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CLASSIFICATION OF LOGARITHMIC FANO THREEFOLDS

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Introduction

Throughout this paper varieties are defined over a fixed algebraically closed field of characteristic zero.

A logarithmic Fano threefold is defined to be a pair (V, D) of a smooth projective threefold V and a reduced divisor D with normal crossings on V, satisfying the following condition:

$-K_V - D$ is ample.

This is one of the extensions of the notion of Fano threefold.

Biregular classification of Fano threefolds was completed by Iskovskih-Shokurov ([8] or [9], see also [24]) in case $B_2(V) = 1$ or index$(V) \geqslant 2$, and by Mori-Mukai ([16],[17]) in case $B_2(V) \geqslant 2$. In case $B_2(V) \geqslant 2$, there exist at least two extremal rays and the classification of extremal rational curves, by S. Mori ([15]), plays an essential role.

In [7], S. Iitaka observed that the classical classification theory of complete algebraic varieties can be extended to the classification theory of open algebraic varieties. Inspired by this theory, the author extends the definition of Fano variety to the case of non-singular pair (V, D) and classifies them. The name "logarithmic" is derived from Iitaka's theory, since he makes use of logarithmic differential forms to define many invariants of this pair.

The purpose of this paper is to classify logarithmic Fano threefolds (V, D) with non-zero boundaries D (cf. Reid [21], p. 10, Problem 2).

Fundamental tools are Norimatsu vanishing theorem, Tsunoda's logarithmic cone theorem, Mori's theory of extremal rational curves on a threefold and some ampleness criteria for the logarithmic anti-canonical divisor, which will be explained in section 1.

In section 2 we study some general properties of logarithmic Fano varieties (V, D) of arbitrary dimension. In particular, the Picard group of V is a free $\boldsymbol{Z}$-module of rank $B_2(V)$ (Lemma 2.3) and the boundary D is strongly connected (Lemma 2.4).

Any component Δ of D of logarithmic Fano variety (V, D) with $D \neq 0$ is also a logarithmic Fano variety with boundary $(D - \Delta)|_{\Delta}$. In order to determine bounaries of logarithmic Fano threefolds, we classify logarithmic Del Pezzo surfaces in section 3.

In section 4 we prove the existence of an extremal rational curve ℓ with $(D \cdot \ell) > 0$ as a key lemma. Moreover all the types of ℓ are F, E_2, D_3, D_2 or C_2.

Roughly speaking logarithmic Fano threefolds (V, D) with $D \neq 0$ are classified into the following five types:

(i) V is $\boldsymbol{P}^3$, Q_2, V_1, V_2, V_3, V_4 or V_5 of [9, I, Theorem 4.2 (ii) and (iv)] of index 2. Letting H be an ample generator of Pic(V), we have $-K_V > rH$, where r is called the index of V. In this case D is a member of $|tH|$, with $t < r$ (section 6).

(ii) V is a blowing up of $\boldsymbol{P}^3$ at a smooth conic curve or a blowing up of another logarithmic Fano threefold (V', D') at some points on a boundary D'. The number of the points is at most 8. Here V' is $\boldsymbol{P}^3$, Q_2 or Σ_{a_1, a_2} (section 7).

(iii) V is a $\boldsymbol{P}^1$-bundle over a smooth surface which is either a Del Pezzo surface or a geometrically ruled surface Σ_n. One component of D is a birational section of this bundle and another component, if exists, is a geometrically ruled surface formed by fibers of this bundle (section 8).

(iv) V is a quadric fibering over $\boldsymbol{P}^1$ with $B_2(V) = 2$. V is embedded in a $\boldsymbol{P}^3$-bundle over $\boldsymbol{P}^1$ as a smooth divisor. One of the components of D is a horizontal one of this fibering. (In particular V is rational.) Another component, if exists, is a fiber (section 9, 9.2).

(v) V is a $\boldsymbol{P}^2$-bundle over $\boldsymbol{P}^1$, denoted by Σ_{a_1, a_2}. D has one or two horizontal components. Another component, if exists, is a fiber (section 9, 9.1).

We give in section 5 the configurations of boundaries except for type (i) above, which we classify together with V in section 6.

From sections 6 to 9 we classify logarithmic Fano threefolds according to the types of extremal rational curves.

As consequences, we obtain the following results:

(1) In [16], Mori and Mukai have shown that for a Fano threefold V, $B_2(V) \leqslant 10$ and the equality holds if and if $V \cong \boldsymbol{P}^1 \times S_1$, where S_1 is a Del Pezzo surface of degree 1. For a logarithmic Fano threefold (V, D) with $D \neq 0$, $B_2(V) \leqslant 10$. The equality holds if and only if V is a $\boldsymbol{P}^1$-bundle over S_1. In contrast with usual Fano threefolds, there is an infinite series of such (V, D) (cf. 8.1.4).

(2) From the classification theory, for a Fano threefold V, $(-K_V)^3 \leqslant 64$ ([16, Corollary 11]). However there is a logarithmic Fano threefold (V, D) with $(-K_V - D)^3$ arbitrary large.

(3) A logarithmic Fano threefold turns out to be either a rational threefold (cases ii, iii, iv, and v) or a usual Fano threefold (case i). In characteristic zero case, these threefolds are simply connected ([4, 3.3], [20]). Hence for any logarithmic Fano threefold (V, D), V is simply connected.

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§1. Preliminaries

Let V be a smooth projective variety and D a reduced divisor with normal crossings on V. We consider here such a pair (V, D) of V and D, which we call a non-singular pair (of dimension $n = \dim V$). D is occasionally called the boundary of this pair. We denote by K_V the canonical divisor on V.

DEFINITION 1.1: A non-singular pair (V, D) is called a logarithmic Fano variety if $-K_V - D$ is an ample divisor on V.

Especially we call two dimensional logarithmic Fano varieties logarithmic Del Pezzo surfaces and three-dimensional ones logarithmic Fano threefolds.

First we recall the following vanishing theorem. In this paper we refer to this theorem as Norimatsu vanishing theorem.

THEOREM 1.2: (*Norimatsu* [19, *Theorem* 1]) *Let (V, D) be a non-singular pair and L an ample divisor on V. Then*

$$H^i(V, \mathcal{O}_V(K_V + D + L)) = 0 \quad \textit{for any } i > 0.$$

Next we explain the cone theorem. We define

$N_1(V) = \{\text{1-cycle on V}\}/\approx \otimes \boldsymbol{R}$,

$NE(V) =$ the convex cone in $N_1(V)$ generated by curves,

$\overline{NE}(V) =$ the closure of $NE(V)$ in $N_1(V)$ with respect to the reql topology.

$\overline{NE}_H(V) = \{Z \in \overline{NE}(V); (H \cdot z) \geqslant 0\}$, *for* $H \in \mathrm{Div}(V) \otimes \boldsymbol{Q}$, where $\approx$ denoted numerical equivalence.

The following theorem is an extension of the theorem (1.4) in [15], where the case D = 0 was treated.

THEOREM 1.3: (*Tsunoda* [22], [23, *p.* 508]) *Let (V, D) be a non-singular pair and L an arbitrary ample **Q**-divisor on V. Then there exist a finite number of (may be singular) rational curves $\ell_1, \ell_2, \dots, \ell_r$ such that*

$$0 < (-K_V - D \cdot \ell_i) < \dim V + 1$$

and

$$\overline{NE}(V)=\boldsymbol{R}_{+}[\ell_1]+\boldsymbol{R}_{+}[\ell_2]+\cdots+\boldsymbol{R}_{+}[\ell_r]+\overline{NE}_{K_V+D+L}(V)$$

where $\boldsymbol{R}_{+}$ *denotes the set of non-negative real numbers.*

We refer to the above theorem as logarithmic cone theorem.

We give here a short presentation of Mori's theory of the classification of extremal ratinal curves on a threefold. In general

THEOREM 1.4: (*Contraction Theorem*). *Let* (V, D) *be a non-singular pair and let* $R=\boldsymbol{R}_{+}[\ell]$ *be an extremal ray of* $\overline{NE}(V)$. *Then there exists a (unique) projective morphism onto a normal projective variety* W, *say* $f\colon V\to W$, *satisfying the following conditions*:

(1) $f_*\mathcal{O}_V=\mathcal{O}_W$,

(2) $0\to \mathrm{Pic}(W)\overset{f^*}{\to}\mathrm{Pic}(V)\overset{(\cdot\ell)}{\to}\boldsymbol{Z}$ *is exact,*

(3) *for an irreducible curve* C, $f(C)$ *is a point if and only if* $[C]\in R$.

The above morphism is called the contraction of R *or* ℓ *and denoted by* cont_R *or* cont_ℓ.

Note that the cone theorem and the contraction theorem holds true in more general setting (Kawamata, Reid, Shokurov, Jânos Kollár).

Mori determined cont_R in case of $\dim V\leqslant 3$. For details we refer to [14], [15] or [17]. In this paper we only need the following facts.

For an extremal rational curve ℓ we let $\mu(R)=\mu(\ell)=\min\{(-K_V\cdot C);\ C$ is an irreducible rational curve with $[C]\in R=\boldsymbol{R}_{+}[\ell]\}$. The following lemma extracts the facts from [17, pp. 107–109].

LEMMA 1.5: *Let* ℓ *be an extremal rational curve on a threefold* V *with* $\mu(\ell)=(-K_V\cdot l)\geqslant 2$. *Then* ℓ *is one of the following types*:

Type F: *In this case*, V *is one of the Fano threefolds with* $B_2(V)=1$, *which are called Fano threefolds of first species* ([9]). *In this case*, $\mu(l)$ *takes any value between* 2 *and* 4.

Type E_2: *In this case*, W *is a smooth threefold and* $\mathrm{cont}_\ell\colon V\to W$ *is a blowing up at a smooth point of* W. *The exceptional divisor* E *is isomorphic to* $\boldsymbol{P}^2$ *and* ℓ *is a line of* $\boldsymbol{P}^2$ *with* $(E\cdot\ell)=-1$. *In this case* $\mu(\ell)=2$.

Type D_2: *in this case*, $\mathrm{cont}_\ell\colon V\to W$ *is a quadric bundle over a smooth curve* W, *i.e., and fiber is isomorphic to an irreducible quadric surface in* $\boldsymbol{P}^3$ *and* ℓ *is a generatrix. In this case* $\mu(\ell)=2$.

Type D_3: *In this case*, $\mathrm{cont}_\ell\colon V\to W$ *is a* $\boldsymbol{P}^2$*-bundle over a smooth curve* W *and* ℓ *is a line on a fiber. In this case*, $\mu(\ell)=3$.

Type C_2: *In this case*, $\mathrm{cont}_\ell\colon V\to W$ *is an étale* $\boldsymbol{P}^1$*-bundle over a smooth surface* W *and* ℓ *is a fiber of this bundle. In this case*, $\mu(\ell)=2$.

Finally we give a numerical criterion of ampleness for the logarithmic anti-canonical divisor $-K_V - D$.

LEMMA 1.6: ([12, (*ii*) of theorem]). *Let* (V, D) *be a non-singular pair of dimension* 3 *and suppose that* $\kappa(-K_V - D, V) \geqslant 0$. *Then* $-K_V - D$ *is ample if and only if* $-K_V - D$ *is numerically positive, i.e.* $(-K_V - D \cdot C) > 0$ *for all curves* C *on* V.

§2. General properties of logarithmic Fano varieties of arbitrary dimension

LEMMA 2.1: *Let* (V, D) *be a logarithmic Fano variety. Then*

(a) $H^i(m(K_V + D)) = 0$ *for any* $m \geqslant 1$ *and* $i < \dim V$,

(b) $H^i(-m(K_V + D)) = 0$ *or any* $m \geqslant 0$ *and* $i > 0$,

(c) $P_m(V) = h^0(mK_V) = 0$ *for any* $m > 0$, *i.e.* $\kappa(V) = -\infty$, *where* $\kappa(V)$ *denotes the Kodaira dimension of* V (*cf.* [6, §10]),

(d) *If* $D \neq 0$, *then* $H^i(-D) = 0$ *for any* $i \geqslant 0$.

PROOF: (a) follows from Kodaira vanishing theorem;

(b) follows frm $H^i(-m(K_V + D)) = H^i(K_V + D + (m+1)(-K_V - D)) = 0$ by Norimatsu vanishing theorem;

(c) is clear since $h^0(mK_V) \leqslant h^0(m(K_V + D)) = 0$ by (a);

(d) follows from $H^i(-D) = H^i(K_V + (-K_V - D)) = 0$ for any $i > 0$. $H^0(-D) = 0$ is obvious Q.E.D.

COROLLARY 2.2: (a) $\mathrm{Alb}(V) = 0$.

(b) $\mathrm{Pic}(V) \cong H^2(V, \boldsymbol{Z})$. *In particular*, $\rho(V) = B_2(V)$, *where* $\rho(V)$ *is the Picard number of* V.

(c) $\chi(\mathcal{O}_V) = 1$.

PROOF: (a) is immediate, because $H^0(\Omega_V^1) = H^1(\mathcal{O}_V) = 0$;

(b) follows from the exponential sequence

$$0 \to \boldsymbol{Z} \to \mathcal{O}_V \overset{\text{exp.}}{\to} \mathcal{O}_V^* \to 1$$

and the fact that $H^1(\mathcal{O}_V) = H^2(\mathcal{O}_V) = 0$;

(c) is obvious. Q.E.D.

LEMMA 2.3: *For a logarithmic Fano variety* (V, D), $\mathrm{Pic}(V)$ *is torsion free*.

PROOF: (cf. [8, 4.10]). We suppose that L is a torsion divisor. By Keliman's numerical criterion for ampleness, $mL - K_V - D$ is an ample divisor for any $m \geqslant 0$ ([11]). By Norimatsu vanishing theorem,

$$H^i(mL) = H^i(K_V + D + (mL - K_V - D)) = 0$$

for any $i > 0$. Thus we obtain $h^0(mL) = \chi(mL)$.

On the other hand, since L is numerically equivalent to zero, we have $\chi(mL) = \chi(\mathcal{O}_V) = 1$. Hence $|mL| \neq \emptyset$ for all $m \geqslant 1$. In particular, $|L| \neq \emptyset$ and this implies that $L \sim 0$. Q.E.D.

LEMMA 2.4: *(a) D is connected, moreover*

(a′) $D_i \cap D_j \neq \emptyset$, for any i and j (in this case D is called strongly connected).

(b) $s \leqslant \dim V$, where s is the number of irreducible components of D, i.e. $D = D_1 + \cdots + D_s$.

PROOF: (a) By the standard exact sequence

$$0 \to \mathcal{O}_V(-D) \to \mathcal{O}_V \to \mathcal{O}_D \to 0$$

and the fact that $H^0(-D) = H^1(-D) = 0$, we have $h^0(\mathcal{O}_D) = h^0(\mathcal{O}_V) = 1$.

(a′) We prove by induction on $\dim V$. The assertion is clear, if $\dim V = 1$. We may assume that $D \neq 0$. Suppose there exist i and j such that $D_i \cap D_j = \emptyset$. Since D is connected, after renumbering D_i's, we may assume that there exist D_1, D_2 and D_3 such that $D_1 \cap D_2 \neq \emptyset$ and $D_1 \cap D_3 \neq \emptyset$, but $D_2 \cap D_3 = \emptyset$. Let $\Gamma_\ell = D_\ell|_{D_1}$ for $\ell \neq 1$. Then $\Gamma_2 + \cdots + \Gamma_s = (D - D_1)|_{D_1}$ is a divisor with normal crossings on D_1. Since

$$(-K_V - D_1 - \cdots - D_s)|_{D_1} = -K_{D_1} - \Gamma_2 - \cdots - \Gamma_s$$

is an ample divisor on V, $(D_1, \Gamma_2 + \cdots + \Gamma_s)$ is a logarithmic Fano variety of dimension $\dim V - 1$. By induction hypothesis, $\Gamma_2 \cap \Gamma_3 \neq \emptyset$, hence $D_2 \cap D_3 \neq \emptyset$. This is a contradiction. (b) By (a'), $D_1 \cap \cdots \cap D_s \neq \emptyset$ if $D \neq 0$. Since D is a divisor with only normal crossings,

$$\dim(D_1 \cap \ldots \cap D_s) = \dim V - s,$$

hence $n \geqslant s$. Q.E.D.

§3. Classification of logarithmic Del Pezzo surfaces

In this section we always assume that (V, D) denotes a logarithmic Del Pezzo surface.

LEMMA 3.1: *The* Δ-*genus of* V (*cf*. [3, *Definition* 1.4]) *with respect to the ample divisor* $-K_V - D$ *is as follows*:

(*a*) *If* $D = 0$, *then* $\Delta(V, -K_V) = 1$.

(*b*) *If* $D \neq 0$, *then* $\Delta(V, -K_V - D) = 0$.

PROOF: By the definition of Δ-genus.

$$\Delta(V, -K_V - D) = 2 + (-K - D)^2 - h^0(-K - D).$$

By Lemma 2.2,

$$H^i(-K - D) = 0 \quad \text{for } i = 1 \text{ and } 2.$$

Hence,

$$\begin{aligned} h^0(-K - D) &= \chi(-K - D) \\ &= 1/2(-K - D \cdot -2K - D) + \chi(\mathcal{O}_V) \\ &= (-K - D)^2 - 1/2(K + D \cdot D) + 1 \end{aligned}$$

by Riemann-Roch Theorem. We have

$$\Delta(V, -K - D) = 1 + 1/2(K + D \cdot D).$$

if $D \neq 0$, then $(-K - D \cdot D) > 0$ and $\Delta(V, -K - D) < 1$. This implies that $\Delta(V, -K - D) = 0$. Q.E.D.

Now we can classify logarithmic Del Pezzo surfaces (V, D). We may assume that $D \neq 0$. Otherwise V is a (classical) Del Pezzo surface. The structure of Del Pezzo surfaces are well known (see, for example [13]).

Using Fujita's classification theorem of polarized varieties of Δ-genera zero ([3, pp. 107–110]), we have $(V, -K_V - D)$ *as follows*:

(*a*) $(V, -K - D) \cong (\boldsymbol{P}^2, H)$ *where* H *is a line on* $\boldsymbol{P}^2$,

(*b*) $(V, -K - D) \cong (\boldsymbol{P}^1 \times \boldsymbol{P}^1, s + f)$ where s is a section and f a fiber of the trivial bundle $\boldsymbol{P}^1 \times \boldsymbol{P}^1 \to \boldsymbol{P}^1$,

(c) $(V, -K - D) \cong (\boldsymbol{P}^2, 2H)$ or $(\boldsymbol{P}(\mathcal{O}_{\boldsymbol{P}^1}(a_1) \oplus \mathcal{O}_{\boldsymbol{P}^1}(a_2)), \mathcal{O}_{\boldsymbol{P}}(1))$.

REMARK. When we treat the two dimensional polarized varieties of Δ – genera zero, the case (b) happens to be a special case of (c).

3.1. *Case* (a). In this case, $V \cong \boldsymbol{P}^2$ and $D \sim 2H$, where $\sim$ means linear equivalence on V. If D is irreducible, then D is a smooth conic. If D has

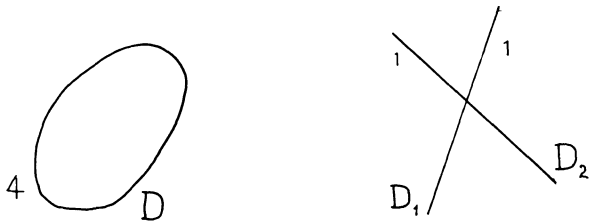


Fig. 1

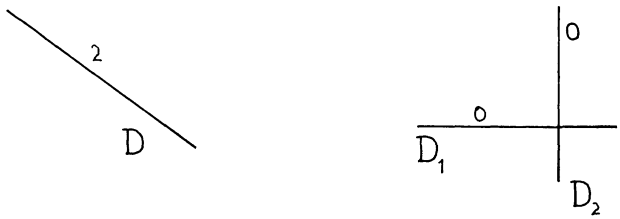


Fig. 2

two irreducible components D_1 and D_2, then both D_1 and D_2 are lines on $\boldsymbol{P}^2$ (see Fig. 1).

3.2. *Case* (b). In this case, $V \cong \boldsymbol{P}^1 \times \boldsymbol{P}^1$ and $D \sim f + s$. If D is irreducible, then D is a smooth section linearly equivalent to $s + f$. If D has two components, D_1 and D_2, then D_1 is a section s and D_2 is a fiber f (see fig. 2).

3.3. *case* (c). If $V \cong \boldsymbol{P}^2$, then $D \sim H$. Hence D is a line (see Fig. 3).

If $V \cong \boldsymbol{P}(\mathcal{O}_{\boldsymbol{P}^1}(a_1) \oplus \mathcal{O}_{\boldsymbol{P}^1}(a_2))$, then V is a geometrically ruled surface and the properties of such surfaces are known (see, for example, [5, Chap. V]).

Let $n = |a_1 - a_2|$. Then $V \cong \boldsymbol{P}(\mathcal{O}_{\boldsymbol{P}^1} \oplus \mathcal{O}_{\boldsymbol{P}^1}(-n))$, denoted by Σ_n. On Σ_n, we have a unique section Δ such that $(\Delta)^2 = -n$ and a fiber ℓ.

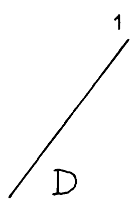


Fig. 3

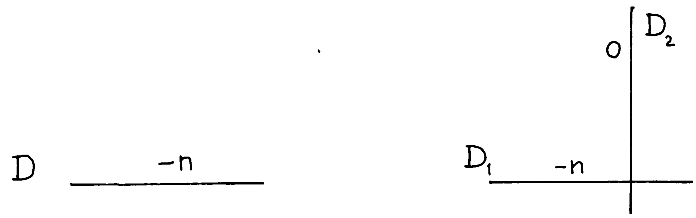


Fig. 4

LEMMA 3.2: ([5, p. 380]). *Let Σ_n, Δ and ℓ be as above. Then $\alpha\Delta + \beta\ell$ is ample if and only if $\alpha > 0$ and $\beta > \alpha n$.*

Since $-K_V - D \sim \mathcal{O}_{\boldsymbol{P}}(1)$ is ample $-K_V \sim \mathcal{O}_{\boldsymbol{P}}(2) +$ fibers, we have $(-K - D \cdot \ell) > 0$ and $(D \cdot \ell) > 0$. Hence $(D \cdot \ell) = 1$. In this case, one of the components of D is a section, say D_1.

First we suppose that $D = D_1$. Then D_1 can be written as $D_1 \sim \Delta + m\ell$, where $m = 0$ or $m \geqslant n$. Since $-K_V - D \sim \Delta + (n + 2 - m)\ell$ is ample, we have $n + 2 - m > n$ by Lemma 3.2.

If $m = 0$, then $D = \Delta$ and n can take any non-negative integer. Hence, $V \cong \Sigma_n$ and $D = \Delta$ (see Fig. 4).

If $m \geqslant n$, then $n = 1$ or 0. In case $n = 0$, we may assume $m = 1$, because the case $m = 0$ is the above. Hence, $V \cong \Sigma_0$ and $D \sim \Delta + \ell$. This case is treated in 3.2. In case $n = 1$, $m = 1$ and V is isomorphic to Σ_1 and $D \sim \Delta + \ell$ (see Fig. 5).

Next we consider the case where D is a sum of a section D_1 and a fiber $D_2 = \ell$. Then D_1 is linearly equivalent to $\Delta + m\ell$ where $m = 0$ or $m \geqslant n$. Since $-K_V - D \sim \Delta + (n + 1 - m)\ell$ is ample, we have $n + 1 - m > n$.

If $m = 0$, then $-K - D$ is always ample for any $n \geqslant 0$. Hence $V \cong \Sigma_n$, $D_1 = \Delta$ and $D_2 = \ell$ (see Fig. 4).

If $m \geqslant n$, then $n = 0$ and $m = 0$. This case is contained in the above case.

3.4. Summarizing the above result, a logarithmic Del Pezzo surface is one of the following:

(i) $V \cong \boldsymbol{P}^2$, $D = D_1$ where D_1 is a line.

(ii) $V \cong \boldsymbol{P}^2$, $D = D_1 + D_2$ where each D_i is a line.

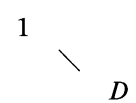


Fig. 5

(iii) $V \cong \boldsymbol{P}^2$, $D = D_1$ where D_1 is a smooth conic.
(iv) $V \cong \Sigma_n = \boldsymbol{P}(\mathcal{O}_{\boldsymbol{P}^1} \oplus \mathcal{O}_{\boldsymbol{P}^1}(-n))$, $D = D_1$ where D_1 is a section with $(D_1)^2 = -n$.
(v) $V \cong \Sigma_n$, $D = D_1 + D_2$ where D_1 is a section with $(D_1)^2 = -n$ and D_2 is a fiber.
(vi) $V \cong \Sigma_1$, $D = D_1$ where D_1 is a section with $(D_1)^2 = 1$.
(vii) $V \cong \Sigma_0$, $D = D_1$ where D_1 is a section with $(D_1)^2 = 2$.

REMARK: The logarithmic Del Pezzo surfaces over an algebraically closed field of positive characteristic are the same as over characteristic zero. But we will not treat this in this paper.

§4. Extremal rational curves on a logarithmic Fano threefold

We can apply the logarithmic cone theorem (1.3) to a logarithmic Fano threefold (V, D). Since $-K_V - D$ is ample, we can take in 1.3 $L = \epsilon(-K_V - D)$, with $\epsilon \in \boldsymbol{Q}_+$ small; then $\overline{NE}_{K_V+D+L}(V) = 0$. Hence $NE(V)$ is a polyhedral cone, i.e.

$$NE(V) = \overline{NE}(V) = \boldsymbol{R}_+[\ell_1] + \cdots + \boldsymbol{R}_+[\ell_r],$$

where the ℓ_i are extremal rational curves. We may assume that each ℓ_i satisfies $\mu(\ell_i) = (-K_V \cdot \ell_i)$. The next lemma is a key lemma of this paper.

LEMMA 4.1: *Let (V, D) be a logarithmic Fano threefold with $D \neq 0$. Then there exists an extremal rational curve ℓ_i on V such that*

$$(-K_V \cdot \ell_i) > (D \cdot l_i) > 0.$$

In particular, $(-K_V \cdot l_i) \geqslant 2$.

PROOF: Since $-K_V - D$ is ample and $NE(V) = \boldsymbol{R}_+[\ell_1] + \cdots + \boldsymbol{R}_+[\ell_r]$, we have

$$(-K_V - D)^2 \approx a_1\ell_1 + \cdots + a_r\ell_r,$$

where $a_i \geqslant 0$ for all i. Since D is a non-zero effective divisor,

$$0 < D \cdot (-K_V - D)^2 = a_1(D \cdot \ell_1) + \cdots + a_r(D \cdot \ell_r).$$

Hence one of $(D \cdot \ell_i)$'s must be positive, say $(D \cdot \ell_1) > 0$. Moreover, we have $(-K_V - D \cdot \ell_1) > 0$ and therefore $(-K_V \cdot \ell_1) \geqslant 2$. Q.E.D.

REMARK: If ℓ is an extremal rational curve satisfying the conditions of Lemma 4.1 for a logarithmic Fano threefold (V, d) with $D \neq 0$, then by Lemma 1.5 ℓ is of type F, E_2, D_3, D_2, or C_2.

§5. Classification of boundaries of logarithmic Fano threefolds

Let (V, D) be a logarithmic Fano threefold with $D \neq 0$. In section 4, we have shown that there is an extremal rational curve ℓ with $(-K_V \cdot \ell) > (D \cdot \ell) > 0$ and the type of ℓ is F, E_2, D_3, D_2 or C_2. Here we will classify the possibilities of D according to the type of ℓ.

5.1. *Type* E_2. In this case, $(-K_V \cdot \ell) = 2$. Hence, $(D \cdot \ell) = 1$.

First we consider the case where the exceptional divisor E is a component of D, say D_1. Then

$$(D_1 + \cdots + D_s \cdot \ell) = (D_1 \cdot \ell) + (\Gamma_2 + \cdots + \Gamma_s \cdot \ell)_{D_1}.$$

Here, $\Gamma_i = D_i|_{D_1}$ are the double curves of D lying on D_1. Since $(D_1 \cdot \ell) = -1$, we have $(\Gamma_2 + \cdots + \Gamma_s \cdot \ell)_{D_1} = 2$. Thus $\Gamma_2 + \cdots + \Gamma_s$ is linarly equivalent to a conic on $D_1 \cong \boldsymbol{P}^2$.

From the classification of logarithmic Del Pezzo surfaces, the possibilities of D in this case are as follows:

(i) $D = D_1 + D_2$, where $D_1 \cong \boldsymbol{P}^2$ and $D_2 \cong \Sigma_2$,

(ii) $D = D_1 + D_2 + D_3$, where $D_1 \cong \boldsymbol{P}^2$, $D_2 \cong \Sigma_1$ and $D_3 \cong \Sigma_1$ (see Fig. 6).

PROOF: If the double curve $\Gamma = D_2 \cap D_1$ is a smooth conic on D_1, then we have

$$(\Gamma)^2_{D_2} = (D_1|_{D_2} \cdot D_1|_{D_2} \cdot D_1|_{D_2})_{D_2}$$

$$= (D_1 \cdot D_1 \cdot D_2) = (\Gamma \cdot D_1) = -2.$$

Thus D_2 is isomorphic to Σ_2 (case (i)).

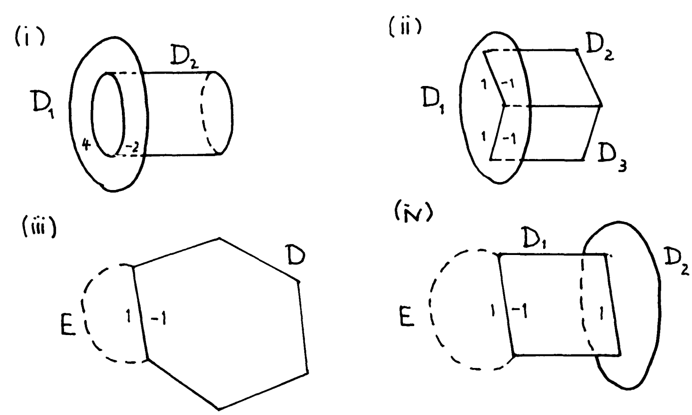


Fig. 6

If the double curve $\Gamma_2 + \Gamma_3 = (D_2 + D_3)|_{D_1}$ is composed of distinct two lines, then

$$(\Gamma_2)^2_{D_2} = (\Gamma_2 \cdot D_1) = -1$$

and similarly $(\Gamma_3)^2_{D_3} = -1$. Hence both D_2 and D_3 are isomorphic to Σ_1 (case (ii)). Q.E.D.

Next we consider the case where the exceptional divisor E is not a component of D. In this case, we have

$$(D|_E \cdot \ell)_E = (D \cdot \ell) = 1.$$

Hence $D|_E$ is a line on $E \cong \boldsymbol{P}^2$. The possibilities of D are as follows:

(iii) $D = D_1$ where $D_1 \cong$ a Del Pezzo surface and $E|_D$ is an exceptional curves of the first kind on D.

(iv) $D = D_1 + D_2$, where $D_1 \cong \Sigma_1$. We cannot determine D_2 immediately but will classify it in 7.4. (see Fig. 6).

5.2. *Type C_2.* Let f: $V \to W$ be a $\boldsymbol{P}^1$-bundle over a smooth projective surface W induced by ℓ. In this case $(-K_V \cdot \ell) = 2$ and thus $(D \cdot \ell) = 1$. This implies that D contains a (birational) section D_1 of f, i.e., $f|_{D_1}$: $D_1 \to W$ is a birational morphism. And the other components D_j satisfy $(D_j \cdot \ell) = 0$; hence, for any $j \geqslant 2$, D_j is a ruled surface formed by fibers of f. Note that D_1 is a geometrically ruled surface or a Del Pezzo surface and W is the image of D_1. Hence W is either a geometrically ruled surface or a Del Pezzo surface. If D_1 contains no exceptional curve of the first kine, then $f|_{D_1}$: $D_1 \to W$ is an isomorphism (see section 8).

Letting $\Gamma_i = D_i|_{D_1}$, we see that the possibilities of D are as follows:

(i) $D = D_1$ where D_1 is a Del Pezzo surface.

(ii) $D = D_1 + D_2$ where $D_1 \cong \boldsymbol{P}^2$.

(iii) $D = D_1 + D_2 + D_3$ where $D_1 \cong \boldsymbol{P}^2$.

(iv) $D = D_1 + D_2$ here $D_1 \cong \Sigma_n$ and $(\Gamma_2)^2 = -n$.

(v) $D = D_1 + D_2 + D_3$ where $D_1 \cong \Sigma_n$, $(\Gamma_2)^2 = -n$ and $(\Gamma_3)^2 = 0$.

(vi) $D = D_1 + D_2$ where $D_1 \cong \Sigma_0$ and $(\Gamma_2)^2 = 2$.

(vii) $D = D_1 + D_2$ where $D_1 \cong \Sigma_1$ and $(\Gamma_2)^2 = 1$ (see Fig. 7).

5.3. *Type D_2 or $_3$.* In this case ℓ induces a quadric fibering or a $\boldsymbol{P}^2$-bundle f: $V \to W$ over a smooth curve W and $B_2(V) = \rho(V) = 2$.

CLAIM: $W \cong \boldsymbol{P}^1$.

PROOF: By Mori's theory, we have

$$f_* \mathcal{O}_V = \mathcal{O}_W \quad \text{if} \quad i = 0$$
$$R^i f_* \mathcal{O}_V = 0 \quad \text{if} \quad i \geqslant 1.$$

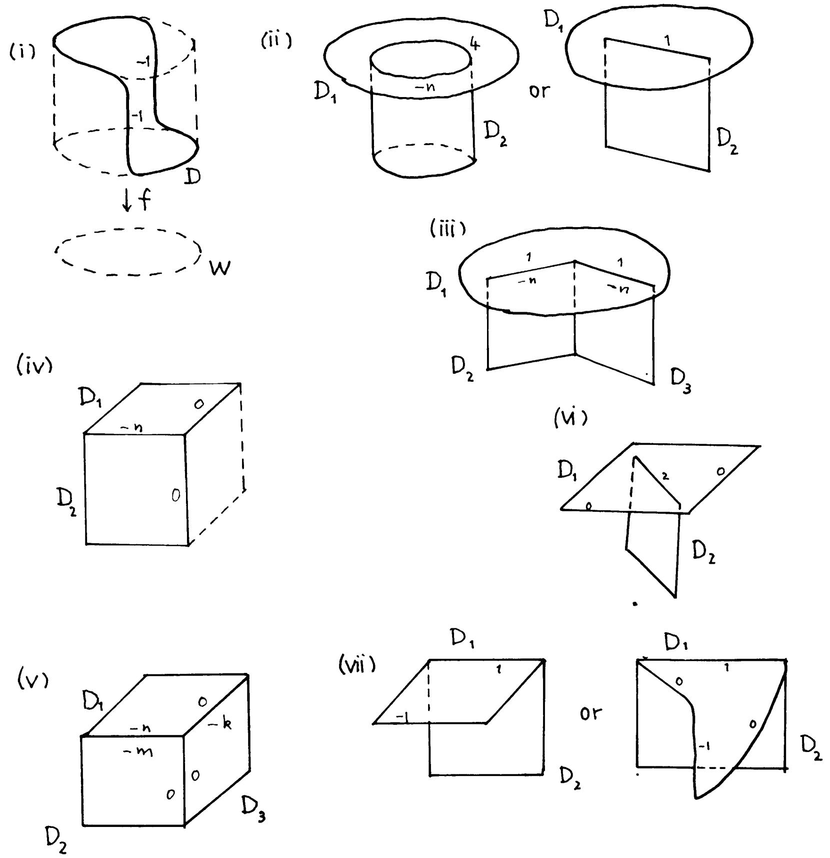


Fig. 7

Hence, $H^1(W, \mathcal{O}_W) = H^1(V, \mathcal{O}_V) = 0$, i.e. genus $(W) = 0$. Q.E.D.

First we consider the case where f is a quadric fibering. Since $(-K_V \cdot \ell) = 2$, we have $(D \cdot \ell) = 1$. This implies that D contains a horizontal component D_1. A general fiber F is isomorphic to $\boldsymbol{P}^1 \times \boldsymbol{P}^1$. 1-cycles corresponding to $\boldsymbol{P}^1 \times pt$ or $pt \times \boldsymbol{P}^1$ on F are numerically equivalent to ℓ on V. Thus $D_w := D_1 \cap f^{-1}(w)$ is a hyperplane section of $f^{-1}(w)$ as a quadric surface in $\boldsymbol{P}^3$ (see Fig. 8).

The possibilities of D are as follows:

(i) $D = D_1$ where D_1 is a Del Pezzo surface except for $\boldsymbol{P}^2$.

(ii) $D = D_1 + D_2$ where $D_1 \cong \Sigma_0$ and $D_2 \cong \Sigma_0$ (see Fig. 9).

Next we assume that f is a $\boldsymbol{P}^2$-bundle over $\boldsymbol{P}^1$. Then $(-K_V \cdot \ell) = 3$ and $(D \cdot \ell) = 2$ or 1. The possibilities of D are as follows:

(i) $D = D_1$ where $D_1 \cong \Sigma_0$ or Σ_1 and $(D_1 \cdot \ell) = 1$.

(ii) $D = D_1$ where $D_1 \cong$ a Del Pezzo surface except for $\boldsymbol{P}^2$ and $(D_1 \cdot \ell) = 2$.

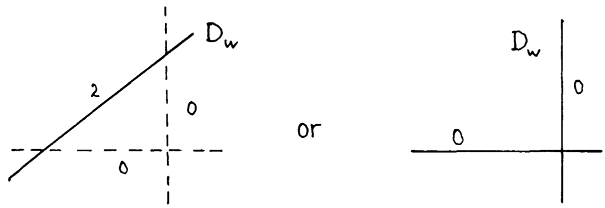


Fig. 8

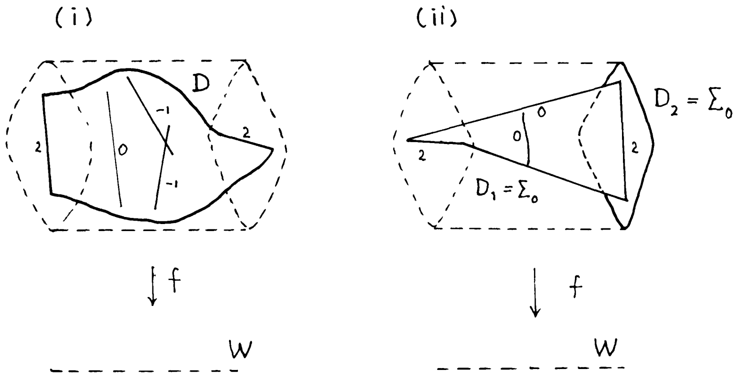


Fig. 9

(Note that $D_1 \cap f^{-1}(w)$ is a conic in $f^{-1}(w) \cong \boldsymbol{P}^2$. When this conic degenerates into two lines, they will become two exceptional curves on D_1.)

(iii) $D = D_1 + D_2$ where $(D_1 \cdot \ell) = (D_2 \cdot \ell) = 1$.

(iv) $D = D_1 + D_2$ where $D_1 \cong \Sigma_0$, $D_2 \cong \boldsymbol{P}^2$ and $(D_1 \cdot \ell) = 1$.

(v) $D = D_1 + D_2$ where $D_1 \cong \Sigma_0$, $D_2 \cong \boldsymbol{P}^2$ and $(D_1 \cdot \ell) = 2$.

(vi) $D = D_1 + D_2 + D_3$ where $(D_1 \cdot \ell) = (D_2 \cdot \ell) = 1$ (see Fig. 10).

5.4. *Type F*. In this case, V is a Fano threefold with $B_2 = 1$. We shall classify D together with V in the next section.

§6. Classification of logarithmic Fano threefolds when V are Fano threefolds with $B_2 = 1$

Let V be a Fano threefold with $B_2 = 1$. Then there exists an ample divisor H which generates Pic(V). Hence $-K_V \sim rH$ for some integer r and r is called the index of V.

Likewise D can be written as sH for $s > 0$. Since $-K_V - D$ is ample, $r > s$. In particular, we have $r \geqslant 2$.

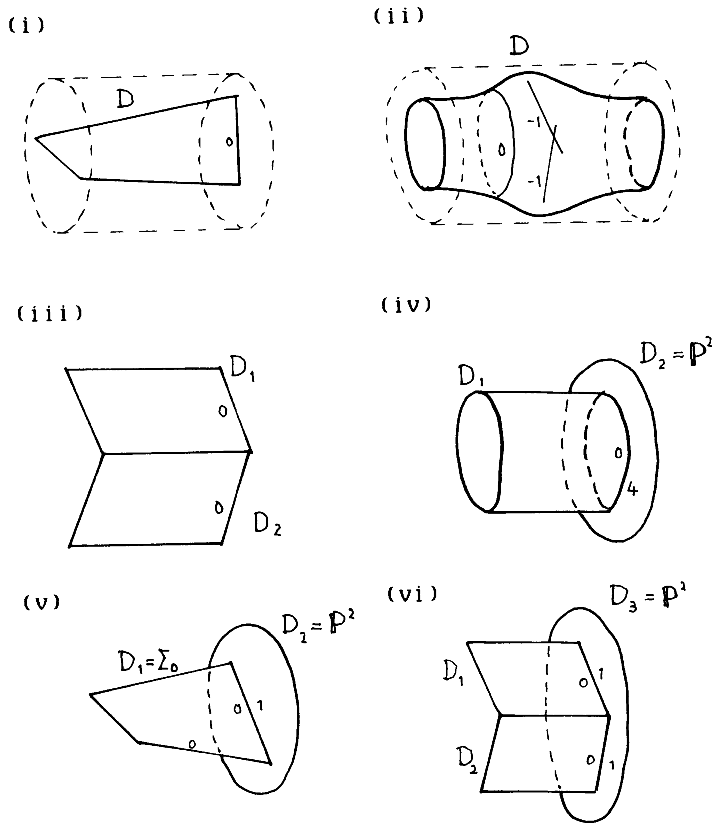


Fig. 10

Using the classification theory of Fano threefolds with index $\geqslant 2$ and $B_2 = 1$ by Iskovskih, we can classify (V, D) in the following way:

6.1. $r = 4$, i.e. $V \cong \boldsymbol{P}^3$. Since $-K_V \sim 4H$ where H is a hyperplane, D is linearly equivalent to H, $2H$ or $3H$.

Hence D is one of the following (see Fig. 11):

(i) $D = D_1$ where D_1 is a smooth cubic surface.

(ii) $D = D_1 + D_2$ where D_1 is a smooth quadric surface and D_2 is a plane.

(iii) $D = D_1 + D_2 + D_3$ where each D_i is a plane.

(iv) $D = D_1$ where D_1 is a smooth quadric surface.

(v) $D = D_1 + D_2$ where each D_i is a plane.

(vi) $D = D_1$ where D_1 is a plane.

6.2. $r = 3$, i.e. $V \cong Q_2 \subset \boldsymbol{P}^4$ that is a smooth quadric hypersurface in $\boldsymbol{P}^4$. In this case $D \sim H$ or $2H$ where H is the restriction of a hyperplane of

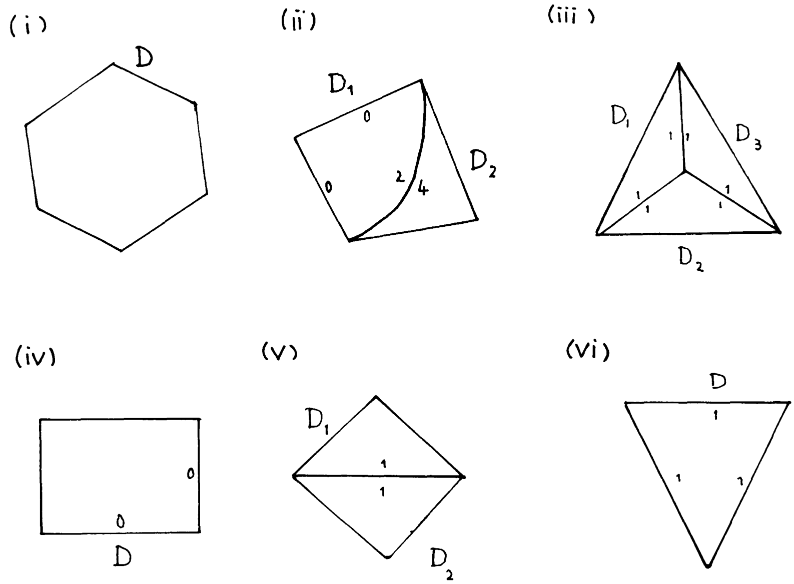


Fig. 11

$\boldsymbol{P}^4$ to Q_2. Note that each member of $|H|$ is irrducible, since H is a generator of $\mathrm{Pic}(Q_2)$.

Hence, D is one of the following (see Fig. 12):

(i) $D = D_1$ where D_1 is a smooth quartic surface in $\boldsymbol{P}^4$.

(ii) $D = D_1 + D_2$ where each D_i is a smooth quadric surface.

(iii) $D = D_1$ where D_1 is a smooth quadric surface.

6.3. $r = 2$. In this case, there are 5 different types of Fano threefolds, namely V_1, V_2, V_3, V_4 and V_5, and $|H|$ has a smooth member ([8, Proposition 5.1.], see also [18, p. 57]).

Conversely, for $i = 1, 2, 3, 4$ or 5, let $V = V_i$ and let D be a smooth member of $|H|$. Then (V, D) is a logarithmic Fano threefold.

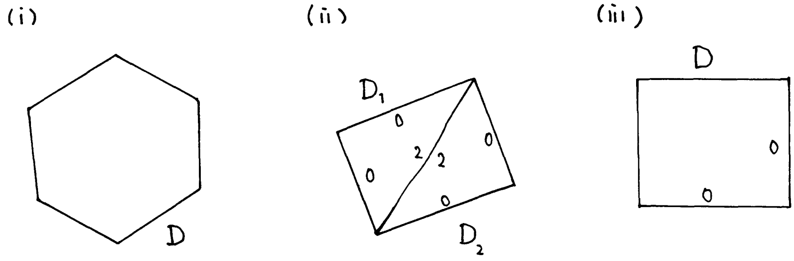


Fig. 12

§7. Classification of logarithmic Fano threefolds having extremal rational curves of type E_2

We have already detrmined the possibilities of D in 5.1. V will be classified according to the type of D.

7.1. *Case where D is of type (i) in* 5.1. We can write

$$(-K_V - D)^2 \approxeq a_1\ell_1 + a_2\ell_2 + \cdots + a_r\ell_r$$

where $\ell_1 = \ell$ and $a_i \geqslant 0$ for any i. Since we have

$$0 < D_1 \cdot (-K - D)^2 = -a_1 + a_2(D.\ell_2) + \cdots + a_r(D_1 \cdot \ell_r),$$

we may assume that $(D_1 \cdot \ell_2) > 0$.

First we consider the case where $(D_2 \cdot \ell_2) \geqslant 0$. In this case ℓ_2 also satisfies the conditions of Lemma 4.1 and we can apply the results of sections 5 and 6 to determine the type of ℓ_2. Assume ℓ_2 of type E_2. Then the exceptional divisor E associated to the contraction of ℓ_2 satisfies $E \cong \boldsymbol{P}^2$ and $E|_E \cong \mathcal{O}_E(-1)$. E cannot be a component of D, since $D_2 \cong \Sigma_2$ and D_1 is the exceptional divisor associated to ℓ_1. On the other hand, in the case of $E \not\subset D$, observing the configurations of D in Fig. 6, we derive a contradiction. If ℓ_2 is of type D_2 or D_3, then we have a morphism f: $V \to W$ onto a smooth curve. Since $(D_1 \cdot \ell_2) > 0$, it follows that $f(D_1) = W$. But this contradicts $D_1 \cong \boldsymbol{P}^2$. Since $\rho(V) \geq 2$, ℓ_2 cannot be of type F. Hence the remaining case is the case of type C_2. Then ℓ_2 induces on V a $\boldsymbol{P}^1$-bundle structure f: $V \to W$. Since $D_1 \cong \boldsymbol{P}^2$, we have $W \cong \boldsymbol{P}^2$ by 5.2.

Let $\mathscr{E} = f_*\mathcal{O}_V(D_1)$. Then $V \cong \boldsymbol{P} = \boldsymbol{P}(\mathscr{E})$ and $D_1 \sim \mathcal{O}_{\boldsymbol{P}}(1)$ (cf. section 8). From the exact sequence

$$0 \to \mathcal{O}_V \to \mathcal{O}_V(D_1) \to \mathcal{O}_{D_1}(D_1) \to 0$$

and the fact $D_1|_{D_1} \sim -\ell_1$, we have

$$0 \to \mathcal{O}_{\boldsymbol{P}^2} \to \mathscr{E} \to \mathcal{O}_{\boldsymbol{P}^2}(-1) \to 0 \quad \text{(exact)} \tag{$*$}$$

on $\boldsymbol{P}^2$. Since $(*)$ splits on $\boldsymbol{P}^2$, we obtain $\mathscr{E} \cong \mathcal{O}_{\boldsymbol{P}^2} \oplus \mathcal{O}_{\boldsymbol{P}^2}(-1)$, and $D_1 \sim \mathcal{O}_{\boldsymbol{P}}(1)$. Conversely, for $V = \boldsymbol{P}(\mathcal{O}_{\boldsymbol{P}^2} \oplus \mathcal{O}_{\boldsymbol{P}^2}(-1))$, we have a smooth divisors D_1 in $|\mathcal{O}_{\boldsymbol{P}}(1)|$ by $(*)$ and $D_2 \in |2F|$ crossing normally. By Lemma 1.6, we easily see that $-K_V - D_1 - D_2 \sim D_1 + 2F$ is ample. Thus (V, D) is a logarithmic Fano threefold.

Next we consider the case where $(D_2 \cdot \ell_2) < 0$. Any curve on D_2 is algebraically equivalent to $\alpha\Gamma + \beta L$, where L is a fiber and Γ is a

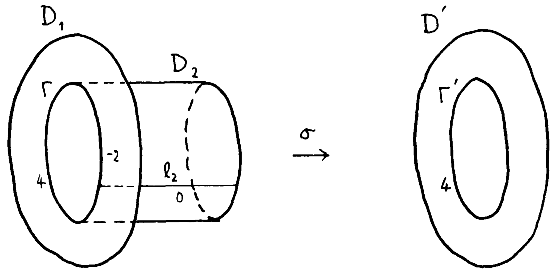


Fig. 13

minimal section of D_2. Since $\Gamma \approx 2\ell_2$, ℓ_2 is a fiber of D_2 and $(D_1 \cdot \ell_2) = 1$. Let r be $-(D_2 \cdot \ell_2)$, that is a positive integer. We have

$$0 < D_1 \cdot (-K_V - D)^2 = -a_1 + a_2 + a_3(D_1 \cdot \ell_3) + \cdots + a_r(D_1 \cdot \ell_r)$$

and

$$0 < D_2 \cdot (-K_V - D)^2 = 2a_1 - ra_2 + a_3(D_2 \cdot \ell_3) + \cdots + a_r(D_2 \cdot \ell_r).$$

Any effective 1-cycle on D is algebraically equivalent to $\alpha\ell_1 + \beta\ell_2$ for some non-negative α, β. If $(\ell_j \cdot D_i) < 0$ for some $j \geqslant 3$ and $i = 1, 2$, then $\ell_j \subset D_1$ or $\ell_j \subset D_2$. In both cases $\ell_j \approx \alpha\ell_1 + \beta\ell_2$ for some $\alpha, \beta \geqslant 0$. Since ℓ_j is extremal, $\ell_j \in \boldsymbol{R}_+[\ell_1]$ or $\boldsymbol{R}_+[\ell_2]$. But this is a contradiction, hence $(D_i \cdot \ell_j) \geqslant 0$. If $(D_i \cdot \ell_j) > 0$ for some i and j, then ℓ_j satisfies the condition of Lemma 4.1. Applying the classification of D in sections 5 and 6, we can conclude that $\ell_j = \ell_1$ or $\ell_j = \ell_2$. This is a contradiction. Hence $(D_i \cdot \ell_j) = 0$ for any $j \geqslant 3$ and $i = 1$ or 2.

Thus we have $-a_1 + a_2 > 0$ and $2a_1 - ra_2 > 0$. Hence $r = 1$ and therefore ℓ_2 is an extremal rational curve of type E_1 ([14, p. 81]).

Let σ: $V \to V'$ be the contraction of ℓ_2, $D' = \sigma(D_1)$ and $\Gamma' = \sigma(\Gamma)$ where $\Gamma = D_1 \cap D_2$ (see Fig. 13).

CLAIM: *Let (V', D') be as above. Then $-K_{V'} - D'$ is an ample divisor*

PROOF. Since σ is a surjective morphism, we have

$$NE(V') = \boldsymbol{R}_+[\ell_1'] + \boldsymbol{R}_+[\ell_3'] + \cdots + \boldsymbol{R}_+[\ell_r']$$

where $\ell_i' = \sigma_*(\ell_i)$ for any $i \neq 2$.

By Kleiman [11], it suffices to show $(-K' - D' \cdot C') > 0$ for any irreducible curve C' on V', where K' denotes $K_{V'}$.

By adjunction formula,

$$\sigma^*(-K'-D') \sim -K_V - D + D_2.$$

Let C' be an irreducible curve on V'. If $C' \neq \Gamma'$, then there exists an irreducible curve C on V such that $C \not\subset D_2$ and $\sigma_* C = C'$. In this case,

$$\begin{aligned}(-K'-D' \cdot C') &= (-K_V - D + D_2 \cdot C)\\ &= (-K_V - D \cdot C) + (D_2 \cdot C)\\ &> 0.\end{aligned}$$

If $C' = \Gamma'$, then $(-K'-D' \cdot \Gamma') = (-K_{D'} \cdot \Gamma')_{D'} > 0$, because $-K_{D'}$ is an ample divisor on $D' \cong \boldsymbol{P}^2$.

Hence $-K'-D'$ is an ample divisor. Q.E.D.

Thus (V', D') is also a logarithmic Fano threefold whose boundary D' is isomorphic to $\boldsymbol{P}^2$. After observing Fig. 7 and Fig. 11, we see that V' is either $\boldsymbol{P}^3$ or a $\boldsymbol{P}^1$-bundle over $\boldsymbol{P}^2$.

First we consider the case when $V' \cong \boldsymbol{P}^3$. Then V is obtained from $\boldsymbol{P}^3$ by blowing up a smooth conic curve Γ' on a plane D' and $D_2 = \sigma^{-1}(\Gamma')$ and D_1 is the proper transform of D'.

Conversely, for such a V and such D_1, D_2 with $D = D_1 + D_2$, we have, by adjunction formula, $-K_V - D \sim 3D_1 + D_2$.

Since $-K_V - D$ is effective and

$$(3D_1 + D_2 \cdot a_1\ell_1 + a_2\ell_2) = a_1 + a_2,$$

$-K_V - D$ is an ample divisor by Lemma 1.6.

Next we consider the case in which V' is a $\boldsymbol{P}^1$-bundle over $\boldsymbol{P}^2$. Since $(D' \cdot \ell_1') = (D_1 + D_2 \cdot \ell_1) = 1$, we have $V' \cong \boldsymbol{P}(\mathcal{O}_{\boldsymbol{P}^2} \oplus \mathcal{O}_{\boldsymbol{P}^2}(1))$ and $D' \sim \mathcal{O}_{\boldsymbol{P}}(1)$. V is obtained from V' by blowing up a smooth conic Γ' on D'. Let $D'' = f^{-1}(f(\Gamma'))$, where f is a $\boldsymbol{P}^1$-fibering of V. Then we have

$$-K_V - D \sim D_1 + D_2 + D_3$$

where D_3 is the proper transform of D''. But in this case, we see $(D_1 + D_2 + D_3 \cdot \ell_3) = 0$ where ℓ_3 is a fiber of D_3. Hence $-K_V = D$ is not an ample divisor in this case, this case cannot occur.

7.2. *Case where D is of type* (*ii*) *in* 5.1. As in 7.1, we have

$$(-K_V - D)^2 \approx a_1\ell_1 + a_2\ell_2 + \cdots + a_r\ell_r.$$

and

$$0 < D_1 \cdot (-K_V - D)^2 = -a_1 + a_2(D_1 \cdot \ell_2) + \cdots + a_r(D_1 \cdot \ell_r).$$

Hence, we may assume that $(D \cdot \ell_2) > 0$.

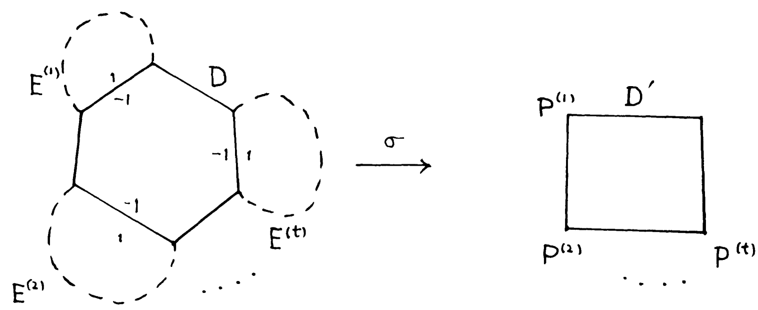


Fig. 14

If $(D_2 \cdot \ell_2) < 0$, then ℓ_2 is algebraically equivalent to a fiber of D_2. In this case ℓ_2 is also algebraically equivalent to a fiber of D_3, and therefore $(D_2 \cdot \ell_2) = 0$. But this is a contradiction, hence $(D_2 \cdot \ell_2) \geqslant 0$. In a similar way $(D_3 \cdot \ell_2) \geqslant 0$.

Thus $(-K \cdot \ell_2) > (D \cdot \ell_2) > 0$ and it follows that $(-K \cdot \ell_2) \geqslant 2$. The same argument as in 7.1 shows that ℓ_2 induces on V a $\boldsymbol{P}^1$-bundle structure over $\boldsymbol{P}^2$ with D_1 as a section. Thus $V \cong \boldsymbol{P}(\mathcal{O}_{\boldsymbol{P}^2} \oplus \mathcal{O}_{\boldsymbol{P}^2}(-1))$, $D_1 \sim \mathcal{O}_{\boldsymbol{P}}(1)$, $D_2 \sim F$ and $D_3 \sim F$. Conversely, consider such V and D_1, D_2, D_3 with $D_2 \neq D_3$. Then letting $D = D_1 + D_2 + D_3$, (V, D) turns out to be a logarithmic Fano threefold, since $-K - D \sim D_1 + 2F$ is ample by Lemma 1.6.

7.3. *Case where D is of type* (*iii*) *in* 5.1. In this case the double curve $E \cap D$ is an exceptional curve of the first kind on D and a line on $E \cong \boldsymbol{P}^2$.

Let $E^{(1)}, \cdots, E^{(t)}$ be all exceptional divisors of the type E_2 such that each intersection of $E^{(i)}$ with D is a line on $E^{(i)}$ and lies on D as an exceptional curve of the first kind. Then $E^{(1)}, \cdots, E^{(t)}$ are all disjoint and can be contracted to smooth points. Let $\sigma\colon V \to V'$ be the contraction of the $E^{(i)}$ and $D' := \sigma(D)$ (see Fig. 14).

Since $\sigma|_D\colon D \to D'$ is a contraction of exceptional curves of the first kind, D' is also a smooth Del Pezzo surface.

CLAIM. (V', D') *is a logarithmic Fano threefold.*
PROOF: Let C' be an irreducible curve on V'. Then there exists a curve C on V such that $\sigma_* C = C'$ and $C \not\subset E^{(i)}$ for any $i = 1, \cdots, t$. In particular $NE(V') = \sigma_*(NE(V))$ is also a polyhedral cone. By the ramification formula in case of point blowing up, we have

$$\sigma'(-K_V' - D') \sim -K_V - D + E^{(1)} + \cdots + E^{(t)}.$$

Then

$$(-K_{V'} - D' \cdot C')_{V'} = \big(-K_V - D + E^{(1)} + \cdots + E^{(t)} \cdot C\big)_V$$

$$> (E^{(1)} + \cdots + E^{(t)} \cdot C) \geqslant 0.$$

Hence, $-K_{V'} - D'$ is an ample divisor by Kleiman's criterion. Q.E.D.

Thus (V', D') is a logarithmic Fano threefold such that $D'(=D'_1)$ is a Del Pezzo surface. And there exists no extremal rational curve of type E_2 on V'. By Lemma 4.1, we can find an extremal rational curve ℓ' on V' such that $(-K_{V'} \cdot \ell') > (D' \cdot \ell') > 0$. By assumption, ℓ' is not of type E_2. If ℓ' is of type C_2, then ℓ' is a fiber of a $\boldsymbol{P}^1$-bundle and satsisfies the relation: $(-K_{V'} - D' \cdot \ell') = 1$. Let C be the strict transform of C' which is a fiber of the fibering and passes through $P := \sigma(E)$. Then

$$(-K_V - D \cdot C) = \big(\sigma^*(-K_{V'} - D') - E \cdot \sigma^* C' - \ell_1\big)$$

$$= 1 - 1 = 0.$$

This contradicts the ampleness of $-K_V - D$.

Using the same argument, we can show that ℓ' is neither of type D_2 nor of type D_3 with $(D' \cdot \ell') = 2$.

Suppose that V' is a Fano threefold with index 2. Since $-K_{V'} - D'(\sim D')$ is ample, we have

$$-K_V - D \sim \sigma^*(D') - E^{(1)} - \cdots - E^{(t)} \sim D$$

If $-K_V - D$ is ample, then $-K \sim 2D$ is alsoample. Thus V may be a Fano threefold with index 2 and $B_2(V) \geqslant 2$.

By the classification of Fano threefolds, V is

$$\boldsymbol{P}^1 \times \boldsymbol{P}^1 \times \boldsymbol{P}^1,\ \boldsymbol{P}(\Theta_{\boldsymbol{P}^2}) \quad \text{or} \quad \boldsymbol{P}\big(\mathcal{O}_{\boldsymbol{P}^2} \oplus \mathcal{O}_{\boldsymbol{P}^2}(-1)\big).$$

But these threefolds cannot be obtained by blowing up another Fano threefold with index 2; hence this is not the case.

In the remaining cases, V' is $\boldsymbol{P}^3$, Q_2 or a $\boldsymbol{P}^2$-bundle over $\boldsymbol{P}^1$ with $(D' \cdot \ell') = 1$. We can see that (V, D) is obtained from one of the following (V', D'), by blowing up points on D' such that the proper transform D of D' is a Del Pezzo surface.

(i) $V' \cong \boldsymbol{P}^3$ and $D' \cong$ a plane or a smooth quadric surface.

(ii) $V' \cong Q_2$ and $D' \cong$ a smooth quadric surface.

(iii) $V' \cong$ a $\boldsymbol{P}^2$-bundle over $\boldsymbol{P}^1$ and $D' \cong$ a Del Pezzo surface with $(D' \cdot \ell') = 1$ where ℓ' is a line on a fiber $\cong \boldsymbol{P}^2$ (see 9.1).

Conversely, V and D obtained in the above way satisfies the conditions of logarithmic Fano threefold by Lemma 1.6.

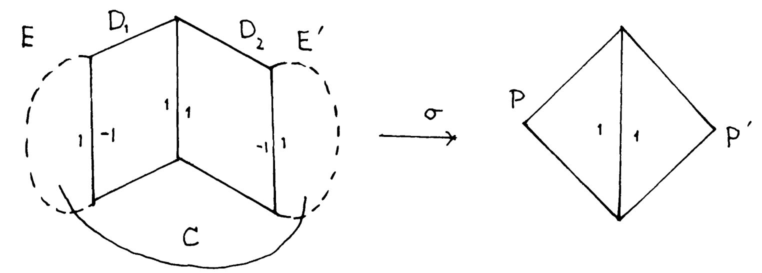


Fig. 15

7.4. *Case where D is of type* (*iv*) *in* 5.1. As in 7.3, E can be blown down to a smooth point P. Let σ: $V \to V'$ be the contraction of E, $D_1' := \sigma(D_1)$ and $D_2' := \sigma(D_2)$. Then (V', D'), where $D' = D_1' + D_2'$, is a logarithmic Fano threefold. By Lemma 4.1, there is an extremal rational curve ℓ' with

$$(-K_{V'} \cdot \ell') > (D' \cdot \ell') > 0.$$

The same argument as in 7.3 shows that ℓ' is not of type C_2. Since $D_1' \cong \boldsymbol{P}^2$, ℓ' cannot be of type D_2.

If V' is a Fano threefold with $B_2 = 1$, then V' is $\boldsymbol{P}^3$ from the configuration of D'. In this case $V \simeq \boldsymbol{P}(\mathcal{O}_{\boldsymbol{P}^2} \oplus \mathcal{O}_{\boldsymbol{P}^2}(-1))$ and D_2' is isomorphic to $\boldsymbol{P}^2$, because D_2 is disjoint from E and is a section of $\boldsymbol{P}^1$-bundle structure on V. D_1 is the proper transform of a plane passing through P in $\boldsymbol{P}^3$. Conversely, V with $D = D_1 + D_2$ obtained in the above manner is a logarithmic Fano threefold.

If V' is a $\boldsymbol{P}^2$-bundle over $\boldsymbol{P}^1$, then D_1' is a fiber. From the configuration of D' and the result in 9.1.4 below, we see $V' \cong \Sigma_{0,\alpha}$, $D_1' \sim F$ and $D_2' \sim H - \alpha F$. In this case we obtain $-K_V - D \sim D_1 + D_2 + \sigma^*(\alpha D_1')$, which is ample if $\alpha > 0$. Hence such a (V, D) is a logarithmic Fano threefold.

If ℓ' is of type E_2. Then the exceptional divisor E' associated with ℓ' intersects D_2' at the exceptional curve of the first kind. Note that E' is disjoint from E. Hence, $D_2' \cong \Sigma_1$ and E' can be blown down to a smooth point. Let σ': $V' \to V''$ be the contraction of E'. Then $D'' = \sigma'(D')$ is a sum of two copies of $\boldsymbol{P}^2$. Hence, $V'' \cong \boldsymbol{P}^3$ and D'' is a sum of two copies of planes (see Fig. 15). Let C be the strict transform of the line which passes through both P and $P' = \sigma'(E')$. Then we can see that $(-K_V - D \cdot C) = 0$. Hence, this case doesn't occur.

§8. Classification of logarithmic Fano threefolds having extremal rational curves of type C_2

Let (V, D) be a logarithmic Fano threefold having an extremal rational cure ℓ of type C_2. As we have seen in 5.2, ℓ induces on V a $\boldsymbol{P}^1$-bundle structure $f\colon V \to W$ with a birational section $D_1 \subset D$. Let $\mathscr{E}$ be $f_*\mathcal{O}_V(D_1)$, $c_1 := c_1(\mathscr{E})$ and $c_2 := c_2(\mathscr{E})$. Then $\mathscr{E}$ is a vector bundle of rank 2 on W associated with f, i.e. $V \cong \boldsymbol{P} = \boldsymbol{P}(\mathscr{E})$ and $\mathcal{O}(D_1) \sim \mathcal{O}_{\boldsymbol{P}}(1)$.

The restriction of f to D_1, denoted $g\colon D_1 \to W$, is a birational morphism. If D_1 has no exceptional curves of the first kind, then g is an isomorphism and D_1 becomes a section, i.e. D_1 defines a section of $\mathscr{E}$ without zeros.

Let $\gamma_1 \cdots \gamma_s$ be all fibers f lying on D_1. By the Hirsch formula (cf. [5, p. 429]), we have in $A(V)$

$$D_1|_{D_1} = g^*c_1 - \gamma_1 - \gamma_2 - \cdots - \gamma_s$$

with $s = c_2$. This also holds in $A(D_1)$. Since D_1 is a rational surface, we have on D_1

$$D_1|_{D_1} \sim g^*c_1 - \gamma_1 - \gamma_2 - \cdots - \gamma_s$$

with $s = c_2$.

Now we classify (V, D) according to the type of D in 5.2.

8.1. *Case in which D is of type* (i) *in* 5.2. Then $D = D_1$, that is a Del Pezzo surface. The birational morphsm $g\colon D_1 \to W$ is a succession of blowing ups with center at $P_1, \cdots, P_{c_2}$ where $P_i = f(\gamma_i)$. Note that the P_i are in general position, since the P_i are isolated simple zeros of a section of $\mathscr{E}$ defined by D_1.

If $c_2 = 0$, then g is an isomorphism and D_1 induces a subbundle of $\mathscr{E}$.

If $c_2 > 0$, then let $\tau\colon W' \to W$ be a blowing up with center at P_i's, $\sigma\colon V' = V \times_W W' \to V$ and D' be the proper transform of D by σ. Then W' is isomorphic to D and $f' := f \times_W W'\colon V' \to W'$ is a $\boldsymbol{P}^1$-bundle over W', where D' is a section (see Fig. 16).
In this case, σ is a succession of blowing ups with center at $\gamma_i = f^{-1}(P_i)$ for $i = 1, \cdots, c_2$. Let $E_i = \sigma^{-1}(\gamma_i)$, which is isomorphic to $\boldsymbol{P}^1 \times \boldsymbol{P}^1$. Then

$$\begin{aligned} D'|_{D'} &\sim \left(\sigma^*(D) - E_1 - \cdots - E_{c_2}\right)|_{D'} \\ &\sim g'^*c_1 - 2\gamma_1 - \cdots - 2\gamma_{c_2}, \qquad (**) \end{aligned}$$

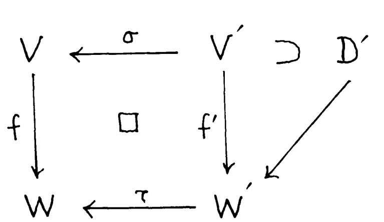


Fig. 16

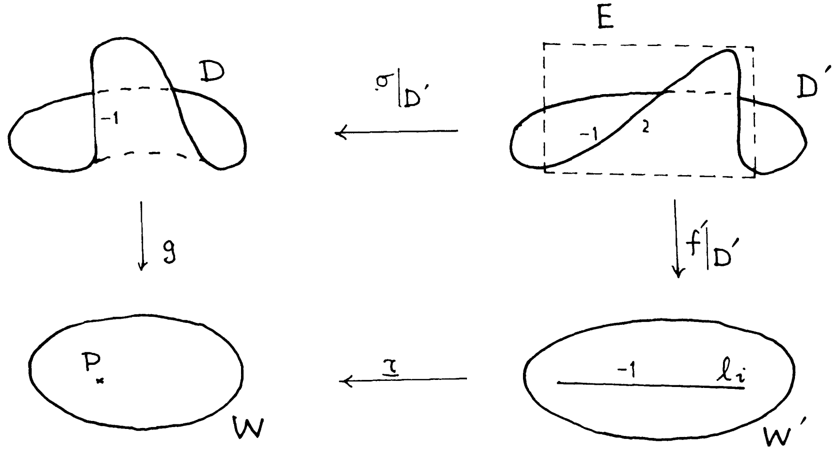


Fig. 17

where $g' = \tau \cdot f'|_{D'}$: $D' \to W$ is a birational morphism which coincides with the birational morphism g: $D \to W$. We may identify D' with W' by $f'|_{D'}$, g' with τ and D' with D by $\sigma|_{D'}$, respectively. Let $\mathscr{E}' = f'_*\mathcal{O}_{V'}(D')$. Then by (* *), we have the following exact sequence on $W' = D'$:

$$0 \to \mathcal{O}_{W'} \to \mathscr{E}' \to \mathcal{O}_{W'}(\tau^* c_1 - 2\gamma_1 - \cdots - 2\gamma_{c_2}) \to 0.$$

$\Gamma_i = D'|_{E_i}$ is a smooth section with $(\Gamma_i)^2 = 2$ on $E_i \cong \boldsymbol{P}^1 \times \boldsymbol{P}^1$ for any $i = 1, \cdots, c_2$, since $(\Gamma_i)^2_{E_i} = D'^2 \cdot E_i = (\sigma^* D \cdot \Gamma_i) - (E \cdot \Gamma_i) = 1 - (-1) = 2$ (see Fig. 17).
The restriction to $\gamma_i \subset W'$ of the above sequence induces

$$0 \to \mathcal{O}_{\gamma_i} \to \mathscr{E}'|_{\gamma_i} \to \mathcal{O}_{\gamma_i}(2) \to 0.$$

This doesn't split, since otherwise $(\Gamma_i^2)_{E_i} = 0$. Thus $\mathscr{E}'$ gives rise to a non-zero element of

$$\mathrm{Ext}^1_{W'}\big(\mathcal{O}_{W'}(\tau'^* c_1 - 2\gamma_1 - \cdots - 2\gamma_{c_2}), \mathcal{O}_{W'}\big).$$

Moreover, $\mathscr{E}'|_{\gamma_i} = \mathcal{O}_{\boldsymbol{P}^1}(1) \oplus \mathcal{O}_{\boldsymbol{P}^1}(1)$ in $\mathrm{Ext}^1(\mathcal{O}_{\gamma_i}(2), \mathcal{O}_{\gamma_i}) \cong k$.
We shall examine according to the type of W.

8.1.1. *Case where* $W \cong \boldsymbol{P}^2$. Let L be a line on W passing through only one of P_i's, say P_1. Then $V_L = f^{-1}(L)$ is a geometrically ruled surface. By the ampleness of $-K_V - D$,

$$(-K_V - D)|_{V_L} \sim -K_{V_L} - L_1$$

is also ample, where L_1 is the proper transform of L on D by g: $D \to W$. Hence (V_L, L_1) is a logarithmic Del Pezzo surface. By section 3, $(L_1)^2_{V_L} \leqslant 2$.

Since $g^*L \sim L_1 + \gamma_1$ and deg $c_1 = (c_1 \cdot L) = (D \cdot g^*L)$, we have

$$\deg c_1 = (D \cdot L_1 + \gamma_1)_V = (L_1 + \gamma_1)^2_{V_L} = (L_1)^2_{V_L} + 2 \leqslant 4.$$

Let L' be a line on W' passing through two of P_i's, say P_1 and P_2, and let $V_{L'} = f^{-1}(L')$. Then $g^*L \sim L'_1 + \gamma_1 + \gamma_2$ where L'_1 is the proper transform of L' on D by g.

Since $L'_1 + \gamma_1 + \gamma_2 \sim L_1 + \gamma_1$, we have

$$\deg c_1 = (D \cdot L'_1 + \gamma_1 + \gamma_2) = (L'_1 + \gamma_1 + \gamma_2)^2_{V_{L'}} = (L'_1)^2_{V_{L'}}.$$

Recalling deg $c_1 \leqslant 4$, we have $(L'_1)^2_{V_{L'}} \leqslant 0$ and obtain the following Table 1.

(1) *Case in which* deg $c_1 = 4$. Let f_1: $V_1 \to W_1$ be a $\boldsymbol{P}^1$-bundle over $W_1 \cong \Sigma_1$, obtained from V by blowing up γ_1. From Table 1, $V_M = f_1^{-1}(M)$ is isomorphic to Σ_0 for all fibers M on Σ_1. Hence V_1 is isomorphic to a trivial bundle, i.e., $V_1 = \Sigma_1 \times \boldsymbol{P}^1$. This implies that $V \cong \boldsymbol{P}^2 \times \boldsymbol{P}^1$. Let $H = \boldsymbol{P}^2 \times pt$ and $F = \text{line} \times \boldsymbol{P}^1$. Since $(H \cdot F \cdot D) = 2$, we have $D \sim H + 2F$. On the other hand, $-K_V \sim 2H + 2F$. It follows that

$$(-K_D)^2_D = (H + F)^2 \cdot (H + 2F) = 5.$$

Hence, $c_2 = 9 - 5 = 4$. This implies that $\mathscr{E}$ is isomorphic to $\mathcal{O}_{\boldsymbol{P}^2}(2) \oplus \mathcal{O}_{\boldsymbol{P}^2}(2)$ and $D \sim \mathcal{O}_{\boldsymbol{P}}(1)$ on $\boldsymbol{P}(\mathscr{E})$.

Conversely, let $V = \boldsymbol{P}^1 \times \boldsymbol{P}^2$, then $\mathcal{O}_{\boldsymbol{P}}(1)$ is very ample and hence we can choose a smooth member D of $|\mathcal{O}_{\boldsymbol{P}}(1)|$. The ampleness of $-K_V - D \sim D - F \sim H + F$ is clear, and the above $(\boldsymbol{P}^2 \times \boldsymbol{P}^1, D)$ is a logarithmic Fano threefold.

(2) *Case in which* deg $c_1 = 3$. First we note that

$$-K_V \sim 2\mathcal{O}_{\boldsymbol{P}}(1) + f^*(-c_1 - K_W),$$

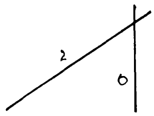
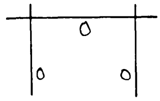
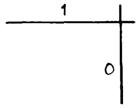
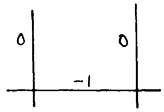
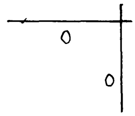
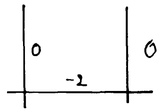
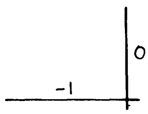
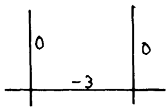
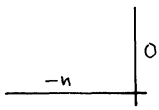
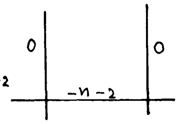
for a $\boldsymbol{P}^1$-bundle $V \cong \boldsymbol{P}(\mathscr{E})$ over W with $c_1 = c_1(\mathscr{E})$. In this case

$$-K_V - D \sim D$$

is an ample divisor. Hence, $-K_V \sim 2D$ is also ample. Thus V is a Fano threefold with index 2 and $B_2 \geqslant 2$. By [9, I, 4.2], V is isomorphic to either $V_7 = \boldsymbol{P}(\mathcal{O}_{\boldsymbol{P}^2} \oplus \mathcal{O}_{\boldsymbol{P}^2}(-1))$ or $V_6 = \boldsymbol{P}(\Theta_{\boldsymbol{P}^2})$.

If $V \cong V_7$, then $\mathscr{E} \cong \mathcal{O}_{\boldsymbol{P}^2}(1) \oplus \mathcal{O}_{\boldsymbol{P}^2}(2)$ and $D \sim \mathcal{O}_{\boldsymbol{P}}(1)$ on $\boldsymbol{P}(\mathscr{E})$. Since $(-K_D)^2_D = (D)^3 = 2$, D is a Del Pezzo surface of degree 7.

Table 1.

$\deg c_1$	$(V_L, D\vert_{V_L})$	$(V_L', D\vert_{V_L'})$
4	Σ_0 	Σ_0 
3	Σ_1 	Σ_1 
2	Σ_0 	Σ_2 
1	Σ_1 	Σ_3 
$-n$ $n \geq 0$	Σ_n 	Σ_{n+2} 

Conversely, for $V = \boldsymbol{P}(\mathcal{O}_{\boldsymbol{P}^2}(1) \oplus \mathcal{O}_{\boldsymbol{P}^2}(2)) \cong V_7$, $|\mathcal{O}_{\boldsymbol{P}}(1)|$ is a very ample linear system and therefore there is a smooth member D in $|\mathcal{O}_{\boldsymbol{P}}(1)|$ and $-K_V - D \sim D$ is ample. Hence the above (V_7, D) is a logarithmic Fano threefold.

If $V \cong V_6$, then we can choose a smooth member D in $|-1/2 \cdot K_V|$. Since $(-K_D)_D^2 = (D^3) = 6$, D is a Del Pezzo surface of degree 6. We obtain a logarithmic Fano threefold (V_6, D).

(3) *Case in which* $\deg c_1 = 2$. Since $(D \cdot \gamma) = 1$, $(D \cdot L_1) = 1$ and $(D \cdot L_1') = 0$, D is a semi-positive divisor. In particular, $0 \leqslant (D^3) = c_1^2 - c_2$; hence, $c_2 \leqslant 4$. Since $-K \sim (-K - D) + D \sim 2D + F$ is an ample divisor, V is a Fano threefold with index 1 such that V has a $\boldsymbol{P}^1$-bundle structure over $\boldsymbol{P}^2$.

It follows from a result of Dëmin [1, Theorem 1] that there are five types of such Fano threefolds.

If $\mathscr{E}$ is decomposable, then

$$V \cong \boldsymbol{P}(\mathcal{O}_{\boldsymbol{P}^2}(1) \oplus \mathcal{O}_{\boldsymbol{P}^2}(1)) \quad \text{and} \quad D \sim \mathcal{O}_{\boldsymbol{P}}(1)$$

or

$$V \cong \boldsymbol{P}(\mathcal{O}_{\boldsymbol{P}^2} \oplus \mathcal{O}_{\boldsymbol{P}^2}(2)) \quad \text{and} \quad D \sim \mathcal{O}_{\boldsymbol{P}}(1).$$

These are logarithmic Fano threefolds.

If $\mathscr{E}$ is unstable, then deg $c_1 = c_2 = 2$. V is isomorphic to the blow up of a quadric Q_2 with center at a line. A smooth member D in $|\mathcal{O}_{\boldsymbol{P}}(1)|$ is a Del Pezzo surface of degree 2, since $(-K_D)_D^2 = (D \cdot F)^2 \cdot D = 7$. Conversely, by the same method as in Lemma 8.1 below, we obtain a smooth mmber D in $|\mathcal{O}_{\boldsymbol{P}}(1)|$, where $\mathscr{E}$ is an unstable vector bundle of rank 2 on $\boldsymbol{P}^2$ with deg $c_1 = c_2 = 2$. In this manner we obtain a logarithmic Fano threefold.

If $\mathscr{E}$ is stable, then $V \cong V_{30} \subset \boldsymbol{P}^{17}$ or $V \cong V_{38} \subset \boldsymbol{P}^{21}$ (for notations, see [1]). If $V \cong V_{30}$, then deg $c_1 = 2$ and $c_2 = 4$. In this case $(-K_D)_D^2 = (D + F)^2 \cdot D = 5$.. If $V \cong V_{38}$, then deg $c = 2$ and $c_2 = 3$. In this case we see $(-K_D)_D^2 = (D+F)^2 \cdot D = 6$. Here we obtain two logarithmic Fano threefolds.

(4) *Case in which* deg $c_1 = 1$. In this case we choose a point P_0 on W such that any line L_0 through P_0 contains at most one zero of D. Since $L_0 \sim L + \gamma$, we have

$$(L_0)^2_{V_{L_0}} = 1.$$

Hence, V_{L_0} is isomorphic to Σ_1 for any L_0. Let $V_0 \to V$ be a blowing up with center at $\gamma_0 = f^{-1}(P_0)$. Then V_0 is a $\boldsymbol{P}^1$-bundle over $W_0 \cong \Sigma_1$, denoted f_0: $V_0 \to W_0$. By Table 1, $V_M = f^{-1}(M)$ is isomorphic to Σ_1 for any fiber M on W_0.

In general, the exceptional curve of the first kind on a surface is stable under deformations ([10]). Thus, there is a section H_0 composed of exceptional curves of the first kind on $V_M \cong \Sigma_1$. It is easy to see that $H_0|_{H_0} \sim -\Delta - M$, where Δ is a section with $(\Delta)^2 = -1$. Hence $V_0 \cong \boldsymbol{P}(\mathcal{O}_{\Sigma_1} \oplus \mathcal{O}_{\Sigma_1}(-\Delta - M))$. This implies that $V \cong \boldsymbol{P}(\mathcal{O}_{\boldsymbol{P}^2} \oplus \mathcal{O}_{\boldsymbol{P}^2}(-1))$. Since deg $c_1 = 1$, we see $\mathscr{E} \cong \mathcal{O}_{\boldsymbol{P}^2} \oplus \mathcal{O}_{\boldsymbol{P}^2}(1)$ and $D \sim \mathcal{O}_{\boldsymbol{P}}(1)$ on $\boldsymbol{P}(\mathscr{E})$ and we obtained a logarithmic Fano threefold. In this case $c_2 = 0$.

(5) *Case in which* deg $c_1 \leqslant 0$. The following lemma is due to T. Fujita.

LEMMA 8.1: *If* $c_1 \leqslant 0$ *and* $0 \leqslant c_2 \leqslant 8$, *then there exist a vector bundle* $\mathscr{E}$ *of rank* 2 *on* $\boldsymbol{P}^2$ *and a smooth divisor* D *on* $V = \boldsymbol{P}(\mathscr{E})$ *such that*

(i) $c_1(\mathscr{E}) = c_1$ *and* $c_2(\mathscr{E}) = c_2$,
(ii) $D \sim \mathcal{O}_{\boldsymbol{P}(\mathscr{E})}(1)$,
(iii) *D is a Del Pezzo surface with* $(K_D)_D^2 = 9 - c_2$.

PROOF: If $c_2 = 0$, then $\mathscr{E}$ can be chosen as

$$\mathscr{E} \cong \mathcal{O}_{\boldsymbol{P}^2} \oplus \mathcal{O}_{\boldsymbol{P}^2}(-n)$$

where $n = -c_1 \geqslant 0$ and D is a unique member of $|\mathcal{O}_{\boldsymbol{P}}(1)|$.

We may assume that $c_2 > 0$. Let $\tau\colon S \to \boldsymbol{P}^2$ be a succession of blowing ups with center at c_2 points, $P_1, \cdots, P_{c_2}$, on $\boldsymbol{P}^2$ such that S is a Del Pezzo surface. Let $\gamma_i = \tau^{-1}(P_i)$ be the exceptional curve on D and h be the total transform of a line on $\boldsymbol{P}^2$.

It is suficient to find an element $\mathscr{E}'$ of

$$\operatorname{Ext}^1_S\big(\mathcal{O}_S(\tau^* c_1 - 2\gamma_1 - \cdots - 2\gamma_{c_2}),\ \mathcal{O}_S\big)$$

such that the restriction $\mathscr{E}'|_{\gamma_i}$ is isomorphic to $\mathcal{O}_{\boldsymbol{P}^2}(1) \oplus \mathcal{O}_{\boldsymbol{P}^2}(1)$ for any $i = 1, \cdots, c_2$.

The extension

$$0 \to \mathcal{O}_S \to \mathscr{E}' \to \mathcal{O}_S(\tau^* c_1 - 2\gamma_1 - \cdots - 2\gamma_{c_2}) \to 0$$

corresponds to a section D' of $\mathscr{E}'$ such that

$$D'|_{D'} \sim c_1 - 2\gamma_1 - \cdots - 2\gamma_{c_2}$$

where we identify D' with S and denote $\tau^* c_1$ by c_1. Then there exists a vector bundle $\mathscr{E}$ on $\boldsymbol{P}^2$ such that

$$\tau^*\mathscr{E} \cong \mathscr{E}' \otimes \mathcal{O}_S(-\gamma_1 - \cdots - \gamma_{c_2}).$$

This means that $V' \cong \boldsymbol{P}(\mathscr{E}')$ is contractible along each divisor $E' = \boldsymbol{P}(\mathscr{E}'|_{\gamma_i}) \cong \boldsymbol{P}^1 \otimes \boldsymbol{P}^1$ and V' is transformed into $\boldsymbol{P}(\mathscr{E})$. Let $\sigma\colon V' \to V$ denote the contraction. Since $\Gamma_i = E' \cap D'$ is a section of E' with respect to the $\boldsymbol{P}^1$-fibering

$$\sigma|_{E'}\colon E' \cong \boldsymbol{P}^1 \times \boldsymbol{P}^1 \to f^{-1}(P_i) \cong \boldsymbol{P}^1,$$

we see $\sigma|_{D'}\colon D' \to D$ is an isomorphism. It is clear that V and D satisfy the desired conditions.

Note that

$$\operatorname{Ext}^1\big(\mathcal{O}_S(c_1 - 2\gamma_1 - \cdots - 2\gamma_{c_2}),\ \mathcal{O}_S\big)$$
$$\cong H^1\big(S,\ \mathcal{O}_S(-c_1 + 2\gamma + \cdots + 2\gamma_{c_2})\big).$$

We have the following two exact sequences:

$$0 \to \mathcal{O}_S(-c_1 + 2\gamma_1 + \cdots + \gamma_i + \cdots + 2\gamma_{c_2})$$
$$\to \mathcal{O}_S(-c_1 + 2\gamma_1 + \cdots + 2\gamma_{c_2})$$
$$\to \mathcal{O}_{\gamma_i}(-2) \to 0,$$

and

$$0 \to \mathcal{O}_S(-c_1 + 2\gamma_1 + \cdots + \hat{\ }_i + \cdots + 2\gamma_{c_2})$$
$$\to \mathcal{O}_S(-c_1 + 2\gamma_1 + \cdots + \gamma_i + \cdots + 2\gamma_{c_2})$$
$$\to \mathcal{O}_{\gamma_i}(-1) \to 0$$

where $\hat{\ }_i$ denotes the removal of γ_i. Since $-c_1$ is semi-positive, both

$$H^2(S, \mathcal{O}_S(-c_1 + 2\gamma_1 + \cdots + \hat{\ }_i + \cdots + 2\gamma_{c_2}))$$

and

$$H^2(S, \mathcal{O}_S(-c_1 + 2\gamma_1 + \cdots + \gamma_i + \cdots + 2\gamma_{c_2}))$$

vanish. It follows that the sequence

$$\mathrm{Ext}^1(\mathcal{O}_S(c_1 - 2\gamma_1 - \cdots - 2\gamma_{c_2}), \mathcal{O}_S)$$
$$\to \mathrm{Ext}^1(\mathcal{O}_{\gamma_i}(2), \mathcal{O}_{\gamma_i}) \to 0$$

is exact for any $i = 1, \cdots, c_2$. Hence we can find the desired $\mathscr{E}'$. Q.E.D.

In this case, i.e. in the case of deg $c_1 \leqslant 0$, we have the following

CLAIM: *Let V and D be as in Lemma* 8.1. *Then* $-K_V - D$ *is an ample divisor.*

PROOF: By the formula in p. 105 [25] we have

$$-K_V - D \sim D + (3 - \deg c_1)F.$$

Since $3 - \deg c_1 > 0$, $-K_V - D$ is effective and therefore we can apply Lemma 1.6.

Let C be an irreducible curve. If $(D \cdot C) < 0$, then C is contained in

D. Since D is a Del Pezzo surface, we have

$$(-K_V - D \cdot C) = (-K_D \cdot C)_D > 0.$$

If $(D \cdot C) \geqslant 0$, then $(D + (3 - c_1)F \cdot C) > 0$ is clear. Q.E.D.

8.1.2. *Case in which* $W \cong \boldsymbol{P}^1 \times \boldsymbol{P}^1$. Let L and M be two fibers of W. Then $c_1 \sim \alpha L + \beta M$ where α and β are integers. In this case we have

$$D|_D \sim \alpha g^*L + \beta g^*M - \gamma_1 - \cdots - \gamma_{c_2}$$

on D. We can choose L and M passing through none of $P_1, \cdots, P_{c_2}$. Let $V_L = f^{-1}(L)$ and $V_M = f^{-1}(M)$.

Since $(-K_V - D)|_{V_L} \sim -K_{V_L} - g^*L$ is ample, we have

$$\beta = (D \cdot g^*L) = (g^*L)^2_{V_L} \leqslant 2.$$

In a similar way, we have

$$\alpha = (D \cdot g^*M) = (g^*M)^2_{V_M} \leqslant 2.$$

If L passes through one of the zero points of a birational section D, then $g^*L \sim L_1 + \gamma_1$ on D with $(L_1)^2_D = -1$ (see Fig. 18).
Let $V_{L_1} = f^{-1}(f(L_1))$. Then $-K_V - D|_{VL_1} \sim -K_{V_{L_1}} - L_1 - \gamma_1$ is ample and therefore $(L_1)^2_{V_{L_1}} \leqslant 0$. In this case we have

$$(L_1 + \gamma_1)^2_{V_{L_1}} = 2 + (L_1)^2_{V_{L_1}}$$

and

$$(L_1 + \gamma_1)^2_{V_{L_1}} = (L_1 + \gamma_1 \cdot D) = (g^*L \cdot D) = \beta.$$

Hence $(L_1)^2_{V_{L_1}} = \beta - 2$. In a similar way $(M_1)^2_{V_{M_1}} = \alpha - 2$ where $g^*M \sim M_1 + \gamma_2$ with $(M_1)^2_D = -1$.

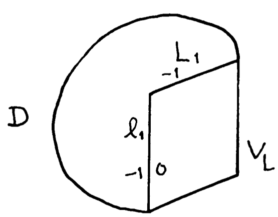


Fig. 18

Table 2.

c_1	V_L	V_{L_1}	V_M	V_{M_1}
$2L+2M$	Σ_0	Σ_0	Σ_0	Σ_0
$2L+M$	Σ_1	Σ_1	Σ_0	Σ_0
$2L$	Σ_0	Σ_2	Σ_0	Σ_0
$L+M$	Σ_1	Σ_1	Σ_1	Σ_1
L	Σ_0	Σ_2	Σ_1	Σ_1
$-nL-mM$				
$n, m \geqslant 0$	Σ_m	Σ_{m+2}	Σ_n	Σ_{n+2}

Thus we have the following Table 2.

Now we examine each case, separately.

(1) *Case where* $c_1 \sim 2L+2M$. From Table 2, it is easy to see that $V \cong \boldsymbol{P}^1 \times \boldsymbol{P}^1 \times \boldsymbol{P}^1$ and $D \sim H + V_L + V_M$ where H is a section with $H|_H \sim 0$. Hence we have

$$(-K_D)_D^2 = (H + V_L + V_M)^3 = 6$$

and $c_2 = 2$. Conversely, the above $(\boldsymbol{P}^1 \times \boldsymbol{P}^1 \times \boldsymbol{P}^1, D)$ is a logarithmic Fano threefold.

(2) *Case where* $c_1 \sim 2L+M$. In this case we have $V \cong \boldsymbol{P}(\mathcal{O}_{\Sigma_0} \oplus \mathcal{O}_{\Sigma_0}(-M))$ and $D \sim H + V_L + V_M$ where H is a section composed of the exceptional curve of the first kind on each $V_L \cong \Sigma_1$ or $V_{L_1} \cong \Sigma_1$. Since $H|_H \sim -M$, we have

$$(-K_D)_D^2 = (H + V_L + 2V_M)^2 \cdot (H + V_L + V_M) = 7.$$

Hence we have $c_2 = 1$ and we can write

$$V \cong \boldsymbol{P}\big(\mathcal{O}_{\Sigma_0}(L+M) \oplus \mathcal{O}_{\Sigma_0}(L)\big) \textit{ and } D \sim \mathcal{O}_{\boldsymbol{P}}(1).$$

Conversely, V with D obtained in this way is a logarithmic Fano threefold.

(3) *Case where* $c_1 \sim 2L$. Since both V_L and V_M are siomorphic to Σ_0, we have $V \cong \boldsymbol{P}^1 \times \boldsymbol{P}^1 \times \boldsymbol{P}^1$ and $D \sim H + V_L$ where H is a section with $H|_H \sim 0$. Since $(-K_D)_D^2 = (H + V_L + 2V_M)^2 \cdot (H + V_L) = 8$, we have $c_2 = 0$. Hence V is isomorphic to $\boldsymbol{P}(\mathcal{O}_{\Sigma_0}(L) \oplus \mathcal{O}_{\Sigma_0}(L))$ and $D \sim \mathcal{O}_{\boldsymbol{P}}(1)$.

Conversely, V with D obtained in this manner is a logarithmic Fano threefold.

(4) *Case where* $c_1 \sim L+M$. There is a section H_1 formed by the exceptional curve on $V_M \cong \Sigma_1$. Since $H_1 \cdot V_L$ is a section on $V_L \cong \Sigma_1$, we have $H_1|_{H_1} \sim -L-M$ or $-L+\alpha M$ where $\alpha \geqslant 1$.

Suppose first that $H_1|_{H_1} \sim -L+\alpha M$. It is easy to see that

$$D \sim H_1 + V_L - \alpha V_M$$

on $V \cong \boldsymbol{P}(\mathscr{E}_1)$ and therefore $\mathscr{E} \cong \mathscr{E}_1 \otimes \mathscr{O}_{\Sigma_0}(L-\alpha M)$.

Since $c_1(\mathscr{E}) = -L+\alpha M + 2(L-\alpha M) = L - \alpha M$, this case cannot occur when $\alpha \geqslant 1$.

Next we consider the case when $H_1 | H_1 \sim -L-M$. Then we have

$$\mathscr{E}_1 := f_* \mathscr{O}_V(H_1) \cong \mathscr{O}_{\Sigma_0} \oplus \mathscr{O}_{\Sigma_0}(-L-M).$$

Since $D \sim H_1 + V_L + V_M$ on $V \cong \boldsymbol{P}(\mathscr{E}_1)$, we have $\mathscr{E} \cong \mathscr{O}_{\Sigma_0} \oplus \mathscr{O}_{\Sigma_0}$.

Conversely, for $V = \boldsymbol{P}(\mathscr{O}_{\Sigma_0} \oplus \mathscr{O}_{\Sigma_0})$ with $D \in |\mathscr{O}_{\boldsymbol{P}}(1)|$, (V, D) is a logarithmic Fano threefold.

(5) *Case where* $c_1 \sim L$. As in (4), there is a section H_1 formed by the exceptional curve on$V_M \cong \Sigma_1$. In this case $H_1|_{H_1}$ is linearly equivalent to either $-L$ or $-L+2M$.

If $H_1|_{H_1} \sim -L$, then we have $f_*\mathscr{O}_V(H_1) \cong \mathscr{O}_{\Sigma_0} \oplus \mathscr{O}_{\Sigma_0}(-L)$, because $\operatorname{Ext}^1(\mathscr{O}_{\Sigma_0}(-L), \mathscr{O}_{\Sigma_0}) = H^1(\mathscr{O}_{\Sigma_0}(L)) = 0$.

Suppose that $H_1|_{H_1} \sim -L+2M$. Then

$$D \sim H_1 + V_L + V_M \quad on \quad V \cong \boldsymbol{P}(\mathscr{E}_1)$$

where $\mathscr{E}_1 = f_*\mathscr{O}_V(H_1)$. Since $(-K_D)_D^2 = 7$, we have $c_2 = 1$. Since $\mathscr{E}_1|_L \cong \mathscr{O}_{\boldsymbol{P}^1}(1) \oplus \mathscr{O}_{\boldsymbol{P}^1}(1)$ and

$$\operatorname{Ext}^1\big(\mathscr{O}_{\Sigma_0}(-L+2M), \mathscr{O}_{\Sigma_0}\big) \to \operatorname{Ext}^1\big(\mathscr{O}_L(2), \mathscr{O}_L\big) \to 0$$

is surjective, $\mathscr{E}_1$ is not decomposable. Hence $\mathscr{E} = f_*\mathscr{O}_V(D) \cong \mathscr{E}_1 \otimes \mathscr{O}_{\Sigma_0}(L - M)$ is not decomposable.

Conversely, for $V = \boldsymbol{P}(\mathscr{E})$ and $D \in |\mathscr{O}_{\boldsymbol{P}}(1)|$ as above, the ampleness of $-K_V - D \sim H_1 + 2V_L + V_M$ on $V \cong \boldsymbol{P}(\mathscr{E}_1)$ follows from Lemma 1.6; hence we obtain two logarithmic Fano threefolds.

(6) *Case where* $c_1 \sim -nL-mM$ *where* $n \geqslant 0$ *and* $m \geqslant 0$. In this case $-c_1$ is a semi-positive divisor on Σ_0. As in Lemma 8.1, we can construct a vector bundle $\mathscr{E}$ with a birational section D such that $c_1(\mathscr{E}) \sim -nL-mM$, $c_2 = 8 - (K_D)_D^2$ and D is smooth.

If $c_1 = 0$, then we see $-K_V - D \sim D + 2V_L + 2V_M$. Since $(-K_V - D)^2 \cdot D_1 = 4 - c_2 > 0$. we have $c_2 \leqslant 3$ in this case.

Conversely, for such V and D, we can verify that $-K_V - D$ is ample by Lemma 1.6. Hence $V \cong \boldsymbol{P}_{\Sigma_0}(\mathscr{E})$ with $D \sim \mathscr{O}_{\boldsymbol{P}}(1)$, where $-c_1$ is semi-positive, is a logarithmic Fano threefold.

If $c_1 \neq 0$, then $0 \leqslant c_2 \leqslant 7$.

Table 3.

c_1	V_Δ	V_M	V_{M_1}
$2\Delta+2M$	Σ_0	Σ_0	Σ_0
$2\Delta+M$	Σ_1	Σ_0	Σ_0
2Δ	Σ_2	Σ_0	Σ_0
$\Delta+M$	Σ_0	Σ_1	Σ_1
Δ	Σ_1	Σ_1	Σ_1
$-m\Delta-nM$			
$m\geqslant n\geqslant 0$	Σ_{m-n}	Σ_m	Σ_{m+2}

If $c_1=0$, then $0\leqslant c_2\leqslant 3$.

8.1.3. *Case in which* $W\cong\Sigma_1$. We can use the same methods as in 8.1.2 to obtain the following Table 3.

We obtain the following four types of logarithmic Fano threefolds where we always assume that $D\sim\mathcal{O}_{\boldsymbol{P}}(1)$.

(1) *Case where* $c_1\sim 2\Delta+2M$. $V\cong\boldsymbol{P}(\mathcal{O}_{\Sigma_1}(\Delta+M)\oplus\mathcal{O}_{\Sigma_1}(\Delta+M))$, $c_2=1$.

(2) *Case where* $c_1\sim 2\Delta+M$. $V\cong\boldsymbol{P}(\mathcal{O}_{\Sigma_1}(\Delta+M)\oplus\mathcal{O}_{\Sigma_1}(\Delta))$, $c_2=0$.

(3) *Case where* $c_1\sim\Delta+M$. $V\cong\boldsymbol{P}(\mathcal{O}_{\Sigma_1}(\Delta+M)\oplus\mathcal{O}_{\Sigma_1})$, $c_2=0$.

(4) *Case where* $c_1\sim -n\Delta-mM$, *where* $m\geqslant n\geqslant 0$. In this case $-c_1$ is semi-positive.

If $n=0$, then $V\cong\boldsymbol{P}(\mathcal{O}_{\Sigma_1}\oplus\mathcal{O}_{\Sigma_1}(-mM))$ and $c_2=0$.

If $n>0$, then $V\cong\boldsymbol{P}(\mathcal{E})$ with this c_1 and $0\leqslant c_2\leqslant 7$.

8.1.4. *Case in which* W *is a Del Pezzo surface except for* $\boldsymbol{P}^2$, Σ_0 *or* Σ_1. Note that $NE(W)$ is a polyhedral cone generated by the exceptional curves of the first kind L on W

Since f is an isomorphism around $g^{-1}(L)$,

$$(-K_V-D)|_{V_L}\sim -K_{V_L}-g^*L-\gamma$$

is an ample divisor. Hence we have

$$D_1\cdot g^*L=(g^*L)^2_{V_L}\cong 0.$$

This implies that $-c_1$ is semi-positive.

LEMMA 8.2: *Let* V *be a* $\boldsymbol{P}^1$*-bundle* $\boldsymbol{P}(\mathcal{E})$ *over a Del Pezzo surface* W *such that* $-c_1$ *is semi-positive. Suppose that there exists a smooth member* D *in* $|\mathcal{O}_{\boldsymbol{P}}(1)|$, *which is a Del Pezzo surface.*

Then $-K_V-D$ *is an ample divisor.*

PROOF: Since $-K_V - D \sim D + f^*(-K_W - c_1)$ and $-K_W - c_1$ is ample, we have $\kappa(-K_V - D, V) \geqslant 0$. It is easy to see that

$$(-K_V - D \cdot C) > 0$$

for any irreducible curve C on V (cf. Claim in 8.1.1). Hence, $-K_V - D$ is ample by Lemma 1.6. Q.E.D.

Using the same method as in 8.1.1, we can prove the existence of a vector bundle $\mathscr{E}$ over W such that $-c_1$ is semi-positive and $|\mathcal{O}_{\boldsymbol{P}}(1)|$ contains a Del Pezzo surface D.
Summarizing this, we have

$$V \cong \boldsymbol{P}_W(\mathscr{E}) \text{ and } D \sim \mathcal{O}_{\boldsymbol{P}}(1)$$

where $-c_1(\mathscr{E})$ is semi-positive and $c_2 = (-K_W)^2_W - (-K_D)^2_D$.

In particular if $c_2 = 0$, then $\mathscr{E} \cong \mathcal{O}_W \oplus \mathcal{O}_W(-\Gamma)$ with a semi-positive divisor Γ on W.

Conversely, for such V with D, we obtain a logarithmic Fano threefold.

8.2. *Case in which D is of type (ii) and (iii) in* 5.2. Since $D - D_1$ consists of fibers of $\boldsymbol{P}^1$-bundle, $D - D_1$ is a semi-positive divisor. Thus

$$-K_V - D_1 \sim -K_V - D + (D - D_1)$$

is an ample divisor.

Hence, V should be among those of V in 8.1.1. In addition, $-K_V - D_1 - (D - D_1)$ is ample.

Thus we have logarithmic Fano threefolds (V, D) as follows:

(1) $V \cong \boldsymbol{P}(\mathcal{O}_{\boldsymbol{P}^2} \oplus \mathcal{O}_{\boldsymbol{P}^2}(-a))$, $a \geqslant 0$, $D_1 \sim \mathcal{O}_{\boldsymbol{P}}(1)$, $D_2 \sim 2F$ such that $D_2 \cong \Sigma_{2a}$.

(2) $V \cong \boldsymbol{P}(\mathcal{O}_{\boldsymbol{P}^2} \oplus \mathcal{O}_{\boldsymbol{P}^2}(-a))$, $a \geqslant 0$, $D_1 \sim \mathcal{O}_{\boldsymbol{P}}(1)$, $D_2 \sim F$ and $D_3 \sim F$ such that $D_2 \cong D_3 \cong \Sigma_a$.

(3) $V \cong \boldsymbol{P}(\mathcal{O}_{\boldsymbol{P}^2} \oplus \mathcal{O}_{\boldsymbol{P}^2}(-a))$, $a \geqslant -1$, $D_1 \sim \mathcal{O}_{\boldsymbol{P}}(1)$, $D_2 \sim F$ such that $D_2 \cong \Sigma_a$, if $a \geqslant 0$ or $D_2 \cong \Sigma_1$, if $a = -1$.

8.3. *Case in which D is of type (iv) in* 5.2. In this case $g\colon D_1 \to W$ is an isomorphism. Let M be a fiber and Δ a section with $(\Delta)^2_{D_1} = -n$ on $D_1 \cong \Sigma_n$. Since $(-K_V - D)|_{V_M} \sim -K_{V_M} - M - \gamma$ is ample, we have

$$(D_1 \cdot M) = (M)^2_{V_M} \leqslant 0$$

where $V_M = f^{-1}(f(M))$. Let $k = -(M)^2_{V_M}$. Since $D_2 = V_\Delta = f^{-1}(f(\Delta))$ and $-K_V - D|_{V_\Delta} \sim -K_{V_\Delta} - \Delta$ are ample, the types of (V_Δ, Δ) must be

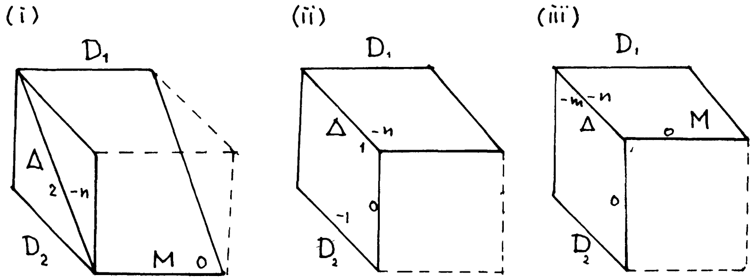


Fig. 19

one of 3 types (see Fig. 19):
(i) $V_\Delta \cong \Sigma_0$, $(\Delta)^2_{V_\Delta} = 2$,
(ii) $V_\Delta \cong \Sigma_1$, $(\Delta)^2_{V_\Delta} = 1$,
(iii) $V_\Delta \cong \Sigma_m$, $(\Delta)^2_{V_\Delta} = -m$ where $m \geqslant 0$.
We obtain the following 3 types of logarithmic Fano threefolds.

Case (i). It is easy to see that

$$D_1 \mid_{D_1} \sim -k\Delta + (2 - kn)M.$$

Hence $c_1 \sim -k\Delta + (2 - kn)M$, where we identify D_1 with W. Since $-K_V - D_1 \sim D_1 + (2+k)V_\Delta + (k + kn)V_M$, we have

$$8 = \left(-K_{D_1}\right)^2_{D_1} = (-K_V - D_1)^2 \cdot D_1 = 8 + 3kn + 2n + k^2.$$

This implies that $k = n = 0$. Hence

$$V \cong \boldsymbol{P}^1 \times \boldsymbol{P}^1 \times \boldsymbol{P}^1,\ D_1 \sim H_1 + V_M \text{ and } D_2 = V_\Delta$$

where H_1 is a section with $H_1 \mid_{H_1} \sim 0$.

Case (ii). In this case.

$$D_1 \mid_{D_1} \sim -k\Delta + (1 - kn)M.$$

As in (1) we can see that $n = k = 0$. Hence we have

$$V \cong \boldsymbol{P}\left(\mathcal{O}_{\Sigma_0} \oplus \mathcal{O}_{\Sigma_0}(M)\right).$$

Case (iii). In this case,

$$D_1 \mid_{D_1} \sim -k\Delta + (-kn - m)M,$$

where $m \geqslant 0$ and $k \geqslant 0$. By the same reason as in (ii), we have $n = k = 0$. Hence $V \cong \boldsymbol{P}^1 \times \boldsymbol{P}^1 \times \boldsymbol{P}^1$.

8.4. *Case in which D is of type* (v) *in* 5.2. In this case, we see

$$D_1 \mid_{D_1} \sim -k\Delta - (kn+m)M,$$

where $-k = (M)^2_{D_3} \leqslant 0$ and $-m = (\Delta)^2_{D_2} \leqslant 0$. Since

$$8 = \left(-K_{D_1}\right)^2_{D_1} = \left(-K_V - D_1\right)^2 \cdot D_1 = 8 + k^2 n + 3nk,$$

we have $n = k = 0$. Hence, we have a logarithmic Fano threefold $V \cong \boldsymbol{P}(\mathcal{O}_{\Sigma_0} \oplus \mathcal{O}_{\Sigma_0}(-mM))$ with $D = D_1 + D_2 + D_3$, where $D_1 \in |\mathcal{O}_{\boldsymbol{P}}(1)|$, $D_2 = f^{-1}(f(\Delta))$ and $D_3 = f^{-1}(f(M))$.

8.5. *Case in which D is of type* (vi) *in* 5.2. Since we have

$$D_1 \mid_{D_1} \sim -aL - bM,$$

where $a \geqslant 0$ and $b \geqslant 0$, we obtain a logarithmic Fano threefold $V \cong \boldsymbol{P}(\mathcal{O}_{\Sigma_0} \oplus \mathcal{O}_{\Sigma_0}(-al - bM))$ with $D = D_1 + D_2$, where $D_1 \sim \mathcal{O}_{\boldsymbol{P}}(1)$ and $D_2 \in |V_L + V_M|$.

8.6. *Case in which D is of type* (vii) *in* 5.2. We choose V among those of V in 8.1.1, where W is $\boldsymbol{P}^2$ or Σ_1 and we obtain the following two logarithmic Fano threefolds.

Case where $W \cong \Sigma_1$: $V \cong \boldsymbol{P}(\mathcal{O}_{\Sigma_1} \oplus \mathcal{O}_{\Sigma_1}(-a\Delta + (-a-b)M))$ with $D = D_1 + D_2$, where $D_1 \in |\mathcal{O}_{\boldsymbol{P}}(1)|$ and $D_2 = V_\Delta$.

Case where $W \cong \boldsymbol{P}^2$: $V \cong \boldsymbol{P}(\mathcal{O}_{\boldsymbol{P}^2}(1) \oplus \mathcal{O}_{\boldsymbol{P}^2}(1))$ with $D = D_1 + D_2$, where $D_1 \in |\mathcal{O}_{\boldsymbol{P}}(1)|$ and D_2 is a fiber over a line which doesn't contain a fiber of D_1.

§9. Classification of logarithmic Fano threefolds having extremal rational curves of type D_2 or D_3

9.1. *Case in which* ℓ *is of type* D_3. The case in which ℓ is of type D_2 will be classified using the similar arguments as in the case of type D_3. So we shall first consider the case of type D_3.

By a theorem of Grothendieck, V is written as $\Sigma_{a_1, a_2} = \boldsymbol{P}(\mathcal{O}^1_{\boldsymbol{P}} \oplus \mathcal{O}_{\boldsymbol{P}^1}(a_1) \oplus \mathcal{O}_{\boldsymbol{P}^1}(a_2))$ where $0 \leqslant a_1 \leqslant a_2$ are integers. Let H be a tautological divisor and F a fiber of Σ_{a_1, a_2}.

The following facts are easily shown.

(1) $Bs\,|H| = \emptyset$, hence we may assume H is smooth (Bertini).

(2) $\alpha H + \beta F$ is ample if and only if $\alpha > 0$ and $\beta > 0$.

(3) $-K_V \sim 3H + (2 - a_1 - a_2)F$.
(4) $(H)^3 = a_1 + a_2$ and $(H^2 \cdot F) = 1$.

9.1.1. *Case where D is of type* (i) *in* Fig. 10 *in* 5.3. In this case, we write $D = D_1 \sim H + dF$ for some d. Since $|(H|_{D_1})| \subset |H||_{D_1}$ is base point free, there exists a smooth curve Γ in $|(H|_{D_1})|$. Since

$$(\Gamma \cdot L)_{D_1} = \left(H|_{D_1} \cdot F|_{D_1}\right)_{D_1} = 1,$$

Γ turns out to be a section of D_1 as a ruled surface.

First we treat the case where $D_1 \cong \Sigma_0$. $\Gamma \sim \Delta + nL$ for some $n \geqslant 0$. Since

$$2n = (\Gamma)^2_{D_1} = \left(H|_{D_1} \cdot H|_{D_1}\right)_{D_1} = H^2 \cdot (H + dF) = a_1 + a_2 + d,$$

we have $a_1 + a_2 + d = 2n \geqslant 0$. By ampleness of $-K_V - D_1 \sim 2H + (2 - a_1 - a_2 - d)F$, we have $a_1 + a_2 + d \leqslant 1$. Hence $n = 0$ and $a_1 + a_2 + d = 0$. Since

$$H^0\left(\boldsymbol{P}^1, \mathcal{O}_{\boldsymbol{P}^1}(d) \oplus \mathcal{O}_{\boldsymbol{P}^1}(d + a_1) \oplus \mathcal{O}_{\boldsymbol{P}^1}(d - a_2)\right) = H^0(H + dF) \neq 0$$

and

$$d \leqslant d + a_1 \leqslant d + a_2,$$

we have $d + a_2 \geqslant 0$. It follows that $a_1 = d + a_2 = 0$. Hence we have a logarithmic Fano threefold $V \cong \Sigma_{0, a_2}$ with $D = D_1 \in |H - a_2 F|$.

Next we consider the case where $D_1 \cong \Sigma_1$. Since $(\Gamma)^2_{D_1} = -1 + 2n = a_1 + a_2 + d$, we have $n = 1$. There are two cases:

Case (1). $a_1 = 0$ and $a_2 + d = 1$. Taking f_* of the exact sequence:

$$0 \to \mathcal{O}_V \to \mathcal{O}_V(D_1) \to \mathcal{O}_{D_1}(D_1) \to 0,$$

we have the following exact sequence:

$$0 \to \mathcal{O}_{\boldsymbol{P}^1} \to \mathcal{E} \otimes \mathcal{O}_{\boldsymbol{P}^1}(d) \to \mathcal{F} \otimes \mathcal{O}_{\boldsymbol{P}^1}(d) \to 0$$

where $\mathcal{F} = f_* \mathcal{O}_{D_1}(H)$.
By assumption, $\mathcal{F}$ can be written as $\mathcal{O}_{\boldsymbol{P}^1}(a) \oplus \mathcal{O}_{\boldsymbol{P}^1}(a+1)$ for some integer a. Since $f_* \mathcal{O}_V(H) = \mathcal{O}_{\boldsymbol{P}^1} \oplus \mathcal{O}_{\boldsymbol{P}^1}(a_1) \oplus \mathcal{O}_{\boldsymbol{P}^1}(a_2)$, we have an exact sequence:

$$0 \to \mathcal{O}_{\boldsymbol{P}^1} \to \mathcal{O}_{\boldsymbol{P}^1}(1 - a_2) \oplus \mathcal{O}_{\boldsymbol{P}^1}(1 - a_2) \oplus \mathcal{O}_{\boldsymbol{P}^1}(1)$$

$$\to \mathcal{O}_{\boldsymbol{P}^1}(1 - a_2) \oplus \mathcal{O}_{\boldsymbol{P}^1}(2 - a_2) \to 0. \qquad (*)$$

Comparing the above sequences, we have $a=0$. Since

$$\operatorname{Ext}^1(\mathcal{O}_{\boldsymbol{P}^1}(1-a_2)\oplus\mathcal{O}_{\boldsymbol{P}^1}(2-a_2),\ \mathcal{O}_{\boldsymbol{P}^1})$$

$$\cong H^1(\boldsymbol{P}^1,\ \mathcal{O}_{\boldsymbol{P}^1}(a_2-1)\oplus\mathcal{O}_{\boldsymbol{P}^1}(a_2-2))\neq 0,$$

($*$) is not splitting and therefore $a_2=0$ and $d=1$. Hence we have a logarithmic Fano threefold $V\cong\boldsymbol{P}^2\times\boldsymbol{P}^1$ with $D=D_1\in|H+F|$.

Case (2). $a_1=1$ and $d=-a_2\leqslant -1$. By the same arguments as in case (1), we have a logarithmic Fano threefold $V\cong\Sigma_{1,a_2}$ with $D=D_1\in|H-a_2F|$.

9.1.2. *Case where D is of type (ii) in* Fig. 10 *in* 5.3. In this case, we write $D\sim 2H+dF$ for some d. Since

$$-K_V-D_1\sim H+(2-a_1-a_2-d)F$$

is ample, we have $a_1+a+d\leqslant 1$. In particular $d\leqslant 1$.

If $d=1$, then $a_1=a_2=0$; hence $V\cong\boldsymbol{P}^1\times\boldsymbol{P}^2$. Since $-K_V-D_1\sim H+F$, we have

$$(-K_{D_1})^2_{D_1}=(H+F)^2\cdot(2H+F)=5.$$

Hence, D_1 is obtained from $\boldsymbol{P}^1\times\boldsymbol{P}^1$ by blowing up 3 points.

Conversely, for such V with $D=D_1$, $(V,\ D)$ is a logarithmic Fano threefold. If $d=0$, then $(a_1,\ a_2)=(0,1)$ or $(0,0)$.

Case (i): $a_1=a_2=0$. In this case,

$$(-K_{D_1})^2_{D_1}=(H+2F)^2\cdot(2H)=8.$$

Hence, we have a logarithmic Fano threefold $V\cong\boldsymbol{P}^1\times\boldsymbol{P}^2$ with $D=D_1\cong\boldsymbol{P}^1\times\boldsymbol{P}^1$.

Case (ii): $a_1=0<a_2=1$. In this case,

$$(-K_{D_1})^2_{D_1}=(H+F)^2\cdot(2H)=6.$$

Hence, D_1 is a 2 point blowing up of $\boldsymbol{P}^1\times\boldsymbol{P}^1$ and $V\cong\Sigma_{0,1}$. For such V with $D=D_1$, $(V,\ D)$ is a logarithmic Fano threefold.

In order to study the case of $d<0$, we construct a curve C in the following may (due to T. Fujita).

Let C be a section of Σ_{a_1, a_2} defined by the following exact sequence:

$$0 \to \mathcal{O}_{\boldsymbol{P}^1}(a_1) \oplus \mathcal{O}_{\boldsymbol{P}^1}(a_2) \to \mathcal{O}_{\boldsymbol{P}^1} \oplus \mathcal{O}_{\boldsymbol{P}^1}(a_1) \oplus \mathcal{O}_{\boldsymbol{P}^1}(a_2) \to \mathcal{O}_{\boldsymbol{P}^1} \to 0.$$

Let H_1 be a divisor corresponding to the following exact sequence:

$$0 \to \mathcal{O}_{\boldsymbol{P}^1}(a_1) \to \mathcal{O}_{\boldsymbol{P}^1} \oplus \mathcal{O}_{\boldsymbol{P}^1}(a_1) \oplus \mathcal{O}_{\boldsymbol{P}^1}(a_2) \to \mathcal{O}_{\boldsymbol{P}^1} \oplus \mathcal{O}_{\boldsymbol{P}^1}(a_2) \to 0,$$

and H_2 a divisor defined by the following exact sequence:

$$0 \to \mathcal{O}_{\boldsymbol{P}^1}(a_2) \to \mathcal{O}_{\boldsymbol{P}^1} \oplus \mathcal{O}_{\boldsymbol{P}^1}(a_1) \oplus \mathcal{O}_{\boldsymbol{P}^1}(a_2) \to \mathcal{O}_{\boldsymbol{P}^1} \oplus \mathcal{O}_{\boldsymbol{P}^1}(a_1) \to 0.$$

Then we have $C = H_1 \cdot H_2$. Since $H_1 \sim H - a_1 F$ and $H_2 \sim H - a_2 F$, we have

$$(H \cdot C) = (H \cdot H_1 \cdot H_2) = H^3 - a_1 H^2 \cdot F - a_2 H^2 \cdot F = 0.$$

Hence, $H|_C \sim \mathcal{O}_{\boldsymbol{P}^1}$.

Note that the inclusion $C \subset H_1$ is determined by the following exact sequence:

$$0 \to \mathcal{O}_{\boldsymbol{P}^1}(a_1) \to \mathcal{O}_{\boldsymbol{P}^1} \oplus \mathcal{O}_{\boldsymbol{P}^1}(a_1) \to \mathcal{O}_{\boldsymbol{P}^1} \to 0.$$

The similar result holds for $C \subset H_2$ if we replace a_1 with a_2. The normal bundle of C in Σ_{a_1, a_2} is

$$N_{C/\Sigma_{a_1, a_2}} \cong \mathcal{O}_{\boldsymbol{P}^1}(-a_1) \oplus \mathcal{O}_{\boldsymbol{P}^1}(-a_2).$$

We continue to examine the case in which $d < 0$. Let C be the above curve. Then $\mathcal{O}(D)|_C \cong \mathcal{O}(H + dF)|_C \cong \mathcal{O}_{\boldsymbol{P}^1}(d)$, i.e., $(D \cdot C) = d < 0$. This implies that C is contained in D. From the standard exact sequence:

$$0 \to N_{C/D} \to N_{C/V} \to N_{D/V}|_C \to 0,$$

we have $a_1 + d \geqslant 0$. In particular, $0 \leqslant a_1 \leqslant 1$. Next we take H_2 as above. Then $D|_{H_2}$ is a non-zero effective divisor. Since

$$D|_{H_2} \sim 2H|_{H_2} + dF|_{H_2} \sim 2\Delta_\infty + 2a_1 L + dL,$$

we have $2a_1 + d \geqslant 0$. Hence $a_1 = 1$. In this case, we have either $d = -2$ and $a_2 \geqslant 2$, or $d = -1$ and $a \geqslant 1$.

If $d = -1$, then $a_2 = 1$, since $-K_V - D_1 \sim H + (2 - a_2)F$ is an ample divisor. But in this case $|2H - F|$ contains no irreducible member. This case doesn't occur.

If $d = -2$, then $-K_V - D_1 \sim H + (3 - a_2)F$ is ample. By the same reason as above, we have $a_2 = 2$ and therefore $V \cong \Sigma_{1,2}$. On the other hand, since C is contained in D, we have

$$(-K_V - D_1 \cdot C) = (H + F \cdot C) = 1$$

That is, $(-K_{D_1} \cdot C)_{D_1} = 1$ and therefore C is an exceptional curve on D_1. Since $(-K_{D_1})^2 = 8$, D_1 is Σ_1. On the other hand, $H|_{D_1}$ is a smooth curve on Σ_1 with $(H|_{D_1})^2 = 4$. But this is a contradiction.

As a consequence, the case where $d < 0$ doesn't occur.

9.1.3. *Case where D is of type (iii) in* Fig. 10 *in* 5.3. In this case, we write

$$D_1 \sim H + d_1 F \text{ where } D_1 \cong \Sigma_{k_1} \text{ for some } d_1$$

and

$$D_2 \sim H + d_2 F \text{ where } D_2 \cong \Sigma_{k_1} \text{ for some } d_2.$$

Letting $\Gamma = D_1 \cdot D_2$, we classify $D = D_1 + D_2$ into the following 6 sub-cases:

Case (i): $D_1 \cong \Sigma_{n_1}$, $D_2 \cong \Sigma_{n_2}$. $(\Gamma)^2_{D_1} = -n_1$, $(\Gamma)^2_{D_2} = -n_2$.
Case (ii): $D_1 \cong \Sigma_{n_1}$, $D_2 \cong \Sigma_1$. $(\Gamma)^2_{D_1} = -n_1$, $(\Gamma)^2_{D_2} = 1$.
Case (iii): $D_1 \cong D_2 \cong \Sigma_1$. $(\Gamma)^2_{D_1} = (\Gamma)^2_{D_2} = 1$.
Case (iv): $D_1 \cong \Sigma_1$, $D_2 \cong \Sigma_0$. $(\Gamma)^2_{D_1} = 1$, $(\Gamma)^2_{D_2} = 2$.
Case (vi): $D_1 \cong D_2 \cong \Sigma_0$. $(\Gamma)^2_{D_1} = (\Gamma)^2_{D_2} = 2$.

Note that by the ampleness of

$$-K_V - D \sim H + (2 - a_1 - a_2 - d_1 - d_2)F,$$

we have $a_1 + a_2 + d_1 + d_2 \leqslant 1$.

Let Δ_1 be a section of D_i with $(\Delta_i)^2 = -k_i$ and L_i be a fiber on D_i for $i = 1, 2$.

We shall examine the above 6 cases:

Case (i). We may assume that $k_i = n_i$ and $d_2 \geqslant d_1$. From the exact sequence on D_1:

$$0 \to \mathcal{O}_{D_1} \to \mathcal{O}_{D_1}(D_2) \to \mathcal{O}_\Gamma(-n_1) \to 0$$

where $n_1 \geqslant 0$, we have $f_* \mathcal{O}_{D_1}(D_2) \cong \mathcal{O}_{P^1} \oplus \mathcal{O}_{P^1}(-n_1)$. On the other hand, from the exact sequence on V:

$$0 \to \mathcal{O}_V(D_2 - D_1) \to \mathcal{O}_V(D_2) \to \mathcal{O}_{D_1}(D_2) \to 0,$$

we have the following exact sequence on $\boldsymbol{P}^1$:

$$0 \to \mathcal{O}_{\boldsymbol{P}^1}(d_2 - d_1) \to \mathcal{E} \otimes \mathcal{O}_{\boldsymbol{P}^1}(d_2) \to \mathcal{O}_{\boldsymbol{P}^1} \oplus \mathcal{O}_{\boldsymbol{P}^1}(-n_1) \to 0.$$

Since

$$\operatorname{Ext}^1(\mathcal{O}_{\boldsymbol{P}^1}(-d_2) \oplus \mathcal{O}_{\boldsymbol{P}^1}(-d_2 - n_1), \mathcal{O}_{\boldsymbol{P}^1}(-d_1))$$
$$= H^1(\boldsymbol{P}^1, \mathcal{O}_{\boldsymbol{P}^1}(-d_1) \oplus \mathcal{O}_{\boldsymbol{P}^1}(n_1 + d_2 - d_1)) = 0,$$

we have $\mathcal{E} \cong \mathcal{O}_{\boldsymbol{P}^1}(-d_2) \oplus \mathcal{O}_{\boldsymbol{P}^1}(-d_1) \oplus \mathcal{O}_{\boldsymbol{P}^1}(-n_1 - d_2)$. By the fact that $-n_1 - d_2 \leqslant -d_2 \leqslant -d_1$, we have $a_1 = -d_2$, $a_2 = -d_1$ and $d_2 = d_1$. The ampleness of $-K_V - D_1 - D_2$ is verified by virtue of Lemma 1.6. Hene we have a logarithmic Fano threefold $V \cong \Sigma_{a_1, a_2}$ with $D = D_1 + D_2$, where $D_1 \sim H - a_2 F$ and $D_2 \sim H - a_2 F$ and both a_1 and a are arbitrary non-negative integers with $a_1 \leqslant a_2$.

Case (*ii*). We may assume that $k_1 = n_1$ and $k_2 = 1$. Then we have $f_* \mathcal{O}_{D1}(D_2) \cong \mathcal{O}_{\boldsymbol{P}^1} \oplus \mathcal{O}_{\boldsymbol{P}^1}(-n_1)$ and $f_* \mathcal{O}_{D_2}(D_1) \cong \mathcal{O}_{\boldsymbol{P}^1} \oplus \mathcal{O}_{\boldsymbol{P}^1}(1)$. There are two exact sequence:

$$0 \to \mathcal{O}_{\boldsymbol{P}^1}(d_2 - d_1) \to \mathcal{E} \otimes \mathcal{O}_{\boldsymbol{P}^1}(d_2) \to \mathcal{O}_{\boldsymbol{P}^1} \oplus \mathcal{O}_{\boldsymbol{P}^1}(-n_1) \to 0$$

and

$$0 \to \mathcal{O}_{\boldsymbol{P}^1}(d_1 - d_2) \to \mathcal{E} \otimes \mathcal{O}_{\boldsymbol{P}^1}(d_1) \to \mathcal{O}_{\boldsymbol{P}^1} \otimes \mathcal{O}_{\boldsymbol{P}^1}(1) \to 0.$$

If $d_2 \geqslant d_1$, then the first sequence splits. Thus

$$\mathcal{E} \cong \mathcal{O}_{\boldsymbol{P}^1}(-d_2 - n_1) \oplus \mathcal{O}_{\boldsymbol{P}^1}(-d_1) \oplus \mathcal{O}_{\boldsymbol{P}^1}(-d_1).$$

Hence, $d_2 = -n_1$ and $a_1 - d_2 \leqslant a_2 = -d_1$. From the second sequence, we have $d_1 = 1$ and therefore $a_2 < 0$. But this contradicts $a_2 \geqslant 0$.

If $d_1 \geqslant d_2$, then the second sequence splits. By the same argument as above, we have

$$V \cong \boldsymbol{P}(\mathcal{O}_{\boldsymbol{P}^1}(-d_1) \oplus \mathcal{O}_{\boldsymbol{P}^1}(-d_2) \oplus \mathcal{O}_{\boldsymbol{P}^1}(1 - d_1)).$$

Hence, $d_1 = 0$. By the second sequence, we have $d_2 = -n_1 - 1$. It follows that $a_1 = 1$ and $a_2 = 1 + n_1$. Hence, we have the following logarithmic Fano threefold $V \cong \Sigma_{1, a_2}$ with $D = D_1 + D_2$ such that $D_1 \sim H$ and $D_2 \sim H - a_2 F$.

Case (*iii*). By the symmetry, we may suppose that $d_1 \geqslant d_2$. Using the similar argument as above, we have $d_1 = d_2 = 0$. Hence we have a logarithmic Fano threefold (V, D) such that $V \cong \Sigma_{0,1}$ with $D_1 \sim H$ and $D_2 \sim H$.

Case (*iv*). In this case we have a logarithmic Fano threefold $V \cong \Sigma_{0, a_2}$ with $a_2 \geqslant 1$ and $D = D_1 + D_2$ where $D_1 \sim H + F$ and $D_2 \sim H - a_2 F$.

Case (*v*). In this case, we obtain a logarithmic Fano threefold (V, D) such that $V \cong \Sigma_{0,0} \cong \boldsymbol{P}^1 \times \boldsymbol{P}^2$ with $D_1 \sim H + F$ and $D_2 \sim H$.

Case (*vi*). In this case, we have a following logarithmic Fano threefold $V \cong \Sigma_{1,1}$ with $D_1 \sim H$ and $D_2 \sim H$.

9.1.4. *Case in which D is of type* (*iv*), (*v*) *or* (*vi*) *in* Fig. 10 *in* 5.3. In this case, $-K_V - D_1 \sim -K_V - D + F$ is also ample. Hence, (V, D) are among those in the cases between 9.1.1 and 9.1.3. Thus, we have the following 3 logarithmic Fano threefolds (V, D):

(1) $V \cong \Sigma_{0, a_2}$ with $D_1 \sim H - a_2 F$ and $D_2 \sim F$.
(2) $V \cong \Sigma_{0,0} \cong \boldsymbol{P}^1 \times \boldsymbol{P}^2$ with $D_1 \sim 2H$ and $D_2 \sim F$.
(3) $V \cong \Sigma_{a_1, a_2}$ with $D_1 \sim H - a_2 F$, $D_2 \sim H - a_1 F$ and $D_3 \sim F$.

9.2. *Case in which ℓ is of type D_2*. We see V is embedded in a $\boldsymbol{P}^3$-bundle associated with the vector bundle $\mathscr{E} := f_* \mathcal{O}_V(D_1)$ over $\boldsymbol{P}^1$. Let $X = \boldsymbol{P}(\mathscr{E})$ and $L \in |\mathcal{O}_{\boldsymbol{P}}(1)|$ on X. We can write $X = \Sigma_{a_1, a_2, a_3} \cong \boldsymbol{P}(\mathcal{O}_{\boldsymbol{P}^1} \oplus \mathcal{O}_{\boldsymbol{P}^1}(a_1) \oplus \mathcal{O}_{\boldsymbol{P}^1}(a_2) \oplus \mathcal{O}_{\boldsymbol{P}^1}(a_3))$, where $0 \leqslant a_1 \leqslant a_2 \leqslant a_3$. Let H be a tautological divisor on Σ_{a_1, a_2, a_3}. Then V and D_1 can be written as $V \sim 2H + dF$ on X and $D_1 \sim L|_V$ where $L \sim H + eF$ for some d and e.

Note that the situation in this case is quite similar to that in 9.1.2.

Let C be a section defined by the following exact sequence:

$$0 \to \mathcal{O}_{\boldsymbol{P}^1}(a_1) \oplus \mathcal{O}_{\boldsymbol{P}^1}(a_2) \oplus \mathcal{O}_{\boldsymbol{P}^1}(a_3)$$
$$\to \mathcal{O}_{\boldsymbol{P}^1} \oplus \mathcal{O}_{\boldsymbol{P}^1}(a_1) \oplus \mathcal{O}_{\boldsymbol{P}^1}(a_2) \oplus \mathcal{O}_{\boldsymbol{P}^1}(a_3) \to \mathcal{O}_{\boldsymbol{P}^1} \to 0.$$

Then $(H \cdot C) = 0$.

First assume that $d < 0$. Then C is contained in V. By the surjectivity of $N_{C/X} \to N_{V/X}|_C$, we have $a_1 + d \geqslant 0$. (We note that if $e < 0$, then we have $a_1 + e \geqslant 0$.) On the other hand, we have $a_3 + e \geqslant 0$, because L is effectie. Since $(-K_V - D \cdot C) > 0$, we have $2 - a_1 - a_2 - a_3 - d - e > 0$ (resp. $1 - a_1 - a_2 - a_3 - d - e > 0$), if $D = D_1$ (resp. $D = D_1 + D_2$ where D_2 is a fiber). But these imply contradiction. Consequently, we have $d \geqslant 0$.

Since

$$(-K_{D_1})^2_{D_1} = (-K_X - V - L)^2 \cdot V \cdot L$$

$$= (H + (2 - a_1 - a_2 - a_3 - d - e)F)^2 \cdot (2H + dF) \cdot (H + eF)$$

$$= 8 - 2(a_1 + a_2 + a_3) - 3d - 2e > 0$$

and $e \geqslant -a_3$, we have $d = 2$, 1 or 0.

If $d = 2$, then $a_1 = a_2 = a_3 = e = 0$ and $(-K_{D_1})^2 = 2$. Hence $X \cong \boldsymbol{P}^1 \times \boldsymbol{P}^2$, $V \sim 2(H + F)$ and $D = D_1 \sim H|_V$.

Conversely, for such V and D, (V, D) is a logarithmic Fano threefold.

If $d = 1$, then by using the above formulae, we obtain the following types of logarithmic Fano threefolds:

(1) $X \cong \boldsymbol{P}^1 \times \boldsymbol{P}^3$, $V \sim 2H + F$ with $D = D_1 \sim H + eF|_V$ where $e = 0$, 1 or 2.

In this case D_1 is a Del Pezzo surface of degree 8, 3 or 1.

(2) $X \cong \Sigma_{0,0,1}$, $V \sim 2H + F$ with $D = D_1 \sim H + eF|_V$, where $e = 0$ or 1.

The degree of a Del Pezzo surface D_1 is 3 or 1.

(3) $X \cong \Sigma_{0,0,2}$, $V \sim 2H + F$ with $D = D_1 \sim H|_V$.

(4) $X \cong \Sigma_{0,1,1}$, $V \sim 2H + F$ with $D = D_1 \sim H|_V$.

(5) $X \cong \Sigma_{1,1,1}$, $V \sim 2H + F$ with $D = D_1 \sim (H - F)|_V$.

In the above 3 cases, D_1 is a Del Pezzo surface of degree 1.

(6) Among the above cases where $e > 0$, we have another boundary $D_2 \sim F|_V$. There are 5 such types.

Note that in the above cases V is a very ample divisor on X. Hence $B_2(V) = B_2(X) = 2$ by the Lefschetz hyperplane section theorem.

If $d = 0$, we can calculate the possible values of (a_1, a_2, a_3) by using the above formulae.

The case where $a_1 = a_2 = a_3 = 0$ is excluded, since otherwise V is realized as a $\boldsymbol{P}^1$-bundle over $\boldsymbol{P}^1 \times \boldsymbol{P}^1$ and therefore $B_2 = 3$.

Hence we may assume that $a_1 + a_2 + a_3 > 0$. Since $|2H|$ is base point free, we can choose a smooth member V in $|2H|$. Note that by Kodaira Vanishing we have $H^i(\mathcal{O}_X(-V)) = 0$ for $i < 4$. This implies that $h^0(\mathcal{O}_V) = 1$ and therefore V is irreducible. For such V with $D = D_1$ or $D_1 + D_2$ where $D_2 = F|_V$, $-K_V - D$ is ample by Lemma 1.6. Thus, we have the following logarithmic Fano threefolds:

(1) $X \cong \Sigma_{1,1,2}$, $V \sim 2H$ with $D = D_1 \sim (H - F)|_V$.

(2) $X \cong \Sigma_{1,1,1}$, $V \sim 2H$ with $D = D_1 \sim (H - F)|_V$ or $H|_V$.

(3) $X \cong \Sigma_{0,0,3}$, $V \sim 2H$ with $D = D_1 \sim H|_V$.

(4) $X \cong \Sigma_{0,1,2}$, $V \sim 2H$ with $D = D_1 \sim H|_V$.

(5) $X \cong \Sigma_{0,1,1}$, $V \sim 2H$ with $D = D_1 \sim H + F\,|_V$ or $H\,|_V$.
(6) $X \cong \Sigma_{0,0,2}$, $V \sim 2H$ with $D = D_1 \sim H + F\,|_V$ or $H\,|_V$.
(7) $X \cong \Sigma_{0,0,1)}$, $V \sim 2H$ with $D = D_1 \sim H + 2F\,|_V$, $H + F\,|_V$ or $H\,|_V$.
(8) Among the above cases where $e > 0$, we have another boundary $D_2 \sim F\,|_V$. There are 3 such types.

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