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A new geometric invariant associated to the space of flat connections

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Introduction

This work grew out of an attempt to generalize the Chern-Simons secondary characteristic classes.

Let G be a real Lie group with finitely many connected components (e.g. a compact Lie group) and let $\pi: E \to M$ be a principal G-bundle (in the smooth category). Let $\mathscr{C}$ be the space of all the (smooth) connections on E and let $\mathscr{F}$ be the subspace of the flat connections on E. We put the Fréchet (also called C^∞) topology (in fact a Fréchet manifold structure) on $\mathscr{C}$ and the subspace topology on $\mathscr{F}$. We will be mainly concerned with the case when $\mathscr{F}$ is non-empty. Let $\Delta_\bullet(\mathscr{C})$ (resp. $\Delta_\bullet(\mathscr{F})$) denote the smooth singular chain complex of $\mathscr{C}$ (resp. $\mathscr{F}$) and let $\Lambda_{\mathrm{dR}}(M)$ be the de Rham complex of M (cf. §§1.1–1.2). We fix a G-invariant homogeneous polynomial P of degree k on $\mathfrak{g} := \mathrm{Lie}\ G$.

The aim of this paper is to define a certain functorial chain map $\bar{\chi}_P: \Delta_\bullet(\mathscr{C}) \to \Lambda^{2k-\bullet}(M)$ and study its properties:

Given a smooth singular n-simplex (i.e. a smooth map) $\sigma: \Delta^n \to \mathscr{C}$, we construct a 'tautological' connection v_σ (cf. §2.1) on the bundle $\pi \times \mathrm{Id}: E \times \Delta^n \to M \times \Delta^n$; which is so defined that its restriction to the bundle $E \times \{t\} \to M \times \{t\}$ (for any $t \in \Delta^n$) is the connection $\sigma(t)$. Now $\bar{\chi}_P(\sigma)$ is defined by $(-1)^{n(n+1)/2} \int_{\Delta^n} P(\Omega_\sigma)$, where Ω_σ is the curvature of v_σ. Let $T_k(\mathscr{F}) \subset \Delta_\bullet(\mathscr{F})$ denote the truncated chain complex; given by $(T_k(\mathscr{F}))_n = \Delta_n(\mathscr{F})$ for $n < k$, and equals zero for $n \geqslant k$. Then our next observation is that the map $\bar{\chi}_P$ factors through $S_k(\mathscr{C}, \mathscr{F}) := \Delta_\bullet(\mathscr{C})/T_k(\mathscr{F})$. In particular the factored map gives rise to (our basic) map (in homology) $\chi_{P,\bullet}: H_\bullet(S_k(\mathscr{C}, \mathscr{F})) \to H_{\mathrm{dR}}^{2k-\bullet}(M)$. From the contractibility of $\mathscr{C}$, it is easy to see that, $H_n(S_k(\mathscr{C}, \mathscr{F})) = 0$ for $n > k$; equals $H_n(\mathscr{C}, \mathscr{F})$ for $n < k$; and $n = k$ is the 'critical dimension' where it is 'essentially' isomorphic with the $(k-1)$-cycles in $\Delta_\bullet(\mathscr{F})$ (cf. Theorem 2.6).

We also define a map $\bar{\psi}_P: \Delta_{\bullet-1}(\mathscr{F}) \to \Lambda^{2k-\bullet}(E)$ by $\bar{\psi}_P(\sigma) = (-1)^{n(n+1)/2} \int_{\Delta^{n-1}} TP(v_\sigma)$ (for σ a smooth singular $n-1$-simplex in $\mathscr{F}$), where $TP(v_\sigma)$ is the Chern-Simons secondary form (cf. Lemma 3.1). In general $\bar{\psi}_P$ is not a chain map, but its restriction to $T_k(\mathscr{F})$ is indeed a chain map giving rise to a map in homology $\psi_{P,\bullet}: H_{\bullet-1}(T_k(\mathscr{F})) \to H_{dR}^{2k-\bullet}(E)$ (cf. Proposition 3.3). Of course the homology of $T_k(\mathscr{F})$ has the following description: $H_n(T_k(\mathscr{F})) = H_n(\mathscr{F})$ for $n < k-1$; is zero for $n \geqslant k$; and for (the 'critical dimension') $n = k-1$ is isomorphic with the space of $(k-1)$-cycles in $\Delta_\bullet(\mathscr{F})$. By the contractibility of $\mathscr{C}$, we show that the map ψ_P is essentially the same map as χ_P (cf. Proposition 3.5). The definition of the map $\psi_{P,\bullet}$ lends itself immediately to conclude that the map $\psi_{P,1}$ is nothing but the Chern-Simons secondary invariant (cf. Proposition 3.8 and Remarks 3.9). Hence the map $\psi_{P,\bullet}$ (and consequently $\chi_{P,\bullet}$) provides a generalization of the Chern-Simons invariant for the flat connections. A similar generalization for more general class of connections (e.g. the class $\mathscr{C}_P$ defined in Section 3.7) does not seem to be possible, as is shown by Example 4.2.

Finally we identify the map $\psi_{P,\bullet}$ with a map given by the slant product in topology (and used by Thom in the study of topology of 'mapping spaces'), in the case when M is simply connected. Observe that, in this case, the existence of a flat connection implies that the bundle is trivial. More generally, for the trivial bundle $\pi: M \times G \to M$ with connected base M, let $\mathscr{G}_0 = \mathrm{Map}_0(M, G)$ denote the space of all the pointed smooth maps: $M \to G$. Of course $\mathscr{G}_0$ can also be thought of as the based gauge group of the bundle π (cf. §5.16). The gauge group acts on $\mathscr{F}$ and hence taking the orbit of the trivial connection on π, we get an embedding $\alpha: \mathscr{G}_0 \hookrightarrow \mathscr{F}$ (which is an isomorphism in the case M is simply connected). For any P as above, let TP be the universally transgressive cohomology class in $H_{dR}^{2k-1}(G)$ as defined in [CS]. Let $ev^*(TP)$ be the pull back class via the evaluation map $ev: \mathscr{G}_0 \times M \to G$. Now we define a map $\xi_{P,\bullet}: H_{\bullet-1}(\mathscr{G}_0) \to H_{dR}^{2k-\bullet}(M)$ by $\xi_{P,n}(\sigma) = (-1)^{n(n+1)/2}\int_\sigma ev^*(TP)$, for any $\sigma \in H_{n-1}(\mathscr{G}_0)$. We prove (cf. Theorem 5.7) that the composite map $s^* \circ \psi_{P,n} \circ \alpha_*: H_{n-1}(T_k(\mathscr{G}_0)) \to H_{dR}^{2k-n}(M)$ factors through $H_{n-1}(\mathscr{G}_0)$ and moreover it (the factored map) is equal to the map $\xi_{P,n}$ for all $n \leqslant k$ and (of course) $H_{n-1}(T_k(\mathscr{G}_0)) = 0$ for $n > k$; where α_* is induced from the embedding α and s^* is induced from the section $s(x) = (x, e)$ of π. From this identification, using Thom's theorem on the topology of space of maps from a finite C–W complex to a $K(\pi, m)$ (cf. Theorem 5.10), we show that (when G and M are both compact and connected) for any 'primitive generator' $P_j \in I^{k_j}(G)$, the map $\chi_{P_j,n}$ is surjective onto $H_{dR}^{2k_j-n}(M)$ for any $1 \leqslant n \leqslant k_j$ (cf. Theorem 5.14 and its Corollary 5.15). Consequently our basic invariant $\chi_{P_j,n}$ in general is 'highly' non-vanishing, for any $1 \leqslant n \leqslant k_j$.

As a trivial consequence of the above results, we obtain a result due to Quillen which asserts that in general 'roughly half' (for a precise result see Theorem 5.18) the generators of $H_{dR}(\mathscr{G}_0^e)$ can be represented by left invariant forms, where $\mathscr{G}_0^e$ is the identity component of $\mathscr{G}_0$. It may be mentioned that, as proved in [Ku], in general 'very few' generators of $H_{dR}(\mathscr{G}_0^e)$ can be represented by bi-invariant forms.

Even though we have no definite questions, on the basis of some formal similarities, it is intriguing to ask if there is any relationship with our work and the cyclic homology of A. Connes.

Roughly the contents of the paper are as follows: Section 1 is devoted to preliminaries and setting up the notation to be used in the paper. We define our basic map $\chi = \chi_P$ in Section 2 and collect its various general properties in Theorem 2.6. We define the map $\psi = \psi_P$ in Section 3 and show, in Proposition 3.5, that the map ψ is 'essentially' the same as the map χ. We give two examples in Section 4 to show that some of the restrictions in the earlier sections are essential. Finally in Section 5 we identify the map ψ with the 'slant product' in the special case when the base M is simply connected.

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1. Preliminaries and notation

(1.1) Unless otherwise stated, by a *manifold* we will mean a smooth (i.e. infinitely differentiable) finite dimensional paracompact real manifold without boundary. The *smooth de Rham complex* of a manifold E is denoted by $\Lambda(E) := \oplus_{n=0}^{\dim E} \Lambda^n(E)$, where $\Lambda^n(E)$ is the space of all the real valued smooth n-forms on E. The *de Rham cohomology* of E is denoted by $H_{\mathrm{dR}}(E)$.

Bundles, maps, differential forms will all be assumed to be smooth. Unless indicated otherwise, vector spaces and linear maps will be assumed to be over the field of real numbers $\mathbb{R}$.

We reserve the notation G for a *real* (*not necessarily compact*) *Lie group with finitely many connected components* and its (real) *Lie algebra* is denoted by $\mathfrak{g}$. For any $k \geqslant 0$, we denote by $I^k(G) := [S^k(\mathfrak{g}^*)]^G$ the space of *G-invariant homogeneous polynomials of degree k on* $\mathfrak{g}$, where G acts on the k-th symmetric power $S^k(\mathfrak{g}^*)$ by the adjoint representation. We set $I(G) = \oplus_{k=0}^{\infty} I^k(G)$.

Let $\pi\colon E \to M$ be a principal G-bundle (in the smooth category), and let $\mathscr{C} = \mathscr{C}(E)$ be the set of all the (smooth) *connections* on E and $\mathscr{F} = \mathscr{F}(E)$ be the subset of all the *flat connections*. Of course the set $\mathscr{F}$ could, in general, be empty;

but we will mainly be interested in the case when $\mathscr{F}$ is non-empty. Since the set $\mathscr{C}$ can (and often will) also be thought of as a certain subset of the $\mathfrak{g}$-valued 1-forms on E, we can put the *Fréchet topology* on $\mathscr{C}$ (cf. [M; Example 2.3]); which makes $\mathscr{C}$ into a *Fréchet manifold* (in fact an affine space). It is easy to see that, under this topology, $\mathscr{F}$ is a closed subspace of $\mathscr{C}$. *From now on we will always endow $\mathscr{C}$ with the Fréchet topology and $\mathscr{F}$ with the subspace topology.*

(1.2) For any $n \geqslant 1$, let $\Delta^n = \{(t_1, \ldots, t_n) \in \mathbb{R}^n \colon \Sigma t_i \leqslant 1$ and all the t_i's are $\geqslant 0\}$ be the *standard n-simplex* in $\mathbb{R}^n$. We set Δ^0 to be the one point space $\{0\}$. For a topological space X, by a *continuous singular n-simplex* in X, one means a continuous map $\sigma\colon \Delta^n \to X$. If X is a (smooth) Fréchet manifold, then by a *smooth singular n-simplex* in X we mean a continuous map $\sigma\colon \Delta^n \to X$ such that there exists an open set $\Delta^n \subset U \subset \mathbb{R}^n$ (U depending on σ) and an extension of σ to a smooth map: $U \to X$. We shall denote by $\Delta_n^{\mathrm{cont}}(X)$ (resp. $\Delta_n^{\infty}(X)$), the vector space (over $\mathbb{R}$) freely generated by all the continuous (resp. smooth) singular n-simplexes in X. If A is a closed subspace of X, then, by definition, $\Delta_n^{\infty}(A) = \Delta_n^{\infty}(X) \cap \Delta_n^{\mathrm{cont}}(A)$. We further define $\Delta_\bullet^{\infty}(X, A) = \Delta_\bullet^{\infty}(X)/\Delta_\bullet^{\infty}(A)$. Of course $\Delta_\bullet^{\infty}(X, A)$ is a subcomplex of $\Delta_\bullet^{\mathrm{cont}}(X, A) := \Delta_\bullet^{\mathrm{cont}}(X)/\Delta_\bullet^{\mathrm{cont}}(A)$.

The following result is well known and follows from the standard sheaf theoretic argument (see, e.g., [W; Chapter 5]).

(1.3) LEMMA. *Let X be a paracompact Fréchet manifold. Then the inclusion $\Delta_\bullet^{\infty}(X) \hookrightarrow \Delta_\bullet^{\mathrm{cont}}(X)$ induces isomorphism in homology.*

In particular if A is a submanifold of X, then the inclusion $\Delta_\bullet^{\infty}(X, A) \hookrightarrow \Delta_\bullet^{\mathrm{cont}}(X, A)$ induces an isomorphism in homology. (Use the five lemma.) $\square$

From now on, in this paper, we will only be considering (for a Fréchet manifold X and a closed subspace A) $\Delta_\bullet^{\infty}(X, A)$ and we will simply denote it by $\Delta_\bullet(X, A)$ and its homology by $H_\bullet(X, A)$.

Unless stated otherwise homologies, cohomologies will be taken with real coefficients.

(1.4) DEFINITION. Given a chain complex $C = \sum_n C_n$ (i.e. equipped with a differential ∂ of degree -1) and an integer k, we define the k-*th truncation* $T_k(C)$ of C as the subcomplex defined by:

$$\begin{aligned}(T_k C)_n &= 0 \quad \text{for } n \geqslant k,\\ &= C_n \quad \text{for } n < k.\end{aligned}$$

Given a pair of chain complexes $C' \subseteq C$, we define the k-*th prolongation* $S_k(C, C')$ of the quotient complex C/C' as the quotient complex $C/T_k(C')$. We of course have the following surjective maps of chain complexes:

$$C \to S_k(C, C') \to C/C'. \qquad \square$$

We record the following trivial lemma, for subsequent use in the paper, which follows from appropriate long exact homology sequences:

(1.5) LEMMA. *With the notation as above*:

(a) $H_n(T_k C) = 0$ *for* $n \geqslant k$;
$H_n(T_k C) \approx H_n(C)$ *for* $n \leqslant k - 2$; *and*
$H_{k-1}(T_k C) = Z_{k-1}(C)$, *where* $Z_{k-1}(C)$ *is the space of cycles in* C_{k-1}.

(b) $H_n(S_k(C, C')) \approx H_n(C)$ *for* $n \geqslant k + 1$;
$H_n(S_k(C, C')) \approx H_n(C/C')$ *for* $n \leqslant k - 1$.

(c) *Let* $D' \subset D$ *be chain complexes with a chain map* $f: C \to D$, *such that* $f(C') \subset D'$. *Then, for any* k, *one has induced chain maps*

$$: T_k(C) \to T_k(D) \quad \textit{and}$$
$$: S_k(C, C') \to S_k(D, D').$$

(d) *Let* $C = \sum_{n=0}^{\infty} C_n$ *be a chain complex which is acyclic in positive dimensions* (*i.e.* $H_n(C) = 0$, *for all* $n > 0$). *Then for any subcomplex* $C' \subset C$ *and any positive integer* k:

$$H_n(S_k(C, C')) \approx H_{n-1}(T_k C') \quad \textit{for } n \geqslant 2;$$
$$H_1(S_k(C, C')) \approx \operatorname{Ker} i, \textit{ where } i \textit{ is the map}: H_0(T_k C') \to H_0(C); \textit{ and}$$
$$H_0(S_k(C, C')) \approx \operatorname{coker} i. \qquad \square$$

2. Construction of the basic map

As in Section 1, *G denotes any real Lie group with finitely many connected components and with* (*real*) *Lie algebra* $\mathfrak{g}$.

(2.1) *A basic construction.* Fix a principal G-bundle π: $E \to M$. Given a smooth map $\sigma: \Delta^n \to \mathscr{C}$ (cf. §§1.1–1.2), we define a 'tautological' connection $v = v_\sigma$ on the principal G-bundle $\pi \times \mathrm{Id}: E \times \Delta^n \to M \times \Delta^n$, as given by the $\mathfrak{g}$-value 1-form on $E \times \Delta^n$ defined below:

$$v|_{(e,t)} = \sigma(t)|_e, \quad \text{for } (e, t) \in E \times \Delta^n. \qquad (\mathrm{I}_1)$$

In other words, for $X = X_1 + X_2 \in T_{(e,t)}(E \times \Delta^n) \cong T_e(E) \oplus T_t(\Delta^n)$ (where T denotes the tangent space)

$$v(X) = \sigma(t)(X_1). \tag{I_2}$$

For a form $\omega = \omega^{p,q}$ (of type (p, q)) on the product manifold $M \times \Delta^n$, by $\int_{\Delta^n}\omega$ we mean the p-form $\int_{\Delta^n}\omega^{p,q}$ on M provided $q = n$ and 0 otherwise. We adopt the convention that $\int_{\Delta^n}\omega_1 \otimes \omega_2 = \omega_1 \int_{\Delta^n}\omega_2$, for ω_1 a form on M and ω_2 a form on Δ^n.

Fix a $P \in I^k(G)$ for some $k \geqslant 1$ (cf. §1.1), and $n \geqslant 0$. Now we define our basic map $\bar{\chi} = \bar{\chi}_{p,n}: \Delta_n(\mathscr{C}) \to \Lambda^{2k-n}(M)$ as given by:

$$\bar{\chi}(\sigma) = (-1)^{n(n+1)/2}\int_{\Delta^n}P(\Omega_\sigma), \quad \text{for } \sigma: \Delta^n \to \mathscr{C} \tag{I_3}$$

and extend linearly, where $\Delta_n(\mathscr{C}) = \Delta_n^\infty(\mathscr{C})$ is as defined in Section 1.2, Ω_σ denotes the curvature of the connection $v = v_\sigma$ (defined by (I_2)) on $\pi \times \mathrm{Id}: E \times \Delta^n \to M \times \Delta^n$, and $P(\Omega_\sigma)$ denotes the evaluation of the invariant polynomial P on Ω_σ.

We will show below in Proposition (2.4) that $\bar{\chi}(\sigma)$ is a chain map. But before that we need to prove the following two lemmas:

The curvature form Ω_σ, which is a $\mathfrak{g}$-valued 2-form on $E \times \Delta^n$, can of course be decomposed in terms of types on the product manifold $E \times \Delta^n$ as

$$\Omega_\sigma = \Omega^{0,2} + \Omega^{1,1} + \Omega^{2,0}. \tag{I_4}$$

Now we have the following:

(2.2) LEMMA. *With the notation as in* (I_4):

(a) $\Omega^{0,2} = 0$ *and* (b) $\Omega|^{2,0}_{(e,t)} = \Omega(\sigma(t))|_e$, *where* $\Omega(\sigma(t))$ *is the curvature of the connection* $\sigma(t)$ *on the principal* G*-bundle* $\pi: E \to M$.

In particular if $\sigma(t)$ *is a flat connection for all* $t \in \Delta^n$, *then* Ω_σ *is a form of type* $(1, 1)$.

Proof. The assertion (a) follows trivially from the definition of the connection v together with the Cartan structure equation [cs; Identity 2.10]:

$$\Omega_\sigma = \mathrm{d}v + \tfrac{1}{2}[v, v]. \tag{I_5}$$

The assertion (b) follows from the 'functoriality' of the curvature. □

We have the following immediate consequence of Stokes' theorem:

(2.3) LEMMA. *Let* X *be a manifold without boundary and let* Y *be a compact oriented manifold (possibly) with boundary* ∂Y. *Then for any differential m-form*

ω on $X \times Y$, we have:

$$\int_Y \mathrm{d}\omega = (-1)^{m+1-\dim Y} \int_{\partial Y} \omega + \mathrm{d}_X \int_Y \omega,$$

where d *(resp.* d_X*) denotes the total exterior differentiation for $X \times Y$ (resp. only in the direction of X).*

In particular if $\partial Y = \emptyset$, $\int_Y \mathrm{d}\omega = \mathrm{d}_X(\int_Y \omega)$.

Proof. Clearly $\mathrm{d}\omega = \mathrm{d}_X\omega + \mathrm{d}_Y\omega$. Integrating and using Stokes' theorem, we obtain:

$$\int_Y \mathrm{d}\omega = (-1)^p \int_{\partial Y} \omega + \mathrm{d}_X \int_Y \omega, \text{ for any form } \omega \text{ on } X \times Y \text{ of type } (p,.).$$

But

$$\int_{\partial Y} \omega = 0, \text{ unless } n - p = \dim Y - 1. \qquad \square$$

Now we come to the following:

(2.4) PROPOSITION. *Let $\pi: E \to M$ be a principal G-bundle and let $P \in I^k(G)$ (for some $k \geqslant 1$). Then $\bar{\chi} = \bar{\chi}_{P,\bullet}: \Delta_\bullet(\mathscr{C}) \to \Lambda^{2k-\bullet}(M)$* (cf. §2.1) *is a chain map, i.e., the following diagram is commutative:*

$$\begin{array}{ccc} \Delta_n(\mathscr{C}) & \xrightarrow{\bar{\chi}_{P,n}} & \Lambda^{2k-n}(M) \\ \partial\big\downarrow & & \big\downarrow \mathrm{d} \\ \Delta_{n-1}(\mathscr{C}) & \xrightarrow{\bar{\chi}_{P,n-1}} & \Lambda^{2k-n+1}(M), \end{array}$$

where $\mathrm{d} = \mathrm{d}_M$ is the exterior differentiation.

Moreover $\bar{\chi}$ factors through $S_k(\Delta_\bullet(\mathscr{C}), \Delta_\bullet(\mathscr{F}))$ (cf. §§1.2 and 1.4), *where $\mathscr{F}$ is the space of all the flat connections on E.*

Proof. The fact that $\bar{\chi}$ is a chain map follows from Stokes' theorem:

For $\sigma: \Delta^n \to \mathscr{C}$,

$$\mathrm{d}_M(\bar{\chi}(\sigma)) = (-1)^{n(n+1)/2}\, \mathrm{d}_M \int_{\Delta^n} P(\Omega_\sigma)$$

$$= (-1)^{n(n-1)/2} \int_{\partial\sigma} P(\Omega_\sigma)|_{M \times \partial\sigma}, \text{ by Lemma (2.3); since } P(\Omega_\sigma) \text{ is a closed form on } M \times \Delta^n$$

$$= \bar{\chi}(\partial\sigma).$$

Now let $\sigma: \Delta^n \to \mathscr{F}$ be a map, for some $n \leqslant k - 1$. Then we want to prove that $\bar{\chi}(\sigma) = 0$:

$$\bar{\chi}(\sigma) := (-1)^{n(n+1)/2} \int_{\Delta^n} P(\Omega_\sigma).$$

But by Lemma (2.2), $P(\Omega_\sigma)$ is a form of type (k, k) on $M \times \Delta^n$ and hence n being $< k$, $\bar{\chi}(\sigma) = 0$. □

The following lemma follows easily from the functoriality of Ω_σ.

(2.5) LEMMA. *With the notation as in the above proposition, the map* $\bar{\chi}_{P,\bullet}: \Delta_\bullet(\mathscr{C}) \to \Lambda^{2k-\bullet}(M)$ *is functorial, in the sense that given a G-bundle map* $\bar{f}$

$$\begin{array}{ccc} E' & \xrightarrow{\bar{f}} & E \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ M' & \xrightarrow{f} & M, \end{array}$$

the following diagram is commutative:

$$\begin{array}{ccc} \Delta_\bullet(\mathscr{C}) & \xrightarrow{\bar{\chi}_{P,\bullet}} & \Lambda^{2k-\bullet}(M) \\ \downarrow & & \downarrow \\ \Delta_\bullet(C') & \xrightarrow{\bar{\chi}_{P,\bullet}} & \Lambda^{2k-\bullet}(M'); \end{array}$$

where $\mathscr{C}'$ is the space of all the connections on E', and the vertical maps are the canonical maps. □

We accumulate various general properties of the map $\bar{\chi}$ in the following:

(2.6) THEOREM. *With the notation and assumptions as in Proposition* (2.4), *we have:*

(a) *The map* $\bar{\chi}_{P,\bullet}: \Delta_\bullet(\mathscr{C}) \to \Lambda^{2k-\bullet}(M)$ *is a functorial chain map* (*in the sense explained in Sections* 2.4 *and* 2.5), *which factors through* $S_k(\mathscr{C}, \mathscr{F}) := S_k(\Delta_\bullet(\mathscr{C}), \Delta_\bullet(\mathscr{F}))$.

In particular, taking homology, we obtain our basic map $\chi_{P,n}: H_n(S_k(\mathscr{C}, \mathscr{F})) \to H_{\mathrm{dR}}^{2k-n}(M)$, for any n.

(b) *The map* $\chi_{P,0}$ *is the zero map provided* $\mathscr{F} \neq \emptyset$.

(c) (1) $H_n(S_k(\mathscr{C}, \mathscr{F})) = 0$ *for all* $n > k$,

(2) $H_n(S_k(\mathscr{C}, \mathscr{F})) \approx H_n(\mathscr{C}, \mathscr{F})$ *for all* $n < k$, *and*

(3) $H_k(S_k(\mathscr{C}, \mathscr{F})) \approx \ker(\partial: \Delta_{k-1}(\mathscr{F}) \to \Delta_{k-2}(\mathscr{F}))$, *if* $k > 1$,
$\approx \{\Sigma_{\sigma \in \mathscr{F}} n_\sigma \sigma \in \Delta_0(\mathscr{F})$ *such that* $\Sigma n_\sigma = 0\}$ *if* $k = 1$.

Proof. (a) has already been proved in Sections 2.4 and 2.5.

By Chern-Weil theory, for any connection θ with curvature Ω the cohomology class of $P(\Omega)$ is independent of θ. In particular, since by assumption $\mathscr{F} \neq \emptyset$, (b) follows,

(c) is an immediate consequence of Lemma (1.5), if we observe that the space $\mathscr{C}$, being smoothly contractible (in fact an affine space), is acyclic in positive dimensions. □

(2.7) REMARK. We will see in Section 4.1 that the map $\chi_{P,n}: H_n(S_k(\mathscr{C}, \mathscr{F})) \to H_{\mathrm{dR}}^{2k-n}(M)$ does not, in general, factor through $H_n(\mathscr{C}, \mathscr{F})$ for $n = k$. But if we assume that M is simply connected, it does factor through $H_k(\mathscr{C}, \mathscr{F})$; as we will see in Section 5.

3. An alternative description of the map χ_P

We recall the definition of the Chern-Simons secondary characteristic forms:

(3.1) LEMMA [CS; Proposition 3.2]. *Fix* $P \in I^k(G)$ *and any connection* θ *on the principal* G*-bundle* $\pi: E \to M$. *Define a (functorial) form* $TP(\theta)$ *on* E *by*:

$$TP(\theta) = \sum_{i=0}^{k-1} A_i P(\theta \wedge [\theta, \theta]^i \wedge \Omega^{k-i-1}),$$

where Ω *is the curvature and* $A_i := (-1)^i k!(k-1)!/2^i(k+i)!(k-1-i)!$ *is the constant as in* [CS; §3].

Then $dTP(\theta) = P(\Omega)$. □

(3.2) Let $\pi: E \to M$ be a principal G-bundle which admits a flat connection. We fix an invariant polynomial $P \in I^k(G)$ (for $k > 0$) and define, for any $n \geqslant 1$, a map $\psi = \psi_{P,n}: \Delta_{n-1}(\mathscr{F}) \to \Lambda^{2k-n}(E)$ as follows:

$$\psi(\sigma) = (-1)^{n(n+1)/2} \int_{\Delta^{n-1}} TP(v_\sigma), \qquad \text{for any } \sigma: \Delta^{n-1} \to \mathscr{F} \qquad (\mathrm{I}_6)$$

and extend linearly to the whole of $\Delta_{n-1}(\mathscr{F})$; where v_σ is the connection on $\pi \times \mathrm{Id}: E \times \Delta^{n-1} \to M \times \Delta^{n-1}$ defined in Section 2.1.

In general the map ψ is not a chain map. But we have the following:

(3.3) PROPOSITION. *With the notation as above, the map* $\psi_{P,\bullet}$ *is a chain map restricted to the subcomplex* $T_k(\mathscr{F}) := T_k(\Delta_\bullet(\mathscr{F}))$ *of* $\Delta_\bullet(\mathscr{F})$ (cf. §1.4).

We denote the map $\bar{\psi}|_{T_k(\mathscr{F})}$ by $\bar{\psi}$ and denote the induced map in homology by $\psi = \psi_{P,n}: H_{n-1}(T_k(\mathscr{F})) \to H^{2k-n}_{\mathrm{dR}}(E)$.

Proof. We need to prove that $\mathrm{d}_E(\bar{\psi}(\sigma)) = \bar{\psi}(\partial\sigma)$, for any (smooth) $\sigma: \Delta^{n-1} \to \mathscr{F}$ and $n \leqslant k$:

Let ν_σ be the connection on $E \times \Delta^{n-1}$ with curvature Ω_σ as in Section 2.1. By Stokes' theorem (Lemma 2.3)

$$\mathrm{d}_E \int_{\Delta^{n-1}} TP(\nu_\sigma) = \int_{\Delta^{n-1}} \mathrm{d}(TP(\nu_\sigma)) - (-1)^{n+1} \int_{\partial\Delta^{n-1}} TP(\nu_\sigma),$$

where d is the exterior differentiation on $E \times \Delta^{n-1}$

$$= \int_{\Delta^{n-1}} P(\Omega_\sigma) - (-1)^{n+1} \int_{\partial\Delta^{n-1}} TP(\nu_\sigma) \quad \text{(by Lemma 3.1).}$$

But by Lemma (2.2), Ω_σ is a form of type (1,1) and hence $P(\Omega_\sigma)$ is a form of type (k, k). Hence, n being $\leqslant k$ by assumption, $\int_{\Delta^{n-1}} P(\Omega_\sigma) = 0$. □

REMARK. By Lemma (1.5), $H_{n-1}(T_k(\mathscr{F})) = 0$ for all $n > k$; $H_{n-1}(T_k(\mathscr{F})) \approx H_{n-1}(\mathscr{F})$ for $n < k$; and $H_{k-1}(T_k(\mathscr{F})) \approx$ cycles in $\Delta_{k-1}(F)$.

(3.4) DEFINITION. By the definitin of S_k, there is an exact sequence of chain complexes:

$$0 \to T_k(\mathscr{F}) \to \Delta(\mathscr{C}) \to S_k(\mathscr{C}, \mathscr{F}) \to 0.$$

In particular there is a boundary operator

$$\delta_n = \delta_{k,n}: H_n(S_k(\mathscr{C}, \mathscr{F})) \to H_{n-1}(T_k(\mathscr{F})), \text{ for any } n.$$

The following proposition connects the map $\psi_{P,n}$ (defined in Proposition 3.3) with the map $\chi_{P,n}$ (defined in Theorem 2.6).

(3.5) PROPOSITION. *With the assumptions as in Section 3.2, we have the following commutative diagram for any $n \geqslant 0$:*

$$\begin{array}{ccc} H_n(S_k(\mathscr{C}, \mathscr{F})) & \xrightarrow{\chi_{P,n}} & H^{2k-n}_{\mathrm{dR}}(M) \\ \big\downarrow {\scriptstyle (-1)^n \delta_n} & & \big\downarrow {\scriptstyle \pi^*} \\ H_{n-1}(T_k(\mathscr{F})) & \xrightarrow{\psi_{P,n}} & H^{2k-n}_{\mathrm{dR}}(E), \end{array}$$

where π^* is induced from the map $\pi: E \to M$.

Further the map δ_n is an isomorphism for all $n \geqslant 2$, and δ_1 is injective with its image precisely equal to the kernel of the map $i: H_0(T_k(\mathscr{F})) \to H_0(\mathscr{C})$.

Proof. Since $H_n(S_k(\mathscr{C}, \mathscr{F})) = 0$ *for all* $n > k$ (by Theorem 2.6c), it suffices to assume that $n \leqslant k$. Take a n-cycle $\bar{\sigma} \in S_k(\mathscr{C}, \mathscr{F})$; represented by an element $\sigma \in \Delta_n(\mathscr{C})$ such that $\partial\sigma \in \Delta_{n-1}(\mathscr{F})$. Then

$$\pi^*\chi_{P,n}(\bar{\sigma}) = \pi^*\left((-1)^{n(n+1)/2} \int_\sigma P(\Omega_\sigma)\right)$$

$$= (-1)^{n(n+1)/2} \int_\sigma \mathrm{d}(TP(v_\sigma)) \qquad \text{(by Lemma 3.1),}$$

where d is the exterior differentiation on $E \times \Delta^n$

$$= (-1)^{(n(n+1)/2)+n} \int_{\partial\sigma} TP(v_\sigma) + (-1)^{n(n+1)/2}\, \mathrm{d}_E \int_\sigma TP(v_\sigma).$$

This proves the commutativity of the diagram. Rest of the assertions follow from Lemma (1.5) □

(3.6) REMARKS. (a) By the Chern-Weil theory and the Leray-Serre spectral sequence, it follows that the map π^* is injective in the case when G is a compact connected Lie group and the principal G-bundle $\pi: E \to M$ admits a flat connection. In particular, in this case, the map $\chi_{P,n}$ is determined by the map $\psi_{P,n}$.

(b) The above proposition relates our invariant $\chi_{P,1}$ with the Chern-Simons secondary invariant defined in [CS] (see also Proposition 3.8). In fact we arrived at our invariants χ in an attempt to generalize the Chern-Simons invariant.

(3.7) DEFINITION. Fix $P \in I^k(G)$ and let $\mathscr{C}_P \subset \mathscr{C}$ be the closed subspace of all the connections θ on the principal G-bundle $\pi: E \to M$ such that the characteristic form $P(\Omega(\theta)) \equiv 0$, where $\Omega(\theta)$ is the curvature corresponding to the connection θ. (We do not require here that E admits a flat connection.)

(3.8) PROPOSITION. *With the notation as above, the chain map* $\bar{\chi}_{P,\bullet}: \Delta_\bullet(\mathscr{C}) \to \Lambda^{2k-\bullet}(M)$ (*defined in* §2.1) *factors through* $S_1(\mathscr{C}, \mathscr{C}_P) := S_1(\Delta_\bullet(\mathscr{C}), \Delta_\bullet(\mathscr{C}_P))$; and hence, taking homology, we get a map $\hat{\chi}_{P,1}: H_1(S_1(\mathscr{C}, \mathscr{C}_P)) \to H^{2k-1}_{\mathrm{dR}}(M)$.

Similarly the restriction of the map $\bar{\psi}_{P,\bullet}: \Delta_{\bullet-1}(\mathscr{C}_P) \to \Lambda^{2k-\bullet}(E)$ (*defined in* §3.2 *for* $\mathscr{C}_P$ *replaced by* $\mathscr{F}$; *but we take the same definition*) *to* $T_1(\mathscr{C}_P) := T_1(\Delta_\bullet(\mathscr{C}_P))$ *is a chain map*; inducing the map $\hat{\psi}_{P,1}: H_0(T_1(\mathscr{C}_P)) \to H^{2k-1}_{\mathrm{dR}}(E)$.

Further the following diagram is commutative:

$$\begin{array}{ccc} H_1(S_1(\mathscr{C}, \mathscr{C}_P)) & \xrightarrow{\hat{\chi}_{P,1}} & H^{2k-1}_{\mathrm{dR}}(M) \\ \downarrow{\scriptstyle -\delta_1} & & \downarrow{\scriptstyle \pi^*} \\ H_0(T_1(\mathscr{C}_P)) \approx \Delta_0(\mathscr{C}_P) & \xrightarrow{\hat{\psi}_{P,1}} & H^{2k-1}_{\mathrm{dR}}(E), \end{array}$$

where δ_1 *is the boundary map as in Section* 3.4.

Proof. The map $\bar{\chi}_{P,\bullet}$ factors through $S_1(\mathscr{C}, \mathscr{C}_P)$; follows from the definition of $\mathscr{C}_p$. The restriction of the map $\bar{\psi}_{P,\bullet}$ to $T_1(\mathscr{C}_P)$ is a chain map is vacuously true. Commutativity of the diagram follows from an identical proof given to prove Proposition (3.5). □

(3.9) REMARKS. With the notation as in the above proposition:

(a) $H_n(S_1(\mathscr{C}, \mathscr{C}_P)) = 0$ for all $n \geqslant 2$, and the map $\hat{\chi}_{P,0}$ is the zero map provided the space $\mathscr{C}_P \neq \emptyset$.
(b) By the very definition, for any connection $\theta \in \mathscr{C}_P$, $\hat{\psi}_{P,1}(\theta)$ is the secondary Chern-Simons invariant of θ (cf. [CS; §3]).
(c) The map $\bar{\chi}_{P,\bullet}: \Delta_\bullet(\mathscr{C}) \to \Lambda^{2k-\bullet}(M)$ does not, in general, factor through $S_j(\mathscr{C}, \mathscr{C}_P)$ for any $j > 1$ (cf. Example 4.2); in contrast to the situation when $\mathscr{C}_P$ is replaced by $\mathscr{F}$. □

As an immediate corollary of Proposition (3.8), we derive the following result due to Chern-Simons:

(3.10) COROLLARY [CS; Theorem 3.9]. *Let $\pi: E \to M$ be any principal G-bundle* and $P \in I^k(G)$. *Assume that* $\dim M < 2k - 1$. *Then the form $TP(\theta)$ is closed for any $\theta \in \mathscr{C}$ (i.e. $\mathscr{C} = \mathscr{C}_P$) and moreover the cohomology class $[TP(\theta)] \in H^{2k-1}_{\mathrm{dR}}(E)$ is independent of θ.*

Proof. Of course $\mathscr{C} = \mathscr{C}_P$, since $\dim M < 2k$ (in fact $< 2k - 1$). Moreover, by assumption, $H^{2k-1}_{\mathrm{dR}}(M) = 0$ and hence by Proposition (3.8), $\hat{\psi}_{P,1}$ (Image δ_1) $= 0$. But for any two connections $\theta, \theta' \in \mathscr{C}$, $\theta - \theta'$ belongs to the image δ_1. This proves the corollary. □

4. Some examples

We give below two examples substantiating our Remarks (2.7) and 3.9(c) respectively:

(4.1) EXAMPLE. *The following example shows that the map $\chi_{p,n}$:*

> $H_n(S_k(\mathscr{C}, \mathscr{F})) \to H^{2k-n}_{\mathrm{dR}}(M)$ *does not factor through $H_n(\mathscr{C}, \mathscr{F}))$ in general, if* $n = k$ (cf. Theorem 2.6):

We take G to be the circle S^1 and consider the trivial G-bundle $\pi: M \times G \to M$. Further we take for P the first Chern polynomial. Since π is a trivial bundle, the space of connections $\mathscr{C}$ can be identified with the space of all the Lie S^1-valued 1-forms on the base M (Lemma 5.1). We identify Lie S^1 with $\mathbb{R}$ and take a closed

1-form ω on M. Define $\sigma: I \to \mathscr{C}$ by $\sigma(t) = t\omega$ where $I := \Delta^1$. Clearly $\Omega_\sigma = -\omega \wedge dt$ and hence $\int_I P(\Omega_\sigma) = -\omega$. But ω being a closed form, $\sigma(I) \subset \mathscr{F}$; in particular $\sigma \in H_1(S_1(\mathscr{C}, \mathscr{F}))$ which is zero considered as an element in $H_1(\mathscr{C}, \mathscr{F})$. So any M with $H^1_{\mathrm{dR}}(M) \neq 0$ provides a desired example. In fact $\chi_{P,1}(H_1(S_1(\mathscr{C}, \mathscr{F}))) = H^1_{\mathrm{dR}}(M)$, whereas $H_1(\mathscr{C}, \mathscr{F})$ is zero; because G being abelian $\mathscr{F}$ is a linear subspace of $\mathscr{C}$ (in particular is connected). $\square$

(4.2) EXAMPLE. *The following example shows that the map* $\bar{\chi}_{P,\bullet}: \Delta_\bullet(\mathscr{C}) \to \Lambda^{2k-\bullet}(M)$ *does not, in general, factor through* $S_2(\mathscr{C}, \mathscr{C}_P)$ (*cf.* *Proposition* 3.8) *in contrast to the situation when* $\mathscr{C}_P$ *is replaced by* $\mathscr{F}$:

We take $G = \mathrm{SU}(2)$ and the trivial G-bundle $\pi: \mathbb{R}^3 \times \mathrm{SU}(2) \to \mathbb{R}^3$. We take, for P, the second Chern polynomial, i.e., the 'determinant'. Obviously, base being three dimensional, $\mathscr{C} = \mathscr{C}_P$. Any connection on the bundle π is given by the matrix

$$\begin{pmatrix} i\alpha & \beta + i\gamma \\ -\beta + i\gamma & -i\alpha \end{pmatrix},$$

where α, β, γ are arbitrary (real valued) 1-forms on $\mathbb{R}^3$ (Lemma 5.1). Define a map $\sigma: \Delta^2 \to \mathscr{C}$ by

$$\sigma(s,t) = \begin{pmatrix} i\alpha_{(s,t)} & 0 \\ 0 & -i\alpha_{(s,t)} \end{pmatrix} \qquad \text{for } (s,t) \in \Delta^2, \tag{I$_7$}$$

where $\alpha_{(\cdot,\cdot)}$ is a two parameter family of 1-forms on $\mathbb{R}^3$. As in Section 2.1, σ defines a connection ν_σ on $\pi \times \mathrm{Id}: E \times \Delta^2 \to \mathbb{R}^3 \times \Delta^2$ (where $E = \mathbb{R}^3 \times SU(2)$) with curvature Ω_σ given by

$$\Omega_\sigma = \begin{pmatrix} \mathrm{i}\mathrm{d}\alpha & 0 \\ 0 & -\mathrm{i}\mathrm{d}\alpha \end{pmatrix},$$

where d denotes the total differentiation on $\mathbb{R}^3 \times \Delta^2$ and $\alpha := \alpha_{(\cdot,\cdot)}$ is thought of as a 1-form on $\mathbb{R}^3 \times \Delta^2$.

Identify $\partial\Delta^2$ with S^1 and set $\tilde{\alpha}_\theta = \sin(2\pi\theta)\,\mathrm{d}y + x\cos(2\pi\theta)\,\mathrm{d}z$, for $\theta \in \mathbb{R}$; where (x, y, z) are the standard coordinates on $\mathbb{R}^3$. Then $\tilde{\alpha}_\theta$ is a family of 1-forms on $\mathbb{R}^3$ parametrized by $\partial\Delta^2 \approx S^1$, and moreover this family can be extended smoothly to Δ^2 to get a 1-form α on $\mathbb{R}^3 \times \Delta^2$. By direct calculation, one obtains $\mathrm{d}\tilde{\alpha} \wedge \mathrm{d}\tilde{\alpha} = 4\pi\cos^2 2\pi\theta\,\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \wedge \mathrm{d}\theta$ as forms on $\mathbb{R}^3 \times S^1$. Hence with this choice of α in (I_7), we obtain

$$\begin{aligned} d_{\mathbb{R}^3}\int_{\Delta^2}\det\Omega_\sigma &= d_{\mathbb{R}^3}\int_{\Delta^2} d\alpha\wedge d\alpha \\ &= \int_{S^1} d\tilde{\alpha}\wedge d\tilde{\alpha} \quad \text{(by Lemma 2.3)} \\ &= 4\pi\int_0^1 \cos^2 2\pi\theta\, dx\wedge dy\wedge dz\wedge d\theta \\ &= 4\pi\left(\int_0^1 \cos^2 2\pi\theta\, d\theta\right) dx\wedge dy\wedge dz \\ &\neq 0. \end{aligned}$$

Hence $\bar{\chi}_{P,2}(\sigma) := -\int_{\Delta^2}\det\Omega_\sigma$ is not a closed form, but $\partial\sigma\in\Delta_1(\mathscr{C}) = \Delta_1(\mathscr{C}_P)$ and hence σ is a cycle of $S_2(\mathscr{C},\mathscr{C}_P)$. In particular the map $\bar{\chi}_{P,2}$ does not factor through $S_2(\mathscr{C},\mathscr{C}_P)$. □

5. Determination of the map ψ in the case M is simply connected

Let us recall the following well known

(5.1) LEMMA. *Any* $\mathfrak{g}$*-valued one-form* θ *on a differentiable manifold* X *gives rise to a unique connection* $\hat{\theta}$ *on the trivial G-bundle* $\pi: X\times G\to X$ *satisfying* $s^*(\hat{\theta}) = \theta$, *where* s *is the trivial section of* π *(defined by* $s(x) = (x, e)$*).*

In fact

$$\hat{\theta}_{(x,g)}(v+w) = \operatorname{Ad} g^{-1}(\theta_x(v)) + L'_{g^{-1}}(w) \tag{I_8}$$

for $(x,g)\in X\times G, v\in T_x(X)$, *and* $w\in T_g(G)$; *where* $L_{g^{-1}}: G\to G$ *is the left multiplication by* g^{-1} *and* $L'_{g^{-1}}$ *denotes its derivative.* □

(5.2) DEFINITION. Let M and N be smooth manifolds with a smooth map $f: M\times N\to G$ and let ω be the Maurer-Cartan form on G. Then the pull back form $\theta = \theta(f) := f^*(\omega)$ can of course be decomposed as $\theta = \theta_1 + \theta_2$, where θ_1 (resp. θ_2) is a one-form on $M\times N$ of type (1, 0) (resp. (0, 1)). By the above lemma, the forms $\theta; \theta_1; \theta_2$ give rise to the connections $\hat{\theta}; \hat{\theta}_1$; and $\hat{\theta}_2$ respectively on the trivial bundle $\pi: M\times N\times G\to M\times N$.

Also recall that, to any $P\in I^k(G)$, there is associated a real valued bi-invariant form TP on G as in [CS; Identity 3.10].

With the notation as above, we have the following:

(5.3) PROPOSITION. *Assume that* $\dim N < k$. *Then* $f^*(TP) = s^*(TP(\hat{\theta}_1)) + \mathrm{d}_{M \times N}(s^*(\int_0^1 TP(\hat{\Theta})))$, *as forms on* $M \times N$; *where recall that* $TP(\hat{\theta}_1)$ *is as defined in Lemma* (3.1), *s is the trivial section of* π, *and* $\hat{\Theta}$ *is the connection on the G-bundle* $\mathrm{Id} \times \pi \colon \mathbb{R} \times M \times N \times G \to \mathbb{R} \times M \times N$ *defined in the proof.*

Proof. Define a one-parameter family of $\mathfrak{g}$-valued one-forms on $M \times N$ by $\Theta_t = \theta_1 + t\theta_2$. By Lemma (5.1), this gives rise to a one-parameter family of connections $\hat{\Theta}_t$ and hence a connection $\hat{\Theta}$ on the G-bundle $\mathrm{Id} \times \pi \colon \mathbb{R} \times M \times N \times G \to \mathbb{R} \times M \times N$. Let $\Omega = \Omega_{\Theta}$ be the corresponding curvature. Then

$$\int_0^1 P(\Omega) = \int_0^1 \mathrm{d}(TP(\hat{\Theta})) \quad \text{(by Lemma 3.1)},$$

where d is the total differentiation on $\mathbb{R} \times M \times N \times G$

$$= \int_0^1 \mathrm{d}_{\mathbb{R}}(TP(\hat{\Theta})) - \mathrm{d}_{M \times N \times G}\left(\int_0^1 TP(\hat{\Theta})\right)$$

$$= TP(\hat{\Theta}_1) - TP(\hat{\Theta}_0) - \mathrm{d}_{M \times N \times G}\left(\int_0^1 TP(\hat{\Theta})\right),$$

i.e.

$$\int_0^1 P(\Omega) = TP(\hat{\theta}) - TP(\hat{\theta}_1) - \mathrm{d}_{M \times N \times G}\left(\int_0^1 TP(\hat{\Theta})\right). \tag{I$_9$}$$

On the other hand, define a G-bundle automorphism F of the bundle $\mathrm{Id} \times \pi$ (inducing identity on the base) by $F(t, m, n, g) = (t, m, n, f(m, n)\cdot g)$, for $t \in \mathbb{R}$, $m \in M$, $n \in N$, and $g \in G$.

Further we define a one-parameter family $\Delta_t := (t - 1)\delta$ of $\mathfrak{g}$-valued one-forms on $M \times N$, where δ is given by:

$$\delta_{(m,n)}(v + w) = R'_{f(m,n)^{-1}}((f')_{(m,n)}(w)), \tag{I$_{10}$}$$

for $(m, n) \in M \times N$, $v \in T_m(M)$, and $w \in T_n(N)$; where R_g is the right multiplication by g and $(f')_{(m,n)}$ denotes the derivative of f at (m, n). This again gives rise to a one-parameter family of connections $\hat{\Delta}_t$ on the bundle π and hence a connection $\hat{\Delta}$ on the bundle $\mathrm{Id} \times \pi$. Now we claim that

$$F^*(\hat{\Delta}) = \hat{\Theta}: \tag{I$_{11}$}$$

For $t \in \mathbb{R}$, $m \in M, n \in N, v \in T_m(M)$, and $w \in T_n(N)$ we have

$$(s^* \bar{F}^* \hat{\Delta}_t)_{(m,n)}(v + w) = (F_1^*(\hat{\Delta}_t)_{(m,n)}(v + w),$$

where $F_1 : M \times N \to M \times N \times G$ is defined by $F_1(m, n) = (m, n, f(m, n))$ and $\bar{F}$ is the bundle automorphism of π induced by F

$$\begin{aligned}
&= (\hat{\Delta}_t)_{(m,n,f(m,n))}(v + w + (f')_{(m,n)}(v + w)) \\
&= (t - 1)\operatorname{Ad}(f(m, n)^{-1})(\delta_{(m,n)}(w)) \\
&\quad + L'_{f(m,n)^{-1}}(f')_{(m,n)}(v + w) \\
&= (t - 1)\operatorname{Ad}(f(m, n)^{-1})(R'_{f(m,n)^{-1}}(f')_{(m,n)}(w)) \\
&\quad + L'_{f(m,n)^{-1}}(f')_{(m,n)}(v + w)
\end{aligned}$$

(by the definition of δ)

$$\begin{aligned}
&= tL'_{f(m,n)^{-1}}(f')_{(m,n)}(w) + L'_{f(m,n)^{-1}}(f')_{(m,n)}(v) \\
&= (\theta_1 + t\theta_2)_{(m,n)}(v + w) \\
&= (s^* \hat{\Theta}_t)_{(m,n)}(v + w).
\end{aligned}$$

By the uniqueness (cf. Lemma 5.1), the above identity proves (I_{11}).

By (I_{11}) we get

$$P(\Omega) = P(\Omega'), \text{ where } \Omega' \text{ is the curvature of } \hat{\Delta}. \tag{I_{12}}$$

By an analogue of Lemma (2.2), we can write (as forms on $\mathbb{R} \times M \times N$):

$$P(\Omega') = \sum_{q \geqslant k} \eta^{p,q},$$

where $\eta^{p,q}$ is a form of type (p, q) on $(\mathbb{R} \times M) \times N$.

But, by assumption, $\dim N < k$ and hence

$$P(\Omega') = 0 \tag{I_{13}}$$

Further by [CS; §3],

$$s^*(TP(\hat{\theta})) = f^*(TP). \tag{I_{14}}$$

Now combining $(\mathrm{I}_9), (\mathrm{I}_{12}), \mathrm{I}_{13}$–$\mathrm{I}_{14}$, we get the proposition. □

(5.4) Let $\pi: M \times G \to M$ be the trivial bundle, with connected base M. Fix (any) base point $m_0 \in M$ and let $\mathscr{G}_0 = \mathrm{Map}_0(M, G)$ denote the set of all the smooth maps $f: M \to G$ such that $f(m_0) = e$. Define a map $\alpha: \mathscr{G}_0 \to \mathscr{F}$ by $\alpha(f) = \hat{f}^*(\varepsilon)$, for any $f \in \mathscr{G}_0$; where $\mathscr{F}$ (as earlier) is the space of flat connections on π, ε is the trivial connection on π, and $\hat{f}$ is the G-bundle map: $M \times G \to M \times G$ got from f by $\hat{f}(m, g) = (m, f(m) \cdot g)$.

Let us recall the following well known lemma (see, e.g., [K; Chapter 3, Theorem 4]):

(5.5) LEMMA. *The map α, defined above, is injective. Moreover if, in addition, M is simply connected then α is bijective.* □

As in [M; Example 1.3], the space $\mathscr{G}_0$ has a canonical Fréchet manifold structure (in fact a Fréchet Lie group structure) and under this manifold structure the map $\alpha: \mathscr{G}_0 \to \mathscr{F} \subset \mathscr{C}$ is a smooth embedding. (Recall that $\mathscr{C}$ is endowed with the Fréchet manifold structure as in Section 1.1.)

(5.6) DEFINITION. Let X and Y be smooth (possibly Fréchet) manifolds. Then there is a map (analogue of the slant product in topology):

$$\int : \Delta(X) \otimes (\Lambda(X) \otimes \Lambda(Y)) \to \Lambda(Y),$$

defined by

$$\sigma \otimes (\omega \otimes \eta) \to \left(\int_\sigma \omega \right) \cdot \eta;$$

for $\sigma \in \Delta(X)$, $\omega \in \Lambda(X)$, and $\eta \in \Lambda(Y)$.

It is obvious how to extend $\int$ to a map (again denoted by)

$$\int : \Delta(X) \otimes \Lambda(X \times Y) \to \Lambda(Y).$$

As is customary, we denote $\int \sigma \otimes \omega$ by $\int_\sigma \omega$. By Stokes' theorem, this map gives rise to a map

$$\left[\int \right] : H_n(X) \otimes H^k_{\mathrm{dR}}(X \times Y) \to H^{k-n}_{\mathrm{dR}}(Y).$$

In particular; take $X = \mathscr{G}_0, Y = M$ and consider the evaluation map $ev: \mathscr{G}_0 \times M \to G$. Fix an invariant polynomial $P \in I^k(G)$ and $n \geqslant 1$; and define a map

$$\xi_{P,n}\colon H_{n-1}(\mathscr{G}_0) \to H_{\mathrm{dR}}^{2k-n}(M)$$

by

$$\xi_{P,n}(\sigma) = (-1)^{n(n+1)/2}\left[\int\right](\sigma \otimes ev^*(TP))$$

$$=(-1)^{n(n+1)/2}\int_\sigma ev^*(TP), \quad \text{for } \sigma \in H_{n-1}(\mathscr{G}_0).$$

(5.7) THEOREM.(*)(**) *Let $\pi\colon M \times G \to M$ be the trivial bundle with connected base M. Fix a $P \in I^k(G)$. Then the following diagram is commutative for all $n \geqslant 1$:*

$$\begin{tikzcd} H_{n-1}(T_k(\mathscr{G}_0)) \arrow[r] \arrow[d] & H_{n-1}(T_k(\mathscr{F})) \arrow[r, "\psi_{P,n}"] & H_{\mathrm{dR}}^{2k-n}(E) \arrow[dl, "s^*"] \\ H_{n-1}(\mathscr{G}_0) \arrow[r, "\xi_{P,n}"] & H_{\mathrm{dR}}^{2k-n}(M), & \end{tikzcd}$$

where $\psi_{P,n}$ (resp. $\xi_{p,n}$) is as defined in Proposition (3.3) (*resp. Definition* 5.6), *the notation $\mathscr{G}_0$ is as in Section* 5.4, *the unlabeled maps are induced from the canonical inclusions* (cf. §5.4), *$E = M \times G$, and s is the trivial section of π.*

Proof. For $n > k$, since H_{n-1} $(T_k(\mathscr{G}_0)) = 0$, there is nothing to prove. So assume that $n \leqslant k$:

Fix a $(n-1)$-cycle $\sigma = \sum n_i\sigma_i \in T_k(\mathscr{G}_0)$. Then $\psi_{P,n}[\sigma]$ is, by definition, the cohomology class of $\sum n_i(-1)^{n(n+1)/2}\int_{\sigma_i} TP(v_{\sigma_i})$, where v_{σ_i} is the connection on the principal G-bundle $\pi \times \mathrm{Id}\colon M \times G \times \Delta^{n-1} \to M \times \Delta^{n-1}$ defined in Section 2.1.

On the other hand, by the exponential correspondence, the singular $(n-1)$-simplex $\sigma_i\colon \Delta^{n-1} \to \mathscr{G}_0$ can also be thought of as a (smooth) map $\bar{\sigma}_i\colon M \times \Delta^{n-1} \to G$. As in Definition 5.2, the map $\bar{\sigma}_i$ gives rise to connections $\widehat{\theta(\bar{\sigma}_i)}, \widehat{\theta_1(\bar{\sigma}_i)}$, and $\widehat{\theta_2(\bar{\sigma}_i)}$. By the definition, we have the equality of connections:

$$\widehat{\theta_1(\bar{\sigma}_i)} = v_{\sigma_i}. \tag{I$_{15}$}$$

(*) This was asked as a question by M.S. Narasimhan.

(**) This theorem can easily be generalized to an arbitrary principal G-bundle $\pi\colon E \to M$ by considering the canonical map $\mathscr{G} \times E \to G$ and pulling back the bundle π to the total space E to reduce to the trivial bundle case; where $\mathscr{G}$ is the gauge group of the bundle π.

Let $\tilde{s}: M \times \Delta^{n-1} \to M \times G \times \Delta^{n-1}$ be the trivial section of $\pi \times \mathrm{Id}$. Now by Proposition (5.3), we get:

$$\bar{\sigma}_i^*(TP) = \tilde{s}^*(TP(\widehat{\theta_1(\bar{\sigma}_i)})) + \mathrm{d}_{M \times \Delta^{n-1}}\left(\tilde{s}^*\left(\int_0^1 TP(\Theta(\sigma_i))\right)\right), \qquad (\mathrm{I}_{16})$$

where $\widehat{\Theta(\sigma_i)}$ is the connection on the bundle : $\mathbb{R} \times M \times G \times \Delta^{n-1} \to \mathbb{R} \times M \times \Delta^{n-1}$; given by the one parameter family of connections $\widehat{\theta_1(\sigma_i) + t\theta_2(\sigma_i)}$.

Hence by $\mathrm{I}_{15} - \mathrm{I}_{16}$ and Lemma (2.3), we obtain:

$$\int_{\Delta^{n-1}} \bar{\sigma}_i^*(TP) = s^*\left(\int_{\Delta^{n-1}} TP(\nu_{\sigma_i})\right) + \mathrm{d}_M \int_{\Delta^{n-1}} \left(\tilde{s}^*\left(\int_0^1 TP(\Theta(\sigma_i))\right)\right) \qquad (\mathrm{I}_{17})$$

$$+(-1)^n \int_{\partial\Delta^{n-1}} \left(\tilde{s}^*\left(\int_0^1 TP(\Theta(\sigma_i))\right)\right).$$

But $TP(\widehat{\Theta(\sigma_i)})|_{\mathbb{R} \times M \times G \times \partial\Delta^{n-1}} = TP(\widehat{\Theta(\partial\sigma_i)})$ and hence, $\partial\sigma$ being 0 (by assumption), we get

$$\sum_i n_i \int_{\partial\Delta^{n-1}} \left(\tilde{s}^*\left(\int_0^1 TP(\widehat{\Theta(\sigma_i)})\right)\right) = 0. \qquad (\mathrm{I}_{18})$$

Now combining I_{17}–I_{18}, we get the theorem. □

(5.8) COROLLARY. *With the assumptions as in the above theorem, the following diagram is commutative for any $n \geqslant 2$:*

$$\begin{tikzcd}
H_{n-1}(T_k(\mathscr{G}_0)) \arrow[r] \arrow[d] & H_{n-1}(T_k(\mathscr{F})) \arrow[r, "\psi_{P,n}"] & H_{\mathrm{dR}}^{2k-n}(E) \arrow[dll, "\pi^*"] \\
H_{n-1}(\mathscr{G}_0) \arrow[r, "\xi_{P,n}"] & H_{\mathrm{dR}}^{2k-n}(M), &
\end{tikzcd}$$

where π^* is induced from the projection.

In particular, in the case when M has finite fundamental group, the map $\psi_{P,n}$ factors through $H_{n-1}(\mathscr{F})$ for all $n \geqslant 2$.

Of course if n = 1 and k > 1 then $H_{n-1}(T_k(\mathscr{F})) \approx H_{n-1}(\mathscr{F})$ (use Lemma 1.5).

Proof. By Proposition (3.5) the image of $\psi_{P,n}$ lies inside the image of π^* for any $n \geqslant 2$, and hence the commutativity of the diagram follows from Theorem (5.7). The second assertion follows immediately from Lemma (5.5) for simply connected M. The general case (i.e. $\pi_1(M)$ finite) can be deduced from the simply connected case by going to the simply connected cover; using functoriality of $\psi_{P,n}$; and the fact that $H_{\mathrm{dR}}^{2k-n}(E)$ injects inside $H_{\mathrm{dR}}^{2k-n}(\tilde{E})$, where $\tilde{E}$ is the pull back of E to $\tilde{M}$. □

(5.9) REMARK. In the case when $n = k = 1$ it can be easily seen, taking M to be a point, that the map $\psi_{P,1}$ does not factor through $H_0(\mathscr{F})$ in general. □

We recall the following well known result due to R. Thom:

(5.10) THEOREM. *Let π be an abelian group, $m \geqslant 1$ an integer, and X a finite C–W complex. Then there exists a homotopy equivalence*

$$\beta\colon \mathrm{Map}^{\mathrm{cont}}(X, K(\pi, m)) \to \prod_{p=0}^{m} K(H^p(X, \pi), m - p),$$

where $\mathrm{Map}^{\mathrm{cont}}(X, K(\pi, m))$ *denotes the space of all the continuous maps*: $X \to K(\pi, m)$ *with compact-open topology.* □

(5.11) Denote by $\mathrm{Map}_0^{\mathrm{cont}}(X, K(\pi, m))$ the subspace of pointed maps, i.e., the continuous maps: $X \to K(\pi, m)$ which take a fixed base point in X to a fixed base point in $K(\pi, m)$. Let $ev\colon \mathrm{Map}_0^{\mathrm{cont}}(X, K(\pi, m)) \times X \to K(\pi, m)$ denote the evaluation map. The cohomology $H^m(K(\pi, m), \pi)$ has a canonical 'm-characteristic element' c (given by the identity map of $K(\pi, m)$ to itself). For any $0 \leqslant p \leqslant m$, taking the slant product (cf. [S; Chapter 6, Section 1]), we get a map

$$\beta_p\colon H_{m-p}(\mathrm{Map}_0^{\mathrm{cont}}(X, K(\pi, m)), \mathbb{Z}) \to H^p(X, \pi)$$

defined by $\beta_p(x) = ev^* c/x$.

Also define a map $\tilde{\beta}_p\colon \pi_{m-p}(\mathrm{Map}_0^{\mathrm{cont}}(X, K(\pi, m))) \to H^p(X, \pi)$; as the composite of the Hurewicz homomorphism

$$\pi_{m-p}(\mathrm{Map}_0^{\mathrm{cont}}(X, K(\pi, m))) \to H_{m-p}(\mathrm{Map}_0^{\mathrm{cont}}(X, K(\pi, m)), \mathbb{Z}) \text{ with the map } \beta_p.$$

Actually we will be interested in the following consequence of Thom's Theorem (5.10); which can be easily deduced by considering the isomorphism induced from β at the homotopy groups' level:

(5.12) THEOREM. *Let the notation be as above and assumptions as in Theorem* (5.10). *Assume, in additon, that X is connected. Then, for any $0 < p \leqslant m$, the map $\tilde{\beta}_p\colon \pi_{m-p}(\mathrm{Map}_0^{\mathrm{cont}}(X, K(\pi, m))) \to H^p(X, \pi)$ is an isomorphism.*

In particular the map β_p itself is surjective. □

From now on, till the end of the paper, we assume that G is a compact connected Lie group of rank l.

(5.13) Let $\{p_1, \ldots, p_l\}$ be a fixed homogeneous basis of primitive elements $\in H^*(G, \mathbb{Q})$. As is well known, there are certain ('primitive') polynomials $\{P_1, \ldots, P_l\}, P_j \in I^{k_j}(G)$ (obtained by 'transgressing' p_j), such that the class $TP_j = p_j \in H^{2k_j - 1}(G, \mathbb{Q})$, for all $1 \leqslant j \leqslant l$. (Recall that TP_j is defined in [CS;

Identity 3.10]; cf. §5.2.) Moreover there is a rational homotopy equivalence

$$\mu\colon G \to \prod_{j=1}^{l} K(\mathbb{Q}, 2k_j - 1),$$

such that, for any j, the canonical $(2k_j - 1)$-characteristic $c_j \in H^{2k_j-1}(K(\mathbb{Q}, 2k_j - 1), \mathbb{Q})$ lifts under μ to the class TP_j.

Also recall the definition of $\mathscr{G}_0$ and the map α from Section 5.4; and the map $\psi_{P_j,n}$ from Proposition (3.3). Now we are in position to prove the following:

(5.14) THEOREM. *Let $\pi\colon M \times G \to M$ be the trivial bundle with connected base M which is assumed to be the homotopy type of a finite C–W complex. Let $P_j (1 \leqslant j \leqslant l)$ be any primitive polynomial in $I^{k_j}(G)$ as above. Then the composite map*

$$s^* \circ \psi_{P_j,n} \circ \alpha_*\colon H_{n-1}(T_{k_j}(\mathscr{G}_0)) \xrightarrow{\alpha_*} H_{n-1}(T_{k_j}(\mathscr{F})) \xrightarrow{\psi_{P_j,n}} H_{\mathrm{dR}}^{2k_j-n}(E) \xrightarrow{s^*} H_{\mathrm{dR}}^{2k_j-n}(M)$$

is surjective for any $1 \leqslant n \leqslant k_j$ (and of course for $n > k_j$ is the zero map); where α_* (resp. *s^**) *is induced from the map α* (resp. *the trivial section s*).

Proof. Since the canonical map: $H_{n-1}(T_{k_j}(\mathscr{G}_0)) \to H_{n-1}(\mathscr{G}_0)$ is surjective for any $n \leqslant k_j$; by Theorem (5.7) it suffices to prove that the map $\xi_{P_j,n}\colon H_{n-1}(\mathscr{G}_0) \to H_{\mathrm{dR}}^{2k_j-n}(M)$ (defined in §5.6) is surjective:

The evaluation map $ev\colon \mathrm{Map}_0^{\mathrm{cont}}(M, G) \times M \to G$ decomposes, under the rational homotopy equivalence μ, as the product of the evaluation maps:

$$\prod_{j=1}^{l} (\mathrm{Map}_0^{\mathrm{cont}}(M, K(\mathbb{Q}, 2k_j - 1)) \times M) \to \prod_{j=1}^{l} K(\mathbb{Q}, 2k_j - 1).$$

Now the theorem follows from Thom's theorem (5.12) (since $\mu^*(c_j) = TP_j$; cf. §5.13), together with the (standard) fact that the canonical inclusion $\mathscr{G}_0 \hookrightarrow \mathrm{Map}_0^{\mathrm{cont}}(M, G)$ is a homotopy equivalence. □

(5.15) COROLLARY. *With the notation and assumptions as in the above theorem; our basic map $\chi_{P_j,n}\colon H_n(S_{k_j}(\mathscr{C}, \mathscr{F})) \to H_{\mathrm{dR}}^{2k_j-n}(M)$ (cf. Theorem 2.6) is surjective for any $1 \leqslant n \leqslant k_j$.*

In particular, taking a suitable M, we get that $\chi_{P_j,n}$ is non-vanishing for any $1 \leqslant n \leqslant k_j$.

Of course, by Theorem (2.6), $\chi_{P_j,n}$ is the zero map for $n = 0$ or any $n > k_j$.

Proof. The corollary follows trivially from the above theorem together with Proposition (3.5), for any $2 \leqslant n \leqslant k_j$; since, in this range, the map δ_n is an isomorphism (cf. Proposition 3.5). For $n = 1$; by the description of the image of the map δ_1 as given in Proposition (3.5), it suffices to observe that for any $\sigma \in H_0(T_{k_j}(\mathscr{F}))$, there exists a suitable multiple $r\varepsilon$ as of the trivial connection ε(on the bundle $M \times G \to M$) such that $\sigma - r\varepsilon \in \operatorname{Ker} i$ (where i is the map of Proposition 3.5) and moreover $s^* o \psi_{P_j,1}(\varepsilon) = 0$. □

(5.16) Consider the based gauge group of the trivial bundle $\pi: M \times G \to M$ with connected base M (where the based gauge group is the group of all the G-bundle smooth automorphisms inducing identity at the base and preserving a fixed point $(m_0, e) \in M \times G$). Of course the based gauge group can be identified with $\mathscr{G}_0 = \mathrm{Map}_0(M, G)$. From now on we will often make this identification. Let $\mathscr{G}_0^e$ denote the identity component of $\mathscr{G}_0$.

Fix $P \in I^k(G)$. Then, as in Definition (5.6), the evaluation map $ev: \mathscr{G}_0^e \times M \to G$ gives rise to a map $\gamma_{P,n}: H_{n-1}(M) \to H_{\mathrm{dR}}^{2k-n}(\mathscr{G}_0^e)$, defined by

$$\sigma \in H_{n-1}(M) \mapsto \int_\sigma ev^*(TP).$$

We recall the following well known result; which is a consequence of Thom's Theorem (5.10) and the fact that $\mathscr{G}_0^e$ is a topological group, in particular an H-space:

(5.17) THEOREM. *Let $\pi: M \times G \to M$ be the trivial bundle with M and G both compact connected, and let $\mathscr{G}_0^e$ be as above. Then $H_{\mathrm{dR}}^*(\mathscr{G}_0^e, \mathbb{R})$ is a freely generated algebra (in the graded sense) on the span of*

$$\{\gamma_{P_j,n}(H_{n-1}(M))\}_{\substack{1 \leqslant j \leqslant l \\ 2 \leqslant n \leqslant 2k_j - 1}} \subset H_{\mathrm{dR}}^*(\mathscr{G}_0^e, \mathbb{R}),$$

where P_j are the primitive polynomials as in Section 5.13. (Of course degree $\gamma_{P_j,n}(H_{n-1}(M)) = 2k_j - n$).

Moreover the map $\gamma_{P_j,n}$ is injective, for any $2 \leqslant n \leqslant 2k_j - 1$. □

Let $L(\mathscr{G}_0)$ denote the Lie algebra of $\mathscr{G}_0$ and $H^*(L(\mathscr{G}_0))$ denote the continuous Lie algebra cohomology with respect to the Fréchet topology on $L(\mathscr{G}_0)$.

Now we are in position to prove the following theorem as an easy consequence of Theorem (5.7):

(5.18) THEOREM*. *With the notation and assumptions as in the above theorem, the image of the canonical map $H^*(L(\mathscr{G}_0)) \to H_{\mathrm{dR}}^*(\mathscr{G}_0^e)$ contains at least the span of (and hence algebra generated by) $\{\gamma_{P_j,n}(H_{n-1}(M))\}_{\substack{1 \leqslant j \leqslant l \\ 2 \leqslant n \leqslant k_j}}$.*

(*) Even though we have not seen this result in the published form, we learnt it from D. Quillen's course in the spring–1985 at MIT; where he proved this result. In fact we were greatly influenced by his proof in proving our Proposition (5.3).

Proof. Consider the trivial bundle $\hat{\pi}: \mathscr{G}_0^e \times G \to \mathscr{G}_0^e$ and let $\hat{\mathscr{F}}$ be the space of flat connections on $\hat{\pi}$. By Proposition (3.3) we get a map, for any $P \in I^k(G)$,

$$s^* \circ \psi_{P,n}: H_{n-1}(T_k(\hat{\mathscr{F}})) \to H_{\mathrm{dR}}^{2k-n}(\mathscr{G}_0^e),$$

where s is the trivial section of $\hat{\pi}$. (Even though $\mathscr{G}_0^e$ is not a finite dimensional manifold but only a Fréchet manifold, the constructions go through without any difficulty.)

We have a canonical map $i: M \to \mathrm{Map}_0(\mathscr{G}_0^e, G)$, got from the evaluation map $ev: \mathscr{G}_0 \times M \to G$. Hence there is a canonical map $\bar{\imath}_*: H_{n-1}(T_k(M)) \to H_{n-1}(T_k(\hat{\mathscr{F}}))$ and therefore, on composition with $s^* \circ \psi_{P,n}$, we get a map: $H_{n-1}(T_k(M)) \to H_{\mathrm{dR}}^{2k-n}(\mathscr{G}_0^e)$. By the very definition of the map $\bar{\psi}_{P,n}$(cf. §3.2), it is easy to see that the image of $T_k(M)$ in $\Lambda(\mathscr{G}_0^e)$ under $s^* \circ \bar{\psi}_{P,n}$ consists of (smooth) left invariant forms. In particular the image of $H_{n-1}(T_k(M))$ in $H_{\mathrm{dR}}^{2k-n}(\mathscr{G}_0^e)$ is represented by (smooth) left invariant forms, i.e., by continuous $(2k-n)$-cocycles of the Lie algebra $L(\mathscr{G}_0)$. But, by Theorem (5.7), the following diagram is commutative:

$$\begin{tikzcd}
H_{n-1}(T_k(M)) \arrow[r, "s^* \circ \psi_{P,n} \circ \bar{\imath}_*"] \arrow[d] & H_{\mathrm{dR}}^{2k-n}(\mathscr{G}_0^e) \\
H_{n-1}(M) \arrow[r, "i_*"] & H_{n-1}(\mathrm{Map}_0(\mathscr{G}_0^e, G)). \arrow[u, "\xi_{P,n}"']
\end{tikzcd}$$

Now take $P = P_j$ (for any $1 \leqslant j \leqslant l$) of degree k_j. Then the canonical map $H_{n-1}(T_{k_j}(M)) \to H_{n-1}(M)$ is of course surjective for any $n \leqslant k_j$ (and 0 otherwise). Moreover $\gamma_{P_j,n} = \xi_{P_j,n} \circ i_*$ and hence any element in the span of $\{\gamma_{P_j,n}(H_{n-1}(M))\}_{\substack{1 \leqslant j \leqslant l \\ 2 \leqslant n \leqslant k_j}}$ can be represented by a continuous cocycle of the Lie algebra $L(\mathscr{G}_0)$. □

Note added in proof

Let M be a (smooth) compact oriented Riemann surface with genus $\geqslant 2$ equipped with the Poincaré metric, and take $G = SU(2)$ with the bi-invariant metric induced from the Killing form. We consider the trivial G-bundle $\pi: M \times G \to M$. Given a smooth function $f: M \to G$, we associate a map $\sigma = \sigma_f: I \to \mathscr{C}$ by setting $\sigma(t)$ to be the connection on π determined by the 1-form $\frac{1}{2}(\theta + (\cos 2\pi t)\theta + (\sin 2\pi t) * \theta)$ on M (cf. Lemma 5.1); where $\theta = \theta(f)$ is as in Definition (5.2), and $*$ is the Hodge star operator. Observe that $\sigma(o) = \sigma(1) = \hat{\theta}$.

By a result of N. Hitchin (cf. his preprint 'Harmonic maps from T^2 into S^3') the function f is harmonic (i.e. is a critical point for the energy function) if and only if $\sigma(I) \subset \mathscr{F}$. So in this case (considering σ as an element of $H_1(T_2(\mathscr{F}))$), we get $s^*\psi_{P,2}(\sigma) \in H^2_{dR}(M) \approx \mathbf{R}$; where P is the second Chern polynomial (cf. Example 4.2), the map ψ is defined in Proposition (3.3), and s is the trivial section of π. We have obtained the following result (the proof of which will appear elsewhere):

THEOREM. With the notation and assumptions as above, assume that $f: M \to G$ is a harmonic map. Then

$$s^*\psi_{P,2}(\sigma) = -E(f),$$

where $E(f) := \frac{1}{2}\int_M \|df\|^2$ is the energy of f. □

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