}(z,\tau) = \sum_{\lambda \in \mathscr{C}_t^{(J)}} p^{(J)}(\lambda)\Lambda_J(z,\tau) \prod_{j=1}^{n-1} (\alpha_j^{p^\kappa} \zeta^{r_j})^{\gamma_j z}, \tag{2.72}$$

$$\phi_{J,t}(z,\tau) = \sum_{\lambda \in \mathscr{C}_t^{(J)}} p^{(J)}(\lambda)\Lambda_J(z,\tau) \prod_{j=1}^{n} (\alpha_j^{p^\kappa} \zeta^{r_j})^{\lambda_j z}. \tag{2.73}$$

LEMMA 2.2. *For every* $t \in \mathcal{T}^{(J)}, \tau = (\tau_0, \ldots, \tau_{n-1})$ *with* $|\tau| \leqslant T$ *and* $y \in \mathbb{Q}$ *with* $y > 0$ *and* $\mathrm{ord}_p\, y \geqslant 0$, *we have*

$$\mathrm{ord}_p(\phi_{J,t}(y,\tau) - f_{J,t}(y,\tau)) \geqslant U - \frac{T\log(L_{-1}+1)}{\log p} - \mathrm{ord}_p\, b_n.$$

Proof. Similar to the proof of Lemma 3.2 of Yu [21].

LEMMA 2.3. *For* $k = 0, 1, \ldots, n$ *we have*

$$\phi_J(s,\tau) = 0 \qquad \textit{for} \quad 1 \leqslant s \leqslant q^{J+k}S, \qquad (s,q) = 1,$$

$$|\tau| \leqslant \left(1 - \frac{k}{n+1}\left(1 - \frac{1}{q}\right)\right) q^{-J}T. \tag{2.74}$$

Proof. By (2.67)–(2.70), every $\lambda \in \mathscr{C}^{(J)}$ determines uniquely $t = t(\lambda_1, \ldots, \lambda_n) \in \mathcal{T}^{(J)}$ and $k = k(\lambda_1, \ldots, \lambda_n)^\dagger \in \mathbb{Z}$ such that

$$r_1\lambda_1 + \cdots + r_n\lambda_n = r^{(J)} + tG_1 + kG_0. \tag{2.75}$$

Let $h = h(\lambda_1, \ldots, \lambda_n, d_0, s)$ be defined by (2.61). Thus by (2.75), (2.21), (2.6), (2.61), we get

$$\begin{aligned} \alpha_0^{d_0} \zeta^{(r_1\lambda_1 + \cdots + r_n\lambda_n)s} &= \alpha_0^{d_0 + p^v sk(\lambda_1,\ldots,\lambda_n)} \zeta^{G_1 st(\lambda_1,\ldots,\lambda_n)} \zeta^{sr^{(J)}} \\ &= \alpha_0^{h(\lambda_1,\ldots,\lambda_n,d_0,s)} \zeta^{G_1 st(\lambda_1,\ldots,\lambda_n)} \zeta^{sr^{(J)}}. \end{aligned} \tag{2.76}$$

We now prove that (2.74) is equivalent to the statement that for every $t \in \mathcal{T}^{(J)}$ we

---

†Of course, t and k are not necessarily the same as that in (2.60); however we still use these notations, because no confusion will be caused.

have

$$\phi_{J,t}(s,\tau)=0 \quad \text{for} \quad 1\leqslant s\leqslant q^{J+k}S,\ (s,q)=1,$$

$$|\tau|\leqslant\left(1-\frac{k}{n+1}\left(1-\frac{1}{q}\right)\right)q^{-J}T. \tag{2.77}$$

By the identity

$$\phi_J(z,\tau)=\sum_{t\in\mathscr{T}^{(J)}}\phi_{J,t}(z,\tau), \tag{2.78}$$

(2.77) implies (2.74) at once. Conversely, by (2.78), (2.76) and the fact that

$$\zeta^{G_1st},\quad t=0,1,\ldots,q^{\mu-u}-1$$

are linearly independent over K, which has been established in the proof of Lemma 2.1, we see that (2.74) implies (2.77). Thus

$$\text{(2.74) is equivalent to (2.77).} \tag{2.79}$$

In the sequel, let t denote an arbitrarily fixed element of $\mathscr{T}^{(J)}$. By the main inductive hypothesis for J and by (2.79), we see that (2.77) with $k=0$ is true. We now assume (2.77) is valid for some k with $0\leqslant k\leqslant n$. We shall prove it for $k+1$ if $k<n$ and include the case $k=n$ for later use. Thus we see, by Lemma 2.2, that

$$\operatorname{ord}_p f_{J,t}(s,\tau)\geqslant U-\frac{T\log(L_{-1}+1)}{\log p}-\operatorname{ord}_p b_n$$

$$\text{for}\quad 1\leqslant s\leqslant q^{J+k}S,\qquad (s,q)=1,$$

$$|\tau|\leqslant\left(1-\frac{k}{n+1}\left(1-\frac{1}{q}\right)\right)q^{-J}T. \tag{2.80}$$

Note that by (2.24) and (2.17), the p-adic function

$$\prod_{j=1}^{n-1}(\alpha_j^{p^\kappa}\zeta^{r_j})^{\gamma_j p^{-\theta}z}$$

is normal, where θ is given by (1.2) and can be written as $\theta=l/m$ with l,m being coprime positive integers, and $p^\theta:=\beta^l$ with $\beta\in\mathbb{C}_p$ being a fixed mth root of p.

Further by (1.14) and (2.3) we see that

$$p^{(L_{-1}+1)(L_0+1)\theta}((L_{-1}+1)!)^{L_0+1}\Lambda_J(p^{-\theta}z,\tau)$$

is a normal function, whence so is

$$p^{(L_{-1}+1)(L_0+1)(\theta+1/(p-1))}\Lambda_J(p^{-\theta}z,\tau).$$

Thus, by (2.72) we see that

$$F_{J,t}(z,\tau):=p^{(L_{-1}+1)(L_0+1)(\theta+1/(p-1))}f_{J,t}(p^{-\theta}z,\tau)$$
$$\text{for}\quad |\tau|\leqslant\left(1-\frac{k+1}{n+1}\left(1-\frac{1}{q}\right)\right)q^{-J}T \tag{2.81}$$

are normal functions. We now apply Lemma 1.2 to each function in (2.81), taking

$$R=q^{J+k}S,\qquad M=\left[\frac{1}{n+1}\left(1-\frac{1}{q}\right)q^{-J}T\right]+1. \tag{2.82}$$

By an argument similar to the proof of (3.74) in Yu [21], using Lemma 2.6 of [21], we deduce from (2.80) that

$$\min_{\substack{1\leqslant s\leqslant R,(s,q)=1\\ m=0,\dots,M-1}}\left\{\operatorname{ord}_p\left(\frac{1}{m!}\frac{d^m}{dz^m}F_{J,t}(sp^\theta,\tau)\right)+m\theta\right\}$$
$$\geqslant U+(L_{-1}+1)(L_0+1)\left(\theta+\frac{1}{p-1}\right)-\frac{T\log(L_{-1}+1)}{\log p}-$$
$$-\left(\frac{1}{n+1}\left(1-\frac{1}{q}\right)q^{-J}T+1\right)\operatorname{ord}_p b_n$$
$$>U-\left\{\frac{\log(h_0+1)}{h_0}+\frac{1}{n+1}\left(1-\frac{1}{q}\right)\left(1+\frac{1}{h_8}\right)\right\}\frac{2+1/(p-1)}{q^{n+1}}\cdot\frac{1}{c_1c_3}U$$
$$\text{for}\quad |\tau|\leqslant\left(1-\frac{k+1}{n+1}\cdot\left(1-\frac{1}{q}\right)\right)q^{-J}T, \tag{2.83}$$

where the second inequality follows from (2.42) and (2.43).

On the other hand, by (2.82), (2.44), (2.45), we see that

$$\left(1-\frac{1}{q}\right)RM\theta + M\,\mathrm{ord}_p(R!) + (M-1)\frac{\log R}{\log p}$$

$$\leqslant \left(\frac{1}{n+1}\left(1-\frac{1}{q}\right)q^{-J}T+1\right)q^{J+k}S\left(\left(1-\frac{1}{q}\right)\theta+\frac{1}{p-1}\right)+$$

$$+\frac{1}{n+1}\cdot\left(1-\frac{1}{q}\right)q^{-J}T\cdot\frac{\log(q^{J+k}S)}{\log p}$$

$$<\left(1+\frac{1}{h_8}\right)\left(1-\frac{1}{q}\right)\left(2+\frac{1}{p-1}-\frac{1}{q}\right)\frac{1}{c_1}U+$$

$$+\left(1-\frac{1}{q}\right)\left(\frac{1}{n}+\frac{\log h_0}{h_0}+\frac{\log q}{qh_0}+\eta\right)\cdot\frac{2+1/(p-1)}{(n+1)q^{n+1}}\cdot\frac{1}{c_1c_3}U. \tag{2.84}$$

Now we see from (2.83), (2.84), (2.29) that each $F_{J,t}(z,\tau)$ in (2.81) satisfies the condition (1.5) with R, M given by (2.82). Thus by Lemma 1.2 and (2.81) we obtain

$$\mathrm{ord}_p\, f_{J,t}\left(\frac{s}{q},\tau\right)=\mathrm{ord}_p\, F_{J,t}\left(\frac{s}{q}p^{\theta},\tau\right)-(L_{-1}+1)(L_0+1)\left(\theta+\frac{1}{p-1}\right)$$

$$\geqslant\left(1-\frac{1}{q}\right)RM\theta-(L_{-1}+1)(L_0+1)\left(\theta+\frac{1}{p-1}\right)$$

$$>\left(1-\frac{1}{q}\right)^2\frac{1}{n+1}q^kST\theta-(L_{-1}+1)(L_0+1)\left(\theta+\frac{1}{p-1}\right)$$

$$\text{for}\quad s\in\mathbb{Z},\quad |\tau|\leqslant\left(1-\frac{k+1}{n+1}\left(1-\frac{1}{q}\right)\right)q^{-J}T. \tag{2.85}$$

By the second inequality in (2.83), we have

$$U-\frac{T\log(L_{-1}+1)}{\log p}-\mathrm{ord}_p b_n+(L_{-1}+1)(L_0+1)\left(\theta+\frac{1}{p-1}\right)$$

$$>U-\frac{2+1/(p-1)}{q^{n+1}}\cdot\frac{1}{c_1c_3}U\cdot\left\{\frac{\log(h_0+1)}{h_0}+\frac{1}{n+1}\left(1-\frac{1}{q}\right)\left(1+\frac{1}{h_8}\right)\right\}.$$

Observe that the right-hand side of the above inequality is, by (2.29), not less than the extreme right-hand side of (2.84). Hence by Lemma 2.2 (note that $\mathrm{ord}_p(s/q) \geqslant 0$ by (2.3)), by the above observation and by (2.84), (2.82), we get

$$\mathrm{ord}_p\left(\phi_{J,t}\left(\frac{s}{q},\tau\right) - f_{J,t}\left(\frac{s}{q},\tau\right)\right) \geqslant U - \frac{T\log(L_{-1}+1)}{\log p} - \mathrm{ord}_p\, b_n$$

$$> \left(1-\frac{1}{q}\right)RM\theta - (L_{-1}+1)(L_0+1)\left(\theta + \frac{1}{p-1}\right)$$

$$> \left(1-\frac{1}{q}\right)^2 \cdot \frac{1}{n+1} q^k ST\theta - (L_{-1}+1)(L_0+1)\left(\theta + \frac{1}{p-1}\right)$$

$$\text{for} \quad s \geqslant 1, \qquad |\tau| \leqslant T. \tag{2.86}$$

On combining (2.85) with (2.86), and utilizing (2.33) and (2.37), we obtain

$$\mathrm{ord}_p\, \phi_{J,t}\left(\frac{s}{q},\tau\right) \geqslant \min\left\{\mathrm{ord}_p\, f_{J,t}\left(\frac{s}{q},\tau\right), \mathrm{ord}_p\left(\phi_{J,t}\left(\frac{s}{q},\tau\right) - f_{J,t}\left(\frac{s}{q},\tau\right)\right)\right\}$$

$$> \left(1-\frac{1}{q}\right)^2 \cdot \frac{1}{n+1} q^k ST\theta - (L_{-1}+1)(L_0+1)\left(\theta + \frac{1}{p-1}\right)$$

$$> \frac{1}{c_1} U q^{k-n}\left\{\left(1-\frac{1}{q}\right)^2\left(1 - \frac{1}{c_3(n+1)}\right)\left(1-\frac{1}{h_1}\right) - \right.$$

$$\left. - \frac{1}{q^{k+1}}\left(1+\frac{1}{h_4}\right)\left(1+\frac{1}{p-1}\right)\frac{1}{f_k}\cdot\frac{1}{c_4}\right\}$$

$$\text{for} \quad s \geqslant 1, \qquad |\tau| \leqslant \left(1 - \frac{k+1}{n+1}\left(1-\frac{1}{q}\right)\right) q^{-J} T. \tag{2.87}$$

From now on we assume $0 \leqslant k < n$.

On the other hand, by Lemma 1.6 and (2.50), (2.76) we see that for any fixed $t \in \mathscr{T}^{(J)}$ and for $1 \leqslant s \leqslant q^{J+k+1}, (s,q) = 1, |\tau| \leqslant (1 - (1 - 1/q)(k+1)/$

$(n+1))q^{-J}T$, we have

$$\zeta^{-G_1 st}\zeta^{-sr^{(J)}}q^{J\cdot 2(L_{-1}+1)(L_0+1)}(v(L_{-1}+1))^{\tau_0}\phi_{J,t}(s,\tau)$$
$$= \sum_{\lambda\in\mathscr{C}_t^{(J)}}\sum_{d_0,d} p^{(J)}(\lambda,d_0,d)q^{2J(L_{-1}+1)(L_0+1)}\times$$
$$\times (v(L_{-1}+1))^{\tau_0}\Lambda_J(s,\tau)\alpha_0^{h(\lambda_1,\ldots,\lambda_n,d_0,s)}\prod_{j=1}^{n}\alpha_j^{p^\kappa\lambda_j s+k_j d}$$
$$=: Q_{J,t;s,\tau}(\alpha_0,\alpha_1,\ldots,\alpha_n), \tag{2.88}$$

with $Q_{J,t;s,\tau}(x_0,x_1,\ldots,x_n)$ being in $\mathbb{Z}[x_0,x_1,\ldots,x_n]$ and having degree at most

$$p^\kappa L_j^{(J)}q^{J+k+1}S + D \leqslant p^\kappa q^{k+1}SL_j + D$$

in x_j ($1\leqslant j\leqslant n$). Note that by the main inductive hypothesis for J and Lemmas 1.6, 1.7, we have, for $\lambda\in\mathscr{C}_t^{(J)}, 0\leqslant d<D_0, 1\leqslant d\leqslant D_1, 1\leqslant s\leqslant q^{J+k+1}S$, $(s,q)=1, |\tau|\leqslant(1-(1-1/q)(k+1)/(n+1))q^{-J}T$, the following estimates:

$$|p^{(J)}(\lambda,d_0,d)|\leqslant X_0^{1/(c_0-1)},\quad q^{2J(L_{-1}+1)(L_0+1)}\leqslant L_n^{2(L_{-1}+1)(L_0+1)},$$

$$|\Delta(q^{-J}s+\lambda_{-1};L_{-1}+1,\lambda_0+1,\tau_0)|\leqslant\left(e\left(2+\frac{q^{k+1}S}{L_{-1}+1}\right)\right)^{(L_{-1}+1)(L_0+1)}$$
$$\leqslant\left(e\left(2+\frac{S}{L_{-1}+1}\right)\right)^{q^{k+1}(L_{-1}+1)(L_0+1)},$$

$$(v(L_{-1}+1))^{\tau_0}\prod_{j=1}^{n-1}|\Delta(b_n\lambda_j-b_j\lambda_n;\tau_j)|$$
$$\leqslant 3^{(L_{-1}+1)\tau_0}e^{T-\tau_0}\left(1+\frac{(n-1)(B_nL_1+B'L_n)}{T-\tau_0}\right)^{T-\tau_0}$$
$$\leqslant 3^{(L_{-1}+1)T}\left(1+\frac{(n-1)(B_nL_1+B'L_n)}{T}\right)^{T}.$$

By the above estimates and by (2.50), (2.88), the length of $Q_{J,t;s,\tau}(x_0,x_1,\ldots,x_n)$ is at most

$$D \prod_{j=-1}^{n} (L_j + 1) \cdot X_0^{1/(c_0-1)} L_n^{2(L_{-1}+1)(L_0+1)} \left(e\left(2 + \frac{S}{L_{-1}+1}\right)\right)^{q^{k+1}(L_{-1}+1)(L_0+1)} \cdot$$

$$\cdot 3^{(L_{-1}+1)T} \left(1 + \frac{(n-1)(B_n L_1 + B' L_n)}{T}\right)^T.$$

Now we assume that there exist s, τ with

$$1 \leqslant s \leqslant q^{J+k+1} S, \quad (s, q) = 1, \quad |\tau| \leqslant \left(1 - \frac{k+1}{n+1}\left(1 - \frac{1}{q}\right)\right) q^{-J} T$$

such that

$$\phi_{J,t}(s, \tau) \neq 0,$$

and we proceed to deduce a contradiction. By Lemma 1.3 (note $h(\alpha_0) = 0$), by the definition of X_0 (see (2.31)), and by (2.88), (2.34)–(2.36), (2.38)–(2.41), we see that the assumption $\phi_{J,t}(s, \tau) \neq 0$ implies that

$$\begin{aligned}
&\operatorname{ord}_p \phi_{J,t}(s, \tau) \leqslant \operatorname{ord}_p Q_{J,t;s,\tau}(\alpha_0, \alpha_1, \ldots, \alpha_n) \\
&\leqslant \frac{D}{e_{\mathfrak{p}} f_{\mathfrak{p}} \log p} \cdot \left\{\log\left(D \prod_{j=-1}^{n} (L_j + 1)\right) + \frac{1}{c_0 - 1} \log X_0 + \log 3 \cdot T(L_{-1} + 1) + \right. \\
&\quad + T \log\left(1 + \frac{(n-1)(B_n L_1 + B' L_n)}{T}\right) + 2(L_{-1} + 1)(L_0 + 1) \log L_n + \\
&\quad + q^{k+1}(L_{-1} + 1)(L_0 + 1) \log\left(e\left(2 + \frac{S}{L_{-1}+1}\right)\right) + \\
&\quad \left. + p^{\kappa} q^{k+1} S \sum_{j=1}^{n} L_j V_j + nD \max_{1 \leqslant j \leqslant n} V_j \right\} \\
&\leqslant q^{k-n} \cdot \frac{q^{n+1} D}{e_{\mathfrak{p}} f_{\mathfrak{p}} \log p} \left\{\frac{1}{q}\left(1 + \frac{1}{c_0 - 1}\right)\left[\log\left(D \prod_{j=-1}^{n} (L_j + 1)\right) + nD \max_{1 \leqslant j \leqslant n} V_j\right] + \right. \\
&\quad + \left(1 + \frac{1}{q(c_0 - 1)}\right) p^{\kappa} S \sum_{j=1}^{n} L_j V_j + \frac{1}{q}\left(1 + \frac{1}{c_0 - 1}\right) \log 3 \cdot T(L_{-1} + 1) +
\end{aligned}$$

$$+\frac{1}{q}\left(1+\frac{1}{c_0-1}\right)T\log\left(1+\frac{(n-1)(B_nL_1+B'L_n)}{T}\right)+$$

$$+\left(1+\frac{1}{q(c_0-1)}\right)(L_{-1}+1)(L_0+1)\log\left(e\left(2+\frac{S}{L_{-1}+1}\right)\right)+$$

$$+\frac{1}{q}\cdot 2(L_{-1}+1)(L_0+1)\log L_n\Bigg\}$$

$$<\frac{1}{c_1}Uq^{k-n}\left\{\left(1+\frac{1}{c_0-1}\right)\left(\frac{1}{h_6}+\frac{1}{h_7}\right)c_1+\left(1+\frac{1}{c_0-1}\right)\frac{1}{c_2}+\right.$$

$$+\left(\frac{1}{q}+\frac{1}{c_0-1}\right)\left(\log 3\cdot\left(1+\frac{1}{h_0}\right)+1\right)\left(2+\frac{1}{p-1}\right)\frac{1}{c_3}+$$

$$+\left(1+\frac{1}{c_0-1}\right)\left(1+\frac{1}{h_4}\right)\frac{1}{n}\cdot\frac{1}{c_4}+$$

$$\left.+\frac{1}{q}\left(1+\frac{1}{h_4}\right)\left(4+\frac{1}{2^{10}qnD}+\frac{2\log h_5}{h_0}\right)\frac{1}{c_4}\right\}.$$

This together with (2.28) implies that

$$\operatorname{ord}_p\phi_{J,t}(s,\tau)<\frac{1}{c_1}Uq^{k-n}\left\{\left(1-\frac{1}{q}\right)^2\left(1-\frac{1}{c_3(n+1)}\right)\left(1-\frac{1}{h_1}\right)-\right.$$

$$-\left(1+\frac{1}{h_4}\right)\left(1+\frac{1}{p-1}\right)\frac{1}{q^{n+1}f_{\mathfrak{p}}}\cdot\frac{1}{c_4}-$$

$$\left.-\left(1-\frac{1}{q}\right)\left(1+\frac{1}{h_4}\right)\left(4+\frac{1}{2^{10}qnD}+\frac{2\log h_5}{h_0}\right)\frac{1}{c_4}\right\}. \tag{2.89}$$

On noting

$$\left(1+\frac{1}{p-1}\right)\frac{1}{q^{n+1}f_{\mathfrak{p}}}+\left(1-\frac{1}{q}\right)\left(4+\frac{1}{2^{10}qnD}+\frac{2\log h_5}{h_0}\right)$$

$$>\left(1+\frac{1}{p-1}\right)\frac{1}{q^{n+1}f_{\mathfrak{p}}}+4\left(1-\frac{1}{q}\right)>\left(1+\frac{1}{p-1}\right)\frac{1}{q^{k+1}f_{\mathfrak{p}}},$$

we see that (2.89) yields

$$\operatorname{ord}_p \phi_{J,t}(s,\tau) < \frac{1}{c_1} U q^{k-n} \left\{ \left(1 - \frac{1}{q}\right)^2 \left(1 - \frac{1}{c_3(n+1)}\right)\left(1 - \frac{1}{h_1}\right) - \right.$$

$$\left. - \frac{1}{q^{k+1}}\left(1 + \frac{1}{h_4}\right)\left(1 + \frac{1}{p-1}\right)\frac{1}{f_{\not{p}}} \cdot \frac{1}{c_4} \right\},$$

contradicting (2.87). This contradiction proves that for any fixed $t \in \mathscr{T}^{(J)}$,

$$\phi_{J,t}(s,\tau) = 0 \quad \text{for} \quad 1 \leqslant s \leqslant q^{J+k+1}S, \qquad (s,q) = 1,$$

$$|\tau| \leqslant \left(1 - \frac{k+1}{n+1}\left(1 - \frac{1}{q}\right)\right) q^{-J} T.$$

This fact and (2.78) imply (2.74) for $k + 1$, and the proof of the lemma is thus complete.

LEMMA 2.4. *We have*

$$\phi_J\left(\frac{s}{q}, \tau\right) = 0 \quad \textit{for} \quad 1 \leqslant s \leqslant q^{J+1}S, \qquad (s,q) = 1, \qquad |\tau| \leqslant q^{-(J+1)}T. \tag{2.90}$$

Proof. By (2.78), it suffices to show that for any fixed $t \in \mathscr{T}^{(J)}$, we have

$$\phi_{J,t}\left(\frac{s}{q}, \tau\right) = 0 \quad \text{for} \quad 1 \leqslant s \leqslant q^{J+1}S, \qquad (s,q) = 1, \qquad |\tau| \leqslant q^{-(J+1)}T.$$

We recall that (2.87) holds for $k = n$, that is,

$$\operatorname{ord}_p \phi_{J,t}\left(\frac{s}{q}, \tau\right) > \left(1 - \frac{1}{q}\right)^2 \frac{1}{n+1} q^n ST\theta -$$

$$- (L_{-1} + 1)(L_0 + 1)\left(\theta + \frac{1}{p-1}\right)$$

$$> \frac{1}{c_1} U \left\{ \left(1 - \frac{1}{q}\right)^2 \left(1 - \frac{1}{c_3(n+1)}\right)\left(1 - \frac{1}{h_1}\right) - \right.$$

$$\left. - \left(1 + \frac{1}{h_4}\right)\left(1 + \frac{1}{p-1}\right)\frac{1}{q^{n+1} f_{\not{p}}} \cdot \frac{1}{c_4} \right\}$$

$$\text{for} \quad s \geqslant 1, \qquad |\tau| \leqslant q^{-(J+1)}T. \tag{2.91}$$

For $x, z, z' \in \mathbb{C}_p$ with $\operatorname{ord}_p x > 1/(p-1)$, $\operatorname{ord}_p z \geqslant 0$, $\operatorname{ord}_p z' \geqslant 0$, we have the following identity for p-adic functions

$$(1+x)^{zz'} = ((1+x)^z)^{z'}.$$

(See Hasse [9], p. 273.) Hence we have, by (2.24), (2.25),

$$(\alpha_j^{p^\kappa}\zeta^{r_j})^{\lambda_j(s/q)} = ((\alpha_j^{p^\kappa}\zeta^{r_j})^{1/q})^{\lambda_j s} = (\alpha_j^{1/q})^{p^\kappa\lambda_j s}\zeta^{r_j\lambda_j s} \quad (1 \leqslant j \leqslant n). \tag{2.92}$$

Recall (2.75) and let $h^* = h^*(\lambda_1, \ldots, \lambda_n, d_0, s)$ be the rational integer satisfying

$$h^* \equiv d_0 q + p^v k(\lambda_1, \ldots, \lambda_n)s \pmod{p^v q^{u+1}} \quad \text{and} \quad 0 \leqslant h^* < p^v q^{u+1}. \tag{2.93}$$

By (2.75), (2.23), (2.93) we have for $\lambda \in \mathscr{C}_t^{(J)}$

$$\begin{aligned}\alpha_0^{d_0}\zeta^{(r_1\lambda_1+\cdots+r_n\lambda_n)s} &= \zeta^{(r^{(J)}+tG_1)s}(\alpha_0^{1/q})^{d_0 q + p^v k(\lambda_1,\ldots,\lambda_n)s} \\ &= \zeta^{(r^{(J)}+tG_1)s}(\alpha_0^{1/q})^{h^*(\lambda_1,\ldots,\lambda_n,d_0,s)}.\end{aligned} \tag{2.94}$$

Now by Lemma 1.6, (2.50), (2.92) and (2.94) we see that for any fixed $t \in \mathscr{T}^{(J)}$ and for $1 \leqslant s \leqslant q^{J+1}S$, $(s,q) = 1$, $|\tau| \leqslant q^{-(J+1)}T$, we have

$$\zeta^{-(r^{(J)}+tG_1)s} q^{(J+1)\cdot 2(L_{-1}+1)(L_0+1)}(v(L_{-1}+1))^{\tau_0}\phi_{J,t}\left(\frac{s}{q}, \tau\right)$$

$$= \sum_{\lambda\in\mathscr{C}_t^{(J)}} \sum_{d_0,d} p^{(J)}(\lambda, d_0, d) q^{2(J+1)(L_{-1}+1)(L_0+1)} \times$$

$$\times (v(L_{-1}+1))^{\tau_0}\Lambda_J\left(\frac{s}{q}, \tau\right)(\alpha_0^{1/q})^{h^*(\lambda_1,\ldots,\lambda_n,d_0,s)}.$$

$$\cdot \prod_{j=1}^{n} (\alpha_j^{1/q})^{p^\kappa\lambda_j s + qk_{jd}}$$

$$=: Q^*_{J,t;s,\tau}(\alpha_0^{1/q}, \alpha_1^{1/q}, \ldots, \alpha_n^{1/q}), \tag{2.95}$$

with $Q^*_{J,t;s,\tau}(x_0, x_1, \ldots, x_n)$ being in $\mathbb{Z}[x_0, x_1, \ldots, x_n]$ and having degree at most

$$p^\kappa L_j^{(J)} q^{J+1} S + qD \leqslant q(p^\kappa S L_j + D)$$

in x_j $(1 \leqslant j \leqslant n)$. By the main inductive hypothesis for J and by Lemmas 1.6, 1.7, we have, for $\lambda \in \mathscr{C}_t^{(J)}$, $0 \leqslant d_0 < D_0$, $1 \leqslant d \leqslant D_1$, $1 \leqslant s \leqslant q^{J+1}S$, $(s,q) = 1$, $|\tau| \leqslant$

$q^{-(J+1)}T$, the following estimates

$$|p^{(J)}(\lambda, d_0, d)| \leqslant X_0^{1/(c_0-1)}, \quad q^{2(J+1)(L_{-1}+1)(L_0+1)} \leqslant (qL_n)^{2(L_{-1}+1)(L_0+1)},$$

$$|\Delta(q^{-(J+1)}s + \lambda_{-1}; L_{-1}+1, \lambda_0+1, \tau_0)| \leqslant \left(e\left(2 + \frac{S}{L_{-1}+1}\right)\right)^{(L_{-1}+1)(L_0+1)},$$

$$(\nu(L_{-1}+1))^{\tau_0} \prod_{j=1}^{n-1} |\Delta(b_n\lambda_j - b_j\lambda_n; \tau_j)|$$

$$\leqslant 3^{(L_{-1}+1)\tau_0} e^{(1/q)T-\tau_0}\left(1 + \frac{(n-1)(B_nL^{(J)} + B'L_n^{(J)})}{q^{-(J+1)}T}\right)^{q^{-(J+1)}T}$$

$$\leqslant 3^{(1/q)T(L_{-1}+1)}\left(1 + \frac{(n-1)q(B_nL_1 + B'L_n)}{T}\right)^{(1/q)T},$$

where $L^{(J)} = \max_{1\leqslant j<n} L_j^{(J)}$. By the above estimates and by (2.50), (2.95), the length of $Q_{J,t;s,\tau}^*(x_0, x_1, \ldots, x_n)$ is at most

$$D \prod_{j=-1}^{n} (L_j+1)\cdot X_0^{1/(c_0-1)} 3^{(1/q)T(L_{-1}+1)}\left(1 + \frac{(n-1)q(B_nL_1 + B'L_n)}{T}\right)^{(1/q)T}\cdot$$

$$\cdot\left(e\left(2 + \frac{S}{L_{-1}+1}\right)\right)^{(L_{-1}+1)(L_0+1)} (qL_n)^{2(L_{-1}+1)(L_0+1)}.$$

Now we assume that there exist s, τ satisfying $1 \leqslant s \leqslant q^{J+1}S$, $(s,q) = 1$, $|\tau| \leqslant q^{-(J+1)}T$, such that

$$\phi_{J,t}\left(\frac{s}{q}, \tau\right) \neq 0, \tag{2.96}$$

and we proceed to deduce a contradiction. In Lemma 1.3, let $E = K(\alpha_0^{1/q}, \alpha_1^{1/q}, \ldots, \alpha_n^{1/q})$, $\mathfrak{p}'$ be a prime ideal of the ring of integers in E, lying above $\mathfrak{p}$. Thus

$$[E:\mathbb{Q}] = [E:K][K:\mathbb{Q}] = q^{n+1}D$$

by (2.15), and

$$e_{\mathfrak{p}'} \geqslant e_{\mathfrak{p}}, \quad f_{\mathfrak{p}'} \geqslant f_{\mathfrak{p}}.$$

Note that $h(\alpha_j^{1/q}) = (1/q)h(\alpha_j)$ and $h(\alpha_0^{1/q}) = 0$. By Lemma 1.3 and the definition of

X_0 (see (2.31)), and by (2.95), (2.34)–(2.36), (2.38)–(2.41), (2.28) we see that the assumption (2.96) implies that

$$\operatorname{ord}_p \phi_{J,t}\left(\frac{s}{q}, \tau\right) \leqslant \operatorname{ord}_p Q^*_{J,t;s,\tau}(\alpha_0^{1/q}, \alpha_1^{1/q}, \ldots, \alpha_n^{1/q})$$

$$\leqslant \frac{q^{n+1}D}{e_{\not p} f_{\not p} \log p} \cdot \left\{ \log\left(D \prod_{j=-1}^{n} (L_j + 1)\right) + \frac{1}{c_0 - 1} \log X_0 + \log 3 \cdot \right.$$

$$\cdot \frac{1}{q} T(L_{-1} + 1) + \frac{1}{q} T \log\left(1 + \frac{(n-1)q(B_n L_1 + B' L_n)}{T}\right) +$$

$$+ (L_{-1} + 1)(L_0 + 1) \log\left(e\left(2 + \frac{S}{L_{-1} + 1}\right)\right) +$$

$$\left. + 2(L_{-1} + 1)(L_0 + 1) \log(q L_n) + p^\kappa S \sum_{j=1}^{n} L_j V_j + nD \max_{1 \leqslant j \leqslant n} V_j \right\}$$

$$\leqslant \frac{1}{c_1} U \left\{ \left(1 + \frac{1}{c_0 - 1}\right)\left(\frac{1}{h_6} + \frac{1}{h_7}\right) c_1 + \left(1 + \frac{1}{c_0 - 1}\right) \frac{1}{c_2} + \right.$$

$$+ \left(\frac{1}{q} + \frac{1}{c_0 - 1}\right)\left(\log 3 \cdot \left(1 + \frac{1}{h_0}\right) + 1\right)\left(2 + \frac{1}{p-1}\right) \frac{1}{c_3} +$$

$$\left. + \left(1 + \frac{1}{h_4}\right)\left(\left(1 + \frac{1}{c_0 - 1}\right)\frac{1}{n} + 4 + \frac{1}{2^{10} q n D} + \frac{2 \log h_5}{h_0}\right) \frac{1}{c_4} \right\}$$

$$\leqslant \frac{1}{c_1} U \left\{ \left(1 - \frac{1}{q}\right)^2 \left(1 - \frac{1}{c_3(n+1)}\right)\left(1 - \frac{1}{h_1}\right) - \right.$$

$$\left. - \left(1 + \frac{1}{h_4}\right)\left(1 + \frac{1}{p-1}\right) \frac{1}{q^{n+1} f_{\not p}} \cdot \frac{1}{c_4} \right\},$$

contradicting (2.91). This contradiction proves (2.90), whence the lemma follows.

LEMMA 2.5. *The main inductive argument is true for* $J + 1$.

Proof. We first show that

$$[K(\xi^{G_1})(\alpha_1^{1/q}, \ldots, \alpha_n^{1/q}) : K(\xi^{G_1})] = q^n. \tag{2.97}$$

Let $K' = K(\alpha_1^{1/q}, \ldots, \alpha_n^{1/q})$. By (2.15),

$$[K' : K] = q^n. \tag{2.98}$$

From (2.20)–(2.22) we see that ξ^{G_1} is a root of the polynomial

$$x^{q^{\mu-u+1}} - \zeta_{q^u} \in K[x] \subseteq K'[x]. \tag{2.99}$$

By (2.15) we have $[K'(\alpha_0^{1/q}) : K'] = q$. This implies, by Abel's Theorem (see, for instance, Rédei [15], p. 674, Theorem 427), that $\alpha_0 \notin K'^q$, whence

$$\zeta_{q^u} = \alpha_0^{p^v} \notin K'^q, \tag{2.100}$$

since $(p, q) = 1$ by (2.3). Thus by Lemma 1.8, (2.1), (2.3) and (2.100), the polynomial in (2.99) is irreducible in $K'[x]$, that is,

$$[K'(\xi^{G_1}) : K'] = [K(\xi^{G_1}) : K] = q^{\mu-u+1}.$$

This together with (2.98) and the identity

$$[K'(\xi^{G_1}) : K(\xi^{G_1})][K(\xi^{G_1}) : K] = [K'(\xi^{G_1}) : K'][K' : K]$$

yields (2.97).

Write $\sigma = (\sigma_{-1}, \ldots, \sigma_n) \in \mathbb{N}^{n+2}$. By Lemma 2.4 and (2.50) we have

$$\sum_{\sigma \in \mathscr{C}^{(J)}} \sum_{d_0, d} p^{(J)}(\sigma, d_0, d) \xi_{d_0, d} \Delta(q^{-(J+1)}s + \sigma_{-1}; L_{-1} + 1, \sigma_0 + 1, \tau_0) \cdot$$

$$\cdot \prod_{j=1}^{n-1} \Delta(b_n \sigma_j - b_j \sigma_n; \tau_j) \cdot \prod_{j=1}^{n} (\alpha_j^{p^\kappa} \zeta^{r_j})^{\sigma_j(s/q)} = 0$$

$$\text{for} \quad 1 \leqslant s \leqslant q^{J+1}, \qquad (s, q) = 1, \qquad |\tau| \leqslant q^{-(J+1)} T, \tag{2.101}$$

where $\mathscr{C}^{(J)}$ is the set of $\sigma = (\sigma_{-1}, \ldots, \sigma_n)$ satisfying

$$0 \leqslant \sigma_j \leqslant L_j^{(J)} \quad (-1 \leqslant j \leqslant n), \qquad r_1 \sigma_1 + \cdots + r_n \sigma_n \equiv r^{(J)} \pmod{G_1}. \tag{2.65}$$

Every $(\sigma_1, \ldots, \sigma_n)$ satisfying (2.65) can be uniquely written as

$$\sigma_j = \lambda_j^* + q\lambda_j \quad (1 \leqslant j \leqslant n) \tag{2.102}$$

with

$$0 \leqslant \lambda_j^* < q, \qquad 0 \leqslant \lambda_j \leqslant L_j^{(J+1)}(\lambda_1^*, \ldots, \lambda_n^*) := \left[\frac{L_j^{(J)} - \lambda_j^*}{q}\right] \quad (1 \leqslant j \leqslant n). \tag{2.103}$$

By the fact that $(G_1, q) = 1$ (see (2.20)) and by (2.65), (2.102), we see that

$$r_1\lambda_1 + \cdots + r_n\lambda_n \equiv r^{(J+1)}(\lambda_1^*, \ldots, \lambda_n^*) \pmod{G_1}, \tag{2.104}$$

where $r^{(J+1)}(\lambda_1^*, \ldots, \lambda_n^*)$ is the unique solution of the congruence

$$qx \equiv r^{(J)} - (r_1\lambda_1^* + \cdots + r_n\lambda_n^*) \pmod{G_1} \quad \text{with} \quad 0 \leqslant x < G_1.$$

Further, again by (2.20), (2.65), (2.102), we see that every $\sigma \in \mathscr{C}^{(J)}$ determines an unique $g = g(\lambda_1^*, \ldots, \lambda_n^*, \lambda_1, \ldots, \lambda_n) \in \mathbb{Z}$ such that

$$\begin{aligned} r_1\lambda_1^* + \cdots + r_n\lambda_n^* + q(r_1\lambda_1 + \cdots + r_n\lambda_n) &= r_1\sigma_1 + \cdots + r_n\sigma_n \\ &\equiv r^{(J)} + g(\lambda_1^*, \ldots, \lambda_n^*, \lambda_1, \ldots, \lambda_n)G_1 \\ &\quad \pmod{qG} \end{aligned}$$

$$\text{with} \quad 0 \leqslant g(\lambda_1^*, \ldots, \lambda_n^*, \lambda_1, \ldots, \lambda_n) < q^{\mu+1}. \tag{2.105}$$

From this and (2.22) we get

$$\xi^{g(\lambda_1^*, \ldots, \lambda_n)G_1 s} = \zeta^{(r_1\lambda_1 + \cdots + r_n\lambda_n)s}\xi^{(r_1\lambda_1^* + \cdots + r_n\lambda_n^* - r^{(J)})s}. \tag{2.106}$$

Now on recalling the identity

$$(1 + x)^{zz'} = ((1 + x)^z)^{z'}$$

for $x, z, z' \in \mathbb{C}_p$ with $\operatorname{ord}_p x > 1/(p-1)$ and z, z' being integral (see Hasse [9], p. 273), we see, by (2.24), (2.25), (2.105), that

$$\prod_{j=1}^{n} (\alpha_j^{p^\kappa}\zeta^{r_j})^{\sigma_j(s/q)} = \prod_{j=1}^{n} (\alpha_j^{1/q})^{p^\kappa\lambda_j^* s} \cdot \prod_{j=1}^{n} \alpha_j^{p^\kappa\lambda_j s} \cdot \xi^{sr^{(J)}} \cdot \xi^{g(\lambda_1^*, \ldots, \lambda_n)G_1 s}. \tag{2.107}$$

On combining (2.101)–(2.104), (2.107), we obtain

$$\begin{aligned} &\sum_{\lambda_1^*=0}^{q-1} \cdots \sum_{\lambda_n^*=0}^{q-1} \prod_{j=1}^{n} (\alpha_j^{1/q})^{p^\kappa s\lambda_j^*} \sum_{\lambda_{-1}=0}^{L_{-1}^{(J)}} \sum_{\lambda_0=0}^{L_0^{(J)}} \sum_{\lambda_1, \ldots, \lambda_n} \sum_{d_0, d} p^{(J)}(\lambda_{-1}, \lambda_0, \lambda_1^* + \\ &\quad + q\lambda_1, \ldots, \lambda_n^* + q\lambda_n, d_0, d)\xi_{d_0,d} \cdot \Delta(q^{-(J+1)}s + \lambda_{-1}; L_{-1} + 1, \lambda_0 + 1, \tau_0) \\ &\quad \cdot \prod_{j=1}^{n-1} \Delta(q(b_n\lambda_j - b_j\lambda_n) + (b_n\lambda_j^* - b_j\lambda_n^*); \tau_j) \\ &\quad \cdot \prod_{j=1}^{n} \alpha_j^{p^\kappa\lambda_j s} \cdot \xi^{g(\lambda_1^*, \ldots, \lambda_n)G_1 s} = 0 \end{aligned}$$

$$\text{for} \quad 1 \leqslant s \leqslant q^{J+1}S, \qquad (s, q) = 1, \qquad |\tau| \leqslant q^{-(J+1)}T, \tag{2.108}$$

where $\Sigma_{\lambda_1,\ldots,\lambda_n}$ is taken over the range

$$0 \leqslant \lambda_j \leqslant L_j^{(J+1)}(\lambda_1^*,\ldots,\lambda_n^*), \quad r_1\lambda_1 + \cdots + r_n\lambda_n \equiv r^{(J+1)}(\lambda_1^*,\ldots,\lambda_n^*) \pmod{G_1}. \qquad (2.109)$$

By (2.97) and the fact that $(p^\kappa s, q) = 1$ (see (2.3)), we see that the q^n elements

$$\prod_{j=1}^{n} (\alpha_j^{1/q})^{p^\kappa s \lambda_j^*} \quad \text{with} \quad 0 \leqslant \lambda_j^* < q \ (1 \leqslant j \leqslant n)$$
are linearly independent over $K(\xi^{G_1})$. (2.110)

By the main inductive hypothesis for J, there exists a n-tuple $(\lambda_1^*,\ldots,\lambda_n^*)$ with $0 \leqslant \lambda_j^* < q(1 \leqslant j \leqslant n)$, such that the rational integers

$$p^{(J)}(\lambda_{-1},\lambda_0,\lambda_1^* + q\lambda_1,\ldots,\lambda_n^* + q\lambda_n, d_0, d)$$
$$\text{for} \quad 0 \leqslant \lambda_j \leqslant L_j^{(J)} \quad (j = -1, 0), \quad \lambda_1,\ldots,\lambda_n \text{ satisfying (2.109)},$$
$$0 \leqslant d_0 < D_0, \quad 1 \leqslant d \leqslant D_1$$

are not all zero. Fix this n-tuple $(\lambda_1^*,\ldots,\lambda_n^*)$; take
$$r^{(J+1)} := r^{(J+1)}(\lambda_1^*,\ldots,\lambda_n^*);$$

set

$$L_j^{(J+1)} := L_j^{(J)} = L_j \ (j = -1, 0), \qquad L_j^{(J+1)} := L_j^{(J+1)}(\lambda_1^*,\ldots,\lambda_n^*) \quad (1 \leqslant j \leqslant n),$$
$$p^{(J+1)}(\lambda_{-1},\lambda_0,\lambda_1,\ldots,\lambda_n,d_0,d) := p^{(J)}(\lambda_{-1},\lambda_0,\lambda_1^* + q\lambda_1,\ldots,\lambda_n^* + q\lambda_n, d_0, d)$$

and define $\mathscr{C}^{(J+1)}$ to be the set of $\lambda = (\lambda_{-1},\ldots,\lambda_n)$ satisfying

$$0 \leqslant \lambda_j \leqslant L_j^{(J+1)} \quad (-1 \leqslant j \leqslant n), \qquad r_1\lambda_1 + \cdots + r_n\lambda_n \equiv r^{(J+1)} \pmod{G_1}.$$

Obviously, by the choice of the n-tuple $(\lambda_1^*,\ldots,\lambda_n^*)$, $\mathscr{C}^{(J+1)} \neq \emptyset$. By (2.110), (2.106), we obtain from (2.108) that

$$\sum_{\lambda \in \mathscr{C}^{(J+1)}} \sum_{d_0,d} p^{(J+1)}(\lambda, d_0, d)\xi_{d_0,d}\Delta(q^{-(J+1)}s + \lambda_{-1}; L_{-1} + 1, \lambda_0 + 1, \tau_0)\cdot$$
$$\cdot \prod_{j=1}^{n-1} \Delta(q(b_n\lambda_j - b_j\lambda_n) + (b_n\lambda_j^* - b_j\lambda_n^*); \tau_j)\cdot \prod_{j=1}^{n} (\alpha_j^{p^\kappa}\zeta^{r_j})^{\lambda_j s} = 0$$

$$\text{for} \quad 1 \leqslant s \leqslant q^{J+1}S, \quad (s,q) = 1, \quad |\tau| \leqslant q^{-(J+1)}T. \qquad (2.111)$$

By an argument similar to that in the proof of Lemma 3.5 in Yu [21], utilizing Lemma 2.6 of [21], we conclude from (2.111) that

$$\phi_{J+1}(s,\tau)=0$$
$$\text{for } 1\leqslant s\leqslant q^{J+1}S, \qquad (s,q)=1, \qquad |\tau|\leqslant q^{-(J+1)}T.$$

This completes the proof of the lemma.

Thus we have established the main inductive argument for $J=0,1,\ldots,[\log L_n/\log q]+1$.

2.5. Completion of the proof of Proposition 2.1

The assumption that Proposition 2.1 is false, that is, there exist algebraic numbers $\alpha_1,\ldots,\alpha_n$ in K and $b_1,\ldots,b_n\in\mathbb{Z}$ satisfying (2.14)–(2.18), such that

$$\operatorname{ord}_p\Theta\geqslant U,$$

implies that the main inductive argument holds for $J_0=[\log L_n/\log q]+1$, whence we can deduce a contradiction (on utilizing Lemma 2.5 of Yu [21], Lemma 1.6 and (2.46), (2.47); the argument here is completely the same as in Section 3.5 of [21]), thereby proving the Proposition.

2.6. Proof of Theorem 2.1

Now this can be reduced to solving the system of inequalities (2.27)–(2.29). We solve it in the following cases respectively:

(1.a) $p=2,\quad n\geqslant 8$;
(1.b) $p=2,\quad 2\leqslant n\leqslant 7$;
(2.a) $p>2,\quad n\geqslant 8$;
(2.b) $p>2,\quad 2\leqslant n\leqslant 5$;
(2.c) $p>2,\quad n=6,7$.

Case (1.a). $p=2, n\geqslant 8$.
In this case $q=3, f_{\mathfrak{p}}\geqslant 2$ (see (2.2)), $c_0=17, c_2=\frac{8}{3}$. We have the following

estimates:

$$\frac{1}{h_0} \leqslant 1.08736 \times 10^{-2}, \qquad \frac{1}{h_1} \leqslant 6.48 \times 10^{-27},$$

$$\frac{1}{h_2} \leqslant 1.2 \times 10^{-23}, \qquad 1 + \varepsilon_1 \leqslant 1 + 10^{-22},$$

$$\frac{1}{h_3} \leqslant 4.15 \times 10^{-25}, \qquad 1 + \varepsilon_2 \leqslant 1 + 5 \times 10^{-25},$$

$$(1 + \varepsilon_1)(1 + \varepsilon_2) \leqslant 1 + 1.1 \times 10^{-22},$$

$$\frac{1}{h_4} \leqslant 3.927 \times 10^{-26}, \qquad \log h_5 \leqslant 5.3228576,$$

$$\frac{1}{h_6} \leqslant 1.486 \times 10^{-29}, \qquad \frac{1}{h_7} \leqslant 1.326 \times 10^{-29}.$$

It is easy to verify that

$$c_0 = 17, \quad c_1 = 1.7986328, \quad c_2 = \tfrac{8}{3}, \quad c_3 = 110.8111, \quad c_4 = 187.84615$$

satisfy the system of inequalities (2.27)–(2.29).

Case (1.b). $p = 2$, $2 \leqslant n \leqslant 7$.
In this case $q = 3$, $f_{\mathfrak{p}} \geqslant 2$, $c_0 = 9, c_2 = \frac{18}{5}$. We have the following estimates:

$$\frac{1}{h_0} \leqslant 4.94584 \times 10^{-2}, \qquad \frac{1}{h_1} \leqslant 7.656646 \times 10^{-9},$$

$$\frac{1}{h_2} \leqslant 1.48846 \times 10^{-6}, \qquad 1 + \varepsilon_1 \leqslant 1 + 2.977 \times 10^{-6},$$

$$\frac{1}{h_3} \leqslant 3.06267 \times 10^{-8}, \qquad 1 + \varepsilon_2 \leqslant 1 + 3.063 \times 10^{-8},$$

$$(1 + \varepsilon_1)(1 + \varepsilon_2) \leqslant 1 + 3.00764 \times 10^{-6},$$

$$\frac{1}{h_4} \leqslant 4.82116 \times 10^{-8}, \qquad \log h_5 \leqslant 4.6310664,$$

$$\frac{1}{h_6} \leqslant 3.994 \times 10^{-11}, \qquad \frac{1}{h_7} \leqslant 8.191 \times 10^{-11}.$$

It is easy to verify that

$$c_0 = 9, \quad c_1 = 1.8412753, \quad c_2 = \tfrac{18}{5}, \quad c_3 = 46.503685, \quad c_4 = 79.452008$$

satisfy the system of inequalities (2.27)–(2.29).

Case (2.a). $p > 2, n \geqslant 8$.

In this case $q = 2$, $f_{\mathfrak{p}} \geqslant 1$, $D/q^u \geqslant \frac{1}{2}$ (see (2.7)), $c_0 = 17$, $c_2 = 5$. We have the following estimates:

$$\frac{1}{h_0} \leqslant 0.011272, \qquad \frac{1}{h_1} \leqslant 7.13 \times 10^{-26},$$

$$\frac{1}{h_2} \leqslant 1.027 \times 10^{-22}, \qquad 1 + \varepsilon_1 \leqslant 1 + 8.22 \times 10^{-22},$$

$$\frac{1}{h_3} \leqslant 4.57 \times 10^{-24}, \qquad 1 + \varepsilon_2 \leqslant 6 \times 10^{-24},$$

$$(1 + \varepsilon_1)(1 + \varepsilon_2) \leqslant 1 + 8.3 \times 10^{-22},$$

$$\frac{1}{h_4} \leqslant 2.325 \times 10^{-25}, \qquad \log h_5 \leqslant 4.9272357,$$

$$\frac{1}{h_6} \leqslant 1.4 \times 10^{-28}, \qquad \frac{1}{h_7} \leqslant 1.3 \times 10^{-28}.$$

It is easy to verify that

$$c_0 = 17, \quad c_1 = 0.4100107 \cdot \left(2 + \frac{1}{p-1}\right),$$

$$c_2 = 5, \quad c_3 = 63.710446 \cdot \left(2 + \frac{1}{p-1}\right), \quad c_4 = 227.85949$$

satisfy the system of inequalities (2.27)–(2.29).

Case (2.b). $p > 2$, $2 \leqslant n \leqslant 5$.

In this case $q = 2$, $f_{\mathfrak{p}} \geqslant 1, D/q^u \geqslant \frac{1}{2}, c_0 = 9, c_2 = 7$. We have the following estimates:

$$\frac{1}{h_0} \leqslant 0.051525, \qquad \frac{1}{h_1} \leqslant 4.1008 \times 10^{-8},$$

$$\frac{1}{h_2} \leqslant 6.88933 \times 10^{-6}, \qquad 1 + \varepsilon_1 \leqslant 1 + 1.3779 \times 10^{-5},$$

$$\frac{1}{h_3} \leqslant 1.64032 \times 10^{-7}, \qquad 1 + \varepsilon_2 \leqslant 1 + 1.641 \times 10^{-7},$$

$$(1 + \varepsilon_1)(1 + \varepsilon_2) \leqslant 1 + 1.39432 \times 10^{-5},$$

$$\frac{1}{h_4} \leqslant 9.19912 \times 10^{-8}, \qquad \log h_5 \leqslant 4.3384949,$$

$$\frac{1}{h_6} \leqslant 1.761 \times 10^{-10}, \qquad \frac{1}{h_7} \leqslant 3.555 \times 10^{-10}.$$

It is easy to verify that

$$c_0 = 9, \quad c_1 = 0.4296612 \cdot \left(2 + \frac{1}{p-1}\right), \quad c_2 = 7,$$

$$c_3 = 30.649838 \cdot \left(2 + \frac{1}{p-1}\right), \quad c_4 = 118.25702$$

satisfy the system of inequalities (2.27)–(2.29).

Case (2.c). $p > 2$, $n = 6, 7$.

In this case $q = 2$, $f_{\mathfrak{p}} \geqslant 1$, $D/q^u \geqslant \frac{1}{2}$, $c_0 = 9$, $c_2 = \frac{27}{4}$. We have the following estimates:

$$\frac{1}{h_0} \leqslant 0.0154283, \qquad \frac{1}{h_1} \leqslant 8.398 \times 10^{-20},$$

$$\frac{1}{h_2} \leqslant 9.523 \times 10^{-17}, \qquad 1 + \varepsilon_1 \leqslant 1 + 5.75 \times 10^{-16},$$

$$\frac{1}{h_3} \leqslant 3.024 \times 10^{-18}, \qquad 1 + \varepsilon_2 \leqslant 1 + 3.04 \times 10^{-18},$$

$$(1 + \varepsilon_1)(1 + \varepsilon_2) \leqslant 1 + 5.8 \times 10^{-16},$$

$$\frac{1}{h_4} \leqslant 1.95 \times 10^{-19}, \qquad \log h_5 \leqslant 3.7861582,$$

$$\frac{1}{h_6} \leqslant 3.71 \times 10^{-22}, \qquad \frac{1}{h_7} \leqslant 3.7 \times 10^{-22}.$$

It is easy to verify that

$$c_0 = 9, \quad c_1 = 0.4099183\cdot\left(2 + \frac{1}{p-1}\right), \quad c_2 = \frac{27}{4},$$

$$c_3 = 31.978249\cdot\left(2 + \frac{1}{p-1}\right), \quad c_4 = 104.3852$$

satisfy the system of inequalities (2.27)–(2.29).

In each of the above cases it is easily seen that

$$(1+\varepsilon_1)(1+\varepsilon_2)c_0c_1c_3c_4 \leqslant c\left(2 + \frac{1}{p-1}\right)^2,$$

where c is the constant given in the statement of Theorem 2.1. Now the Theorem follows from Proposition 2.1 at once.

2.7. Proof of Theorem 2.2 and Corollaries 2.3, 2.4

Proof of Theorem 2.2. Set

$$K' := \mathbb{Q}(\alpha_0, \alpha_1, \ldots, \alpha_n) \subseteq K, \quad D' := [K' : \mathbb{Q}]. \tag{2.112}$$

By (2.1), (2.4), (2.6) we see that K' satisfies (2.1). Denoting by $O_{K'}$ the ring of integers in K', set

$$\mathfrak{p}' = \mathfrak{p} \cap O_{K'}.$$

Then $\mathfrak{p}'$ is a prime ideal of $O_{K'}$, and we define $\operatorname{ord}_{\mathfrak{p}'}\alpha(\alpha \in K')$, $e_{\mathfrak{p}'}$, $f_{\mathfrak{p}'}$ in the way described in Section 0.2. Obviously

$$f_{\mathfrak{p}'} \leqslant f_{\mathfrak{p}}, \tag{2.113}$$

$$\begin{aligned} u' &:= \max\{t \in \mathbb{N} \mid \zeta_{q^t} \in K'\} = u, \\ v' &:= \max\{t \in \mathbb{N} \mid \zeta_{p^t} \in K'\} = v, \end{aligned} \tag{2.114}$$

$$\alpha_0' := e^{2\pi i/(p^{v'}q^{u'})} = \alpha_0, \tag{2.115}$$

$$\operatorname{ord}_{\mathfrak{p}'}\alpha = \frac{e_{\mathfrak{p}'}}{e_{\mathfrak{p}}}\operatorname{ord}_{\mathfrak{p}}\alpha \quad \text{for } \alpha \in K'. \tag{2.116}$$

Let

$$V'_j := \frac{Df_{\wp'}}{D'f_{\wp}} \cdot V_j \quad (1 \leqslant j \leqslant n), \tag{2.117}$$

$$(W^*)' := \max\left\{\log\left(1 + \frac{1}{\rho n}\cdot\frac{f_{\wp'}\log p}{D'}\left(\frac{B_n}{V'_1} + \frac{B'}{V'_n}\right)\right),\right.$$

$$\left.\rho''\log B_0, \frac{f_{\wp'}\log p}{D'}, n\log(2^{11}qnD')\right\}, \tag{2.118}$$

$$(V^*_{n-1})' := \max\left(p^{f_{\wp'}}, (2^{11}qn(D')^2 V'_{n-1})^n\right). \tag{2.119}$$

It is well-known that

$$\frac{f_{\wp}}{f_{\wp'}} \leqslant \frac{e_{\wp}f_{\wp}}{e_{\wp'}f_{\wp'}} \leqslant \frac{D}{D'}. \tag{2.120}$$

By virtue of (2.120) and utilizing (2.8)–(2.10), (2.12), (2.13), (2.117)–(2.119), we see that

$$V'_j \geqslant \max\left(h(\alpha_j), \frac{f_{\wp'}\log p}{D'}\right) \quad (1 \leqslant j \leqslant n), \qquad V'_1 \leqslant \cdots \leqslant V'_{n-1},$$

$$\frac{D'}{f_{\wp'}}V'_j = \frac{D}{f_{\wp}}V_j \quad (1 \leqslant j \leqslant n), \qquad \frac{D'}{f_{\wp'}}(W^*)' \leqslant \frac{D}{f_{\wp}}W^*,$$

$$\frac{e_{\wp}D'}{e_{\wp'}f_{\wp'}}\log(V^*_{n-1})' \leqslant \frac{D}{f_{\wp}}\log V^*_{n-1}. \tag{2.121}$$

On observing further that

$\mathbb{Q}(\alpha'_0, \alpha_1, \ldots, \alpha_n) = K'$ (by (2.112), (2.115)),

$[K'((\alpha'_0)^{1/q}, \alpha_1^{1/q}, \ldots, \alpha_n^{1/q}) : K'] = q^{n+1}$ (by (2.15), (2.115)),

$\mathrm{ord}_{\wp'}\alpha_j = 0 \quad (1 \leqslant j \leqslant n)$ (by (2.16), (2.116)),

we may apply Theorem 2.1 to $\mathrm{ord}_{\wp'}(\alpha_1^{b_1}\cdots\alpha_n^{b_n} - 1)$ with V'_j $(1 \leqslant j \leqslant n)$, $(W^*)'$,

$(V_{n-1}^*)'$ given by (2.117)–(2.119); and on utilizing (2.121), (2.113), (2.114), (2.116), we obtain the inequality stated in Theorem 2.1. This proves Theorem 2.2.

Proof of Corollary 2.3. We remark, by (2.8), (2.11)–(2.13) and the fact $n \geqslant 2$, that in Theorems 2.1 and 2.2 we may choose

$$W^* = \max\left(\log B, n\log(2^{11}qnD), \frac{f_{\mathfrak{p}}\log p}{D}\right).$$

Note further that the constants a, c in the statement of Theorem 2.1 satisfy

$$ca^n \leqslant \begin{cases} 101186.36 \times 5^n, & \text{if } p > 2, \\ 70718.74 \times (\frac{8}{3})^n, & \text{if } p = 2. \end{cases}$$

Now, on noting (2.3), we see that Theorems 2.1 and 2.2 yield the Corollary.

Proof of Corollary 2.4. By (2.15)–(2.18) we may apply Theorems 2.1 and 2.2 with V_n replaced by

$$V_n' := \max\left(V_n, \frac{\delta B}{B_n Z W'}\right).$$

We may also replace B', B_0 in (2.12) by B, B_n, respectively. By the inequalities (2.8), $0 < \delta \leqslant (f_{\mathfrak{p}}\log p/D)Z$, $W' > 1$, we see that

$$\frac{B_n}{V_1} + \frac{B}{V_n'} \leqslant \frac{B_n}{V_1} + \delta^{-1}ZB_nW' \leqslant 2\delta^{-1}ZB_nW'.$$

On recalling (2.12), (2.13), $n \geqslant 2$, it suffices to prove

$$\max\left\{\log\left(1 + \frac{1}{\rho}\delta^{-1}\frac{f_{\mathfrak{p}}\log p}{D}ZB_nW'\right), \log B_n, \frac{f_{\mathfrak{p}}\log p}{D}, n\log(2^{11}qnD)\right\} \leqslant W'.$$

By the assumptions on Z, δ, W', we need only to show that

$$\log\left(1 + \frac{1}{\rho}\psi W'\right) \leqslant W', \tag{2.122}$$

where

$$\psi = \delta^{-1}\frac{f_{\mathfrak{p}}\log p}{D}ZB_n.$$

We need two inequalities, which can be easily verified:

$$\frac{d}{dx}(x - \log(1 + bx)) > 0 \quad \text{for } x \geqslant 1, \tag{2.123}$$

where $b > 0$ is fixed; and

$$g'(x) > 0 \quad \text{for } x \geqslant 10^6 \tag{2.124}$$

with

$$g(x) := \rho' \log x - \log(1 + (\rho'/\rho)x \log x),$$

where

$$\rho = \begin{cases} 5, & \text{if } p > 2, \\ \frac{8}{3}, & \text{if } p = 2 \end{cases} \quad \text{and} \quad \rho' = \begin{cases} 1.0752, & \text{if } p > 2, \\ 1.1114, & \text{if } p = 2. \end{cases}$$

By the hypothesis on W', we have

$$W' \geqslant \rho' \max\{\log \psi, (n/\rho') \log(2^{11}qnD)\}. \tag{2.125}$$

We devide two cases.

(a) $\psi \geqslant (2^{11}qnD)^{n/\rho'}$. By (2.123) and (2.125), to prove (2.122) it suffices to show that $g(\psi) > 0$. By (2.3) and $n \geqslant 2$, $D \geqslant 2$ it is easy to verify that

$$g((2^{11}qnD)^{n/\rho'}) > 0. \tag{2.126}$$

On noting that

$$\psi \geqslant (2^{11}qnD)^{n/\rho'} > 10^6$$

and utilizing (2.124), (2.126), we obtain $g(\psi) > 0$.

(b) $\psi < (2^{11}qnD)^{n/\rho'}$. By (2.123) and (2.125), we see that (2.122) follows from (2.126).

This completes the proof of Corollary 2.4.

3. Propositions for Kummer descent

The condition $\mathrm{ord}_p\, b_n = \min_{1 \leqslant j \leqslant n} \mathrm{ord}_p\, b_j$ yields the sharpest form of the main results of the present paper. When $b_1, \ldots, b_n$ satisfy this condition, to transfer this property during the course of the Kummer descent (see (4.20) and (4.56) in Section 4 below) is somewhat subtle, and some complication, compared with the Kummer descent in the classical theory of linear forms in logarithms, arises from here. The statement (d) of the following Propositions 3.1 and 3.3 is for this purpose. Furthermore, we use the idea in the proof of Lemma 4.1 in Waldschmidt [18] and in the proof of Lemmas 5.1, 5.2 in Lang [10], Chapter XI; and we give refinements in our context, in order to obtain good constants in our estimates for $\mathrm{ord}_{\wp}(\alpha_1^{b_1} \ldots \alpha_n^{b_n} - 1)$.

Let $K, D, p, q, u, v, \alpha_0, \wp, \mathrm{ord}_{\wp}, f_{\wp}$ be defined in Section 0.2. Evidently,

$$2q \leqslant p^v q^u \leqslant 3D. \tag{3.1}$$

Fix

$$l_0 := \frac{2\pi i}{p^v q^u}, \quad V_0 := \frac{1}{p^v q^u D}. \tag{3.2}$$

Recall $\mathscr{L}_K := \{l \in \mathbb{C} \mid e^l \in K\}$ and for $l \in \mathscr{L}_K$

$$V(l) := \max\left\{h(e^l), \frac{|l|}{2\pi D}, \frac{f_{\wp} \log p}{D}\right\}. \tag{3.3}$$

Define

$$\mathscr{L}_{K,\wp} := \{l \in \mathbb{C} \mid e^l \in K,\ \mathrm{ord}_{\wp}(e^l) = 0\}. \tag{3.4}$$

Obviously, $l_0 \in \mathscr{L}_{K,\wp}$. Throughout this Section $l_1, \ldots, l_n$ denote $n(\geqslant 2)$ elements of $\mathscr{L}_{K,\wp}$ such that

$$|\mathrm{Im}\, l_j| \leqslant \pi \quad (1 \leqslant j \leqslant n), \tag{3.5}$$

and $V_1, \ldots, V_n$ denote n real numbers satisfying

$$V_1 \leqslant \cdots \leqslant V_n \tag{3.6}$$

and

$$V_j \geqslant V(l_j) \quad (1 \leqslant j \leqslant n). \tag{3.7}$$

By linear dependence (or independence) of elements of $\mathscr{L}_{K,\wp}$ we mean that over $\mathbb{Q}$. By the rank of a finite set of elements in $\mathscr{L}_{K,\wp}$, we mean the cardinal of a maximal linearly independent subset of the given set.

PROPOSITION 3.1. *Suppose that $l_0, l_1, \ldots, l_n$ are linearly independent. Then there exist $l'_0 = l_0, l'_1, \ldots, l'_n \in \mathscr{L}_{K,\mathfrak{p}}$ and $m_{sj} \in \mathbb{Z}(1 \leqslant s \leqslant n, 0 \leqslant j \leqslant s)$ such that*[†]

(a) $[K((\alpha'_0)^{1/q}, \ldots, (\alpha'_n)^{1/q}) : K] = q^{n+1}$, *where* $\alpha'_j := e^{l'_j}$ $(0 \leqslant j \leqslant n)$,

(b) $V(l'_s) \leqslant \max(V_s, \frac{1}{2}(V_0 + \cdots + V_s))$ $(1 \leqslant s \leqslant n)$,

(c) $l_s = \sum_{j=0}^{s} m_{sj} l'_j$ $(1 \leqslant s \leqslant n)$,

(d) $m_{ss} = q^{w_s}$ *for some* $w_s \in \mathbb{N}$ $(1 \leqslant s \leqslant n)$,

(e) $\max_{1 \leqslant j \leqslant s} |m_{sj}| \leqslant 2((s+1)D^3)^{s+1}(s+1)!\, V_0 V_s^s$ $(1 \leqslant s < n)$,

$\max_{1 \leqslant j \leqslant n} |m_{nj}| \leqslant 4((n+1)D^3)^{n+1} n!\, V_0 V_{n-1}^{n-2} V_n \max(V_n, \frac{1}{2}(nV_{n-1} + V_n))$.

Proof. Let $\mathscr{M} = \mathbb{Z}l_0 + \mathbb{Z}l_1 + \cdots + \mathbb{Z}l_n$ and

$$\mathscr{M}_q = \{l \in \mathscr{L}_{K,\mathfrak{p}} \mid \text{there exists } t \in \mathbb{N} \text{ such that } q^t l \in \mathscr{M}\}.$$

For $l \in \mathscr{M}_q$ write $\bar{l} = l + \mathscr{M} \in \mathscr{M}_q/\mathscr{M}$. Then the order of $\bar{l}$ in $\mathscr{M}_q/\mathscr{M}$ is q^h for some $h \in \mathbb{N}$, and by Lemma 1.11 we see that

$$q^h \leqslant (2(n+1)D^3)^{n+1} V_0 V_1 \ldots V_n. \tag{3.8}$$

Set $q^w := \max\{\text{order of } \bar{l} \mid l \in \mathscr{M}_q\}$, then

$$q^w \mathscr{M}_q \subseteq \mathscr{M}. \tag{3.9}$$

For $s = 0, 1, \ldots, n$, let

$$N_s = \left\{ t \in \mathbb{Z} \mid t > 0, \text{ there exist } t_{sj} \in \mathbb{Z}(0 \leqslant j < s) \text{ such that } \sum_{j=0}^{s-1} t_{sj} l_j + t l_s \in q^w \mathscr{M}_q \right\}.$$

We see, by (3.9), that $q^w \in N_s (0 \leqslant s \leqslant n)$, whence N_s has the least element t_{ss} satisfying $1 \leqslant t_{ss} \leqslant q^w$. We fix $t_{sj} (0 \leqslant j < s)$ such that

$$t_{sj} = 0 \quad (0 \leqslant j < s), \qquad \text{if } t_{ss} = q^w; \tag{3.10}$$

$$-\tfrac{1}{2}q^w < t_{sj} \leqslant \tfrac{1}{2}q^w \quad (0 \leqslant j < s), \quad \text{if } t_{ss} < q^w. \tag{3.11}$$

((3.11) is possible in virtue of the division algorithm.) Then there exist $l'_0, l'_1, \ldots, l'_n$

[†] Here and in the sequel $\alpha^{1/q}$ $(\alpha \in K)$ denotes qth root, which may be chosen in $\mathbb{C}_p$. See also the remark after the statement of Theorem 2.1.

in $\mathcal{M}_q$ such that

$$\sum_{j=0}^{s} t_{sj} l_j = q^w l'_s \quad (0 \leqslant s \leqslant n). \tag{3.12}$$

By the linear independence of $l_0, l_1, \ldots, l_n$ and by the construction, $\{q^w l'_0, \ldots, q^w l'_n\}$ is a basis of $q^w \mathcal{M}_q$, whence $\{l'_0, \ldots, l'_n\}$ is a basis of $\mathcal{M}_q$. Observing $l_s \in \mathcal{M} \subseteq \mathcal{M}_q$ $(0 \leqslant s \leqslant n)$, we see that there exist $m_{sj} \in \mathbb{Z}$ $(0 \leqslant s, j \leqslant n)$ such that

$$l_s = \sum_{j=0}^{n} m_{sj} l'_j. \tag{3.13}$$

On combining (3.12) with (3.13) we get

$$m_{sj} = 0 \quad (0 \leqslant s < n, s < j \leqslant n) \tag{3.14}$$

and

$$t_{ss} m_{ss} = q^w \quad (0 \leqslant s \leqslant n). \tag{3.15}$$

Now (3.13)–(3.15) imply (c) and (d). We assert that

$$t_{00} = q^w,$$

for otherwise we would have, by (3.15), $t_{00} = q^{w_0}$ with $0 \leqslant w_0 < w$, whence, by (3.12), $l_0 = q^{w-w_0} l'_0$ and $\alpha_0 = (\alpha'_0)^{q^{w-w_0}} \in K^q$, a contradiction to the definition of α_0 (see (0.4)–(0.6)). Hence $l_0 = l'_0$, $\alpha_0 = \alpha'_0$. By (d), we see that $t_{ss} < q^w$ implies $t_{ss} \leqslant q^{w-1} \leqslant \frac{1}{2} q^w$. By this observation, (3.10)–(3.12) yield (b). By (3.7), (3.3), Lemma 1.12, (3.2), we have $V_s \geqslant f_{\mathfrak{p}} (\log p)/D > 1/D > V_0$ $(1 \leqslant s \leqslant n)$, whence $\frac{1}{2}(V_0 + V_1 + \cdots + V_s) \leqslant \frac{1}{2}(s+1) V_s$. So from (b) and (3.3), we obtain

$$\max\left(h(e^{l'_s}), \frac{|l'_s|}{2\pi D}\right) \leqslant \tfrac{1}{2}(s+1) V_s \quad (1 \leqslant s \leqslant n).$$

Now Lemma 1.11 together with the above estimates and the linear independence of $l'_0, \ldots, l'_n$ implies (e). It remains to verify (a). Suppose that (a) is false, i.e., $K((\alpha'_0)^{1/q}, \ldots, (\alpha'_n)^{1/q})$ has degree (over K) less than q^{n+1}, we proceed to deduce a contradiction. By Lemma 1.9, we have a relation

$$(\alpha'_0)^{j_0} \ldots (\alpha'_n)^{j_n} \eta^q = 1 \tag{3.16}$$

for some $\eta \in K\backslash\{0\}$ and $j_0, \ldots, j_n \in \mathbb{N}$ with

$$1 \leqslant \max_{0 \leqslant s \leqslant n} j_s < q. \tag{3.17}$$

Note that $\operatorname{ord}_{\mathfrak{p}} \alpha'_s = 0$, since $l'_s \in \mathscr{M}_q \subseteq \mathscr{L}_{K,\mathfrak{p}}$ $(0 \leqslant s \leqslant n)$. Thus by (3.16) we have $\operatorname{ord}_{\mathfrak{p}} \eta = 0$. Let $\lambda \in \mathscr{L}_{K,\mathfrak{p}}$ be such that $e^{\lambda} = \eta$. Now by (3.16) there exists $j \in \mathbb{Z}$ such that

$$j_0 l'_0 + \cdots + j_n l'_n + q\lambda + j \cdot 2\pi i = 0.$$

Write

$$l = -\left(\lambda + j \cdot \frac{2\pi i}{q}\right).$$

Note that $2\pi i/q \in \mathscr{L}_{K,\mathfrak{p}}$, since $\zeta_q \in K$ by (0.3). Thus $l \in \mathscr{L}_{K,\mathfrak{p}}$ and

$$ql = j_0 l'_0 + \cdots + j_n l'_n \in \mathscr{M}_q, \tag{3.18}$$

whence $l \in \mathscr{M}_q$. Therefore there exist $i_0, \ldots, i_n \in \mathbb{Z}$ such that

$$l = i_0 l'_0 + \cdots + i_n l'_n. \tag{3.19}$$

On comparing (3.18) with (3.19), we get, by the linear independence of $l'_0, \ldots, l'_n$,

$$j_s = q i_s \quad (0 \leqslant s \leqslant n),$$

a contradiction to (3.17). This proves (a). The proof of the Proposition is thus complete.

LEMMA 3.1. *Let $l_1, \ldots, l_n, l'_j, \alpha'_j$ $(0 \leqslant j < n)$ be given in Proposition* 3.1 *and its proof. Suppose that*

$$V_n \geqslant n V_{n-1}. \tag{3.20}$$

Suppose further that $l \in \mathscr{L}_{K,\mathfrak{p}}$ and $V > 0$ are such that

$$l_0, \ldots, l_{n-1}, l \text{ are linearly independent}, \tag{3.21}$$

$$|\operatorname{Im} l| \leqslant \pi, \quad V(l) \leqslant V, \tag{3.22}$$

$$n V_{n-1} \leqslant V \leqslant V_n. \tag{3.23}$$

Then there exist $l' \in \mathscr{L}_{K,\not{p}}$ *and* $m_0, \ldots, m_n \in \mathbb{Z}$ *such that*

(a) $[K((\alpha_0')^{1/q}, \ldots, (\alpha_{n-1}')^{1/q}, (\eta')^{1/q}) : K] = q^{n+1}$, *where* $\eta' = e^{l'}$,

(b) $V(l') \leqslant V$,

(c) $l = m_0 l_0' + \cdots + m_{n-1} l_{n-1}' + m_n l'$,

(d) $m_n = q^{v_n}$ *for some* $v_n \in \mathbb{N}$,

(e.1) $\max_{1 \leqslant j \leqslant n} |m_j| \leqslant C_3 D V_0 (DV)^{C_6}$,

(e.2) $\max_{1 \leqslant j \leqslant n} |m_j| \leqslant \left(\dfrac{V}{\max(V(l'), nV_{n-1})} \right)^{C_4} C_5 D V_0 (D V_{n-1})^n$,

where

$$C_3 = 4((n+1)D^2)^{n+1} n!/n^{n-2}, \qquad C_4 = \begin{cases} 2.41, & \text{if } p > 2, \\ 2.71, & \text{if } p = 2, \end{cases}$$

$$C_5 = \begin{cases} C_3 (n+1)^{2n}, & \text{if } p > 2, \\ C_3 (n+1)^{2n} \cdot (\frac{3}{4})^n, & \text{if } p = 2, \end{cases} \qquad C_6 = \max(n, C_4).$$

Proof. Let

$$U_0 = \begin{cases} (n+1)^2 V_{n-1}, & \text{if } p > 2, \\ \frac{3}{4}(n+1)^2 V_{n-1}, & \text{if } p = 2 \end{cases} \tag{3.24}$$

and (if $V_n > U_0$)

$$k_0 = \begin{cases} D(V_n - U_0), & \text{if } D(V_n - U_0) \in \mathbb{Z}, \\ [D(V_n - U_0)] + 1, & \text{otherwise.} \end{cases} \tag{3.25}$$

We proceed to prove the following assertions:

(P_0) The Lemma holds if $nV_{n-1} \leqslant V \leqslant \min(U_0, V_n)$;

and (if $V_n > U_0$)

(P_k) The Lemma holds if $U_0 + (k-1)/D < V \leqslant \min(U_0 + k/D, V_n)$ $(k = 1, \ldots, k_0)$.

We now show (P_0). On applying Proposition 3.1 to $l_0, \ldots, l_{n-1}, l$ and $V_1, \ldots, V_{n-1}, V$, we see that there exist $l' \in \mathscr{L}_{K,\not{p}}$ and $m_0, \ldots, m_n \in \mathbb{Z}$ such that (a), (c), (d) hold and that

$$V(l') \leqslant \max(V, \tfrac{1}{2}(V_0 + \cdots + V_{n-1} + V)) \leqslant V$$

(by the fact that $nV_{n-1} \leqslant V$), whence (b) is valid. Further by Lemma 1.11 and by (c), (b), (3.22), (b) of Proposition 3.1, the inequality $nV_{n-1} \leqslant V$ and the linear independence of $l'_0, \ldots, l'_{n-1}, l$, we have

$$\begin{aligned} \max_{1 \leqslant j \leqslant n} |m_j| &\leqslant 4((n+1)D^2)^{n+1} n! DV_0(DV_{n-1})^{n-2}(DV)^2 \\ &\leqslant C_3 DV_0(DV)^n \leqslant C_3 DV_0(DV)^{C_6}, \end{aligned} \tag{3.26}$$

i.e. (e.1) holds. From the second inequality of (3.26) and the assumption $V \leqslant \min(U_0, V_n)$, recalling (3.24), we get

$$\max_{1 \leqslant j \leqslant n} |m_j| \leqslant C_5 DV_0(DV_{n-1})^n.$$

This together with (b) and $nV_{n-1} \leqslant V$ implies (e.2). Thus we see that (P_0) is true. If $V_n \leqslant U_0$, then (P_0) is exactly the Lemma. So we may assume $V > U_0$ and we prove the Lemma by induction on k.

Assuming $(P_0), \ldots, (P_k)(0 \leqslant k < k_0)$, we proceed to show (P_{k+1}). Now

$$U_0 + \frac{k}{D} < V \leqslant \min\left(U_0 + \frac{k+1}{D}, V_n\right). \tag{3.27}$$

If $[K((\alpha'_0)^{1/q}, \ldots, (\alpha'_{n-1})^{1/q}, \eta^{1/q}) : K] = q^{n+1}$, where $\eta = e^l$, then we may take $l' = l$, $\eta' = \eta$, $m_0 = \cdots = m_{n-1} = 0$, $m_n = 1$, whence (P_{k+1}) is trivially true. So we may assume in the sequel

$$[K((\alpha'_0)^{1/q}, \ldots, (\alpha'_{n-1})^{1/q}, \eta^{1/q}) : K] < q^{n+1}. \tag{3.28}$$

Set $K' = K((\alpha'_0)^{1/q}, \ldots, (\alpha'_{n-1})^{1/q})$. By Proposition 3.1, we have $[K' : K] = q^n$. This together with (3.28) yields $[K'(\eta^{1/q}) : K'] < q$. Thus by Lemma 1.9, there exist $\eta_1 \in K \backslash \{0\}$ and $t_0, \ldots, t_{n-1} \in \mathbb{Z}$ with $0 \leqslant t_j < q$ $(0 \leqslant j < n)$ such that

$$\eta = (\alpha'_0)^{t_0} \ldots (\alpha'_{n-1})^{t_{n-1}} \eta_1^q. \tag{3.29}$$

From (3.29) and the fact that $l, l'_0, \ldots, l'_{n-1} \in \mathscr{L}_{K,\wp}$, we see that $\mathrm{ord}_{\wp} \eta_1 = 0$. So there exists $\lambda_1 \in \mathbb{C}$ such that

$$\lambda_1 \in \mathscr{L}_{K,\wp}, \quad e^{\lambda_1} = \eta_1, \quad |\mathrm{Im}\, \lambda_1| \leqslant \pi. \tag{3.30}$$

From (3.29) we get, by $l'_0 = l_0$ and (3.2),

$$\begin{aligned} l &= t_0 l'_0 + \cdots + t_{n-1} l'_{n-1} + q\lambda_1 + t \cdot 2\pi i \\ &= (t_0 + p^v q^u t) l'_0 + t_1 l'_1 + \cdots + t_{n-1} l'_{n-1} + q\lambda_1 \end{aligned} \tag{3.31}$$

for an integer t. Now the linear independence of $l_0, \ldots, l_{n-1}, l$ implies that of $l'_0, \ldots, l'_{n-1}, l$. This together with (3.31) yields the linear independence of $l'_0, \ldots, l'_{n-1}, \lambda_1$, whence

$$l_0, \ldots, l_{n-1}, \lambda_1 \quad \text{are linearly independent.} \tag{3.32}$$

Note that by (3.27) and (3.24) we have

$$nV_{n-1} \leqslant \frac{q+1}{2q} V \leqslant \min\left(U_0 + \frac{k}{D}, V_n\right). \tag{3.33}$$

Next we show that

$$V(\lambda_1) \leqslant \frac{q+1}{2q} V. \tag{3.34}$$

From (3.29) and Proposition 3.1, (b), we see that

$$\begin{aligned} h(\eta_1) &\leqslant \frac{1}{q}\{h(\eta) + (q-1)(h(\alpha'_0) + \cdots + h(\alpha'_{n-1}))\} \\ &\leqslant \frac{1}{q}\{V + (q-1)(V(l'_1) + \cdots + V(l'_{n-1}))\} \\ &\leqslant \frac{1}{q}(V + (q-1)\cdot\tfrac{1}{2}(2 + \cdots + n)V_{n-1}) \\ &\leqslant \frac{1}{q}V + \left(1 - \frac{1}{q}\right)\cdot\tfrac{1}{4}n(n+1)V_{n-1} \\ &\leqslant \frac{q+1}{2q}V, \end{aligned} \tag{3.35}$$

where the last inequality follows from the fact that $V > U_0 + k/D \geqslant U_0$ (see (3.27)). To bound $|\lambda_1|/(2\pi D)$ we estimate $|t|$. By (3.1), (3.2) we have

$$\operatorname{Im} l'_0 = \operatorname{Im} l_0 = \frac{2\pi}{p^v q^u} \leqslant \frac{\pi}{q}. \tag{3.36}$$

From (3.5), (3.10)–(3.12) we get for $1 \leqslant s < n$

$$|\operatorname{Im} l'_s| = |\operatorname{Im} l_s| \leqslant \pi, \quad \text{if } t_{ss} = q^w$$

and

$$|\operatorname{Im} l'_s| \leqslant \frac{1}{q^w} \sum_{j=0}^{s} |t_{sj}|\,|\operatorname{Im} l_j| \leqslant \tfrac{1}{2}(s+1)\pi, \quad \text{if } t_{ss} < q^w.$$

So in any case

$$|\operatorname{Im} l'_s| \leqslant \tfrac{1}{2}(s+1)\pi, \quad (1 \leqslant s < n). \tag{3.37}$$

Thus, by (3.31), (3.22), (3.30), (3.36), (3.37) we get

$$\begin{aligned} |t| &\leqslant \frac{1}{2\pi}\left(|\operatorname{Im} l| + q|\operatorname{Im} \lambda_1| + (q-1)\sum_{s=0}^{n-1} |\operatorname{Im} l'_s|\right) \\ &\leqslant \tfrac{1}{8}(q-1)n(n+1) + \tfrac{1}{4}(q+5) - \frac{1}{2q}. \end{aligned} \tag{3.38}$$

Note that by (3.1), (3.2), $l'_0 = l_0$,

$$\frac{|l'_0|}{2\pi D} = \frac{1}{p^v q^u D} \leqslant \frac{1}{2qD}$$

and by Proposition 3.1, (b),

$$\frac{|l'_s|}{2\pi D} \leqslant V(l'_s) \leqslant \tfrac{1}{2}(s+1)V_s \leqslant \tfrac{1}{2}(s+1)V_{n-1} \quad (1 \leqslant s < n).$$

Thus by (3.31), (3.22), (3.38), (3.24) and the inequalities $n \geqslant 2$, $1/D \leqslant V_{n-1}/(f_{\mathfrak{p}} \log p)$ (*see* (3.7), (3.3)), $f_{\mathfrak{p}} \geqslant 2$ if $p = 2$ (see Lemma 1.12), we get

$$\begin{aligned} \frac{|\lambda_1|}{2\pi D} &\leqslant \frac{1}{q}\left\{V + \frac{q-1}{2qD} + (q-1)\cdot\tfrac{1}{2}(2 + \cdots + n)V_{n-1} + \frac{|t|}{D}\right\} \\ &\leqslant \frac{1}{q}V + \left(1 - \frac{1}{q}\right)V_{n-1}\left\{\tfrac{1}{4}n(n+1) - \tfrac{1}{2} + \right. \\ &\quad \left. + \frac{1}{f_{\mathfrak{p}} \log p}\left(\frac{1}{2q} + \tfrac{1}{8}n(n+1) + \tfrac{1}{4}\cdot\frac{q+5}{q-1} - \frac{1}{2q(q-1)}\right)\right\} \\ &\leqslant \frac{1}{q}V + \frac{1}{2}\left(1 - \frac{1}{q}\right)U_0 \leqslant \frac{q+1}{2q}V. \end{aligned} \tag{3.39}$$

Now, on noting (by (3.33)) $((q+1)/(2q))V \geqslant nV_{n-1} \geqslant f_{\not p} \log p/D$, (3.34) follows from (3.35) and (3.39).

By (3.30), (3.32)–(3.34) we can apply the inductive hypothesis, which states that $(\mathrm{P}_0), \ldots, (\mathrm{P}_k)$ are true, to λ_1 and $((q+1)/(2q))V$, and thus we can find $l' \in \mathscr{L}_{K,\not p}$ and $m'_0, \ldots, m'_n \in \mathbb{Z}$ such that

(a) holds,

(b′) $V(l') \leqslant \dfrac{q+1}{2q} V < V$, whence (b) is valid,

(c′) $\lambda_1 = m'_0 l'_0 + \cdots + m'_{n-1} l'_{n-1} + m'_n l'$,

(d′) $m'_n = q^{v'_n}$ for some $v'_n \in \mathbb{N}$,

(e′.1) $\max\limits_{1 \leqslant j \leqslant n} |m'_j| \leqslant C_3 D V_0 \left(\dfrac{q+1}{2q} D V\right)^{C_6} \leqslant \left(\dfrac{q+1}{2q}\right)^{C_4} C_3 D V_0 (DV)^{C_6}$,

(e′.2) $\max\limits_{1 \leqslant j \leqslant n} |m'_j| \leqslant \left(\dfrac{q+1}{2q}\right)^{C_4} \left(\dfrac{V}{\max(V(l'), nV_{n-1})}\right)^{C_4} C_5 D V_0 (DV_{n-1})^n$.

By (3.31) and (c′) we have

$$l = m_0 l'_0 + \cdots + m_{n-1} l'_{n-1} + m_n l'$$

with

$$\begin{aligned} m_0 &= t_0 + p^v q^u t + q m'_0, \\ m_j &= t_j + q m'_j \quad (1 \leqslant j < n), \\ m_n &= q m'_n = q^{v'_n + 1}. \end{aligned} \tag{3.40}$$

Thus (c), (d) hold. It remains to verify (e.1) and (e.2). We first deal with the case when $p > 2$. So $q = 2$. By (3.1) and the inequalities

$$n \geqslant 2, \quad D \geqslant 2, \quad V > U_0 + \frac{k}{D} \geqslant U_0 = (n+1)^2 V_{n-1}, \quad DV_{n-1} \geqslant f_{\not p} \log p > 1,$$

we get

$$\left(\frac{q+1}{2q}\right)^{C_4} C_3 D V_0 (DV)^{C_6} \geqslant (\tfrac{3}{4})^{2.41} C_3 \cdot \frac{1}{p^v q^u} \cdot ((n+1)^2 DV_{n-1})^n > 10^5.$$

So by (3.40) and (e'.1) we have

$$\begin{aligned}\max_{1\leqslant j\leqslant n}|m_j| &\leqslant q\max_{1\leqslant j\leqslant n}|m'_j|+q-1\\ &\leqslant q\cdot\left(\frac{q+1}{2q}\right)^{C_4}C_3DV_0(DV)^{C_6}\left(1+\frac{q-1}{q}\cdot 10^{-5}\right)\\ &\leqslant 2\cdot(\tfrac{3}{4})^{2.41}\cdot(1+\tfrac{1}{2}\cdot 10^{-5})C_3DV_0(DV)^{C_6}\\ &\leqslant C_3DV_0(DV)^{C_6},\end{aligned}$$

whence (e.1) is valid. It is easy to see that the right-hand side of (e'.2) is at least 10^5. Thus (e.2) can be verified similarly. This completes the case $p > 2$. The verification of (e.1) and (e.2) for the case $p = 2$ is similar, so we omit the details. This establishes the assertion (P_{k+1}). The proof of Lemma 3.1 is thus complete.

PROPOSITION 3.2. *Let* $l_1,\dots,l_n$ *be given in Proposition* 3.1 *and suppose that*

$$V_n \geqslant nV_{n-1}.$$

Then we can replace (e) *for* $s = n$ *in Proposition* 3.1 *by*

$$\text{(e*)}\quad \max_{1\leqslant j\leqslant n}|m_{nj}| \leqslant C_3DV_0(DV_n)^{C_6},$$

$$\max_{1\leqslant j\leqslant n}|m_{nj}| \leqslant \left(\frac{V_n}{\max(V(l'_n), nV_{n-1})}\right)^{C_4}C_5DV_0(DV_{n-1})^n,$$

where $C_3,\dots,C_6$ *are given in Lemma* 3.1.

Proof. Apply Lemma 3.1 to $l = l_n$, $V = V_n$.

Let $r+1$ be the rank of $\{l_0,\dots,l_n\}$. We fix the integers $j_0,\dots,j_r$ with $0 = j_0 < \cdots < j_r \leqslant n$ such that $l_{j_0},\dots,l_{j_r}$ are linearly independent and l_j is linearly dependent on $l_{j_0},\dots,l_{j_s}$ for j with $j_s \leqslant j < j_{s+1}$ ($0 \leqslant s \leqslant r$, $j_{r+1} := n+1$).

PROPOSITION 3.3. *Suppose that*

$$2 \leqslant r < n,\quad j_r = n.$$

Then there exist $l'_0 = l_0, l'_1,\dots,l'_r \in \mathscr{L}_{K,\mathfrak{p}}$ *and rational integers* u_i*'s* (>0) *and* m_{ij}*'s such that*

(a) $[K((\alpha'_0)^{1/q},\dots,(\alpha'_r)^{1/q}):K] = q^{r+1}$, where $\alpha'_j = e^{l'_j}$ $(0 \leqslant j \leqslant r)$.

(b) $V(l'_s) \leqslant \max(V_{n-r+s}, \tfrac{1}{2}(V_0 + sV_{n-r+s})) \leqslant \tfrac{1}{2}(s+1)V_{n-r+s}$ $(1 \leqslant s < r)$,

$V(l'_r) \leqslant \max(V_n, \tfrac{1}{2}(\tfrac{1}{4}r(r+1)V_{n-1} + V_n))$,

(c) $u_i l_i = \sum_{j=0}^{s} m_{ij} l'_j \quad (j_s \leqslant i < j_{s+1}, 0 \leqslant s \leqslant r)$,

(d) $u_i = 1 \quad (i = j_0, \dots, j_r), \qquad (u_i, pq) = 1 \quad (0 \leqslant i \leqslant n), \qquad m_{00} = 1,$

$m_{j_s s} = p^{h_s} q^{w_s}$ for some $h_s, w_s \in \mathbb{N}$ $(1 \leqslant s < r)$,

$m_{nr} = q^{w_r}$ for some $w_r \in \mathbb{N}$,

(e) $$\max\left(\max_{j_s < i < j_{s+1}} u_i, \max_{\substack{j_s \leqslant i < j_{s+1} \\ 1 \leqslant j \leqslant s}} |m_{ij}| \right)$$

$$\leqslant 2((s+1)D^3)^{s+1}(s+1)!\, V_0 V_{n-r+s}^s \quad (1 \leqslant s < r),$$

$$\max_{1 \leqslant j \leqslant r} |m_{nj}| \leqslant 4((r+1)D^3)^{r+1} r!\, V_0 V_{n-1}^{r-2} V_n \times$$

$$\times \max(V_n, \tfrac{1}{2}(\tfrac{1}{4}r(r+1)V_{n-1} + V_n)).$$

Proof. Let $\mathcal{N} = \mathbb{Z}l_{j_0} + \cdots + \mathbb{Z}l_{j_{r-1}}$ and

$$\mathcal{N}_{p,q} = \{l \in \mathcal{L}_{K,\wp} \mid \text{there exist } h', w' \in \mathbb{N} \text{ such that } p^{h'} q^{w'} l \in \mathcal{N}\}.$$

By Lemma 1.11, we see, similarly to the proof of Proposition 3.1, that $\mathcal{N}$ is of finite index in $\mathcal{N}_{p,q}$. Denote by $p^h q^w$ the index, where $h, w \in \mathbb{N}$. Set for $0 \leqslant s < r$

$$N_s = \left\{ t \in \mathbb{Z} \mid t > 0, \text{ there exist } t_{sj} \in \mathbb{Z}\ (0 \leqslant j < s) \text{ such that } \sum_{i=0}^{s-1} t_{si} l_{j_i} + t l_{j_s} \in p^h q^w \mathcal{N}_{p,q} \right\}.$$

Obviously $p^h q^w \in N_s$, whence N_s has the least element $t_{ss} \leqslant p^h q^w$ $(0 \leqslant s < r)$. We fix t_{si} $(0 \leqslant i < s)$ such that

$$t_{si} = 0 \quad (0 \leqslant i < s), \qquad \text{if } t_{ss} = p^h q^w, \tag{3.41}$$

$$-\tfrac{1}{2} p^h q^w < t_{si} \leqslant \tfrac{1}{2} p^h q^w \quad (0 \leqslant i < s), \qquad \text{if } t_{ss} < p^h q^w. \tag{3.42}$$

((3.42) is always possible by the division algorithm.) Then there exist $l'_0, \dots, l'_{r-1} \in \mathcal{N}_{p,q}$ such that

$$\sum_{i=0}^{s} t_{si} l_{j_i} = p^h q^w l'_s \quad (0 \leqslant s < r). \tag{3.43}$$

By the linear independence of $l_{j_0}, \ldots, l_{j_{r-1}}$ and by the construction, $\{p^h q^w l'_0, \ldots, p^h q^w l'_{r-1}\}$ is a basis of $p^h q^w \mathcal{N}_{p,q}$, whence $\{l'_0, \ldots, l'_{r-1}\}$ is a basis of $\mathcal{N}_{p,q}$. Observing $l_{j_s} \in \mathcal{N} \subseteq \mathcal{N}_{p,q}$ $(0 \leqslant s < r)$, we see that there exist m_{ij}'s in $\mathbb{Z}$ $(i = j_0, \ldots, j_{r-1})$ such that

$$l_i = \sum_{j=0}^{s} m_{ij} l'_j \quad \text{for } i = j_s \quad \text{with } s = 0, \ldots, r-1. \tag{3.44}$$

Taking $u_i = 1$, we see that (3.44) is exactly (c) for $i = j_0, \ldots, j_{r-1}$. It is easy to see, on combining (3.43) with (3.44), that

$$t_{ss} m_{j_s s} = p^h q^w \quad (0 \leqslant s < r). \tag{3.45}$$

Thus

$$m_{j_s s} = p^{h_s} q^{w_s} \quad \text{for some } h_s, w_s \in \mathbb{N} \quad \text{with } h_s \leqslant h, w_s \leqslant w \quad (0 \leqslant s < r).$$

We assert that $h_0 = w_0 = 0$, for if $h_0 > 0$, then from

$$l_0 = m_{00} l'_0 = p^{h_0} q^{w_0} l'_0$$

we get

$$\zeta_{p^v} = \alpha_0^{q^u} = (\alpha'_0)^{q^{w_0 + u} p^{h_0}} \in K^p \quad (\text{where } \alpha'_0 := e^{l'_0}),$$

a contradiction to (0.5); and if $w_0 > 0$, then we have

$$\zeta_{q^u} = \alpha_0^{p^v} = (\alpha'_0)^{p^{h_0 + v} q^{w_0}} \in K^q,$$

a contradiction to (0.4). Thus $h_0 = w_0 = 0$ and $m_{00} = 1$, $l_0 = l'_0$. For i with $j_s < i < j_{s+1}$ $(0 \leqslant s < r)$, from (3.44) and the fact that l_i is linearly dependent on $l_{j_0}, \ldots, l_{j_s}$, we see that l_i is linearly dependent on $l'_0, \ldots, l'_s$. Let u_i be the least positive integer such that

$$u_i l_i \in \mathbb{Z} l'_0 + \cdots + \mathbb{Z} l'_{r-1} = \mathcal{N}_{p,q},$$

where the equality follows from the fact that $\{l'_0, \ldots, l'_{r-1}\}$ is a basis of $\mathcal{N}_{p,q}$. Then we obtain (c) for i with $j_s < i < j_{s+1}$ $(0 \leqslant s < r)$. From the definitions of u_i and $\mathcal{N}_{p,q}$ we get

$$(u_i, pq) = 1 \quad (j_s < i < j_{s+1}, 0 \leqslant s < r).$$

Now set

$$\mathcal{M} = \mathbb{Z}l'_0 + \cdots + \mathbb{Z}l'_{r-1} + \mathbb{Z}l_n = \mathcal{N}_{p,q} + \mathbb{Z}l_n,$$
$$\mathcal{M}_q = \{l \in \mathcal{L}_{K,\wp} \mid \text{there exists } k' \in \mathbb{N} \text{ such that } q^{k'}l \in \mathcal{M}\}.$$

By Lemma 1.11, $\mathcal{M}$ is of finite index q^k in $\mathcal{M}_q$ for some $k \in \mathbb{N}$. As before, for $s = 0, \ldots, r$ let τ_{ss} be the least positive integer for which there are τ_{si} $(0 \leqslant i < s)$ in $\mathbb{Z}$ such that

$$\sum_{i=0}^{s} \tau_{si} l'_i \in q^k \mathcal{M}_q \quad (0 \leqslant s < r) \tag{3.46}$$

and

$$\sum_{i=0}^{r-1} \tau_{ri} l'_i + \tau_{rr} l_n \in q^k \mathcal{M}_q. \tag{3.47}$$

We fix for $s = 1, \ldots, r$

$$\tau_{si} = 0 \quad (0 \leqslant i < s), \qquad \text{if } \tau_{ss} = q^k, \tag{3.48}$$

$$-\tfrac{1}{2}q^k < \tau_{si} \leqslant \tfrac{1}{2}q^k \quad (0 \leqslant i < s), \qquad \text{if } \tau_{ss} < q^k. \tag{3.49}$$

By (3.46), for s with $0 \leqslant s < r$ there is $l''_s \in \mathcal{M}_q$ such that

$$q^k l''_s = \sum_{i=0}^{s} \tau_{si} l'_i \in \mathcal{N}_{p,q}. \tag{3.50}$$

So by the definition of $\mathcal{N}_{p,q}$, we have $l''_s \in \mathcal{N}_{p,q}$ $(= \mathbb{Z}l'_0 + \cdots + \mathbb{Z}l'_{r-1})$. This and (3.50) yield $q^k \mid \tau_{ss}$. On the other hand, $\tau_{ss} \leqslant q^k$ by definition. Thus, recalling (3.48), we get

$$\tau_{ss} = q^k, \qquad \tau_{si} = 0 \quad (0 \leqslant s < r, 0 \leqslant i < s). \tag{3.51}$$

Denote by $l'_r \in \mathcal{M}_q$ the element such that

$$\sum_{i=0}^{r-1} \tau_{ri} l'_i + \tau_{rr} l_n = q^k l'_r. \tag{3.52}$$

As before, we can see that $l'_0, \ldots, l'_r$ is a basis of $\mathcal{M}_q$. On noting that $l_n \in \mathcal{M} \subseteq \mathcal{M}_q$ and taking $u_n = 1$, we obtain (c) for $i = n$. It is easily seen that $m_{nr} = q^{w_r}$ for some

$w_r \in \mathbb{N}$ with $w_r \leqslant k$ and $\tau_{rr} = q^{k-w_r}$. This completes the proof of (d). By (3.41)–(3.43), (3.45), (3.49), (3.51), (3.52), $\tau_{rr} = q^{k-w_r}$ and the inequalities

$$V_0 = \frac{1}{p^v q^u D} \leqslant \frac{1}{2q} V_j < \tfrac{1}{2} V_j \quad (1 \leqslant j \leqslant n) \quad \text{(by (3.1), (3.2))},$$

$$V_{j_s} \leqslant V_{n-r+s} \quad (1 \leqslant s < r),$$

we obtain (b). Further by Lemma 1.11, (b), (c), the linear independence of $l'_0, \ldots, l'_r$, the definition of u_i, we get (e). Finally, using an argument based on Lemma 1.9 and the fact that $\zeta_q \in K$ (see (0.3)), which is similar to that in the proof of Proposition 3.1, we obtain (a). The proof of the Proposition is complete.

PROPOSITION 3.4. *Suppose that*

$$2 \leqslant r < n, \quad j_r = n, \quad V_n \geqslant \tfrac{1}{4} r(r+1) V_{n-1}.$$

Then the second inequality in (e) *of Proposition* 3.3 *can be replaced by*

$$\text{(e*)} \max_{1 \leqslant j \leqslant r} |m_{nj}| \leqslant C'_3 D V_0 (D V_n)^{C'_6},$$

$$\max_{1 \leqslant j \leqslant r} |m_{nj}| \leqslant \left(\frac{V_n}{\max(V(l'_r), \frac{1}{4} r(r+1) V_{n-1})} \right)^{C_4} C'_5 D V_0 (D V_{n-1})^r,$$

where C_4 is that given in Lemma 3.1,

$$C'_3 = 4^{r-1}(r+1)^3 \frac{r!}{r^{r-2}} D^{2(r+1)},$$

$$C'_5 = \begin{cases} C'_3 (r+1)^{2r}, & \text{if } p > 2, \\ C'_3 (r+1)^{2r} \cdot (\frac{3}{4})^r, & \text{if } p = 2, \end{cases} \quad C'_6 = \max(r, C_4).$$

Proof. Very similar to the proof of Lemma 3.1 and Proposition 3.2. It is easy to write down the proof *mutatis mutandis*, and we omit the details here.

4. Proof of Theorems 1, 1′, Corollaries 1 and 2

Proof of Theorem 1. By Lemma 1.3, (0.14), (0.7)–(0.10) we have

$$\operatorname{ord}_{\mathfrak{p}} \Theta \leqslant \frac{D}{f_{\mathfrak{p}} \log p} (\log 2 + n V_n B). \tag{4.1}$$

Evidently, (0.4), (0.5) imply

$$2q \leqslant p^v q^u \leqslant 3D, \tag{4.2}$$

$$q^u \leqslant 2D \quad \text{if } p > 2; \qquad q^u \leqslant \tfrac{3}{2}D \quad \text{if } p = 2. \tag{4.3}$$

Set

$$l_0 := \frac{2\pi i}{p^v q^u}, \quad V_0 := \frac{1}{p^v q^u D}, \tag{4.4}$$

$$l_j := \log \alpha_j = \log |\alpha_j| + i \arg \alpha_j \tag{4.5}$$

with

$$-\pi < \arg \alpha_j \leqslant \pi.$$

By (4.4), (0.9), (0.7), (4.2), we have

$$V_0 \leqslant \sigma V_j < \tfrac{1}{6} V_j \quad (1 \leqslant j \leqslant n) \tag{4.6}$$

with σ given by (0.12), where the second inequality follows from (0.2) and Lemma 1.12.

Let $r + 1$ be the rank of $\{l_0, l_1, \ldots, l_n\}$ and $j_0, j_1, \ldots, j_r$ be the integers with $0 = j_0 < j_1 < \cdots < j_r \leqslant n$ such that $l_{j_0}, \ldots, l_{j_r}$ are linearly independent and l_j is linearly dependent on $l_{j_0}, \ldots, l_{j_s}$ for j with $j_s \leqslant j < j_{s+1} (0 \leqslant s \leqslant r, j_{r+1} := n + 1)$. We deal with the following eight cases (a)–(h) separately.

(a) $r = n, V_n < nV_{n-1}$.
We shall prove

$$\begin{aligned} \operatorname{ord}_{\mathfrak{p}} \Theta < {} & 2c' \left(\frac{a'}{2}\right)^n (n+1)^{n+2} n^{n+\sigma} \cdot \frac{p^{f_{\mathfrak{p}}} - 1}{q^u} \cdot \left(\frac{2 + 1/(p-1)}{f_{\mathfrak{p}} \log p}\right)^{n+2} \cdot \\ & \cdot D^{n+2} V_1 \ldots V_n \log(D^2 B) \max(n \log(2^{10} qn(n+\sigma) D^2 V), f_{\mathfrak{p}} \log p) \\ =: {} & U_1, \end{aligned} \tag{4.7}$$

where a' and c' are given in Corollary 2.3. By (4.3) and $DV_j \geqslant f_{\mathfrak{p}} \log p \,(1 \leqslant j \leqslant n)$ (see (0.9), (0.7)) we get

$$\frac{D}{f_{\mathfrak{p}} \log p} \log 2 < \tfrac{1}{2} U_1. \tag{4.8}$$

We assert that we may assume

$$B > 4\cdot 10^5 \cdot 20^n (n+1)^{n+2} n^{n-1} \cdot \frac{p^{f_{\mathfrak{p}}}-1}{(f_{\mathfrak{p}} \log p)^2} D^2 V_{n-1}, \tag{4.9}$$

for otherwise we would have, by (4.3), (0.9), (0.7), $D \geqslant 2$ (see (0.3)),

$$\frac{D}{f_{\mathfrak{p}} \log p} n V_n B \leqslant \tfrac{1}{2} U_1; \tag{4.10}$$

and (4.7) would follow from (4.1), (4.8) and (4.10). So in the rest of (a), we can assume (4.9).

Now we apply Proposition 3.1 to $l_1, \ldots, l_n$. On recalling (0.6) and noting, by the fact that $l'_j \in \mathscr{L}_{K,\mathfrak{p}}$, that

$$\operatorname{ord}_{\mathfrak{p}} \alpha'_j = 0 \quad (1 \leqslant j \leqslant n), \tag{4.11}$$

we get

$$\operatorname{ord}_{\mathfrak{p}} \Theta = \operatorname{ord}_{\mathfrak{p}}(\alpha_0^{b'_0}(\alpha'_1)^{b'_1} \ldots (\alpha'_n)^{b'_n} - 1) \leqslant \operatorname{ord}_{\mathfrak{p}}((\alpha'_1)^{b''_1} \ldots (\alpha'_n)^{b''_n} - 1), \tag{4.12}$$

where

$$b'_j = \sum_{s=\max(j,1)}^{n} b_s m_{sj} \quad (0 \leqslant j \leqslant n), \qquad b''_j = p^v q^u b'_j \quad (1 \leqslant j \leqslant n). \tag{4.13}$$

Note that $b''_1, \ldots, b''_n$ are not all zero, since $b'_1, \ldots, b'_n$ are not all zero by the equality $\alpha_0^{b'_0}(\alpha'_1)^{b'_1} \ldots (\alpha'_n)^{b'_n} = \alpha_1^{b_1} \ldots \alpha_n^{b_n}$ and the assumption $r = n$. This fact together with $r = n$ yields

$$(\alpha'_1)^{b''_1} \ldots (\alpha'_n)^{b''_n} \neq 1. \tag{4.14}$$

Further we have

$$[K(\alpha_0^{1/q}, (\alpha'_1)^{1/q}, \ldots, (\alpha'_n)^{1/q}) : K] = q^{n+1}. \tag{4.15}$$

By Proposition 3.1, (b) and (4.6) we see that

$$\begin{aligned} &V(\log \alpha'_1) \leqslant V(l'_1) \leqslant V_1 =: V'_1, \\ &V(\log \alpha'_j) \leqslant V(l'_j) \leqslant \tfrac{1}{2}(j+\sigma) V_j =: V'_j \quad (2 \leqslant j \leqslant n). \end{aligned} \tag{4.16}$$

By Proposition 3.1, (e), (4.4) and the assumption $V_n < nV_{n-1}$, we get

$$\max_{1\leqslant j\leqslant n}|b_j''| \leqslant nBp^v q^u \cdot 4((n+1)D^2)^{n+1} n! n^2 D V_0 (DV_{n-1})^n$$

$$\leqslant 4n^{n+3}((n+1)D^2)^{n+1}(DV_{n-1})^n B =: B''. \tag{4.17}$$

It is easily verified, by (4.9) and the inequality $(x-1)/(\log x)^2 \geqslant 1/2$ for $x > 1$, that

$$(n+1)\log(D^2 B) \geqslant \log B''. \tag{4.18}$$

From (4.9) and the inequalities $n \geqslant 2$, $D \geqslant 2$, $DV_{n-1} \geqslant f_{\wp} \log p$ (see (0.9), (0.7)), (1.18), we see that

$$(n+1)\log(D^2 B) \geqslant \max\left(n\log(2^{11} qnD), \frac{f_{\wp}\log p}{D}\right). \tag{4.19}$$

Observe that we have

$$b_n' = b_n m_{nn} = b_n q^{w_n}$$

by (4.13) and Proposition 3.1, (d). Thus, by (0.2), $\operatorname{ord}_p b_n = \operatorname{ord}_p b_n'$. So by (4.13) we see that

$$\operatorname{ord}_p b_n = \min_{1\leqslant j\leqslant n} \operatorname{ord}_p b_j \quad \text{implies} \quad \operatorname{ord}_p b_n'' = \min_{1\leqslant j\leqslant n} \operatorname{ord}_p b_j''. \tag{4.20}$$

Now by (4.11), (4.14), (4.15) we can apply Corollary 2.3 to $\operatorname{ord}_{\wp}((\alpha_1')^{b_1''}\dots(\alpha_n')^{b_n''}-1)$, and on observing (4.12), (4.16)–(4.20) and using (1.17), we obtain (4.7).

(b) $r = n$, $V_n \geqslant nV_{n-1}$.
We shall prove

$$\operatorname{ord}_{\wp}\Theta < 4c'\left(\frac{a'}{2}\right)^n (n+1)^{n+2} n^{n-1}(n-1)^{\sigma} \cdot \frac{p^{f_{\wp}}-1}{q^u} \cdot \left(\frac{2+1/(p-1)}{f_{\wp}\log p}\right)^{n+2} \cdot$$

$$\cdot D^{n+2} V_1 \dots V_n \log(D^2 B) \max(n\log(2^{10} qn^2 D^2 V), f_{\wp}\log p)$$

$$=: U_2. \tag{4.21}$$

Using (4.1) and arguing as in (a), we may assume

$$B > 8\cdot 10^5 \cdot 20^n (n+1)^{n+2} n^{n-2} \cdot \frac{p^{f_{\wp}}-1}{(f_{\wp}\log p)^2} D^2 V_{n-1}. \tag{4.22}$$

Obviously (4.11)–(4.15), (4.16) (with $1 \leqslant j < n$) and (4.20) are valid in the case (b). By (4.16) with $1 \leqslant j < n$ and (1.17) we have for $n > 2$

$$V'_1 \ldots V'_{n-1} \leqslant \frac{1}{2^{n-2}}(n-1)!(n-1)^{\sigma} V_1 \ldots V_{n-1}; \tag{4.23}$$

and we remark that (4.23) is trivially true for $n = 2$ by (4.16) with $j = 1$. Note also, by Proposition 3.1, (b) and the assumption $V_n \geqslant nV_{n-1}$ we have

$$V(\log \alpha'_n) \leqslant \max(V(l'_n), nV_{n-1}) =: V'_n \leqslant V_n. \tag{4.24}$$

By Propositions 3.1 and 3.2, on noting that

$$C_4 < 3, \quad C_5 \leqslant C_3(n+1)^{2n} \leqslant 4n^2(n+1)^{3n+1} D^{2(n+1)}$$

and using (4.13) and (4.4), we get

$$\max_{1 \leqslant j \leqslant n} |b''_j| \leqslant p^v q^u nB\left(\frac{V_n}{V'_n}\right)^{C_4} C_5 D V_0 (DV_{n-1})^n$$

$$\leqslant 4n^3(n+1)^{3n+1} D^{2(n+1)} (DV_{n-1})^n B\left(\frac{V_n}{V'_n}\right)^3 =: B''. \tag{4.25}$$

By (4.22) and (4.25) it is easily seen that

$$(n+1)\log(D^2 B) + 3\log\left(\frac{V_n}{V'_n}\right) \geqslant \log B''. \tag{4.26}$$

From (4.22), (4.24), (1.18) and the inequalities $n \geqslant 2$, $D \geqslant 2$, $DV_{n-1} \geqslant f_{\mathfrak{p}} \log p$, we have

$$(n+1)\log(D^2 B) + 3\log\left(\frac{V_n}{V'_n}\right) \geqslant \max\left(n\log(2^{11} qnD), \frac{f_{\mathfrak{p}} \log p}{D}\right). \tag{4.27}$$

By (4.26), (4.27), (4.24) and the inequalities $n \geqslant 2$, $D \geqslant 2$, we obtain

$$V'_n \max\left(\log B'', n\log(2^{11} qnD), \frac{f_{\mathfrak{p}} \log p}{D}\right)$$

$$\leqslant V'_n\left((n+1)\log(D^2 B) + 3\log\left(\frac{V_n}{V'_n}\right)\right)$$

$$\leqslant V_n \cdot (n+1)\log(D^2B)\cdot\left(\frac{V_n}{V_n'}\right)^{-1}\left(1+\log\left(\frac{V_n}{V_n'}\right)\right)$$

$$\leqslant (n+1)V_n \log(D^2B). \tag{4.28}$$

Now by (4.11), (4.14), (4.15) we can apply Corollary 2.3 to $\operatorname{ord}_{\not p}((\alpha_1')^{b_1''}\dots(\alpha_n')^{b_n''}-1)$, and on noting (4.12), (4.16) (with $1\leqslant j<n$), (4.23)–(4.25), (4.28), (4.20), we obtain (4.21).

(c) $2\leqslant r<n$, $j_r<n$.
We shall prove

$$\operatorname{ord}_{\not p}\Theta < 2c'\left(\frac{a'}{2}\right)^r (r+1)^{r+2} r^{r+1+\sigma}(n-r+1)\cdot\frac{p^{f_{\not p}}-1}{q^u}\cdot$$
$$\cdot\left(\frac{2+1/(p-1)}{f_{\not p}\log p}\right)^{r+2} D^{r+2}V_{n-r+1}\dots V_n\log(D^2B)\cdot$$
$$\cdot\max(r\log(2^{10}qr(r+1)D^2V_{n-1}), f_{\not p}\log p)$$
$$=: U_3. \tag{4.29}$$

On arguing by (4.1) as in the case (a) and noting $r(n-r+1)\geqslant 2(n-1)\geqslant n$, we may assume

$$B > 4\cdot 10^5\cdot 20^r(r+1)^{r+2}r^r\cdot\frac{p^{f_{\not p}}-1}{(f_{\not p}\log p)^2}\cdot D^2V_{n-1}. \tag{4.30}$$

Define

$$l_s' := l_{j_s}, \qquad \alpha_s' := \alpha_{j_s} \quad (0\leqslant s\leqslant r).$$

Then by the assumption $j_r<n$ we have

$$V(l_s')\leqslant V_{j_s}\leqslant V_{n-r-1+s} =: V_s' \quad (1\leqslant s\leqslant r). \tag{4.31}$$

By Lemma 1.11, we see that there exist $u_j\in\mathbb{Z}, u_j>0$ $(1\leqslant j\leqslant n)$ and $m_{js}\in\mathbb{Z}$ $(1\leqslant j\leqslant n, 0\leqslant s\leqslant r)$ such that

$$u_jl_j = \sum_{s=0}^{r} m_{js}l_s' \quad (1\leqslant j\leqslant n) \tag{4.32}$$

and

$$u_{j_s} = 1 \quad (1 \leqslant s \leqslant r), \qquad \max_{1 \leqslant j \leqslant n} u_j \leqslant (2(r+1)D^2)^{r+1} D V_0 (D V_{n-1})^r,$$

$$m_{j_s i} = \begin{cases} 1, & \text{if } i = s \\ 0, & \text{if } i \neq s \end{cases} \quad (1 \leqslant s \leqslant r), \tag{4.33}$$

$$\max_{\substack{1 \leqslant j \leqslant n \\ 1 \leqslant s \leqslant r}} |m_{js}| \leqslant (2(r+1)D^2)^{r+1} D V_0 (D V_{n-1})^r \cdot \frac{V_n}{V_{n-1}}.$$

Write

$$M := u_1 \dots u_n, \qquad m'_{js} := \frac{M}{u_j} m_{js} \quad (1 \leqslant j \leqslant n,\ 0 \leqslant s \leqslant r).$$

By (4.32), (4.33) we get

$$M l_j = \sum_{s=0}^{r} m'_{js} l'_s \quad (1 \leqslant j \leqslant n),$$

$$\max_{\substack{1 \leqslant j \leqslant n \\ 1 \leqslant s \leqslant r}} |m'_{js}| \leqslant \{(2(r+1)D^2)^{r+1} D V_0 (D V_{n-1})^r\}^{n-r} \cdot \frac{V_n}{V_{n-1}}. \tag{4.34}$$

By (0.13) and (4.34), we have

$$\begin{aligned} \operatorname{ord}_{\mathfrak{p}} \Theta &\leqslant \operatorname{ord}_{\mathfrak{p}}((\alpha_1^{b_1} \dots \alpha_n^{b_n})^M - 1) \\ &= \operatorname{ord}_{\mathfrak{p}}(\alpha_0^{b'_0} (\alpha'_1)^{b'_1} \dots (\alpha'_r)^{b'_r} - 1) \leqslant \operatorname{ord}_{\mathfrak{p}}((\alpha'_1)^{b''_1} \dots (\alpha'_r)^{b''_r} - 1), \end{aligned} \tag{4.35}$$

where

$$b'_s = \sum_{j=1}^{n} b_j m'_{js} \quad (0 \leqslant s \leqslant r), \qquad b''_s = p^v q^u b'_s \quad (1 \leqslant s \leqslant r). \tag{4.36}$$

We assert that we may assume

$$(\alpha'_1)^{b''_1} \dots (\alpha'_r)^{b''_r} \neq 1, \tag{4.37}$$

for otherwise, by (4.36), we would have

$$(\alpha_1^{b_1} \dots \alpha_n^{b_n})^{Mp^v q^u} = 1,$$

whence Lemma 1.3 would yield

$$\operatorname{ord}_{\mathfrak{p}} \Theta \leqslant \frac{D}{f_{\mathfrak{p}} \log p} \log 2 < U_3.$$

Now by (4.36), (4.34), (4.4) we get

$$\max_{1 \leqslant s \leqslant r} |b_s''| \leqslant B \cdot n \{(2(r+1)D^2)^{r+1} (DV_{n-1})^r\}^{n-r} \cdot \frac{V_n}{V_{n-1}} =: B''. \tag{4.38}$$

By (4.30) and by the inequalities

$$2 \leqslant r < n, \quad r^2(n-r) \geqslant 4(n-2), \tag{4.39}$$

it is readily verified that

$$\log(D^2 B'') \leqslant r(n-r+1) \log(D^2 B) + \log\left(\frac{V_n}{V_{n-1}}\right). \tag{4.40}$$

On noting that $D \geqslant 2, r(n-r+1) \geqslant 2(n-1) > 1$ and using (4.40), (4.31) we obtain

$$\begin{aligned} V_n' \log(D^2 B'') &= V_{n-1} \log(D^2 B'') \\ &\leqslant r(n-r+1) V_n \log(D^2 B) \cdot \left(\frac{V_n}{V_{n-1}}\right)^{-1} \left(1 + \log\left(\frac{V_n}{V_{n-1}}\right)\right) \\ &\leqslant r(n-r+1) V_n \log(D^2 B). \end{aligned} \tag{4.41}$$

By (4.31) we have

$$V_1' \dots V_{r-1}' \leqslant V_{n-r} \dots V_{n-2} \leqslant V_{n-r+1} \dots V_{n-1}. \tag{4.42}$$

Now by (4.37) and the linear independence of $l_0, l_1', \dots, l_r'$, we may apply (4.7) and (4.21) to $\operatorname{ord}_{\mathfrak{p}}((\alpha_1')^{b_1''} \dots (\alpha_r')^{b_r''} - 1)$. On observing $U_2 \leqslant U_1$, (4.35), (4.31), (4.41), (4.42), we obtain (4.29).

(d) $2 \leqslant r < n, j_r = n, V_n < \frac{1}{4} r(r+1) V_{n-1}$.

We shall prove

$$\operatorname{ord}_{\mathfrak{p}} \Theta < \tfrac{5}{6}c'\left(\frac{a'}{2}\right)^r (r+1)^{r+2} r^{r+1}(n-r+1)(r-1)^{\sigma} \cdot \frac{p^{f_{\mathfrak{p}}}-1}{q^u} \cdot$$

$$\cdot \left(\frac{2+1/(p-1)}{f_{\mathfrak{p}} \log p}\right)^{r+2} \cdot D^{r+2} V_{n-r+1} \dots V_n \log(D^2 B) \cdot$$

$$\cdot \max(r \log(2^9 q r^2 (r+1) D^2 V), f_{\mathfrak{p}} \log p)$$

$$=: U_4. \tag{4.43}$$

Utilizing (4.1), arguing as in the case (a), noting $r(n-r+1) \geqslant 2(n-1) \geqslant n$, we may assume

$$B > 10^5 \cdot 20^r (r+1)^{r+2} r^r \cdot \frac{p^{f_{\mathfrak{p}}}-1}{(f_{\mathfrak{p}} \log p)^2} D^2 V_{n-1}. \tag{4.44}$$

For i with $j_s \leqslant i < j_{s+1}$ $(0 \leqslant s < r)$ define

$$m_{ij} := 0 \quad (j = s+1, \dots, r).$$

Then by Proposition 3.3, we have

$$u_i l_i = \sum_{j=0}^{r} m_{ij} l'_j \quad (1 \leqslant i \leqslant n).$$

Writing

$$M = u_1 \dots u_n, \quad m'_{ij} = \frac{M}{u_i} m_{ij},$$

we get

$$M l_i = \sum_{j=0}^{r} m'_{ij} l'_j \quad (1 \leqslant i \leqslant n). \tag{4.45}$$

By (0.13) and (4.45) we see that

$$\operatorname{ord}_{\mathfrak{p}} \Theta \leqslant \operatorname{ord}_{\mathfrak{p}}((\alpha_1^{b_1} \dots \alpha_n^{b_n})^M - 1)$$

$$= \operatorname{ord}_{\mathfrak{p}}(\alpha_0^{b'_0}(\alpha'_1)^{b'_1} \dots (\alpha'_r)^{b'_r} - 1) \leqslant \operatorname{ord}_{\mathfrak{p}}((\alpha'_1)^{b''_1} \dots (\alpha'_r)^{b''_r} - 1), \tag{4.46}$$

where

$$b'_j = \sum_{i=1}^{n} b_i m'_{ij} \quad (0 \leqslant j \leqslant r), \qquad b''_j = p^v q^u b'_j \quad (1 \leqslant j \leqslant r). \tag{4.47}$$

By Proposition 3.3, we have

$$[K(\alpha_0^{1/q}, (\alpha'_1)^{1/q}, \ldots, (\alpha'_r)^{1/q}):K] = q^{r+1}, \tag{4.48}$$

$$\operatorname{ord}_{\wp} \alpha'_j = 0 \quad (1 \leqslant j \leqslant r). \tag{4.49}$$

We assert that we may assume

$$(\alpha'_1)^{b''_1} \ldots (\alpha'_r)^{b''_r} \neq 1, \tag{4.50}$$

for otherwise we would have $(\alpha_1^{b_1} \ldots \alpha_n^{b_n})^{Mp^v q^u} = 1$ and Lemma 1.3 would yield

$$\operatorname{ord}_{\wp} \Theta \leqslant \frac{D}{f_{\wp} \log p} \log 2 < U_4.$$

Again by Proposition 3.3, and using (4.6), (4.45), (4.47), (4.4), (4.2) and the assumption $V_n < \frac{1}{4}r(r+1)V_{n-1}$, we get

$$V(l'_1) \leqslant V_{n-r+1} =: V'_1, \qquad V(l'_j) \leqslant \frac{j+\sigma}{2} V_{n-r+j} =: V'_j \quad (2 \leqslant j < r), \tag{4.51}$$

$$V(l'_r) \leqslant \max(V_n,\ \tfrac{1}{2}(\tfrac{1}{4}r(r+1)V_{n-1} + V_n)) \leqslant \tfrac{5}{24}r(r+1)V_n =: V'_r, \tag{4.52}$$

$$\begin{aligned} \max_{1 \leqslant j \leqslant r} |b''_j| &\leqslant Bnp^v q^u \cdot (2(rD^2)^r r! D V_0 (DV_{n-1})^{r-1})^{n-r} \cdot \\ &\quad \cdot \tfrac{1}{4} r^2 (r+1)^2 ((r+1)D^2)^{r+1} r! D V_0 (DV_{n-1})^r \\ &\leqslant B \cdot n(r+1)^{r+3} r^{2r(n-r)+r+2} D^{2r(n-r+1)+2} (DV_{n-1})^{(r-1)(n-r)+r} \\ &=: B''. \end{aligned} \tag{4.53}$$

By (4.44) and the assumption $2 \leqslant r < n$, it is readily seen that

$$r(n-r+1)\log(D^2 B) \geqslant \log B''; \tag{4.54}$$

furthermore, on noting (1.18), we get

$$r(n-r+1)\log(D^2 B) \geqslant \max\left(r\log(2^{11} qrD), \frac{f_{\wp} \log p}{D} \right). \tag{4.55}$$

Observing $j_r = n$, we see that

$$m'_{ir} = \frac{M}{u_i} m_{ir} = 0 \quad (1 \leqslant i < n),$$

whence, by (4.47), Proposition 3.3, (d) and (0.2), we get

$$b'_r = b_n m'_{nr} = b_n u_1 \dots u_n q^{w_r}, \quad \mathrm{ord}_p b'_r = \mathrm{ord}_p b_n.$$

Thus by (4.47) we see that

$$\mathrm{ord}_p b_n = \min_{1 \leqslant j \leqslant n} \mathrm{ord}_p b_j \quad \text{implies} \quad \mathrm{ord}_p b''_r = \min_{1 \leqslant j \leqslant r} \mathrm{ord}_p b''_j. \tag{4.56}$$

By (4.48)–(4.50) we may apply Corollary 2.3 to $\mathrm{ord}_{\not p}((\alpha'_1)^{b''_1} \dots (\alpha'_r)^{b''_r} - 1)$, and on noting (4.46), (4.51)–(4.56), (1.17), we obtain (4.43).

(e) $2 \leqslant r < n, j_r = n, V_n \geqslant \frac{1}{4} r(r+1) V_{n-1}$.
We shall prove

$$\begin{aligned} \mathrm{ord}_{\not p} \Theta &< 4c' \left(\frac{a'}{2}\right)^r (r+1)^{r+1} r^r (n-r+1)(r-1)^\sigma \cdot \frac{p^{f_{\not p}} - 1}{q^u} \cdot \\ &\quad \cdot \left(\frac{2 + 1/(p-1)}{f_{\not p} \log p}\right)^{r+2} \cdot D^{r+2} V_{n-r+1} \dots V_n \log(D^2 B) \cdot \\ &\quad \cdot \max(r \log(2^{10} q r^2 D^2 V), f_{\not p} \log p) \\ &=: U_5. \end{aligned} \tag{4.57}$$

Using (4.1), arguing as in the case (a), and noting $r(n-r+1) \geqslant 2(n-1) \geqslant n$, we may assume

$$B > 8 \cdot 10^5 \cdot 20^r (r+1)^{r+1} r^{r-1} \cdot \frac{p^{f_{\not p}} - 1}{(f_{\not p} \log p)^2} \cdot D^2 V_{n-1}. \tag{4.58}$$

Note, by Proposition 3.3, that (4.45)–(4.49), (4.51) and (4.56) are valid in the present case. Further, by Lemma 1.3, we may assume (4.50). By (4.51) and (1.17) we see that if $r > 2$ then

$$V'_1 \dots V'_{r-1} \leqslant \frac{1}{2^{r-2}} (r-1)!(r-1)^\sigma V_{n-r+1} \dots V_{n-1}, \tag{4.59}$$

and we remark that (4.59) is trivially true if $r = 2$. From Proposition 3.3, (b) and the assumption $V_n \geqslant \frac{1}{4}r(r+1)V_{n-1}$, we get

$$V(l'_r) \leqslant \max(V(l'_r), \tfrac{1}{4}r(r+1)V_{n-1}) =: V'_r \leqslant V_n. \tag{4.60}$$

Note that the constants C_4, C'_5 in Proposition 3.4 satisfy

$$C_4 < 3, \quad C'_5 \leqslant C'_3(r+1)^{2r} \leqslant 4^{r-1}(r+1)^{2r+3}r^2D^{2(r+1)}. \tag{4.61}$$

By Propositions 3.3, 3.4 and on noting (4.45), (4.47), (4.2), (4.4), (4.61) we obtain

$$\begin{aligned}\max_{1\leqslant j\leqslant r}|b''_j| &\leqslant Bnp^v q^u(2(rD^2)^r r! D V_0(DV_{n-1})^{r-1})^{n-r}\cdot\left(\frac{V_n}{V'_r}\right)^{C_4} C'_5 D V_0 (DV_{n-1})^r \\ &\leqslant Bn\cdot 4^{r-1}\cdot r^{2r(n-r)+2}(r+1)^{2r+3}D^{2r(n-r+1)+2}\cdot \\ &\quad \cdot(DV_{n-1})^{(r-1)(n-r)+r}\left(\frac{V_n}{V'_r}\right)^3 \\ &=: B''.\end{aligned} \tag{4.62}$$

By (4.58) and the assumption $2 \leqslant r < n$ it is readily verified that

$$r(n-r+1)\log(D^2B) + 3\log\left(\frac{V_n}{V'_r}\right) \geqslant \log B''. \tag{4.63}$$

Further, by (4.58), (4.60) and (1.18) we have

$$r(n-r+1)\log(D^2B) + 3\log\left(\frac{V_n}{V'_r}\right) \geqslant \max\left(r\log(2^{11}qrD), \frac{f_{\mathfrak{p}}\log p}{D}\right). \tag{4.64}$$

On noting $r(n-r+1) \geqslant 2(n-1) \geqslant 4, D \geqslant 2$ and using (4.60), (4.63), (4.64) we get

$$\begin{aligned}&V'_r \max\left(\log B'', r\log(2^{11}qrD), \frac{f_{\mathfrak{p}}\log p}{D}\right) \\ &\leqslant V'_r\left\{r(n-r+1)\log(D^2B) + 3\log\left(\frac{V_n}{V'_r}\right)\right\} \\ &\leqslant r(n-r+1)V_n\log(D^2B)\cdot\left(\frac{V_n}{V'_r}\right)^{-1}\left(1+\log\left(\frac{V_n}{V'_r}\right)\right) \\ &\leqslant r(n-r+1)V_n\log(D^2B).\end{aligned} \tag{4.65}$$

Now we may apply Corollary 2.3 to $\mathrm{ord}_{\wp}((\alpha'_1)^{b''_1}\dots(\alpha'_r)^{b''_r}-1)$; and on using (4.46), (4.59), (4.65), (4.51), (4.60) and (4.56), we obtain (4.57).

(f) $r = 1, j_1 < n$.

It is easily seen that (4.35) with $\alpha'_1 = \alpha_{j_1}$ and (4.38) are valid in the present case; the latter is just

$$|b''_1| \leqslant Bn(4D^2)^{2(n-1)}(DV_{n-1})^{n-2}DV_n =: B''. \tag{4.66}$$

We may also assume (4.37). On applying Lemma 1.4 to $\mathrm{ord}_{\wp}((\alpha'_1)^{b''_1}-1)$ and utilizing (4.35), (4.66), $h(\alpha'_1) \leqslant V_{n-1}$ and $e_{\wp} \leqslant D$, we get

$$\begin{aligned}\mathrm{ord}_{\wp}\,\Theta &\leqslant \mathrm{ord}_{\wp}((\alpha'_1)^{b''}-1)\\ &\leqslant \frac{D}{f_{\wp}\log p}\{\log(2B'') + (p^{f_{\wp}}-1)(1+1/(p-1))DV_{n-1}\}\\ &\leqslant \frac{D}{f_{\wp}\log p}\{(p^{f_{\wp}}-1)(1+1/(p-1))DV_{n-1} + (n-1)\log(DV_n) +\\ &\quad + \log(D^2B) + (4n-6)\log D + (n-1)\log 16 + \log(2n)\}\\ &< U_1,\end{aligned} \tag{4.67}$$

where U_1 is given in (4.7).

(g) $r = 1,\ j_1 = n$.

By Lemma 1.11 and the fact that l_j is linearly dependent on $l_{j_0} = l_0$ $(1 \leqslant j < n)$, there exist $u_j \in \mathbb{Z}, u_j > 0$, $m_{j0} \in \mathbb{Z}$ $(1 \leqslant j < n)$ such that

$$u_j l_j = m_{j0} l_0, \quad u_j \leqslant 2D^3V_0.$$

Write

$$M = u_1 \dots u_{n-1}, \quad b''_1 = Mp^v q^u b_n.$$

We have

$$\mathrm{ord}_{\wp}\,\Theta \leqslant \mathrm{ord}_{\wp}((\alpha_1^{b_1}\dots\alpha_n^{b_n})^{Mp^vq^u}-1) = \mathrm{ord}_{\wp}(\alpha_n^{b''_1}-1). \tag{4.68}$$

We may assume $\alpha_n^{b''_1} \neq 1$, for otherwise Lemma 1.3 would yield

$$\mathrm{ord}_{\wp}\,\Theta \leqslant \frac{D}{f_{\wp}\log p}\log 2 < U_1.$$

By (4.4), we have

$$|b_1''| = |b_n M p^v q^u| \leqslant B(2D^2)^{n-1} =: B''. \tag{4.69}$$

On applying Lemma 1.4 and using (4.68), (4.69), we obtain

$$\begin{aligned}\operatorname{ord}_{\mathfrak{p}} \Theta &\leqslant \frac{D}{f_{\mathfrak{p}} \log p}\{\log(2B'') + (p^{f_{\mathfrak{p}}} - 1)(1 + 1/(p-1))DV_n\} \\ &\leqslant \frac{D}{f_{\mathfrak{p}} \log p}\{(p^{f_{\mathfrak{p}}} - 1)(1 + 1/(p-1))DV_n + \\ &\quad + \log(D^2 B) + (2n-4)\log D + n \log 2\} \\ &< U_1.\end{aligned} \tag{4.70}$$

(h) $r = 0$.

By the fact that every l_j $(1 \leqslant j \leqslant n)$ is linearly dependent on l_0, we see that $\alpha_1^{b_1} \dots \alpha_n^{b_n}$ is a root of unity. By Lemma 1.3, we get

$$\operatorname{ord}_{\mathfrak{p}} \Theta \leqslant \frac{D}{f_{\mathfrak{p}} \log p} \log 2 < U_1.$$

Note that by the inequalities $DV_j \geqslant f_{\mathfrak{p}} \log p$ $(1 \leqslant j \leqslant n)$ (see (0.7), (0.9)), $n \geqslant 2$, $r(n - r + 1) \leqslant \frac{1}{4}(n+1)^2$, it is readily verified that

$$U_1 \geqslant U_j \quad (2 \leqslant j \leqslant 5). \tag{4.71}$$

On observing (4.71) and the fact that the cases (a)–(h) cover all the possibilities, we complete the proof of Theorem 1.

Proof of Corollary 1. By (0.2)–(0.4), (0.12) and Lemma 1.12, we have

$$\begin{aligned}&u \geqslant 2, \quad 2 + \frac{1}{p-1} \leqslant \frac{9}{4}, \quad \sigma \leqslant 0.155334, \quad \text{if } p \equiv 1 \pmod 4, \\ &u \geqslant 2, \quad 2 + \frac{1}{p-1} \leqslant \frac{5}{2}, \quad \sigma \leqslant 0.1137802, \quad \text{if } p \equiv 3 \pmod 4, \\ &u \geqslant 1, \quad \sigma \leqslant 0.1202248, \quad \text{if } p = 2.\end{aligned} \tag{4.72}$$

Now we prove that $f(x) = x^{x+1+\sigma}/(x+1)^{x+2}$ decreases monotonically for $x \geqslant \frac{3}{2}$. Set $g(x) = \log f(x)$. It suffices to show that

$$g'(x) < 0 \quad \text{for} \quad x \geqslant \tfrac{3}{2}.$$

It is easily verified that

$$\sigma < \tfrac{1}{6} \text{ (by (4.72))}, \quad \log(1+y) \geqslant \tfrac{3}{5}y \quad \text{for} \quad 0 \leqslant y \leqslant \tfrac{2}{3}.$$

Now for $x \geqslant \frac{3}{2}$ we have

$$\begin{aligned} g'(x) &= \frac{\sigma x + 1 + \sigma}{x(x+1)} - \log\left(1 + \frac{1}{x}\right) \\ &\leqslant \frac{\sigma x + 1 + \sigma}{x(x+1)} - \tfrac{3}{5}\cdot\frac{1}{x} \\ &= \frac{1}{x(x+1)}\{-(\tfrac{3}{5} - \sigma)x + \tfrac{2}{5} + \sigma\} \\ &\leqslant \frac{1}{x(x+1)}\{-(\tfrac{3}{5} - \sigma)\cdot\tfrac{3}{2} + \tfrac{2}{5} + \sigma\} \\ &= \frac{1}{x(x+1)}(\tfrac{5}{2}\sigma - \tfrac{1}{2}) < 0. \end{aligned}$$

On noting that $n \geqslant 2$, we get

$$\begin{aligned} (n+1)^{n+2}n^{n+1+\sigma} &= f(n)(n+1)^{2n+4} \leqslant f(2)(n+1)^{2n+4} \\ &= \tfrac{8}{81}\cdot 2^{\sigma}(n+1)^{2n+4}. \end{aligned} \tag{4.73}$$

By (4.72) and (4.73), Corollary 1 follows from Theorem 1 at once.

Proof of Corollary 2. Let

$$k_j := \operatorname{ord}_p \alpha_j, \qquad \alpha'_j = p^{-k_j}\alpha_j \quad (1 \leqslant j \leqslant n). \tag{4.74}$$

Then for $j = 1, \ldots, n$ we have

$$\begin{aligned} &p^{k_j} | p_j \quad \text{and} \quad A_j \geqslant \max(|p^{-k_j}p_j|, |q_j|, p) \quad \text{if } k_j \geqslant 0, \\ &p^{-k_j} | q_j \quad \text{and} \quad A_j \geqslant \max(|p_j|, |p^{k_j}q_j|, p) \quad \text{if } k_j < 0. \end{aligned} \tag{4.75}$$

Now

$$\alpha_1^{b_1}\ldots\alpha_n^{b_n} = p^{k_1b_1+\cdots+k_nb_n}(\alpha'_1)^{b_1}\ldots(\alpha'_n)^{b_n}.$$

From this and (4.74) we get

$$\operatorname{ord}_p\Theta = \begin{cases} 0, & \text{if } k_1b_1 + \cdots + k_nb_n > 0, \\ k_1b_1 + \cdots + k_nb_n, & \text{if } k_1b_1 + \cdots + k_nb_n < 0, \end{cases}$$

and Corollary 2 follows trivially. Thus we may assume $k_1 b_1 + \cdots + k_n b_n = 0$ and we obtain

$$\Theta = (\alpha'_1)^{b_1} \dots (\alpha'_n)^{b_n} - 1. \tag{4.76}$$

On combining (4.74)–(4.76), we may assume in the sequel

$$\mathrm{ord}_p\, \alpha_j = 0 \quad (1 \leqslant j \leqslant n). \tag{4.77}$$

Set

$$K = \begin{cases} \mathbb{Q}(\zeta_4), & \text{if } p > 2, \\ \mathbb{Q}(\zeta_3), & \text{if } p = 2. \end{cases}$$

Thus $D = 2$. Denote by $\mathfrak{p}$ a prime ideal of the ring of integers in K, lying above p. It is well-known that

$$e_{\mathfrak{p}} = 1,$$

$$f_{\mathfrak{p}} = \begin{cases} 1, & \text{if } p \equiv 1 (\mathrm{mod}\, 4), \\ 2, & \text{otherwise.} \end{cases}$$

By (4.77) we have

$$\mathrm{ord}_{\mathfrak{p}}\, \alpha_j = 0 \quad (1 \leqslant j \leqslant n). \tag{4.78}$$

Note that for $j = 1, \dots, n$ we have

$$\log A_j \geqslant \log p \geqslant \frac{f_{\mathfrak{p}} \log p}{D},$$

$$\log A_j \geqslant \log \max(|p_j|, |q_j|) = h(\alpha_j),$$

and

$$|\log \alpha_j| \leqslant \big|\log |\alpha_j|\big| + \pi \leqslant \log \max(|p_j|, |q_j|) + \pi \leqslant \log A_j + \pi,$$

whence

$$\frac{|\log \alpha_j|}{2\pi D} \leqslant \frac{1}{4\pi}(\log A_j + \pi) \leqslant \log A_j.$$

Thus we may take

$$V_j = \log A_j \quad (1 \leqslant j \leqslant n), \qquad V = \log A.$$

By (4.78) and (0.14) we may apply Corollary 1; and by the above observations, Corollary 2 follows from Corollary 1 immediately.

Proof of Theorem 1′. Let

$$k_j := \operatorname{ord}_{\not p_0} \alpha_j, \qquad \alpha'_j := p^{-k_j} \alpha_j^{e_{\not p_0}} \quad (1 \leqslant j \leqslant n). \tag{4.79}$$

We may assume

$$k_1 b_1 + \cdots + k_n b_n = 0, \tag{4.80}$$

for otherwise we would have

$$\operatorname{ord}_{\not p_0} \Theta = \min(k_1 b_1 + \cdots + k_n b_n, 0) \leqslant 0,$$

and the theorem would hold trivially. By (4.79), (4.80) we get

$$\operatorname{ord}_{\not p_0} \Theta \leqslant \operatorname{ord}_{\not p_0}((\alpha_1^{b_1} \dots \alpha_n^{b_n}) e_{\not p_0} - 1) = \operatorname{ord}_{\not p_0}((\alpha'_1)^{b_1} \dots (\alpha'_n)^{b_n} - 1). \tag{4.81}$$

If $e_{\not p_0} > 1$, we may assume further that $(\alpha_1^{b_1} \dots \alpha_n^{b_n}) e_{\not p_0} \neq 1$, for otherwise we would have, by Lemma 1.3,

$$\operatorname{ord}_{\not p_0} \Theta \leqslant \frac{D_0}{f_{\not p_0} \log p} \log 2,$$

whence the theorem would follow at once. Thus in any case we have

$$(\alpha'_1)^{b_1} \dots (\alpha'_n)^{b_n} \neq 1. \tag{4.82}$$

By Lemma 1.3 and by the identity $h(\alpha) = h(1/\alpha)$ for any non-zero algebraic number α, we get

$$|k_j| \leqslant \frac{D_0}{f_{\not p_0} \log p} h(\alpha_j) \leqslant \frac{D_0}{f_{\not p_0} \log p} V_j \quad (1 \leqslant j \leqslant n). \tag{4.83}$$

Thus, by $e_{\not p_0} f_{\not p_0} \leqslant D_0$, we have

$$h(\alpha'_j) \leqslant |k_j| h(p) + e_{\not p_0} h(\alpha_j) \leqslant 2(D_0/f_{\not p_0}) V_j \quad (1 \leqslant j \leqslant n). \tag{4.84}$$

Further, by (4.83) and (0.21), we see that

$$|\log \alpha_j'| = \min_{m \in \mathbb{Z}} |\log \alpha_j' + 2m\pi i| \leqslant |-k_j \log p + e_{\wp_0} \log \alpha_j|$$
$$\leqslant (D_0/f_{\wp_0}) V_j (12D_0 + 1) \quad (1 \leqslant j \leqslant n). \tag{4.85}$$

Now we choose

$$K = \begin{cases} K_0(\zeta_4), & \text{if } p > 2, \\ K_0(\zeta_3), & \text{if } p = 2 \end{cases} \tag{4.86}$$

and let $\wp$ be any prime ideal of the ring of integers in K, such that $\wp \supseteq \wp_0$. Thus

$$D = [K:\mathbb{Q}] = [K:K_0]D_0. \tag{4.87}$$

By Lemma 1.12 and Lemma in the Appendix, we have

$$e_\wp = e_{\wp_0}, \tag{4.88}$$

$$f_\wp = f_0, \tag{4.89}$$

where f_0 is given by (0.20). It is readily verified, by (4.84), (4.85) and (0.21), that

$$\max\left(h(\alpha_j'), \frac{|\log \alpha_j'|}{2\pi D}, \frac{f_\wp \log p}{D} \right) \leqslant 2(D_0/f_{\wp_0}) V_j =: V_j' \quad (1 \leqslant j \leqslant n). \tag{4.90}$$

Now by (4.86), (4.82) and the fact that $\text{ord}_\wp \alpha_j' = \text{ord}_{\wp_0} \alpha_j' = 0$ $(1 \leqslant j \leqslant n)$, which follows from (4.79), we can apply Corollary 1 to $\text{ord}_\wp((\alpha_1')^{b_1} \ldots (\alpha_n')^{b_n} - 1)$; and on utilizing (4.81), (4.87)–(4.90), we obtain Theorem 1′.

REMARK 1. It is easy to verify that if $K = K_0$ with K defined by (4.86), then C_1' can be replaced by $2^n C_1$, where C_1 is given in Corollary 1.

2. Using the argument in the proof of Theorem 1′, we can deduce from Theorem 1, instead of from Corollary 1, a more precise and more sophisticated bound for $\text{ord}_{\wp_0} \Theta$.

5. Proof of Theorems 2 and 2′

Proof of Theorem 2. We record inequalities (5.1)–(5.3) for later use. It is readily verified that

$$\log x \leqslant x^{1/7} \quad \text{for } x \geqslant 10^{10}. \tag{5.1}$$

By $n \geqslant 2$, $D \geqslant 2$, $DV_{n-1} \geqslant f_{\mathfrak{p}} \log p$ and Lemma 1.12 it is easy to see that

$$\frac{D}{f_{\mathfrak{p}} \log p} Q > 10^{10}. \tag{5.2}$$

Recalling $\rho' = 1.0752$ if $p > 2$ and $\rho' = 1.1114$ if $p = 2$, we show

$$\rho' \log\left(\frac{D}{f_{\mathfrak{p}} \log p} Q\right) \geqslant \max\left(n \log(2^{11} qnD), \frac{f_{\mathfrak{p}} \log p}{D}\right). \tag{5.3}$$

We verify the case $p = 2$ and leave the remaining cases to the reader. Now $q = 3$, $\rho' = 1.1114$. By $D \geqslant 2$, $DV_{n-1} \geqslant f_{\mathfrak{p}} \log p$ and Lemma 1.12 we see that for $n \geqslant 8$

$$\left(\frac{2^{11} \cdot 3nD}{(DV_{n-1})^{\rho'}}\right)^n \leqslant \left(\frac{2^{11} \cdot 3nD}{(\log 4)^{\rho'}}\right)^{n+1} \leqslant (10nD)^{2(n+1)\rho'},$$

which implies (5.3); (5.3) for $p = 2$, $2 \leqslant n \leqslant 7$ is readily verified by direct calculation.

Let $r + 1$ be the rank of $\{l_0\, l_1, \ldots, l_n\}$, where l_j is given by (4.4) and (4.5). We fix $0 = j_0 < j_1 < \cdots < j_r \leqslant n$ as in the proof of Theorem 1. We deal with the following eight cases (a)–(h) separately, and we shall freely use the discussion in the corresponding cases (a)–(h) of the proof of Theorem 1.

In the proof of Theorem 2 we always bear the following simple observation in mind that if (0.19) holds for $Z > 0$ and any δ with $0 < \delta \leqslant (f_{\mathfrak{p}} \log p/D)Z$, then so does (0.19) for any $Z'' \geqslant Z$ and any δ'' with $0 < \delta'' \leqslant (f_{\mathfrak{p}} \log p/D)Z''$.

(a) $r = n$, $V_n < nV_{n-1}$.

By (0.15) and (4.20), we have

$$\operatorname{ord}_p b_n'' = \min_{1 \leqslant j \leqslant n} \operatorname{ord}_p b_j''. \tag{5.4}$$

On noting (4.11), (4.14), (4.15), (5.4) we may apply Corollary 2.4 to

$$\operatorname{ord}_{\mathfrak{p}}((\alpha_1')^{b_1''} \ldots (\alpha_n')^{b_n''} - 1).$$

Set

$$\Psi_1 = 2c'\left(\frac{a'}{2}\right)^n (n+1)^{n+1} n^{n+\sigma} \cdot \frac{p^{f_{\mathfrak{p}}} - 1}{q^u} \cdot \left(\frac{2 + 1/(p-1)}{f_{\mathfrak{p}} \log p}\right)^{n+2} \cdot$$

$$\cdot D^{n+2} V_1 \ldots V_n \max(n \log(2^{10} qn^2 D^2 V_{n-1}), f_{\mathfrak{p}} \log p), \tag{5.5}$$

where a', c' are given in Corollary 2.3. By the argument in the proof of Corollary 1, we have

$$\Psi_1 \leqslant \Phi/\rho'. \tag{5.6}$$

By (4.13), Proposition 3.1.(e), (4.4) and $V_n < nV_{n-1}$, we get

$$|b_n''| \leqslant 4n^{n+3}((n+1)D^2)^{n+1}(DV_{n-1})^n B_n = \frac{B_n}{B} B'' =: B_n'', \tag{5.7}$$

where B'' is given by (4.17). Now we take

$$Z = \Phi/V_n, \quad Z' = \Psi_1/V_n'. \tag{5.8}$$

Then by (4.16) and (5.6) we get

$$\frac{Z'}{Z} = \frac{\Psi_1 V_n}{\Phi V_n'} \leqslant \frac{1}{\rho'} < 1. \tag{5.9}$$

It is readily verified, on noting (5.3), (5.7) and (0.18), that for any δ with $0 < \delta \leqslant (f_{\mathfrak{p}} \log p/D)Z$,

$$\max\left\{\rho' \log\left(\delta^{-1}\frac{f_{\mathfrak{p}}\log p}{D} ZB_n''\right), n\log(2^{11}qnD), \frac{f_{\mathfrak{p}}\log p}{D}\right\}$$
$$\leqslant \rho' \log(\delta^{-1}ZB_nQ). \tag{5.10}$$

By (4.12), (4.16), (5.7)–(5.10), an application of Corollary 2.4 yields that for any δ with $0 < \delta \leqslant (f_{\mathfrak{p}} \log p/D)Z$

$$\begin{aligned} \operatorname{ord}_{\mathfrak{p}} \Theta &< \max(\rho'\Psi_1 \log(\delta^{-1}ZB_nQ), \delta B''/B_n'') \\ &\leqslant \max(ZV_n \log(\delta^{-1}ZB_nQ), \delta B/B_n). \end{aligned}$$

This is just (0.19) with $j = n$. Suppose now $1 \leqslant j < n$. We take

$$Z = \frac{15}{7}\cdot\frac{\Phi}{V_j}, \qquad Z' = \frac{\Psi_1}{V_n'}, \qquad B_n'' := B'' \quad (B'' \text{ is given by (4.17)}). \tag{5.11}$$

Then, by (5.6), we have

$$\rho'\Psi_1 \leqslant \Phi = \tfrac{7}{15}ZV_j, \quad \frac{Z'}{Z} < 1. \tag{5.12}$$

On noting (5.3), (5.11) and (4.17), it is easy to see that for any δ with

$$0 < \delta \leqslant (f_{\mathfrak{p}} \log p/D)Z,$$

$$\max\left\{\rho' \log\left(\delta^{-1}\frac{f_{\mathfrak{p}} \log p}{D} ZB_n''\right), n\log(2^{11}qnD), \frac{f_{\mathfrak{p}} \log p}{D}\right\}$$
$$\leqslant \rho' \log(\delta^{-1} ZBQ). \tag{5.13}$$

By (4.12), (4.16), (5.11)–(5.13), on applying Corollary 2.4 to

$$\operatorname{ord}_{\mathfrak{p}}((\alpha_1')^{b_1''} \ldots (\alpha_n')^{b_n''} - 1),$$

we get

$$\begin{aligned}\operatorname{ord}_{\mathfrak{p}} \Theta &< \max(\Psi_1 \rho' \log(\delta^{-1} ZBQ), \delta)\\ &\leqslant \max(\tfrac{7}{15} Z V_j \log(\delta^{-1} ZBQ), \delta).\end{aligned} \tag{5.14}$$

It remains to show that for any δ with $0 < \delta \leqslant (f_{\mathfrak{p}} \log p/D)Z$,

$$\tfrac{7}{15} Z V_j \log(\delta^{-1} ZBQ) \leqslant \max(Z V_j \log(\delta^{-1} Z B_j Q), \delta B/B_j). \tag{5.15}$$

To prove (5.15) we may assume

$$\log\left(\frac{B}{B_j}\right) > \tfrac{8}{7} \log(\delta^{-1} Z B_j Q), \tag{5.16}$$

and it suffices to show that

$$7 Z V_j \log(B/B_j) \leqslant 8 \delta B/B_j. \tag{5.17}$$

Note that, by (5.2) we have

$$\delta^{-1} Z B_j Q \geqslant \frac{D}{f_{\mathfrak{p}} \log p} Q > 10^{10}.$$

Hence we get, by (5.1), (5.16),

$$\frac{\frac{8}{7} B/B_j}{\log(B/B_j)} \geqslant \frac{(\delta^{-1} Z B_j Q)^{8/7}}{\log(\delta^{-1} Z B_j Q)} \geqslant \delta^{-1} Z B_j Q \geqslant \delta^{-1} Z V_{n-1} \geqslant \delta^{-1} Z V_j,$$

whence (5.17) and (5.15). On combining (5.15) with (5.14), we obtain (0.19). This completes the proof in the case (a).

(b) $r = n, V_n \geqslant n V_{n-1}$.

In the present case, (5.4), (4.11), (4.14), (4.15) are also valid. Hence we may apply Corollary 2.4 to $\operatorname{ord}_{\mathfrak{p}}((\alpha_1')^{b_1''} \ldots (\alpha_n')^{b_n''} - 1)$. By (4.13), Proposition 3.2 and (4.4) we get

$$|b_n''| \leqslant 4n^3(n+1)^{3n+1}D^{2(n+1)}(DV_{n-1})^n B_n\left(\frac{V_n}{V_n'}\right)^3 = \frac{B_n}{B}B'' =: B_n'', \tag{5.18}$$

where B'' is given by (4.25). Set

$$\Psi_2 = 4c'\left(\frac{a'}{2}\right)^n (n+1)^{n+1}n^{n-1}(n-1)^{\sigma}\cdot\frac{p^{f_{\mathfrak{p}}}-1}{q^u}\cdot\left(\frac{2+1/(p-1)}{f_{\mathfrak{p}}\log p}\right)^{n+2}\cdot$$
$$\cdot D^{n+2}V_1\ldots V_{n-1}V_n'\max(n\log(2^{10}qn^2D^2V_{n-1}), f_{\mathfrak{p}}\log p).$$

By (5.5), (5.6) we see that

$$\frac{\Psi_2}{\Phi} = \frac{\Psi_2}{\Psi_1}\cdot\frac{\Psi_1}{\Phi} \leqslant \frac{2}{\rho' n}\cdot\frac{V_n'}{V_n}. \tag{5.19}$$

Now take

$$Z = \frac{\Phi}{V_n}, \quad Z' = \frac{\Psi_2}{V_n'}. \tag{5.20}$$

Thus

$$\frac{Z'}{Z} \leqslant \frac{2}{\rho' n} < 1. \tag{5.21}$$

It is easily verified, by (5.3), (5.18) and (0.18), that for any δ with $0 < \delta \leqslant (f_{\mathfrak{p}}\log p/D)Z$,

$$\max\left\{\rho'\log\left(\delta^{-1}\frac{f_{\mathfrak{p}}\log p}{D}ZB_n''\right), n\log(2^{11}qnD), \frac{f_{\mathfrak{p}}\log p}{D}\right\}$$
$$\leqslant \frac{n}{2}\rho'\log\left(\delta^{-1}ZB_nQ\left(\frac{V_n}{V_n'}\right)^3\right). \tag{5.22}$$

By (4.12), (4.16) (with $1 \leqslant j < n$), (4.24), (5.18)–(5.22) and (5.2), an application of

Corollary 2.4 yields that for any δ with $0 < \delta \leqslant (f_{\mathfrak{p}} \log p/D)Z$,

$$\begin{aligned}\mathrm{ord}_{\mathfrak{p}}\Theta &< \max\left(\Psi_2 \cdot \frac{n}{2}\rho' \log(\delta^{-1}ZB_nQ(V_n/V_n')^3), \delta B/B_n\right)\\ &\leqslant \max\left\{\Phi \cdot \frac{V_n'}{V_n}\left(\log(\delta^{-1}ZB_nQ) + 3\log\left(\frac{V_n}{V_n'}\right)\right), \delta B/B_n\right\}\\ &\leqslant \max\{ZV_n \cdot \log(\delta^{-1}ZB_nQ), \delta B/B_n\},\end{aligned}$$

which is just (0.19) for $j = n$. It is readily to verify (0.19) for j with $1 \leqslant j < n$, using the same argument as in the case (a). We omit the details here

REMARK. If (0.15) does not hold, then we have the following result.

Suppose that (0.13) *and* (0.14) *hold. Suppose further that* $r = n$ *and*

$$\begin{aligned}&h = \max\{i \mid 1 \leqslant i \leqslant n, \mathrm{ord}_p b_i = \min_{1 \leqslant k \leqslant n} \mathrm{ord}_p b_k\} < n,\\ &\mathrm{ord}_p b_j = \min_{1 \leqslant k \leqslant n} \mathrm{ord}_p b_k.\end{aligned} \tag{0.15)'$$

Set

$$\Phi' \geqslant C_2(n+1)^{2n+3}\frac{p^{f_{\mathfrak{p}}}}{(f_{\mathfrak{p}}\log p)^{n+2}}D^{n+2}V_1 \ldots V_n \cdot$$

$$\cdot \max\left(\log(2^{10}q(n+1)^2D^2V_n), \frac{f_{\mathfrak{p}}\log p}{n}\right),$$

$$Z = \frac{15}{7} \cdot \frac{\Phi'}{V_j}.$$

Then for any δ *with* $0 < \delta \leqslant (f_{\mathfrak{p}} \log p/D)Z$, *we have*

$$\mathrm{ord}_{\mathfrak{p}}\Theta < \max(ZV_j \log(\delta^{-1}ZB_jQ), \delta B/B_j),$$

where Q *and* C_2 *are given in Theorem* 2.

Proof. By (4.13) and Proposition 3.1, and by the first row of (0.15)′, we see that

$$\max\{i \mid 1 \leqslant i \leqslant n, \mathrm{ord}_p b_i'' = \min_{1 \leqslant k \leqslant n} \mathrm{ord}_p b_k''\} = h. \tag{5.23}$$

Obviously (0.15)′ implies that

$$j \leqslant h. \tag{5.24}$$

In case (a) ($r = n, V_n < nV_{n-1}$), set

$$\Psi_3 = 2c'\left(\frac{a'}{2}\right)^n (n+1)^{n+1} n^{n+\sigma} \frac{p^{f_\mathfrak{p}}-1}{q^u} \cdot \left(\frac{2+1/(p-1)}{f_\mathfrak{p}\log p}\right)^{n+2} \cdot$$
$$\cdot D^{n+2} V_1 \dots V_n \max(n\log(2^{10}q(n+1)^2D^2V_n), f_\mathfrak{p}\log p)$$

and

$$Z' = \frac{\Psi_3}{V'_h}, \quad (V'_h \text{ is given in (4.16)}).$$

Note that by the argument in the proof of Corollary 1, we have

$$\Psi_3 \leqslant \Phi'/\rho'.$$

Further (5.24) gives

$$V_j \leqslant V_h \leqslant V'_h.$$

Thus

$$\frac{Z'}{Z} = \frac{7}{15}\cdot\frac{\Psi_3}{\Phi'}\cdot\frac{V_j}{V'_h} < 1.$$

We have also

$$|b''_h| \leqslant B'' =: B''_h \quad (B'' \text{ is given in (4.17)}).$$

On applying Corollary 2.4 to $\operatorname{ord}_\mathfrak{p}((\alpha'_1)^{b''_1}\dots(\alpha'_n)^{b''_n} - 1)$ and using (4.12), we obtain

$$\operatorname{ord}_\mathfrak{p}\Theta < \max\left\{\Psi_3 \max\left(\rho'\log\left(\delta^{-1}\frac{f_\mathfrak{p}\log p}{D} ZB''_h\right),\right.\right.$$
$$\left.\left. n\log(2^{11}qnD), \frac{f_\mathfrak{p}\log p}{D}\right), \delta\right\}.$$

The rest of the proof is completely the same as in the case (a) with (0.15), so we omit the details. We also leave the verification for the case (b) ($r = n, V_n \geqslant nV_{n-1}$) to the reader.

(c) $2 \leqslant r < n,\ j_r < n$.

In this case we set

$$\alpha'_s := \alpha_{j_s} \quad (1 \leqslant s \leqslant r)$$

and (4.31), (4.35), (4.38) are valid. We may also assume (4.37). Thus we can apply the results proved in the cases (a), (b) and the above remark to $\mathrm{ord}_{\wp}((\alpha'_1)^{b''_1} \cdots (\alpha'_r)^{b''_r} - 1)$, since $\{l_0, \log\alpha'_1, \ldots, \log\alpha'_r\} = \{l_0, l_{j_1}, \ldots, l_{j_r}\}$ has rank $r+1$. Set

$$\Phi^* = C_2^*(r+1)^{2r+3} \cdot \frac{p^{f_\wp}}{(f_\wp \log p)^{r+2}} \times$$

$$\times D^{r+2} V_{n-r} \ldots V_{n-1} \max\left(\log(2^{10} q(r+1)^2 D^2 V_{n-1}), \frac{f_\wp \log p}{r}\right),$$

where C_2^* is obtained by substituting r for n in C_2. Let h with $0 \leqslant h < r$ be such that

$$\mathrm{ord}_p b''_{h+1} = \min_{1 \leqslant i \leqslant r} \mathrm{ord}_p b''_i.$$

Let

$$Z^* = \frac{15}{7} \cdot \frac{\Phi^*}{V'_{h+1}} = \frac{15}{7} \cdot \frac{\Phi^*}{V_{n-r+h}}, \tag{5.25}$$

$$Z = \frac{\Phi}{V_j}, \quad j \text{ fixed with } 1 \leqslant j \leqslant n. \tag{5.26}$$

By the inequality $DV_j \geqslant f_\wp \log p$ $(1 \leqslant j \leqslant n)$ (see (0.7), (0.9)), it is easily verified that

$$\frac{\Phi^*}{\Phi} \leqslant \frac{2}{45} \cdot \frac{1}{(n+1)^2} \cdot \frac{V_{n-r} \ldots V_{n-1}}{V_{n-r+1} \ldots V_n} \tag{5.27}$$

and hence

$$\frac{Z^*}{Z} \leqslant \frac{1}{10} \cdot \frac{1}{(n+1)^2} \cdot \frac{V_j}{V_n}. \tag{5.28}$$

Set $Q^* = p(10rD)^{2(r+1)}(DV_{n-2})^r$. Obviously, $Q^* \leqslant Q$, where Q is given by (0.18).

Note also that

$$|b''_{h+1}| \leqslant B'' =: B''_{h+1},$$

where B'' is given by (4.38). Now by (4.35), (4.37) and on applying the cases (a), (b) and the Remark (below the proof for the case (b)) to $\operatorname{ord}_{\mathfrak{p}}((\alpha'_1)^{b''_1} \ldots (\alpha'_r)^{b''_r} - 1)$, we see that for any δ_1 with $0 < \delta_1 \leqslant (f_{\mathfrak{p}} \log p/D)Z^*$, we have

$$\operatorname{ord}_{\mathfrak{p}} \Theta < \max(Z^* V_{n-r+h} \log(\delta_1^{-1} Z^* B''_{h+1} Q^*), \delta_1),$$

whence for any δ with $0 < \delta \leqslant (f_{\mathfrak{p}} \log p/D)Z$, we have, by (5.28),

$$\begin{aligned}\operatorname{ord}_{\mathfrak{p}} \Theta &< \max(Z^* V_{n-r+h} \log(\delta^{-1} Z B'' Q), \delta) \\ &\leqslant \max\left(\frac{1}{10} \cdot \frac{1}{(n+1)^2} Z V_j \frac{V_{n-1}}{V_n} \log(\delta^{-1} Z B'' Q), \delta\right) \\ &\leqslant \max\left(\frac{1}{10} \cdot Z V_j \frac{V_{n-1}}{V_n} \log\left(\delta^{-1} Z B Q \frac{V_n}{V_{n-1}}\right), \delta\right),\end{aligned} \tag{5.29}$$

where the third inequality follows from

$$\log(\delta^{-1} Z B'' Q) \leqslant (n+1)^2 \log\left(\delta^{-1} Z B Q \frac{V_n}{V_{n-1}}\right),$$

which can be easily verified, using (0.18) and (4.38).

When $j < n$, on noting that (by (5.2))

$$\begin{aligned}\frac{V_{n-1}}{V_n} \log\left(\delta^{-1} Z B Q \frac{V_n}{V_{n-1}}\right) &\leqslant \frac{V_{n-1}}{V_n} \log(\delta^{-1} Z B Q) \cdot \left(1 + \log\left(\frac{V_n}{V_{n-1}}\right)\right) \\ &\leqslant \log(\delta^{-1} Z B Q),\end{aligned}$$

we see, by (5.29) and by an argument similar to the proof of (5.15), that

$$\begin{aligned}\operatorname{ord}_{\mathfrak{p}} \Theta &< \max(\tfrac{1}{10} Z V_j \log(\delta^{-1} Z B Q), \delta) \\ &\leqslant \max(Z V_j \log(\delta^{-1} Z B_j Q), \delta B/B_j).\end{aligned}$$

When $j = n$, we see from (5.29) that

$$\operatorname{ord}_{\mathfrak{p}} \Theta < \max\left\{\tfrac{1}{10} Z V_{n-1}\left(\log(\delta^{-1} Z B Q) + \log\left(\frac{V_n}{V_{n-1}}\right)\right), \delta\right\}. \tag{5.30}$$

Similarly to the proof of (5.15), it is easily seen that

$$\tfrac{7}{15} Z V_{n-1} \log(\delta^{-1} Z B Q) \leqslant \max(Z V_{n-1} \log(\delta^{-1} Z B_n Q),\ \delta B/B_n). \tag{5.31}$$

On combining (5.30) with (5.31) we get

$$\mathrm{ord}_{\mathfrak{p}} \Theta < \max(Z V_n \log(\delta^{-1} Z B_n Q),\ \delta B/B_n).$$

This completes the proof for the case (c).

REMARK. From the proof we see that in the case (c), the hypothesis (0.15) can be omitted.

(d) $2 \leqslant r < n, j_r = n, V_n < \frac{1}{4} r(r+1) V_{n-1}$.

In this case we have (4.48), (4.49) and we may also assume (4.50). By (4.56) and (0.15) we see that

$$\mathrm{ord}_p b_r'' = \min_{1 \leqslant i \leqslant r} b_i''. \tag{5.32}$$

Thus we can apply Corollary 2.4 to $\mathrm{ord}_{\mathfrak{p}}((\alpha_1')^{b_1''} \dots (\alpha_r')^{b_r''} - 1)$. Set

$$\Psi_4 = \tfrac{5}{6} c' \left(\frac{a'}{2}\right)^r (r+1)^{r+2} r^r (r-1)^{\sigma} \cdot \frac{p^{f_{\mathfrak{p}}} - 1}{q^u} \cdot \left(\frac{2 + 1/(p-1)}{f_{\mathfrak{p}} \log p}\right)^{r+2} \cdot$$

$$\cdot D^{r+2} V_{n-r+1} \dots V_n \max(r \log(2^{10} q r^2 D^2 V_{n-1}), f_{\mathfrak{p}} \log p),$$

$$Z' := \frac{\Psi_4}{V_r'} = \frac{24}{5} \cdot \frac{1}{r(r+1)} \cdot \frac{\Psi_4}{V_n}, \quad Z = \frac{\Phi}{V_n}. \tag{5.33}$$

By (5.5), (5.6) and the inequality $D V_j \geqslant f_{\mathfrak{p}} \log p$ $(1 \leqslant j \leqslant n)$, we see that

$$\Psi_4 = \Psi_1(\Psi_4/\Psi_1) \leqslant \frac{5}{6} \cdot \left(2 \cdot \frac{a'}{2} \cdot \left(2 + \frac{1}{p-1}\right) n \rho'\right)^{-1} \Phi \leqslant \Phi/(48 n \rho'). \tag{5.34}$$

Hence

$$Z'/Z \leqslant \Psi_4/\Phi \leqslant 1/(48 n \rho') < 1. \tag{5.35}$$

By (4.47) and Proposition 3.3 we get

$$|b_r''| \leqslant \frac{B_n}{B} \cdot B'' =: B_r'', \tag{5.36}$$

where B'' is given by (4.53). By (5.3), (5.36), (4.53) and (0.18), it is easily seen that for any δ with $0 < \delta \leqslant (f_{\mathfrak{p}} \log p/D)Z$, we have

$$\max\left\{\rho' \log\left(\delta^{-1} \frac{f_{\mathfrak{p}} \log p}{D} Z B_r''\right), r\log(2^{11} qrD), \frac{f_{\mathfrak{p}} \log p}{D}\right\}$$
$$\leqslant 48\rho' n \log(\delta^{-1} Z B_n Q). \tag{5.37}$$

On noting (4.46), (5.33)–(5.37) and on applying Corollary 2.4 to

$$\operatorname{ord}_{\mathfrak{p}}((\alpha_1')^{b_1''} \ldots (\alpha_r')^{b_r''} - 1),$$

we obtain for any δ with $0 < \delta \leqslant (f_{\mathfrak{p}} \log p/D)Z$

$$\begin{aligned}\operatorname{ord}_{\mathfrak{p}} \Theta &< \max(\Psi_4 \cdot 48\rho' n \log(\delta^{-1} Z B_n Q),\ \delta B/B_n)\\ &\leqslant \max(Z V_n \log(\delta^{-1} Z B_n Q),\ \delta B/B_n),\end{aligned}$$

which is exactly (0.19) with $j = n$. The verification of (0.19) for $j < n$ is similar to that in the case (a). We omit the details here.

(e) $2 \leqslant r < n, j_r = n, V_n \geqslant \frac{1}{4}r(r+1)V_{n-1}$.

In this case we have (4.48), (4.49), (5.32) and we may also assume (4.50). Thus we can apply Corollary 2.4 to $\operatorname{ord}_{\mathfrak{p}}((\alpha_1')^{b_1''} \ldots (\alpha_r')^{b_r''} - 1)$. Set

$$\Psi_5 = 4c'\left(\frac{a'}{2}\right)^r (r+1)^{r+1} r^{r-1} (r-1)^{\sigma} \cdot \frac{p^{f_{\mathfrak{p}}} - 1}{q^u} \cdot \left(\frac{2 + 1/(p-1)}{f_{\mathfrak{p}} \log p}\right)^{r+2} \cdot$$
$$\cdot D^{r+2} V_{n-r+1} \ldots V_{n-1} V_r' \max(r \log(2^{10} q r^2 D^2 V_{n-1}),\ f_{\mathfrak{p}} \log p),$$
$$Z' = \frac{\Psi_5}{V_r'}, \quad Z = \frac{\Phi}{V_n}, \quad (V_r' \text{ is given by (4.60)}). \tag{5.38}$$

By (5.5), (5.6) and the inequality $DV_j \geqslant f_{\mathfrak{p}} \log p$ $(1 \leqslant j \leqslant n)$, we see that

$$\frac{\Psi_5}{\Phi} = \frac{\Psi_5}{\Psi_1} \cdot \frac{\Psi_1}{\Phi} \leqslant \frac{1}{10(n+1)n^2 \rho'} \cdot \frac{V_r'}{V_n}, \tag{5.39}$$

whence

$$\frac{Z'}{Z} \leqslant \frac{1}{10(n+1)n^2 \rho'} < 1. \tag{5.40}$$

By (4.47) and Proposition 3.4 we get

$$|b_r''| \leqslant \frac{B_n}{B} \cdot B'' =: B_r'', \tag{5.41}$$

where B'' is given by (4.62). By (5.3), (5.41), (4.62) and (0.18), it is easily verified that for any δ with $0 < \delta \leqslant (f_{\mathfrak{p}} \log p/D)Z$ we have

$$\max\left\{\rho' \log\left(\delta^{-1}\frac{f_{\mathfrak{p}}\log p}{D}ZB''_r\right), r\log(2^{11}qrD), \frac{f_{\mathfrak{p}}\log p}{D}\right\}$$

$$\leqslant 10(n+1)n^2\rho' \log\left(\delta^{-1}ZB_nQ\left(\frac{V_n}{V'_r}\right)^3\right). \tag{5.42}$$

For any δ in the above interval, we have, by (5.2)

$$\log(\delta^{-1}ZB_nQ) \geqslant \log\left(\frac{D}{f_{\mathfrak{p}}\log p}Q\right) > 3,$$

whence

$$\frac{V'_r}{V_n}\log\left(\delta^{-1}ZB_nQ\left(\frac{V_n}{V'_r}\right)^3\right) \leqslant \frac{V'_r}{V_n}\log(\delta^{-1}ZB_nQ)\cdot\left(1+\log\left(\frac{V_n}{V'_r}\right)\right)$$

$$\leqslant \log(\delta^{-1}ZB_nQ). \tag{5.43}$$

On noting (4.46), (5.38)–(5.43), and on applying Corollary 2.4 to

$$\operatorname{ord}_{\mathfrak{p}}((\alpha'_1)^{b''_1}\dots(\alpha'_r)^{b''_r}-1),$$

we obtain for any δ with $0 < \delta \leqslant (f_{\mathfrak{p}} \log p/D)Z$

$$\operatorname{ord}_{\mathfrak{p}}\Theta < \max\left(\Psi_5\cdot 10(n+1)n^2\rho' \log\left(\delta^{-1}ZB_nQ\left(\frac{V_n}{V'_r}\right)^3\right),\ \delta B/B_n\right)$$

$$\leqslant \max\left(\Phi\cdot\frac{V'_r}{V_n}\log\left(\delta^{-1}ZB_nQ\left(\frac{V_n}{V'_r}\right)^3\right),\ \delta B/B_n\right)$$

$$\leqslant \max(ZV_n\log(\delta^{-1}ZB_nQ),\ \delta B/B_n),$$

exactly (0.19) with $j = n$. The verification of (0.19) for $j < n$ is similar to that in the case (a). We omit the details here.

(f) $r = 1, j_1 < n$.

By $DV_j \geqslant f_{\mathfrak{p}} \log p$ $(1 \leqslant j \leqslant n)$ and (0.18), it is readily verified that for any δ with $0 < \delta \leqslant (f_{\mathfrak{p}}\log p/D)Z$,

$$\log(\delta^{-1}ZB_jQ) \geqslant \log p. \tag{5.44}$$

Again by $DV_j \geqslant f_{\not p} \log p$ $(1 \leqslant j \leqslant n)$, and by (5.44), we see that for any δ with $0<\delta\leqslant(f_{\not p}\log p/D)Z$,

$$\frac{D}{f_{\not p}\log p}\left\{(p^{f_{\not p}}-1)\left(1+\frac{1}{p-1}\right)DV_{n-1}+(n-1)\log(DV_n)+\right.$$

$$\left.+(4n-6)\log D+(n-1)\log 16+\log(2n)\right\}$$

$$<\tfrac{5}{6}\Phi\log(\delta^{-1}ZB_jQ)$$

$$=\tfrac{5}{6}ZV_j\log(\delta^{-1}ZB_jQ). \tag{5.45}$$

By (0.16) and $DV_j \geqslant f_{\not p}\log p$ $(1\leqslant j\leqslant n)$, it is easy to see that

$$\frac{D}{f_{\not p}\log p}\leqslant 10^{-4}\Phi\cdot\frac{V_{n-1}}{V_n}\leqslant 10^{-4}\Phi.$$

Obviously, by (0.18),

$$\log(D^2B)\leqslant\log(\delta^{-1}ZBQ).$$

When $j=n$, we have

$$\begin{aligned}\frac{D}{f_{\not p}\log p}\log(D^2B)&\leqslant 10^{-4}\Phi\cdot\frac{V_{n-1}}{V_n}\log(\delta^{-1}ZBQ)\\&=10^{-4}ZV_{n-1}\log(\delta^{-1}ZBQ)\\&<\tfrac{1}{6}\max(ZV_n\log(\delta^{-1}ZB_nQ),\ \delta B/B_n),\end{aligned}\tag{5.46}$$

where the last inequality follows from (5.31). When $j<n$, we see, by an argument similar to the proof of (5.15), that

$$\begin{aligned}\frac{D}{f_{\not p}\log p}\log(D^2B)&\leqslant 10^{-4}\Phi\log(\delta^{-1}ZBQ)\\&=10^{-4}ZV_j\log(\delta^{-1}ZBQ)\\&<\tfrac{1}{6}\max(ZV_j\log(\delta^{-1}ZB_jQ),\ \delta B/B_j).\end{aligned}\tag{5.47}$$

On combining (4.67) and (5.45)–(5.47), we obtain (0.19).

(g) $r=1, j_1=n$.

By (4.70) and by the argument in the case (f), we can easily obtain (0.19).

(h) $r = 0$.

In this case $\alpha_1, \ldots, \alpha_n$ are roots of unity. By Lemma 1.3, we get

$$\operatorname{ord}_{\wp} \Theta \leqslant \frac{D}{f_{\wp} \log p} \cdot \log 2,$$

whence (0.19) is trivially true.

Noting that the cases (a)–(h) cover all the possibilities, the proof of Theorem 2 is complete.

Proof of Theorem 2′. By an argument similar to the proof of Theorem 1′, one can easily deduce Theorem 2′ from Theorem 2. We omit the details here.

REMARK 1. It is easy to see that if $K = K_0$ with K defined by (4.86), then C_1' in the statement of Theorem 2′ can be replaced by $2^n C_1$, where C_1 is given in Corollary 1.

2. From the proof of Theorem 2, it is easily seen that (0.16) in the statement of Theorem 2 can be replaced by $\Phi = \rho' \Psi_1$ with Ψ_1 given by (5.5). Accordingly, on choosing K by (4.86), we can replace Φ in the statement of Theorem 2′ by the quantity $\rho' \Psi_1'$, where Ψ_1' is obtained from Ψ_1 by substituting (in (5.5)) f_0 for $f_{\wp}$, $[K:K_0]D_0$ for D, $(2D_0/f_{\wp_0})V_j$ for V_j $(1 \leqslant j \leqslant n)$.

Appendix

Let p be a prime number, K_0 an algebraic number field and

$$K = \begin{cases} K_0(\zeta_4), & \text{if } p > 2, \\ K_0(\zeta_3), & \text{if } p = 2, \end{cases} \quad \text{with } \zeta_m = e^{2\pi i/m}, \quad m = 3, 4.$$

Let $\wp(\wp_0)$ be a prime ideal of the ring of integers in $K(K_0)$, such that $p \in \wp_0 \subseteq \wp$. Let $\operatorname{ord}_{\wp}, e_{\wp}, f_{\wp}$ be defined as in Section 0.2, and $\operatorname{ord}_{\wp_0}, e_{\wp_0}, f_{\wp_0}$ be defined with respect to K_0 in the similar way. Denote by $\mathbb{F}_{p^k}$ the finite field with p^k elements.

LEMMA. *Suppose that $K \neq K_0$. Then*

$$e_{\wp} = e_{\wp_0}, \quad f_{\wp} = \begin{cases} f_{\wp_0}, & \text{if } p \equiv 1 \pmod 4, \\ \max(f_{\wp_0}, 2), & \text{otherwise.} \end{cases}$$

Proof. For $p = 2$, we have $K = K_0(\zeta_3)$. By the hypothesis, 1, ζ_3 are linearly independent over K_0; and

$$\Delta(1, \zeta_3) = \begin{vmatrix} 1 & \frac{1}{2}(-1+\sqrt{3}i) \\ 1 & \frac{1}{2}(-1-\sqrt{3}i) \end{vmatrix}^2 = -3.$$

Thus

$$\mathrm{ord}_{\wp_0} \Delta(1, \zeta_3) = e_{\wp_0} \mathrm{ord}_2(-3) = 0,$$

whence $\{1, \zeta_3\}$ is an integral basis at $\wp_0$ (see Weiss [19], p. 159, 4-8-8). By [19], p. 169, 4-9-2, we see that Kummer's theorem (i.e. [19], p. 168, 4-9-1) holds for $\wp_0$. Note that the minimal polynomial of ζ_3 over K_0 is $x^2 + x + 1$. It is well-known that the residue class field of K_0 at $\wp_0$ is $\mathbb{F}_{2^{f_{\wp_0}}}$ and that $x^2 + x + 1$ is irreducible in $\mathbb{F}_2[x]$. Thus if $f_{\wp_0} = 1$, we see, by Kummer's theorem, that

$$e_\wp / e_{\wp_0} = e(\wp/\wp_0) = 1, \quad f_\wp / f_{\wp_0} = f(\wp/\wp_0) = 2. \tag{A.1}$$

If $f_{\wp_0} \geqslant 2$, we see, by Lidl and Niederreiter [11], p. 48, 2.14, that $x^2 + x + 1$ splits into two distinct linear factors in $\mathbb{F}_{2^2}[x]$, whence so does it in $\mathbb{F}_{2^{f_{\wp_0}}}[x]$. Thus

$$e_\wp / e_{\wp_0} = e(\wp/\wp_0) = 1, \quad f_\wp / f_{\wp_0} = f(\wp/\wp_0) = 1. \tag{A.2}$$

For $p > 2$, we have $K = K_0(\zeta_4) = K_0(i)$. By the hypothesis, 1, i are linearly independent over K_0; and

$$\Delta(1, i) = \begin{vmatrix} 1 & i \\ 1 & -i \end{vmatrix}^2 = -4.$$

So

$$\mathrm{ord}_{\wp_0} \Delta(1, i) = e_{\wp_0} \mathrm{ord}_p(-4) = 0,$$

whence $\{1, i\}$ is an integral basis at $\wp_0$ (see [19], p. 159, 4-8-8) and Kummer's theorem holds for $\wp_0$. Note that the residue class field of K_0 at $\wp_0$ is $\mathbb{F}_{p^{f_{\wp_0}}}$ and the minimal polynomial of i over K_0 is $x^2 + 1$. It is well-known that if $p \equiv 1 \pmod 4$ then $x^2 + 1$ splits into two distinct linear factors in $\mathbb{F}_p[x]$, whence so does it in $\mathbb{F}_{p^{f_{\wp_0}}}[x]$. By Kummer's theorem, we get (A.2). Note further that if $p \equiv 3 \pmod 4$ then $x^2 + 1$ is irreducible in $\mathbb{F}_p[x]$. An argument similar to that in the case $p = 2$ yields (A.1) if $f_{\wp_0} = 1$ and (A.2) if $f_{\wp_0} \geqslant 2$.

Thus the proof of the lemma is complete.

References

[1] A. Baker, A sharpening of the bounds for linear forms in logarithms II, *Acta Arith.* 24 (1973) 33–36.

[2] A. Baker, The theory of linear forms in logarithms, *Transcendence theory: advances and applications* edited by A. Baker and D.W. Masser, Academic Press, London, 1977, pp. 1–27.

[3] A. Baker and J. Coates, Fractional parts of powers of rationals, *Math. Proc. Camb. Phil. Soc.* 77 (1975) 269–279.

[4] A. Baker and H. Stark, On a fundamental inequality in number theory, *Ann. Math.* 94 (1971) 190–199.

[5] D.W. Boyd, The maximal modulus of an algebraic integer, *Mathematics of Computation* 45 (1985) 243–249.

[6] A. Capelli, Sulla riduttibilità della funzione $x^n - A$ in un campo qualunque di razionalità (Auszug aus einem an Herrn F. Klein gerichteten Briefe), *Mathemat. Annalen* 54 (1901) 602–603.

[7] P.J. Cijsouw and M. Waldschmidt, Linear forms and simultaneous approximations, *Compositio Math.* 37 (1978) 21–50.

[8] E. Dobrowolski, On the maximal modulus of conjugates of an algebraic integer, *Bull. Acad. Polon. Sci.* 26 (1978) 291–292.

[9] H. Hasse, *Number theory*, Springer-Verlag, Berlin, Heidelberg, New York 1980.

[10] S. Lang, *Elliptic curves diophantine analysis*, Grundl. Math. Wissensch. 231, Springer-Verlag, Berlin, 1978.

[11] R. Lidl and H. Niederreiter, *Introduction to finite fields and their applications*, Cambridge University Press, Cambridge, 1986.

[12] J.H. Loxton, M. Mignotte, A.J. van der Poorten and M. Waldschmidt, A lower bound for linear forms in the logarithms of algebraic numbers, *C.R. Math. Acad. Sci. Canada* 11 (1987) 119–124.

[13] K. Mahler, Über transzendente *P*-adische Zahlen, *Compositio Math.* 2 (1935) 259–275.

[14] A.J. van der Poorten, Linear forms in logarithms in the *p*-adic case, *Transcendence theory: advances and applications* edited by A. Baker and D.W. Masser, Academic Press, London, 1977, pp. 29–57.

[15] L. Rédei, *Algebra* Vol. 1, Akadémiai Kiadó, Budapest, 1967.

[16] C.J. Smyth, On the product of the conjugates outside the unit circle of an algebraic integer, *Bull. London Math. Soc.* 3 (1971) 169–175.

[17] R. Tijdeman, On the equation of Catalan, *Acta Arith.* 29 (1976) 197–209.

[18] M. Waldschmidt, A lower bound for linear forms in logarithms, *Acta Arith.* 37 (1980) 257–283.

[19] E. Weiss, *Algebraic number theory*, McGraw Hill, New York, 1963.

[20] Kunrui Yu, Linear forms in logarithms in the *p*-adic case, *New advances in transcendence theory* edited by A. Baker, Cambridge University Press, Cambridge, 1988, pp. 411–434.

[21] Kunrui Yu, Linear forms in *p*-adic logarithms, to appear in *Acta Arith.*; Max-Planck-Institut für Mathematik Bonn Preprint MPI/87-20.