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CHARACTERIZATION OF SUBSPACES AND QUOTIENTS OF NUCLEAR $L_f(\alpha, \infty)$ -SPACES

Heikki Apiola

Introduction

In a series of important papers ([15–20]) D. Vogt and M.J. Wagner study the structure of subspaces and quotients of some classes of nuclear Köthe spaces. They are able to give a complete basis free characterization in the case of stable power series spaces thus completing and generalizing to the basis free setting the study carried out by Alpseymen, Dubinsky, Robinson and Wagner in [2], [4], [5]–[9], [21] and [22].

The method of Vogt and Wagner can be briefly described as a combination of the following steps: (1) Proving that under suitable topological conditions concerning the Fréchet spaces E and F , a short exact sequence of the form $0 \rightarrow E \rightarrow G \rightarrow F \rightarrow 0$ splits. (2) Constructing exact sequences of power series spaces of the form $0 \rightarrow \Lambda_r(\alpha) \rightarrow \Lambda_r(\alpha) \rightarrow (\Lambda_r(\alpha))^{\mathbb{N}} \rightarrow 0$. (3) Using a suitable generalization of the Komura embedding theorem.

The purpose of the present paper is to study to which extent this method can be further generalized to cover a wider class of nuclear Köthe spaces. We first introduce a relation called S concerning pairs of Fréchet spaces and prove two forms of splitting theorems using this relation. These splitting criteria along with suitable generalizations of the two other steps described above allow us to characterize subspaces and quotients of nuclear $L_f(\alpha, \infty)$ -spaces. As a special case, assuming the subspace to have a basis, we get the result of Alpseymen [2] III.1. The quotient space case is new even in the case of a quotient with basis.

We hope that the ideas developed in this paper will find further applications in the structure theory of nuclear Fréchet spaces. Reference [3] can be regarded as an example of such an application, which is also the reason for our stating some of the present results in a more general form than would be necessary for the needs of this paper.

At this point I would like to express my gratitude to Ed Dubinsky for inviting me to Clarkson College and for his kind hospitality during my visit. I am especially happy about his active interest in my work as well as the work of the whole functional analysis group in Finland. I would also

like to thank D. Vogt and M.J. Wagner for providing me with several of their preprints. Needless to say, this paper is strongly influenced by their work.

0. Preliminaries and notation

This paper will deal exclusively with Fréchet spaces, i.e. locally convex metrizable linear spaces. The main interest will be on nuclear and Schwartz spaces, i.e. reduced projective limits of Banach spaces with nuclear, resp. compact linking maps. We shall use the abbreviations (F)-, (NF)-, (FS)-space respectively.

By a *subspace* we shall mean a closed linear subspace which is usually understood to be infinite dimensional. If M is a set in an (F)-space E , we denote by $\overline{\text{sp}}(M)$ its closed linear span. By a *fundamental system of seminorms*, or equivalently a *seminorm basis*, we shall always mean a sequence $\|\cdot\|_1 \leq \|\cdot\|_2 \leq \dots$ such that the corresponding unit balls form a neighborhood basis of the origin. If $\|\cdot\|_j$ is a continuous seminorm on the space E and if U_j is the corresponding unit ball, we shall use the notations: $E_j = \tilde{E}_{U_j}$ = the completion of the normed space $E/\ker \|\cdot\|_j$, E'_j = the dual of E_j = the linear span of U_j^0 in E' . For $f \in E'$ we let $\|f\|'_j = \sup_{x \in U_j} |f(x)| \in [0, \infty]$. The restriction of $\|\cdot\|'_j$ on E'_j defines the usual dual norm.

By the absolute basis theorem, every (NF)-space E with a basis (x_n) can be identified with a Köthe sequence space $K(a)$, where $a = (a_n^k) = (\|x_n\|_k)$. The matrix a is called a *representation* of $(E, (x_n))$. If E is nuclear (resp. Schwartz), the representation a can be so chosen that $(a_n^k/a_n^{k+1})_n \in \ell_1$, (resp. c_0). The natural system of seminorms in $K(a)$ is understood to be $\|\xi\|_k = \sum_n |\xi_n| a_n^k$. In the nuclear case the system $\xi \rightarrow \sup_n |\xi_n| a_n^k$ is equivalent to this. We shall make the slightly unconventional agreement that our representations always correspond to a seminorm basis, which means that the unit balls (rather than the ε -balls) of the corresponding seminorms form a neighborhood basis.

The dual of Köthe space $K(a)$ can be identified with the set of sequences (η_n) such that $|\eta_n| \leq \mu a_n^k$ for some $\mu > 0$ and some k . If $f = (\eta_n) \in K(a)'$, then $\|f\|'_k =$ the infimum of the above μ 's = $\sup_n (|\eta_n|/a_n^k)$, where the conventions $0/0 = 0$ and $\alpha/0 = \infty$ for $\alpha > 0$ are used. Especially, if $f = e_m =$ the m th coordinate vector in $K(a)'$, then $\|f\|'_k = \|e_m\|'_k = 1/a_m^k$.

The Köthe spaces of special interest to us are the $L_f(\alpha, \infty)$ -spaces introduced by Dragilev [10]. Here f is a strictly increasing odd function in the real line, logarithmically convex on the positive axis (i.e. $\log \circ f \circ \exp$ is convex) and $\alpha = (\alpha_n)$ is an increasing unbounded sequence, called an *exponent sequence*. We shall refer to a function f with the above properties as a *Dragilev function*. The space $L_f(\alpha, \infty)$ is then defined to be the Köthe space with respect to the matrix $(e^{f(k\alpha_n)})$. The special case $f = id$

gives us the infinite type power series spaces. Our main interest will be in the case where f is a rapidly increasing Dragilev function, i.e. $\lim_{t \rightarrow \infty} (f(at)/f(t)) = \infty$ for all $a > 1$ and (α_n) is a stable exponent sequence, i.e. $\sup_n (\alpha_{2n}/\alpha_n) < \infty$. The latter condition is equivalent to $L_f(\alpha, \infty)$ being isomorphic to its Cartesian square.

For details of the above discussion and for unexplained standard concepts and notation the reader is referred to [8], [11] and [12].

1. Splitting of exact sequences of Fréchet spaces

1.1. DEFINITION: Let E and F be (F)-spaces with fundamental systems $(\|\cdot\|_k)$ and $(\|\cdot\|_p)$ of seminorms respectively. We say that $(E, F) \in S$ if

$$\begin{aligned} & \exists p \forall j \exists k \forall \ell \forall q \exists r \\ & \ni \|f\|'_k \|x\|_q \leq \|f\|'_j \|x\|_p + \|f\|'_\ell \|x\|_r \quad \forall f \in E'_j \quad \forall x \in F. \end{aligned} \quad (1)$$

The definition is clearly independent of the particular fundamental systems. The following fact is an immediate consequence of the definition:

1.2. PROPOSITION: Let $(E, F) \in S$. If E_1 is isomorphic to a quotient of E and if F_1 is isomorphic to a subspace of F , then $(E_1, F_1) \in S$.

Let us next consider the relation S in the case where either one of the spaces E or F is a Köthe space. If $(K(a), F) \in S$, then choosing for f in (1) the n^{th} coordinate functional with respect to the canonical basis of $K(a)$, we get the following:

$$\begin{aligned} S_1: & \exists p \forall j \exists k \forall \ell \forall q \exists r \\ & \ni \frac{1}{a_n^k} \|x\|_q \leq \frac{1}{a_n^j} \|x\|_p + \frac{1}{a_n^\ell} \|x\|_r \quad \forall x \in F, \forall_n: a_n^j > 0. \\ & \left(\text{equivalently, } \frac{1}{a_n^k} V_q^0 \subset \frac{1}{a_n^j} V_p^0 + \frac{1}{a_n^\ell} V_r^0 \quad \forall_n: a_n^j > 0 \right) \end{aligned}$$

If $(E, K(b)) \in S$, then choosing for x in (1) the n^{th} basis vector of $K(b)$, we get

$$\begin{aligned} S_2: & \exists p \forall j \exists k \forall \ell \forall q \exists r \\ & \ni \|f\|'_k b_n^q \leq \|f\|'_j b_n^p + \|f\|'_\ell b_n^r \quad \forall_n \quad \forall f \in E'_j \end{aligned}$$

(equivalently, $b_n^q U_k \subset b_n^p U_j + b_n^r U_\ell \quad \forall_n$).

In fact S_1 (resp. S_2) is equivalent to S in case E (resp. F) is a Köthe space as will be shown in the next two propositions.

1.3. PROPOSITION: *Let E and F be (F) -spaces and assume E to have an absolute basis. The following conditions are equivalent:*

- (i) $(E, F) \in S$.
- (ii) *There exists an absolute basis (x_n) such that $((E, (x_n)), F) \in S_1$.*
- (iii) *For all absolute bases (x_n) of quotients of E , $((\overline{sp}(x_n), (x_n)), F) \in S_1$.*

PROOF: (i) \Rightarrow (iii) follows from 1.2 and the discussion following it. (iii) \Rightarrow (ii) is trivial. (ii) \Rightarrow (i): Let (a_n^k) be a representation of (x_n) . By the S_1 -assumption,

$$\begin{aligned} & \exists p \forall j \exists k \forall \ell \forall q \exists r \\ & \ni \frac{1}{a_n^k} \|x\|_q \leq \frac{1}{a_n^j} \|x\|_p + \frac{1}{a_n^\ell} \|x\|_r \quad \forall_n, \forall x \in F. \end{aligned}$$

Let $f = (\eta_n) \in K(a)'$. Multiplying both sides of the above inequality by $|\eta_n|$, taking sup over n and remembering that $\|f\|'_k = \sup_n (|\eta_n|/a_n^k)$, we are led to condition S . \square

The proof of the corresponding fact about S_2 is analogous and will be omitted.

1.4. PROPOSITION: *Let E and F be (F) -spaces and assume F to have an absolute basis. The following conditions are equivalent:*

- (i) $(E, F) \in S$.
- (ii) *There exists an absolute basis (y_n) in F such that $(E, (F, (y_n))) \in S_2$.*
- (iii) *For all absolute basic sequences (y_n) in F , $(E, (\overline{sp}(y_n), (y_n))) \in S_2$.*

The next lemma will serve as a preparation for the first splitting theorem, which is one of the two main results of this section.

1.5. LEMMA: *Let $K(c)$ be a Köthe space with a continuous norm and let (V_p) be a neighborhood basis in an (F) -space F . If $(K(c), F) \in S_1$, there exists a representation (a_n^k) of the canonical basis of $K(c)$ such that $\exists p \forall q \forall k \exists r = r(k, q) \ni$*

$$\frac{1}{a_n^k} V_q^0 \subset \frac{1}{a_n^{k-1}} V_p^0 + \frac{1}{a_n^{k+1}} V_{r(k,q)}^0 \quad \forall_n.$$

PROOF: Choose p according to the definition of property S_1 and proceed by induction as follows:

For $j = 0$ choose $k = k_1$ such that $\forall \ell \forall q \exists r(0, \ell, q) \ni$

$$\frac{1}{c_n^{k_1}} V_q^0 \subset \frac{1}{c_n^0} V_p^0 + \frac{1}{c_n^\ell} V_{r(0, \ell, q)}^0 \quad \forall n.$$

Having chosen k_1, \dots, k_{m-1} , take $j = k_{m-1}$ and denote the corresponding k in the definition of S_1 by $k = k_m$. This gives us a sequence (k_i) and a function $r(i, \ell, q)$ such that

$$\frac{1}{c_n^{k_i}} V_q^0 \subset \frac{1}{c_n^{k_{i-1}}} V_p^0 + \frac{1}{c_n^\ell} V_{r(i, \ell, q)}^0 \quad \forall i, \ell, q, n.$$

Now, given i , choose $\ell = k_{i+1}$. Then the matrix $(a'_n) = (c_n^{k_i})$ gives us the desired representation. \square

1.6. THEOREM: Let $0 \rightarrow K(a) \rightarrow G \xrightarrow{T} F \rightarrow 0$ be an exact sequence of (NF) -spaces and assume $K(a)$ to have a continuous norm. If $(K(a), F) \in S_1$, the sequence splits.

PROOF: Identify $K(a)$ with $\text{Ker } T$ and assume that the representation a is chosen according to 1.5. The nuclearity allows us to use the sup-norms, thus $\text{Ker } T = K(a)$ is the reduced projective limit of the Banach spaces

$$K_k = \left\{ \xi: \|\xi\|_k = \sup_n |\xi_n| a_n^k < \infty \right\}.$$

Let $\rho_k: K(a) \rightarrow K_k$ and $\rho_{m,k}: K_m \rightarrow K_k$ for $m > k$ be the inclusion maps. Then we have the canonical commutative diagrams: $\rho_k = \rho_{m,k} \circ \rho_m$ for $m > k$. Our aim is to extend the mappings ρ_k to the whole of G in such a manner that the extended diagram also commutes for $m > k$.

By the nuclearity of $K(a)$, the mapping ρ_k is nuclear for each k , hence admits an extension to G . Let's denote such an extension by F_k and let

$$\tilde{G}_k = \rho_{k+1,k} \circ F_{k+1} - F_k \in L(G, K_k).$$

As $\tilde{G}_k|_{\text{Ker } T} = 0$, there exists $G_k \in L(F, K_k)$ such that $\tilde{G}_k = G_k \circ T$.

Let $f_n(\xi) = \xi_n$ for $\xi \in K_k$. It follows from the nuclearity of F that G_k is nuclear, hence admits a representation

$$G_k x = \sum_n \xi_n a_n(x) y_n,$$

where $(a_n)_n$ is equicontinuous in F' , $(y_n)_n$ bounded in K_k and $(\xi_n) \in \mathcal{L}_1$. From this and the fact that $\|f_n\|_k = 1/a_n^k$ it follows easily that the set $\{a_n^k f_n \circ G_k\}_n$ is equicontinuous in F' for each k . Denote $G_n^k = f_n \circ G_k$. We

shall next construct a neighborhood basis $(V_k)_k$ in F such that $\{a_n^k G_n^k\}_n \subset V_k^0$ and

$$\frac{2^{k+1}}{a_n^k} V_k^0 \subset \frac{1}{a_n^{k-1}} V^0 + \frac{1}{a_n^{k+1}} V_{k+1}^0 \quad \forall k \forall n \quad (1)$$

(where V means the same as V_0). The construction goes as follows: We shall pick a suitable subsequence of our given neighborhood basis (V_p) (and relabel the indices). Start by taking $V = V_p$ according to 1.5. By the equicontinuity of the sequence $\{a_n^1 G_n^1\}_n$ we can pick a neighborhood V_1 such that $\{a_n^1 G_n^1\}_n \subset V_1^0$. Having chosen the neighborhoods V_1, \dots, V_k we apply 1.5 with $V_q = 2^{-k-1} V_k$. The neighborhood $V_{r(k,q)}$ will be denoted by V_{k+1} and it will be chosen small enough to ensure the inclusion $\{a_n^{k+1} G_n^{k+1}\}_n \subset V_{k+1}^0$. This completes the construction.

Formula (1) implies

$$\frac{2}{a_n^k} V_k^0 \subset \frac{2^{-k}}{a_n^{k-1}} V^0 + \frac{1}{a_n^{k+1}} V_{k+1}^0 \quad \forall k \forall n. \quad (2)$$

The rest of the proof is practically the same as Vogt's proof of [15] 1.5, c.f. also [16] 2.2. So we shall only give a brief outline.

The fact that $G_n^k \in (1/a_n^k) V_k^0$ together with (2) enables us to define, for fixed n , inductively a sequence $(A_n^k)_k$ in F' such that

$$G_n^k + A_n^k - A_n^{k+1} \in \frac{2^{-k}}{a_n^{k-1}} V^0. \quad (3)$$

Now define the operator $A_k: F \rightarrow K_k$ by $A_k x = (A_n^k x)_n$. Let $\tilde{A}_k = A_k \circ T$ and $\pi_k = F_k - \tilde{A}_k$. Using (3) we infer that for all m and all $x \in G$ there exists $\lim_{k \rightarrow \infty} \rho_{k,m} \circ \pi_k(x)$ which will be denoted by $\tilde{\pi}_m(x)$. It is now easy to check that the mapping $x \rightarrow (\tilde{\pi}_m x)_m$ defines a continuous projection of G onto $\text{Ker } T$. Thus $\text{Ker } T$ is complemented and so our exact sequence splits. \square

To prove our second splitting theorem we shall again begin with a lemma, which is in a sense dual to Lemma 1.5. The proof is analogous and will hence be omitted.

1.7. LEMMA: *Let E be an (F) -space and let $K(b)$ be a Köthe space with a continuous norm. If $(E, K(b)) \in S_2$, there exists a neighborhood basis $(U_k)_k$ in E such that $\forall k \forall j \exists \ell(k, j) \ni$*

$$2b_n^j U_k \subset b_n^{\ell(k,j)} U_{k+1} + 2^{-k} b_n^0 U_{k-1} \quad \forall n.$$

$((U_k)$ can be chosen as a subsequence of any given neighborhood basis in E .)

We shall give the second splitting theorem in a generality that slightly exceeds the needs of this paper but will be needed in [3].

1.8. THEOREM: Let $0 \rightarrow E \rightarrow G \xrightarrow{q} F \rightarrow 0$ be an exact sequence of (NF)-spaces. Let $K(b)$ be a (not necessarily nuclear) Köthe space with a continuous norm and let H be a subspace of $K(b)$. Assume that $(E, K(b)) \in S_2$. Then for all $\phi \in L(H, F)$ there exists a lifting ψ , that is $\psi \in L(H, G)$ and $\phi = q \circ \psi$.

PROOF: By nuclearity, we can start out with a Hilbertian neighborhood basis (W'_k) in G . Let $U'_k = E \cap W'_k$. By 1.7 we can choose a neighborhood basis (U_k) in E , which is a subsequence of (U'_k) and

$$2b_n^j U_k \subset b_n^{l(k,j)} U_{k+1} + 2^{-k} b_n^0 U_{k-1} \quad \forall k, j, n. \quad (1)$$

Let (W_k) be the corresponding subsequence of the (W'_k) -basis. Taking $V_k = q(W_k)$ and denoting $E_k = \tilde{E}_{U_k}$, $G_k = \tilde{G}_{W_k}$, $F_k = \tilde{F}_{V_k}$, we have the exact sequence

$$0 \rightarrow E_k \rightarrow G_k \xrightarrow{q_k} F_k \rightarrow 0 \quad \forall k.$$

Let now $H \subset K(b)$ and $\phi \in L(H, F)$. We have for each k the following canonical diagram:

$$\begin{array}{ccccc} G_{k+1} & \rightarrow & F_{k+1} & \xleftarrow{\eta_{k+1}} & F & \xleftarrow{\phi} & H \\ & & \downarrow \eta_{k+1,k} & & & & \\ G_k & \rightarrow & F_k & \xleftarrow{\eta_k} & & & \end{array}$$

Define $\phi_k = \eta_k \circ \phi$. By the nuclearity of F , the mapping ϕ_k is nuclear for each k , hence admits a nuclear lifting $\psi_k: H \rightarrow G_k$. Let $\chi_k = \rho_{k+1,k} \circ \psi_{k+1} - \psi_k$. Then $q_k \circ \chi_k = 0$, hence $\text{Im}(\chi_k) \subset E_k$. Now, χ_k is nuclear considered as a mapping into G_k and E_k is complemented in G_k (because G_k is a Hilbert space), thus $\chi_k: H \rightarrow E_k$ is also nuclear. Therefore there exists an extension $\tilde{\chi}_k: K(b) \rightarrow E_k$. Set $x_{n,k} = \tilde{\chi}_k(e_n) \in E_k$, where (e_n) denotes the coordinate basis in $K(b)$. By the continuity of $\tilde{\chi}_k$ there exists a function $k \mapsto m(k)$ such that

$$\|\tilde{\chi}_k(z)\|_k \leq \|z\|_{m(k)} \quad \text{for } z \in K(b).$$

We want to choose a representation (call it (b_n^k) again) such that

$$\|x_{n,k}\|_k \leq b_n^k \quad \text{and} \quad (2)$$

$$2b_n^k U_k \subset b_n^{k+1} U_{k+1} + 2^{-k} b_n^0 U_{k-1} \quad (3)$$

hold for all n and k . So fix n and apply (1) for $j = m(1)$, $k = 1$. Let $v(2) = \max\{\ell(k, j), m(2)\}$. Next apply (1) for $j = v(2)$, $k = 2$ and choose $v(3) = \max\{\ell(k, j), m(3)\}$. Continuing this way and denoting $b_n^{v(k)}$ by b_n^k we get the desired representation. The rest of the proof goes exactly along the lines of the proof of [19] 1.4 and will therefore be omitted. \square

2. Properties $D_3(f)$ and $D_4(f)$

In what follows we shall denote by f a rapidly increasing Dragilev function satisfying the following condition:

$$\exists R > 0 \ni f^{-1}(Mt) \leq RMf^{-1}(t) \quad \forall t \geq 0, \forall M \geq 1. \quad (1)$$

This will cause no loss of generality because of [8] Proposition I.5.4.

2.1 DEFINITION: Let E be an (F) -space with a continuous norm and let $\|\cdot\|_1 \leq \|\cdot\|_2 \leq \dots$ be a fundamental system of norms in E . E is said to have *property $D_3(f)$* if $\exists j \forall M \geq 1 \forall k \exists \ell \ni$

$$Mf^{-1} \log(\|x\|_k / \|x\|_j) \leq f^{-1} \log(\|x\|_\ell / \|x\|_k) \quad \forall x \in E \setminus \{0\}.$$

2.2. DEFINITION: Let E be an (F) -space with a fundamental system of seminorms $\|\cdot\|_1 \leq \|\cdot\|_2 \leq \dots$. E is said to have *property $D_4(f)$* if $\forall j \exists k > j \forall \ell > k \exists M \geq 1 \ni$

$$f^{-1} \log(\|y\|_k' / \|y\|_\ell') \leq Mf^{-1} \log(\|y\|_j' / \|y\|_k') \quad \forall y \in E' \setminus \{0\}$$

It is easy to see that the above definitions are independent of the particular fundamental systems of seminorms. It is also straightforward to verify the following:

2.3. PROPOSITION: *If E has $D_3(f)$ and F is a subspace of E , then F has $D_3(f)$. If E has $D_4(f)$ and F is a quotient space of E , then F has $D_4(f)$.*

If in the above definitions we take E to be a Köthe space $K(a)$ and require the inequalities to be valid for all basis vectors or all coordinate functionals respectively, we are led to the following conditions:

$$d_3(f): \exists j \forall M \geq 1 \quad \forall k \exists \ell \ni$$

$$Mf^{-1} \log(a_n^k / a_n^j) \leq f^{-1} \log(a_n^\ell / a_n^k) \quad \forall n,$$

$$d_4(f): \forall j \exists k \forall \ell \exists M \geq 1 \ni$$

$$f^{-1} \log(a_n^\ell / a_n^k) \leq Mf^{-1} \log(a_n^k / a_n^j) \quad \forall n: a_n^j > 0.$$

The conditions $d_3(f)$ and $d_4(f)$ were introduced by H. Ahonen [1], $d_3(f)$ is the “representation invariant” form of the condition d_f in [2] p. 42.

2.4. PROPOSITION: *Let E be a nuclear (F) -space with a basis and continuous norm. The following conditions are equivalent:*

- (1) E has $D_3(f)$.
- (2) There exists a $d_3(f)$ -basis in E .
- (3) All basic sequences in E are $d_3(f)$.

PROOF: (1) \Rightarrow (3) is obtained immediately from proposition 2.3 and (3) \Rightarrow (2) is trivial. (2) \Rightarrow (1): Let (a_n^k) be a representation of our $d_3(f)$ -basis (x_n) . By nuclearity we can choose the system of seminorms:

$$\|x\|_k = \sup_n |\xi_n| a_n^k = |\xi_{q^k}| a_{q^k}^k \quad \text{for } x = \sum_n \xi_n x_n.$$

(The index q^k exists because $(\xi_n a_n^k)_n \in c_0$.) By the definition of $d_3(f)$, we can choose an index j such that for all $M \geq 1$ and k there exists an l such that the $d_3(f)$ inequality holds. Denote $g = f^{-1} \circ \log$. Now,

$$\begin{aligned} g(\|x\|_k / \|x\|_j) &= g(|\xi_{q^k}| a_{q^k}^k / |\xi_{q^j}| a_{q^j}^j) \\ &\leq g(|\xi_{q^k}| a_{q^k}^k / |\xi_{q^k}| a_{q^k}^k) \leq M^{-1} g(|\xi_{q^k}| a_{q^k}^k / |\xi_{q^k}| a_{q^k}^k) \\ &\leq M^{-1} g(|\xi_{q^l}| a_{q^l}^l / |\xi_{q^k}| a_{q^k}^k) = M^{-1} g(\|x\|_l / \|x\|_k). \quad \square \end{aligned}$$

2.5. PROPOSITION: *Let E be an (FS) -space with an absolute basis. The following conditions are equivalent:*

- (1) E has $D_4(f)$.
- (2) There exists an absolute $d_4(f)$ -basis in E .
- (3) All absolute bases of quotients of E are $d_4(f)$

PROOF: Again, there is only one non-trivial implication: (2) \Rightarrow (1): Let (a_n^k) be a representation of the absolute $d_4(f)$ -basis (x_n) in E . Given j , choose $k > j$ such that for all $l > k$ there exists M such that the $d_4(f)$ -inequality holds. In order to prove the $D_4(f)$ -inequality, we only need to consider those $y \in E'_j$ for which $\|y\|'_j < \infty$. Let $y \in E'$ and $n \in N$. If $a_n^j = 0$ and $y(x_n) \neq 0$, it follows that $\|y\|'_j = \infty$. Hence assuming, as we do, that $\|y\|'_j < \infty$, we have $a_n^j \neq 0$ for all n for which $y(x_n) \neq 0$ and hence the $d_4(f)$ -inequality is valid for all such n . Now,

$$\|y\|'_j = \sup_n (|\eta_n| / a_n^j), \quad \text{where } \eta_n = y(x_n),$$

hence

$$|\eta_n|/a_n^k \leq \|y\|'_j (a_n^j/a_n^k) \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

for all $k > j$ (because of the Schwartz-assumption). Thus for each $k > j$ there is an index q^k such that $\|y\|'_k = |\eta_{q^k}|/a_{q^k}^k$. Let $g = f^{-1} \circ \log$. Then for $j < k < \ell$ and M as above we have

$$\begin{aligned} g(\|y\|'_k/\|y\|'_\ell) &= g(|\eta_{q^k}|a_{q^k}^\ell/|\eta_{q^\ell}|a_{q^k}^k) \\ &\leq g(|\eta_{q^k}|a_{q^k}^\ell|\eta_{q^k}|a_{q^k}^k) \leq \text{Mg}(|\eta_{q^k}|a_{q^k}^k/|\eta_{q^k}|a_{q^k}^k) \\ &\leq \text{Mg}(\|y\|'_j/\|y\|'_k) \end{aligned}$$

and the proof is complete. \square

The next proposition tells us how the properties $D_3(f)$ and $D_4(f)$ are affected by changing the rapidity of growth of the function f . Recall Dragilev's definition [10]: If f and h are Dragilev functions such that $\phi = f^{-1} \circ h$ is a rapidly increasing Dragilev function, we say that h increases more rapidly than f and denote $h > f$.

2.6. PROPOSITION: *If E has $D_3(f)$ and $h < f$, then E has $D_3(h)$. If E has $D_4(f)$ and $h > f$, then E has $D_4(h)$.*

PROOF: The proofs are very similar, so let us only consider the $D_4(f)$ -case. Let $g_{f,N}(t) = \exp f(Nf^{-1} \log t)$ for $N \geq 1$. It suffices to find for a given $N \geq 1$ an $M \geq 1$ such that $g_{h,M}(t) \geq g_{f,N}(t)$ for t sufficiently large. This is done simply by taking M such that $\phi(M) \geq N\phi(1)$, where $\phi = f^{-1} \circ h$ is logarithmically convex by assumption. This implies that $\phi(Ms) \geq N\phi(s)$ for all $s \geq 1$. The rest of the proof is now obvious. \square

Note that the conditions $D_3(\text{id})$ and $D_4(\text{id})$ are the same as the (DN) – and (Ω) – conditions respectively, introduced by Vogt and Wagner in [15] and [20]. The above proposition tells us especially that $D_3(f)$ implies (DN) and (Ω) implies $D_4(f)$, for any Dragilev function f .

We now turn to the study of the connection of the conditions $D_3(f)$ and $D_4(f)$ to the splitting condition S . We shall begin with the following two propositions, whose proofs are very much alike, so we choose to give only the proof of the latter one.

2.7. PROPOSITION. *An (F) -space E has $D_3(f)$ if and only if $\exists j \forall M \geq 1 \forall k \exists \ell \ni$*

$$U_k^0 \subset rU_j^0 + (g_M(r))^{-1}U_\ell^0 \quad \forall r \geq 1$$

(equivalently: $\|\cdot\|_k \leq r\|\cdot\|_j + (g_M(r))^{-1}\|\cdot\|_\ell \quad \forall r \geq 1$), where $g_M(t) = \exp f(Mf^{-1} \log t)$.

2.8. PROPOSITION: An (F) -space E has $D_4(f)$ if and only if $\forall j \exists k \forall \ell \exists M \ni$

$$U_k \subset g_M(r)U_\ell + \frac{1}{r}U_j \quad \forall r \geq 1$$

$$\left(\text{equivalently: } \|\cdot\|'_k \leq g_M(r)\|\cdot\|'_\ell + \frac{1}{r}\|\cdot\|'_j \quad \forall r \geq 1 \right)$$

PROOF: Assume that E has $D_4(f)$. Given j , choose k such that for all ℓ there exists an M such that

$$f^{-1} \log(\|y\|'_k / \|y\|'_\ell) \leq Mf^{-1} \log(\|y\|'_j / \|y\|'_k) \quad \forall y \in E'_j.$$

From this it follows in a straightforward manner that

$$\left((g_M(r))^{-1}U_\ell^0 \right) \cap (rU_j^0) \subset U_k^0 \quad \forall r \geq 1.$$

Taking polars we get

$$U_k \subset \text{closure of } \left(g_M(r)U_\ell + \frac{1}{r}U_j \right) \subset 2g_M(r)U_\ell + \frac{1}{r}U_j \quad \forall r \geq 1.$$

Thus we only need to choose k' such that $U_{k'} \subset \frac{1}{2}U_k$, to complete the first half of the proof. (To see that the statement in parentheses is equivalent is a standard dualization argument.)

Conversely assume that $\forall j \exists k \forall \ell \exists g = g_M \ni$

$$\|\cdot\|'_k \leq g(r)\|\cdot\|'_\ell + \frac{1}{r}\|\cdot\|'_j \quad \forall r \geq 1.$$

Fix j, k, ℓ, M and $y \in E'_j$ and substitute $r = 2\|y\|'_j / \|y\|'_k$ in the above inequality. This yields the $D_4(f)$ -condition where instead of k we take k' such that $2\|y\|'_k \leq \|y\|'_k$ for all y . \square

2.9. PROPOSITION: Let E and F be (F) -spaces such that E has $D_4(f)$ and F has $D_3(f)$. Then $(E, F) \in S$.

PROOF: Let $(\|\cdot\|'_k)$ [resp. $(\|\cdot\|_p)$] be a fundamental system in E' [resp. F]. As F has $D_3(f)$, we can choose p such that $\forall M \geq 1 \forall q \exists s$ such that

$$\|x\|_q \leq r\|x\|_p + (g_M(r))^{-1}\|x\|_s \quad \forall x \in F. \quad (1)$$

Now, E has $D_4(f)$, so given j we can choose k such that for all ℓ there exists $M(\ell, j)$ such that

$$\|y\|'_k / \|y\|'_\ell \leq g_{\ell, j} (\|y\|'_j / \|y\|'_k) \quad \forall y \in E'_j,$$

where $g_{\ell, j} = g_{M(\ell, j)} = \exp f(M(\ell, j)f^{-1} \log)$. Now, for an arbitrary q and for $M = M(\ell, j)$, pick s such that (1) holds. Fix $y \in E'_j$ and substitute $r = \|y\|'_j / \|y\|'_k$. Hence we get:

$$\begin{aligned} \exists p \forall j \exists k \forall \ell \forall q \exists s \ni \\ \|y\|_q &\leq \frac{\|y\|'_j}{\|y\|'_k} \|x\|_p + \left[g_{\ell, j} \left(\frac{\|y\|'_j}{\|y\|'_k} \right) \right]^{-1} \|x\|_s \\ &\leq \frac{\|y\|'_j}{\|y\|'_k} \|x\|_p + \frac{\|y\|'_\ell}{\|y\|'_k} \|x\|_s. \end{aligned}$$

and the proof is complete. \square

This combined with propositions 2.4 and 2.5 yields:

2.10. COROLLARY: *If $K(a)$ has $d_4(f)$ and E has $D_3(f)$, then $(K(a), E) \in S_1$. If E has $D_4(f)$ and $K(b)$ has $d_3(f)$, then $(E, K(b)) \in S_2$.*

NOTE: It looks as if there were an omission in the latter statement because of the nuclearity assumption in 2.4. The statement of 2.10 is however correct as it stands, as can be proved directly.

In abbreviated notation we can write the statements of 2.9 and 2.10 as follows:

$$(D_4(f), D_3(f)) \in S, (d_4(f), D_3(f)) \in S_1, (D_4(f), d_3(f)) \in S_2$$

To end this section, we shall prove the easy halves of our characterization theorems.

2.11. PROPOSITION: *$L_f(\alpha, \infty)$ -spaces have properties $d_3(f)$ and $d_4(f)$.*

PROOF: The $d_3(f)$ -condition can be derived from [2] III.1. (c.f. also [8] 6.2.1) although a direct proof is just as easy and analogous to the $d_4(f)$ -case, which we shall give.

Given j , choose k such that

$$f(k\alpha_n) \geq 2f(j\alpha_n) \quad \text{and} \quad f(k\alpha_n) \geq 2f\left(\frac{k}{2}\alpha_n\right) \quad \text{for all } n.$$

Then, for an arbitrary $\ell > k$ we have

$$\begin{aligned} f^{-1}(f(\ell\alpha_n) - f(k\alpha_n)) &\leq f^{-1}(f(\ell\alpha_n)) \leq 2\ell \frac{k}{2} \alpha_n \\ &\leq 2\ell f^{-1}(\frac{1}{2}f(k\alpha_n)) \leq 2\ell f^{-1}(f(k\alpha_n) - f(j\alpha_n)). \end{aligned}$$

Hence we have $d_4(f)$ with $M(j, \ell) = 2\ell$.

2.12. COROLLARY: *Subspaces of nuclear $L_f(\alpha, \infty)$ -spaces have $D_3(f)$. Quotients of $L_f(\alpha, \infty)$ -spaces have $D_4(f)$.*

PROOF. 2.3, 2.4, 2.5, 2.11.

COROLLARY: *If E has $D_3(f)$, then $(L_f(\alpha, \infty), E) \in S_1$. If E has $D_4(f)$, then $(E, L_f(\alpha, \infty)) \in S_2$.*

3. Characterization of subspaces and quotients of $L_f(\alpha, \infty)$ -spaces

In this section we shall deal exclusively with nuclear, stable $L_f(\alpha, \infty)$ -spaces. We shall say that a space E is $\Lambda(f, \alpha, \mathbb{N})$ -nuclear if it is $\Lambda(a, \mathbb{N})$ -nuclear in the sense of [13], with respect to the matrix $a = (a_n^k)$, where $a_n^k = \exp f(k\alpha_n)$. This is equivalent to the condition $\Delta(E) \supset \Delta(L_f(\alpha, \infty))$, where $\Delta(E)$ is the diametral dimension of E (c.f. [14]). We shall need the following Komura-type theorem due to Ramanujan and Rosenberger [13]: If E is a $\Lambda(a, \mathbb{N})$ -Nuclear Fréchet space, then E is isomorphic to a subspace of $(K(a))^{\mathbb{N}}$ under certain conditions about $K(a)$, which are certainly satisfied by a nuclear, stable $L_f(\alpha, \infty)$ -space. One of the basic facts we shall also need is the existence of an exact sequence

$$0 \rightarrow L_f(\alpha, \infty) \rightarrow L_f(\alpha, \infty) \rightarrow (L_f(\alpha, \infty))^{\mathbb{N}} \rightarrow 0 \quad (1)$$

for any stable, nuclear $L_f(\alpha, \infty)$. This follows from [20] 2.3. (For a proof of a slightly more general result we refer to [3].)

Let's make the following convention concerning the rest of the paper. All $L_f(\alpha, \infty)$ -spaces encountered are nuclear and stable.

3.1. PROPOSITION: *Let E be a $\Lambda(f, \alpha, \mathbb{N})$ -nuclear (F) -space. Then there exists an exact sequence*

$$0 \rightarrow L_f(\alpha, \infty) \rightarrow G \rightarrow E \rightarrow 0,$$

where G is a subspace of $L_f(\alpha, \infty)$.

PROOF: Start out with the exact sequence (1). Embed E as a subspace of

$(L_f(\alpha, \infty))^{\mathbb{N}}$ using the Ramanujan-Rosenberger theorem discussed above. Define G to be the pre-image of E under the quotient map: $L_f(\alpha, \infty) \rightarrow (L_f(\alpha, \infty))^{\mathbb{N}}$. Restricting the quotient map to G gives us the desired exact sequence. \square

We are now ready to prove our first main result.

3.2. THEOREM: *An (F) -space E is isomorphic to a subspace of $L_f(\alpha, \infty)$ if and only if it is $\Lambda(f, \alpha, \mathbb{N})$ -nuclear and has property $D_3(f)$.*

PROOF: The necessity of the conditions follows from 2.12 together with [13] 3.8 and 3.3.

To prove the sufficiency, assume E is $\Lambda(f, \alpha, \mathbb{N})$ -nuclear and has $D_3(f)$. Set up the exact sequence of 3.1. proposition. By 2.13, $(L_f(\alpha, \infty), E) \in S_1$, hence the sequence splits by 1.6. This means that there exists an operator $S: E \rightarrow G$ such that $q \circ S = \text{id}_E$, where q is the quotient map: $G \rightarrow E$. It follows that S is an isomorphism into G , which in turn is a subspace of $L_f(\alpha, \infty)$. This completes the proof. \square

The proof of the quotient space case uses again the same steps with appropriate modifications, as the corresponding proof for quotients of (s) due to Vogt and Wagner [19]. In addition to the previous results we shall need the following fact whose proof is standard and can be found in [19].

3.3. LEMMA: Assume that both the row and the column of the following commutative diagram of (F) -spaces is exact:

$$\begin{array}{ccccccc}
 & & & 0 & & & \\
 & & & \uparrow & & & \\
 0 & \rightarrow & E_1 & \rightarrow & E_2 & \rightarrow & Q \rightarrow 0 \\
 & & & & \uparrow & & \\
 & & & & \swarrow & & \\
 & & & & F_2 & & \\
 & & & & \uparrow & & \\
 & & & & F_1 & & \\
 & & & & \uparrow & & \\
 & & & & 0 & &
 \end{array}$$

Then there exists the following exact sequence:

$$0 \rightarrow F_1 \rightarrow E_1 \oplus F_2 \rightarrow E_2 \rightarrow 0.$$

3.4. THEOREM: *An (F) -space E is isomorphic to a quotient of $L_f(\alpha, \infty)$ if and only if it is $\Lambda(f, \alpha, \mathbb{N})$ -nuclear and has property $D_4(f)$.*

PROOF. The necessity follows again from 2.12 and [13] 3.8 and 3.3.

To prove the sufficiency, assume that E is $\Lambda(f, \alpha, \mathbb{N})$ -nuclear and has

property $D_4(f)$. Using the Ramanujan-Rosenberger embedding theorem we can identify E with a subspace of $(L_f(\alpha, \infty))^{\mathbb{N}}$. Let $Q = (L_f(\alpha, \infty))^{\mathbb{N}}/E$. Now, Q is again $\Lambda(f, \alpha, \mathbb{N})$ -nuclear, hence we can set up the exact sequence of 3.1 with Q in the role of E . Thus we get the following exact row and column:

$$\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \uparrow & & \\
 0 \rightarrow E \rightarrow & (L_f(\alpha, \infty))^{\mathbb{N}} & \rightarrow & Q & \rightarrow & 0 & \\
 & \swarrow \phi & & \uparrow & & & \\
 & & & G & & & \\
 & & & \uparrow & & & \\
 & & & L_f(\alpha, \infty) & & & \\
 & & & \uparrow & & & \\
 & & & 0 & & &
 \end{array}$$

where $G \subset L_f(\alpha, \infty)$. By 2.13, $(E, L_f(\alpha, \infty)) \in S_2$, hence 1.8 Theorem gives us a lifting ϕ . Then 3.3 applies and we get the exact sequence:

$$0 \rightarrow L_f(\alpha, \infty) \rightarrow E \oplus G \rightarrow (L_f(\alpha, \infty))^{\mathbb{N}} \rightarrow 0.$$

Set up the exact sequence (1) as a column and use again 2.13 (together with 2.12) and 1.8 to get the following commutative diagram:

$$\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \uparrow & & \\
 0 \rightarrow L_f(\alpha, \infty) \rightarrow & E \oplus G & \rightarrow & (L_f(\alpha, \infty))^{\mathbb{N}} & \rightarrow & 0 & \\
 & \swarrow & & \uparrow & & & \\
 & & & L_f(\alpha, \infty) & & & \\
 & & & \uparrow & & & \\
 & & & L_f(\alpha, \infty) & & & \\
 & & & \uparrow & & & \\
 & & & 0 & & &
 \end{array}$$

Finally, 3.3 gives us the exact sequence:

$$0 \rightarrow L_f(\alpha, \infty) \rightarrow L_f(\alpha, \infty) \oplus L_f(\alpha, \infty) \rightarrow E \oplus G \rightarrow 0.$$

By our stability assumption, $L_f(\alpha, \infty) \oplus L_f(\alpha, \infty)$ is isomorphic to $L_f(\alpha, \infty)$, so $E \oplus G$ and hence E is a quotient of $L_f(\alpha, \infty)$. \square

We shall conclude the paper by characterizing complemented subspaces of $L_f(\alpha, \infty)$.

3.5. THEOREM: *An (F) -space E is isomorphic to a complemented subspace*

of $L_f(\alpha, \infty)$ if and only if E is $\Lambda(f, \alpha, \mathbb{N})$ -nuclear and has properties $D_3(f)$ and $D_4(f)$.

PROOF: The necessity is clear. To prove the sufficiency, assume E has $D_3(f)$, $D_4(f)$ and is $\Lambda(f, \alpha, \mathbb{N})$ -nuclear. The proof of 3.4 gives us an exact sequence:

$$0 \rightarrow L_f(\alpha, \infty) \rightarrow L_f(\alpha, \infty) \rightarrow E \oplus G \rightarrow 0,$$

where G is a subspace of $L_f(\alpha, \infty)$. Now, E and G have $D_3(f)$, hence also $E \oplus G$ (as can be seen immediately). So the sequence splits by 2.13 and 1.6. Thus, $E \oplus G$ is isomorphic to a complemented subspace of $L_f(\alpha, \infty)$ and so is E . \square

3.6. COROLLARY: E is isomorphic to a complemented subspace of $L_f(\alpha, \infty)$ if and only if E is isomorphic to a subspace and a quotient space of $L_f(\alpha, \infty)$.

Having 3.4 theorem, the question arises of what spaces we get by taking quotients of stable $L_f(\alpha, \infty)$ -spaces. It turns out that if we consider the wider class of Schwartz $L_f(\alpha, \infty)$ -spaces, this gives us all nuclear Fréchet spaces. This is the topic of [3].

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