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# UNIVERSAL CYCLE CLASSES

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## §0. Introduction

The objective of this paper is to prove:

THEOREM 0.1: *For each positive integer $p \geq 1$, there exists a smooth simplicial scheme $BL^p_{\cdot}$, with a smooth, closed subsimplicial scheme $Z^p_{\cdot}$ of codimension p in each degree, having the property that if X is any noetherian scheme and $Y \subset X$ any codimension p subscheme locally a complete intersection in X, then there is an open cover $\{U_\alpha\}$ of X and a map of simplicial schemes*

$$\chi_Y : N_{\cdot}\{U_\alpha\} \to BL^p_{\cdot} \tag{0.2}$$

*such that $\chi_Y^{-1}(Z^p_{\cdot}) = N_{\cdot}\{U_\alpha \cap Y\} \subset N_{\cdot}\{U_\alpha\}$. Furthermore the subscheme $Z^p_{\cdot}$ has cycle classes in three cohomology theories: the K-theoretic version of the Chow ring, étale cohomology and crystalline cohomology, which we may regard as universal cycle classes for local complete intersections.*

Given a pair $Y \subset X$ as above, the universal cycle classes may be pulled back via the classifying map $\chi_Y$ to define cycle classes

$$\gamma[Y] \in \begin{cases} H^p_Y(X, \mathbf{K}_p) & \text{if } X \text{ is defined over a field} \\ H^{2p}_{\mathrm{et},Y}(X, \boldsymbol{\mu}_n^{\otimes p}) & \text{if } 1/n \in \mathcal{O}_X(X) \\ H^{2p}_{\mathrm{crys}}(X/W) & \text{if } X \text{ is defined over a perfect field } k \text{ of characteristic } p > 0, \text{ and } W \text{ is the ring of Witt vectors of } k. \end{cases}$$

One can verify in the first two cases that these cycle classes have good properties, in particular if $X$ is smooth over a field in the Chow ring case or Spec($\mathbb{Z}[1/n]$) in the étale case they coincide with the cycle classes defined in ([21], [7]). In the crystalline case it seems more difficult to compare these classes with those of Berthelot [1], so we do not consider the question (one may easily see that they do agree locally).

We also construct a similar universal cycle class for general codimension two determinental subschemes $Y \subset X$ lying in $H^2_Y(X, \mathbf{K}_2)$. This class defines a Gysin homomorphism $CH^*(X) \to CH^*(Y)$, whose existence has so far only been known (using the grassmannian graph construction of MacPherson [3]) for quasi-projective $X$.

The primary motivation for proving these results is to improve our understanding of intersection theory on singular varieties and schemes, and in particular to explore the possibility that the groups $H^*(\ ,\mathbf{K}_*)$ are the right Chow cohomology groups in the sense of ([10], [11]). One already knows that Quillen $K$-theory may be used to describe intersection theory on smooth varieties over a field.

The idea of the construction of $BL^p_{\cdot}$ came from the work of Toledo and Tong who considered the problem of passing from local cycle classes to global cycle classes in DeRham cohomology, the analogue of the problem considered here of passing from a local class in $H^0(X, \mathbf{H}^p_Y(\mathbf{K}_p))$ (as originally constructed by Bloch for the case $p = 2$ [2]) to a global class in $H^p_Y(X, \mathbf{K}_p)$. I would like to thank David Mumford for drawing my attention to the paper [22], and Spencer Bloch for pointing out errors in the earlier versions of this paper.

§1 outlines the properties of simplicial schemes that we shall be using and §2 gives a brief description of Toledo and Tong's theory of twisted resolutions. In §3 we construct the classifying schemes $BL^p_{\cdot}$ for $p \geq 0$ and describe their properties, while in §4 we define the universal cycle classes referred to in Theorem 0.1, so §3 and §4 constitute the proof of the main theorem. Finally in §5 we consider the determinental case.

As we shall see in Section 3, twisted complexes play a key role in the construction of the classifying space $BL^p_{\cdot}$. However, in this paper I have not attempted a more general examination of their role in $K$-theory. In a future paper I hope to remedy this by describing how twisted complexes may be used to construct elements of a modified Quillen $K$-theory of locally free sheaves and how they may then be used to define Gysin homomorphisms in the $K$-theory of coherent sheaves.

All schemes will be assumed to be separated, noetherian and excellent. A variety over a field will mean a scheme, reduced, irreducible and of finite type over the ground field.

## §1. Cohomology of simplicial schemes

In this section we shall review those facts about the cohomology theory of sheaves on simplicial schemes that we shall need in the main body of the paper. For a more general discussion see [6].

Recall that if **C** is a category, a simplicial object in **C** is a contravariant functor $\Delta \to \mathbf{C}$ where $\Delta$ is the category of finite totally ordered sets. An object $X_. \in [\Delta^{\mathrm{op}}, \mathbf{C}]$ (the category of all such functors) will be described by what it does to the sets $[n] = \{0 < 1 < \ldots < n\}$ and the monotonic morphisms between them (in particular the face maps $d_i : [n] \to [n+1]$ and the degeneracies $s_i : [n+1] \to [n]$; see [20] for details). $[\Delta^{\mathrm{op}}, \mathbf{C}]$ is a category in a natural way: *the category of simplicial objects in* **C** (sometimes denoted $\mathbf{C}^{\Delta^{\mathrm{op}}}$ rather than $[\Delta^{\mathrm{op}}, \mathbf{C}]$). We shall be concerned in this paper with simplicial schemes, i.e. objects in $[\Delta^{\mathrm{op}}, \mathbf{Sch}]$ or more specifically $[\Delta^{\mathrm{op}}, \mathbf{Var}_k]$ the category of simplicial varieties over $k$, where $k$ is some fixed field.

Generally if $X_.$ is a simplicial topological space, by a sheaf on $X_.$ we shall mean a compatible system of sheaves $\mathscr{S}^{\cdot} = \{\mathscr{S}^n\}$, one on each $X_n$, together with morphisms $\mathscr{S}(\tau) : \mathscr{S}^n \to X(\tau)_* \mathscr{S}^m$ for each monotonic $\tau : [n] \to [m]$. For an abelian sheaf $\mathscr{S}^{\cdot}$ on $X_.$ we may define the cohomology groups $H^i(X_., \mathscr{S}^{\cdot})$ as the derived functors of the global section functor

$$\begin{aligned} \Gamma &: \text{Sheaves}/X_. \to \text{Abelian Groups} \\ &: \mathscr{S}^{\cdot} \mapsto \operatorname{Ker}(\Gamma(X_0, \mathscr{S}^0) \rightrightarrows \Gamma(X_1, \mathscr{S}^1)). \end{aligned}$$

We can define these cohomology groups more explicitly as follows. By a Lubkin covering $\mathscr{U}_.$ of $X_.$ shall mean Lubkin covers $\mathscr{U}_n$ of each $X_n$ (see [26] I §5 for a definition of Lubkin cover) such that for $\tau : [n] \to [m]$ $\mathscr{U}_m$ refines $X(\tau)^{-1}\mathscr{U}_n$. Given a sheaf $\mathscr{S}^{\cdot}$ on $X_.$ we obtain a cosimplicial differential complex of abelian groups:

$$C_L^{pq}(X_., \mathscr{U}_., \mathscr{S}^{\cdot}) = C_L^p(X_q, \mathscr{U}_q, \mathscr{S}^q),$$

"Lubkin $p$-chains with coefficients in $\mathscr{S}^q$ with respect to the Lubkin cover $\mathscr{U}_q$ of $X_q$". $C_L^{**}(X_., \mathscr{U}_.; \mathscr{S}^{\cdot})$ is naturally a bicomplex, and we define the cohomology of $\mathscr{S}^{\cdot}$ with respect to $\mathscr{U}_.$, $H^*(X_., \mathscr{U}_.; \mathscr{S}^{\cdot})$ to be the cohomology of the associated total complex. In the situation discussed in this paper, taking the limit over all such covers computes the cohomology of $\mathscr{S}^{\cdot}$. There is an important spectral sequence converging to the cohomology $H^*(X_., \mathscr{S}^{\cdot})$:

$$E_r^{pq}(X_., \mathscr{S}^{\cdot}) \Rightarrow H^{p+q}(X_., \mathscr{S}^{\cdot}).$$

The $E_1$ and $E_2$ terms of $E_r^{**}(X_., \mathscr{S}^{\cdot})$ are:

$$E_1^{pq} = H^q(X_p, \mathscr{S}^p)$$
$$E_2^{pq} = H^p(k \mapsto H^q(X_k, \mathscr{S}^k)).$$

This spectral sequence comes from the 'filtration (in the notation of [13] §4.8),

$$'F_p^* = \sum_{i \geq p} C_L^{pq}(X_., \mathscr{U}^{\cdot}; \mathscr{S}^{\cdot}),$$

of the double complexes computing the Lubkin cohomology of $\mathscr{S}^{\cdot}$.

We can also introduce relative cohomology. If $U_. \hookrightarrow X_.$ is a morphism of simplicial topological spaces such that each $U_n$ is an open subset of $X_n$, then the family $(\{X_n, U_n\})$ defines a Lubkin cover of $X_.$. If $\mathscr{V}_.$ is a refinement of this cover we can define the bicomplex $C^{p,q}(X_., U_.; \mathscr{V}_.; \mathscr{S}^{\cdot})$ of relative cochains to be the kernel of the restriction map:

$$C^{pq}(X_., \mathscr{V}_.; \mathscr{S}^{\cdot}) \to C^{pq}(U_., \mathscr{V}_.|_{U_.}; \mathscr{S}^{\cdot}|_{U_.})$$

where $\mathscr{V}_.|_{U_.}$ is the Lubkin cover of $U_.$ consisting of those members of $\mathscr{V}_.$ contained in $U_.$. The relative cohomology groups $H^p(X_., U_.; \mathscr{S}^{\cdot})$ are by definition the direct limit over all such refinements $\mathscr{V}_.$, of the cohomology of the total complex of $C^{**}(X_., U_.; \mathscr{V}_.; \mathscr{S}^{\cdot})$. Again there is a spectral sequence:

$$E_1^{pq}(X_., U_.; \mathscr{S}^{\cdot}) = H^q(X_p, U_p, \mathscr{S}^p) \Rightarrow H^{p+q}(X_., U_., \mathscr{S}^{\cdot}).$$

If $Y_. \to X_.$ is a closed subsimplicial space (i.e. each $Y_n \hookrightarrow X_n$ is a closed subspace), the complement of which is an open subsimplicial space of $X$, we can regard the relative cohomology $H^*(X_., X_. - Y_., \mathscr{S}^{\cdot})$ as cohomology with supports in $Y$, $H^*_{Y_.}(X_., \mathscr{S}^{\cdot})$.

A type of simplicial topological space that we shall be making much use of later on is the *nerve of an open covering*. Suppose $\mathscr{U} = \{U_\alpha\}$ is a cover of the topological space $X$. Then the nerve $\mathscr{N}_.\mathscr{U}$ of $\mathscr{U}$ is the simplicial space:

$$N_k\mathscr{U} = \coprod_{\alpha_0, \ldots, \alpha_k} U_{\alpha_0} \cap \ldots \cap U_{\alpha_k}.$$

If $\mathscr{F}$ is a sheaf on $X$, then it induces a sheaf $\mathscr{F}^{\cdot}$ on $N_.\mathscr{U}$ ($\mathscr{F}^k = \mathscr{F}|_{N_k\mathscr{U}}$) and $H^p(N_.\mathscr{U}, \mathscr{F}^{\cdot})$ is isomorphic to $H^p(X, \mathscr{F})$. A similar natural isomorphism holds for relative cohomology; if $V \subset X$ is an open subset,

$$H^*(N_.\mathscr{U}, V \cap N_.\mathscr{U}; \mathscr{F}^{\cdot}) \simeq H^*(X, V; \mathscr{F}^{\cdot}).$$

Sometimes we may abuse terminology and speak of morphisms $N_{\bullet}\mathscr{U} \to Y_{\bullet}$ being "morphisms $X \to Y_{\bullet}$" when $\mathscr{U}$ is an open cover of $X$.

In the previous paragraph we saw one typical example of a sheaf on a simplicial topological space, in which any sheaf $n$ on a space $X$ induced a simplicial sheaf $\mathscr{F}^{\bullet}$ on $N_{\bullet}\mathscr{U}$ for any open cover $\mathscr{U}_{\bullet}$ of $X$. In this case there is a very simple relationship between the $\mathscr{F}^k$ for various $k$, specifically $\mathscr{F}^k$ is the restriction to $X_k$ of $\mathscr{F}^0$. A more complicated example is of central importance in the following sections. The Quillen $K$-functors $K_n$ $(n \geq 0)$ give rise to sheaves $\mathbf{K}_n$ in the Zariski topology on any scheme. If $f: X \to Y$ is an arbitrary morphism of schemes then there is a natural map $f^!: \mathbf{K}_n \to f_* \mathbf{K}_n$. It follows that on any simplicial scheme $X_{\bullet}$ $K$-theory gives rise to sheaves $\mathbf{K}_n$ for all $n \geq 0$. However the sheaf $\mathbf{K}_n$ on $X_k$ can in no way be deduced from its counterpart on $X_0$.

Remarks: Some readers may find sheaves on simplicial topological spaces slightly mystifying at first. For example, given $\mathscr{F}^{\bullet}$ on $X_{\bullet}$, the cohomology groups $H^k(X_{\bullet}, \mathscr{F}^{\bullet})$ only depend on the $(k+1)$-skeleton of $X_{\bullet}$; i.e. on the family of spaces and sheaves $(X_i, \mathscr{F}^i)$ for $i = 0 \ldots k+1$ and the maps between them. One can make arbitrary changes in $\mathscr{F}^i$ for $i > k+1$ without changing $H^k(X_{\bullet}, \mathscr{F})$. Another fact to notice is that the process of passing from a sheaf on a space to a sheaf on the nerve of any open cover of that space is not reversible. For example, if we take the trivial cover of $X$ consisting of $X$ itself then $N_{\bullet}\{X\}$ is just the constant simplicial space with all maps the identity:

$$
\begin{array}{ccccccc}
 & \xrightarrow{\text{Id}} & & \longleftarrow & & \longleftarrow & \\
X & \xleftarrow[\text{Id}]{\text{Id}} & X & \longleftarrow & X & \longleftarrow & X \ldots \\
 & \longleftarrow & & \longleftarrow & & \longleftarrow &
\end{array}
$$

and a sheaf on $N_{\bullet}\{X\}$ is a *cosimplicial sheaf* on $X$. However to show that all is not confusion, the following may be interesting.

Example 1.1: First of all a definition: If $X_{\bullet}$ is a simplicial scheme, by a *vector bundle* $V_{\bullet}$ over $X_{\bullet}$ we mean: for each $k \geq 0$, a vector bundle $V_k$ over $X_k$ and for each morphism $\tau: [m] \to [n]$ in $\Delta$ an *isomorphism* $\tau^* V_m \to V_n$. Note that this is not the same as requiring that $V_{\bullet}$ be a sheaf locally free in each degree. On $X_{\bullet}$ we also have a sheaf of groups $\mathbf{GL}_n^{\bullet}$ for each $n \geq 0$ ($\mathbf{GL}_n^k$ is just the sheaf $\mathbf{GL}_n(\mathscr{O}_{X_k})$). The reassuring fact is that vector bundles of rank $n$ over $X_{\bullet}$ are classified up to isomorphism by $H^1(X_{\bullet}, \mathbf{GL}_n^{\bullet})$. To see this one observes first that a vector bundle $V_{\bullet}/X_{\bullet}$ is determined entirely by the data:

(a) A vector bundle $V_0/X_0$

(b) An isomorphism $\alpha : d_0^* V_0 \simeq d_1^* V_1$
such that $d_2^*\alpha \circ d_0^*\alpha = d_1^*\alpha$.

However there exists some Lubkin covering $\{U_.\}$ of $X_.$ such that $V_0$ is trivial on each open set in $\mathscr{U}_0$ and hence there is a 1-cochain $\gamma^{01} \in C^1(X_0, \mathscr{U}_0; \mathbf{GL}_n)$ such that:

(i) $\partial\gamma^{01} = 0$ (where $\partial$ is the coboundary on Lubkin cochains)

We have $\alpha : d_0^* V_0 \simeq d_1^* V_0$ and since $\mathscr{U}_1$ is a common refinement of both $d_0^{-1}\mathscr{U}_0$ and $d_1^{-1}\mathscr{U}_0$ there is a 0-cochain $\gamma^{10}$ in $C^0(X_1, \mathscr{U}_1; \mathbf{GL}_n)$ representing the isomorphism $\alpha$ such that:

(ii) $d_0^*(\gamma^{01}) = d_1^*(\gamma^{01}) \cdot \partial(\gamma^{10})$.

The fact that on $X_2$ we have a commutative triangle of isomorphisms between the three pull backs of $V_0$ corresponds to the following identity between cochains in $C^0(X_2, \mathscr{U}_2; \mathbf{GL}_n)$:

(iii) $d_2^*(\gamma^{10}) \cdot d_0^*(\gamma^{10}) = d_1^*(\gamma^{10})$.

Now one observes that the identities (i), (ii), (iii) are precisely the condition for the pair $(\gamma^{10}, \gamma^{01})$ to define an element of $H^1(X_., \mathbf{GL}_n^{\cdot})$.

## §2. Twisted resolutions

In this section we introduce twisted cocycles and twisted resolutions. Both concepts are due to Toledo and Tong ([22], see also [16]) and the results of this section are adapted from their work.

Let $X_.$ be a simplicial scheme and $E^*$ a complex of coherent locally free sheaves on $X_0$. On $X_n$, for each $i$ $(0 \le i \le n)$ we have a complex $E_i^* = X(\varepsilon_i)^* E^*$ of locally free $\mathcal{O}_{X_n}$ modules where $X(\varepsilon_i) : X_n \to X_0$ is the "$i$-th vertex" map corresponding to the map $\varepsilon_i : [0] \to [n]$ sending 0 to $i$. We define a brigaded module:

$$C^p(X_., \mathrm{End}^q(E)) = \mathrm{Hom}^q_{\mathcal{O}_{X_p}}(E_0^*, E_p^*).$$

($\mathrm{Hom}^q$ = maps of degree $q$) which has an associative product defined by $(f^{p,q} \in C^p(X_., \mathrm{End}^q(E)),\ g^{r,s} \in C^r(X_., \mathrm{End}^s(E)))$ : $f^{p,q} \cdot g^{r,s} = (-1)^{qr}(X(p, \ldots, p+r)^* g^{r,s}) \circ (X(0, \ldots, p)^* f^{p,q}) \in C^{p+r}(X, \mathrm{End}^{q+s}(E))$. (Here and in the future, a multi-index $0 \le \alpha_0 < \ldots < \alpha_m \le 0$ defines a map $\alpha : [m] \to [n]$, $\alpha(i) = \alpha_i$ and $X(\alpha)$ is written $(X(\alpha_0, \ldots, \alpha_n))$. We also have a differential

$$\delta : C^p(X_., \mathrm{End}^q(E)) \to C^{p+1}(X_., \mathrm{End}^q(E))$$
$$\delta = \sum_{i=1}^{p} (-1)^i d_i^*,$$

where $d_i^* : \mathrm{Hom}^q_{\mathcal{O}_{X_p}}(E_0^*, E_p^*) \to \mathrm{Hom}^q_{\mathcal{O}_{X_{p+1}}}(E_0^*, E_{p+1}^*)$ is the natural map (note that for $1 \le i \le n-1$, $\varepsilon_j \circ d_i = \varepsilon_j$ for $j = 0$ or $n$).

Definition 2.1: (1) A *twisting cochain* is an element $a = \sum_{p=0}^{\infty} a^{p,1-p}$ of total degree 1 in $C^*(X_., \mathrm{End}^*(E^\cdot))$ satisfying:

(i) $a^{0,1}$ is the differential of $E^*$.

(ii) $\delta a + a_. a = 0$.

We shall refer to the pair $(E^*, a)$ as a twisted complex.

(2) Let $\mathcal{F}^\cdot$ be a coherent sheaf on $X_.$. A twisted resolution of $\mathcal{F}^\cdot$ is a triple $(e, E^*, a)$ where $E^*$ is a complex of locally free $\mathcal{O}_{X_0}$ modules, with $E^j = 0$ for $j > 0$, $a$ is a twisting cocyle for $E^*$ and $e : \mathrm{E}^\circ \to \mathcal{F}^\circ$ is an augmentation, such that:

(i) $\forall n \ge 0$, $\forall i$ $(0 \le i \le n)$

$$e_i : E_i^* \to \mathcal{F}^n$$

is a resolution of $\mathcal{F}^n$ $(e_i = X(\varepsilon_i)^*((e) \cdot \mathcal{F}(\varepsilon_i)))$.

(ii) The following diagram commutes:

$$\begin{array}{ccc} d_1^* E^* = E_0^* & \xrightarrow{a^{1,0}} & E_1^* = d_0^* E^* \\ \downarrow e_0 & & \downarrow e_1 \\ \mathcal{F}^1 & = & \mathcal{F}^1 \end{array}$$

Note that conditions (i) and (ii) force $a^{1,0}$ to be a weak equivalence of complexes. Since $a$ is a twisted cocycle we have (in $C^2(X_., \mathrm{End}^*(E^\cdot))$):

$$d_2^*(a^{1,0}) \circ d_0^*(a^{1,0}) = d_1^*(a^{1,0}) + a^{2,-1} a^{0,1} + a^{0,1} a^{2,-1},$$

and because $a^{0,1}$ is the differential in $E^*$ this means that $a^{2,-1}$ is a homotopy between $d_2^*(a^{1,0}) \cdot d_0^*(a^{1,0})$ and $d_1^*(a^{1,0})$. Hence the twisted complex $(E^*, a)$ is an approximation "in the derived category" to a complex of vector bundles on $X$ (see Example (1.1): $(E^*, a)$ being a complex of vector bundles on $X$ is equivalent to requiring $a^{p,1-p} = 0$ for $p > 1$). The advantage of twisted resolutions is that even though a coherent sheaf $\mathcal{F}$ that has local resolutions by vector bundles on a scheme $X$

may not have a global resolution, we have:

THEOREM 2.2: *Let X be a scheme and $\mathscr{F}$ a coherent sheaf on X, locally of finite projective dimension. Then there exists an open cover $\{U_\alpha\}$ of X such that there is a twisted resolution of the restriction of $\mathscr{F}$ to $N_.\{U_\alpha\}$.*

PROOF: Choose $\{U_\alpha\}$ an affine cover so that on each $U_\alpha$, $\mathscr{F}|_{U_\alpha}$ has a finite locally free resolution:

$$e_\alpha : E^*_\alpha \to \mathscr{F}|_{U_\alpha}.$$

On each $U_{\alpha_0} \cap U_{\alpha_1}$ two such resolutions are related by a map of complexes

$$a^{1,0}_{\alpha_0\alpha_1} : E^*_{\alpha_0}\Big|_{U_{\alpha_0}\cap U_{\alpha_1}} \to E^*_{\alpha_1}\Big|_{U_{\alpha_0}\cap U_{\alpha_1}}.$$

Proceeding by induction on $n$ we may assume that we have constructed $a^{p,1-p}$ for all $p < n$. First recall that if $K^*, L^*$ are complexes in an abelian category, then the differential in $\mathrm{Hom}^*(K^\cdot, L^\cdot)$ is

$$D(f) = f \circ d_K + (-1)^{|f|+1} d_L \circ f$$

where $|f|$ is the degree of $f$. Hence if $D$ is the differential in $\coprod_{\alpha_0,\ldots,\alpha_p} \mathrm{Hom}^*_{\mathscr{O}_{U_{\alpha_0}\cap\ldots\cap U_{\alpha_p}}}(E_{\alpha_0}, E_{\alpha_p})$ we find that if $f$ is an element of bidegree $(p,q)$ in $C^*(N_.\{U_\alpha\}, \mathrm{End}^*(E^\cdot))$ that

$$D(f) = a^{01} f^{pq} + (-1)^{p+q+1} f^{pq} a^{01}. \tag{2.3}$$

Now consider

$$A_n = \sum_{i=1}^{n-1} a^{i,1-i} a^{n-i,1-n+i} + \delta(a^{n-1,2-n}) \in C^n(N_.\{U_\alpha\}, \mathrm{End}^{2-n}(E^\cdot)).$$

When $n = 2$, $(A_n)_{\alpha\beta\gamma} = a^{1,0}_{\alpha\beta} a^{1,0}_{\beta\gamma} - a^{1,0}_{\alpha\gamma}$ and since $E^*_\alpha|_{U_\alpha\cap U_\beta\cap U_\gamma}$ and $E^*_\gamma|_{U_\alpha\cap U_\beta\cap U_\gamma}$ are both resolutions of $\mathscr{F}$, $(A_n)$ represents zero in $Ext^0_{\mathscr{O}_{U_\alpha\cap U_\beta\cap U_\gamma}}(E_\alpha, E_\gamma)$, hence there exixts an element $f^{2,1}_{\alpha,\beta,\gamma} \in Hom^{-1}_{\mathscr{O}}|_{U_\alpha\cap U_\beta\cap U_\gamma}(E_\alpha, E_\gamma)$ such that $D(f^{2,1}) = A_2$; we set $a^{2,1} = -f^{2,1}$. When $n > 2$, we shall show $D(A_n) = 0$ and then since $\mathrm{Ext}^i_{\mathscr{O}_{N_n\{U_\alpha\}}}(\mathscr{F}|_{N_n\{U_\alpha\}}, \mathscr{F}|_{N_n\{U_\alpha\}}) = 0$ for $i < 0$ there exists an element $f^{n,1-n}$ of

$\mathrm{Hom}^{1-n}_{\mathcal{O}_{N_n\{U_\alpha\}}}(E_0^*, E_n^*)$ such that

$$D(f^{n,1-n}) = A_n.$$

If we set $a^{n,1-n} = -f^{n,1-n}$ then by (2.3) this equation becomes:

$$\sum_{i=1}^{n} a^{i,1-i}a^{n-i,1-n+i} + \delta(a^{n-1,2-n}) = 0$$

and the inductive step is completed. Now

$$\begin{aligned} D(A_n) &= a^{01}A_n - A_n a^{01} \\ &= \sum_{i=1}^{n-1} (a^{0,-1}a^{i,1-i}a^{n-i,1-n+i} - a^{i,1-i}a^{n-i,1-n+i}a^{01}) \\ &\quad + a^{01}\delta(a^{n-1,2-n}) - \delta(a^{n-1,2-n})a^{01}. \end{aligned} \tag{2.4}$$

By the induction hypothesis, for all $p < n$ we have:

$$\sum_{i=1}^{p} a^{i,1-i}a^{p-i,1-p+i} + \delta(a^{p-1,2-p}) = 0. \tag{2.5}$$

It also follows from the definition of $\delta$ and of the product in $C^*(N_.\{U_\alpha\}, \mathrm{End}^*(E^{\cdot}))$ that:

$$\delta(f^{p,q}g^{r,s}) = \delta(f^{p,q})g^{r,s} + (-1)^{p+q}f^{p,q}\delta(g^{r,s}). \tag{2.6}$$

By (2.5)

$$\begin{aligned} &a^{i,1-i}a^{n-i,1-n+i}a^{01} \\ &\quad = -a^{i,1-i}\sum_{j=0}^{n-i-1} a^{j,1-j}a^{n-i-j,1-n+i+j} - a^{i,1-i}\delta(a^{n-i-1,2-n+i}) \\ &\quad = (\sum_{k=0}^{i-1} a^{k,1-k}a^{1-k,1-k+i} + \delta(a^{k-1,2-i}))a^{n-i,1-n+i} \\ &\quad\quad - a^{i,1-i}\sum_{j=1}^{n-i-1} a^{j,1-j}a^{n-i-j,1-n+i+j} - a^{i,1-i}\delta(a^{n-i-1,2-n+i}). \end{aligned}$$

Hence

$$\begin{aligned} D(A_n) &= \sum_{i=1}^{n-1} a^{01}a^{i,1-i}a^{n-i,1-n+i} \\ &\quad - \sum_{i=1}^{n-1}\{\sum_{k=0}^{i-1} a^{k,1-k}a^{i-k,1-k+i} + \delta(a^{i-1,2-i})\}a^{n-i,1-n+i} \end{aligned}$$

$$+\sum_{i=1}^{n-1}\{a^{i,1-i}\sum_{j=1}^{n-i-1}a^{j,1-j}a^{n-i-j,1-n+i+j} - a^{i,1-i}\delta(a^{n-i-1,2-n+i})\}$$
$$+ a^{0,1}\delta(a^{n-1,2-n}) - \delta(a^{n-1,2-n})a^{0,1}.$$

All the terms in this expression sum to zero except those involving $\delta$, and by (2.6) they may be seen to equal

$$\delta(\sum_{i=0}^{n-1} a^{i,1-i}a^{n-1-i,2-n+i} + \delta(a^{n-2,3-n}))$$

which is zero by the induction hypothesis.

This proof is different from that of [22] and has advantage of showing the existence of twisted resolutions for general schemes. Twisted resolutions have advantages even when $X$ is regular (when global locally free resolutions do exist), since twisted resolutions can be constructed using preferred local resolutions such as the Koszul complex. This was Toledo and Tong's original motivation for the definition (see [22] where they use twisted resolutions to prove the Riemann–Roch theorem).

## §3. The Universal Local Complete Intersection

We turn now to the construction, in the category of simplicial schemes, of the "classifying space" $Z^p_{\cdot} \subset BL^p_{\cdot}$. Given a codimension $p$ local complete intersection $Y \to X$, there exists, by definition, an affine open cover $\{U_\alpha\}$ of $X$ such that the ideal of $Y \cap U_\alpha$ is generated by a regular sequence $\{f_1^\alpha, \dots, f_p^\alpha\}$ in $\Gamma(U_\alpha, \mathcal{O}_X)$. $Y \cap U_\alpha$ is then the inverse image of the origin in $\mathbb{A}^p_{\mathbb{Z}}$ under the map

$$f^\alpha = (f_1^\alpha, \dots, f_p^\alpha)^t : U_\alpha \to \mathbb{A}^p_{\mathbb{Z}}.$$

(We shall think of $\mathbb{A}^p$ as the space $\mathbb{M}_{p,1}$ of $p \times 1$ matrices). $f^\alpha$ can be regarded as a "local trivialization" of $Y$, and we set $Z_0 \subset BL^p_0$ equal to $\{0\} \subset \mathbb{A}^p_{\mathbb{Z}}$. Two such local trivializations $f^\alpha$ and $f^\beta$ say, differ by an element $T^{\alpha\beta}$ of $M_{pp}(\mathcal{O}_X(U_\alpha \cap U_\beta))$ such that $f^\alpha = T^{\alpha\beta}f^\beta$. Unlike a transition function between two local trivializations of a vector bundle, $T^{\alpha\beta}$ is unique only up to homotopy. This means that if we regard $T^{\alpha\beta}$ as a map between the Koszul complexes $K(f^\alpha)_*$ and $K(f^\beta)_*$ (whose terms we think of as being composed of row vectors), any $p \times \binom{p}{2}$ matrix $H$ defines a map $K(f^\alpha)_1 \to K(f^\beta)_2$ such that $H \cdot d \cdot f^\beta = 0$ where $d : K(f^\beta)_2 \to K(f^\beta)_1$ is the differential in the Koszul complex and for such an $H$, $T^{\alpha\beta} + H \cdot d$ is an alternative transition matrix between the two local trivializations of $Y$. In order to take into account this lack of uniqueness, the classifying space $BL^p_{\cdot}$ must contain information about all possible transition functions between local trivializations and also about all homotopies between different choices of transition function. In order to construct $Z^p_n \subset BL^p_n$ for all $n$, we must first construct $Z_1 \subset BL^p_1$ so that a morphism $U \to BL^p_1$ transverse to $Z_1$ defines a pair $(f, T)$ where $f = (f_1, \ldots, f_p)^t$ is a regular sequence in $\mathcal{O}_U$ and $T = (t_{ij})$ $1 \le i, j \le p$ is a $p \times p$ matrix of functions in $\mathcal{O}_U$ such that the ideals generated by $f$ and $Tf$ coincide. That we are able to do this, follows from:

LEMMA 3.1: *If $0 < r < s$ let $D \subset \mathbb{M}_{s,r} = \mathrm{Spec}(\mathbb{Z}[x_{ij}]$ $1 \le i \le s$, $1 \le j \le r)$ be the universal determinental subscheme whose ideal is generated by the maximal minors $\Delta_{\underline{k}}(X)$ of the matrix $X = (x_{ij})$; there are $\binom{s}{r}$ such minors, one for each increasing r-tuple $\underline{k} = \{0 \le k_1 < k_2 < \ldots < k_r \le s\}$. Now consider the scheme $\mathbb{M}_{s,r} \times \mathbb{M}_{\binom{s}{r},\binom{s}{r}} = \mathrm{Spec}(\mathbb{Z}[X, Y])$ where $X = (x_{i,j})$ $1 \le i \le s, 1 \le j \le r$; $Y = (y_{k,l})$, $\underline{k} = (0 \le k_1 < \ldots < k_r \le s)$, $\underline{l} = (0 \le l_1 < \ldots < l_r \le s)$ and define $\Gamma$ to be the set of points of this scheme where the ideals $(\Delta(X) = (\{\Delta_{\underline{k}}(X)\}_{\underline{k}})$ and $(Y.\Delta(X)) = (\{\sum_l y_{k,\underline{l}}\Delta(X)_l\}_k)$ coincide. Then $\Gamma$ is a Zariski open subset.*

PROOF: Define the $\mathbb{Z}[X, Y]$ ($X$, $Y$ as above) module $C$ by the exact sequence:

$$0 \to (Y, \Delta(X)) \to (\Delta(X)) \to C \to 0.$$

Then $\Gamma = \mathbb{M}_{s,r} \times \mathbb{M}_{\binom{s}{r},\binom{s}{r}} - \mathrm{Supp}(C)$ and is therefore Zariski open.

In fact, in general if $X$ is any scheme and $A, B$ are $r \times s$ and $s \times t$ matrices respectively of elements of $\Gamma(X, \mathcal{O}_X)$ then the subset of $X$ on which the ideals $(B)$ and $(A, B)$ coincide is Zariski open in $X$.

We can given an explicit description of $X$ as follows. The ideal $(\Delta(X))$ has an explicit resolution which is described in [8]. As in the discussion preceding the lemma, the two ideals $(\Delta(X))$ and $(Y.\Delta(X))$ coincide if and only if there are matrices $Z \in M_{\binom{s}{r},\binom{s}{r}}(\mathbb{Z}[X, Y])$, $H \in M_{\binom{s}{r},\binom{s}{r}r}(\mathbb{Z}[X, Y])$ such that

$$ZY = I + HR \tag{3.2}$$

where $I$ is the $\binom{s}{r} \times \binom{s}{r}$ identity matrix and $R$ is the $\binom{s}{r+1}r \times \binom{s}{r}$

matrix representing the first differential in the Eagon-Northcott resolution of the ideal $(\Delta(X))$. Equation (3.2) can also be written:

$$I = HR - ZY.$$

It is now clear that the existence of Hand $Z$ as above is equivalent to requiring that the $\left(r\binom{s}{r+1}+\binom{s}{r}\right)\times\binom{r}{s}$ matrix $\binom{R}{Y}$ has maximal rank, which is an open condition on $\mathbb{M}_{s,r}\times\mathbb{M}_{\binom{s}{r},\binom{s}{r}}$.

THEOREM 3.3: *For each $p \geq 1$ there exists a simplicial science $BL^p_\bullet$ of finite type over $\mathbb{Z}$ and a closed sub-simplicial scheme $Z_\bullet \subset BL^p_\bullet$ such that*

(i) $Z^p_0 \subset BL^p_0$ *is isomorphic to the pair* $\{0\} \subset \mathbb{A}^p_{\mathbb{Z}}$.

(ii) *for each $n \geq 0$, $BL^p_n$ is smooth over $\mathbb{Z}$.*

(iii) *for each $n \geq 0$, $Z^p_n$ is a complete intersection subscheme of $BL^p_n$ and is smooth over $\mathbb{Z}$.*

(iv) *for each $n \geq 1$ and each $i = 0,\dots,n$, the diagram*

$$\begin{array}{ccc} Z^p_n & \longrightarrow & BL^p_n \\ \downarrow{\scriptstyle d_i} & & \downarrow{\scriptstyle d_i} \\ Z^p_{n-1} & \longrightarrow & BL^p_{n-1} \end{array}$$

*is Cartesian, and $[Z^p_n]$ is the inverse image under $d_i$ of $[Z^p_{n-1}]$ in the sense of algebraic cycles* ([25]).

(v) $\mathcal{O}_{Z^p_\bullet}$ *has a twisted resolution on $BL^p_\bullet$.*

PROOF: We shall construct $BL_n$ by induction on $n$, building up $BL^p_\bullet$ by skelata. For each $n \geq 0$ set

$$P^p_n = \prod_{i=0}^{p} \prod_{0\leq\alpha_0<\dots<\alpha_i\leq n} \mathbb{M}_{p,\binom{p}{i}}.$$

Remember that each multi-index $(\alpha_0,\dots,\alpha_i)$ is to be viewed as an injective monotone map $\alpha:[i]\to[n]$; we shall write $i = |\alpha|$. Given $\alpha$ with $|\alpha| = i$, we write the $p \times \binom{p}{i}$ matrix of coordinates on the $\alpha$ factor of $P^p_n$ as $\eta^\alpha$. For any monotone $\tau:[m]\to[n]$ there is a natural map $P(\tau): \mathrm{P}^p_n \to \mathrm{P}^p_m$ defined by setting

$$P(\tau)^*\eta^\alpha = \begin{cases} \eta^{\tau\cdot\alpha} \text{ if } \tau(\alpha_j) < \tau(\alpha_{j+1}) \text{ for all } j = 0,\ldots,|\alpha|-1. \\ 0_{p,\binom{p}{|\alpha|}} \text{ if } \tau(\alpha_j) = \tau(\alpha_{j+1}) \text{ for some } j = 0,\ldots,|\alpha|-1 \text{ and } \\ |\alpha| > 1. \\ I_{p,p} \text{ if } |\alpha| = 1 \text{ and } \tau(\alpha_0) = \tau(\alpha_1). \end{cases}$$

Clearly $\tau \to P(\tau)$ is a contravariant functor, so the family $P^p_{\cdot} = \{P^p_n\}_{n\geq 0}$ is a simplicial scheme.

$BL^p_{\cdot}$ is going to be constructed as a locally closed subscheme of $P^p_{\cdot}$. For each $n \geq 0$ and each injective monotone map $\alpha:[i] \to [n]$ for $i \leq n$, the restriction of $\eta^\alpha$ to $BL^p_n$, which we shall also write $\eta^\alpha$, is a $p \times \binom{p}{i}$ matrix of functions on $BL^p_n$; in particular for each $j = 0,\ldots,n$ we have the column vector $\eta^j$ and the corresponding Koszul complex $K(\eta^j)_*$ of free $\mathcal{O}_{BL^p_n}$ modules. For each $j = 0,\ldots,n$ the entries of $\eta^j$ will generate the ideal of $Z^p_n \subset BL^p_n$ and the twisted resolution of $\mathcal{O}_{Z^p_{\cdot}}$ will be built out of the Koszul complexes $K(\eta^j)_*$, so for each $j$, $K(\eta^j)_* = (\varepsilon_j)^* K(\eta^0)_*$. The twisting cocycle which in the notation of §2 we would write $a^{*,1-*}$ with $a^{n,1-n}$ a map of degree $1-n$ from $K(\eta^0)_*$ to $K(\eta^n)_*$ on $BL^p_n$ we shall in fact write as $a^{n,1-n} = \Lambda^*\eta^{0,\ldots,n}$ where $\Lambda^i\eta^{0,\ldots,n}: K(\eta^0)_i \to K(\eta^n)_{i+n-1}$, with $\Lambda^1\eta^{0,\ldots,n}$ being defined by the matrix $\eta^{0,\ldots,n}$. This notation is chosen to generalize the $n = 1$ case, where $\Lambda^i\eta^{01}$ will be the usual $i$-th exterior power of $\eta^{01}$. The conditions that the $\Lambda^*\eta^{0,\ldots,n}$ from a twisting cocycle on $BL^p_{\cdot}$ may be expressed by the equations

$$\begin{aligned} D(\Lambda^*\eta^{0,\ldots,n}) &\overset{(\mathrm{def})}{=} \sum_{i=1}^{p} d_i(\eta^0)\Lambda^{i-1}\eta^{0,\ldots,n} + (-1)^{i+1}\Lambda^i\eta^{0,\ldots,n}d_i(\eta^0) \\ &= \sum_{k=1}^{n} \{(-1)^{(1-k)(n-k)}\Lambda^*n^{0,\ldots,k}\Lambda^*\eta^{k,\ldots,n} + (-1)^k\Delta^*\eta^{0,\ldots,\hat{k},\ldots,n}\} \end{aligned} \tag{3.4}$$

where for any multi-index $\alpha:[i] \to [n]$:

$$\Lambda^*\eta^\alpha = BL^p_{\cdot}(\alpha)^*(\Lambda^*\eta^{0,\ldots,i}).$$

It is important to note that if $n > 1$, $\Lambda^*\eta^{0,\ldots,n}$ is not the usual exterior power of $\eta^{0,\ldots,n}$; however since the latter makes no appearance in this paper, this abuse of notation should not cause any confusion.

Turning now to the construction, we can (following Lemma 3.1) define an open sub-simplicial scheme $Q^p_{\cdot} \subset P^p_{\cdot}$ by the condition that for all $n \geq 0$ and all $\sigma:[1] \to [n]$ the ideals $(\eta^{\sigma_1})$ and $(\eta^{\sigma_0,\sigma_1}\eta^{\sigma_1})$ coincide. To check that if $\tau:[m] \to [n]$ is a monotone map, $Q^p_n \subset P(\tau)^{-1}(Q^p_m)$ we need only observe that if the ideals $(\eta^{\sigma_1})$ and $(\eta^{\sigma_0,\sigma_1}\eta^{\sigma_1})$ coincide, so do $(\eta^{\tau(\sigma_1)})$ and $(\eta^{\tau(\sigma_0),\tau(\sigma_1)}\eta^{\tau(\sigma_1)})$.

We now construct by induction on $n \geq 0$, $BL^p_n$ as a closed subscheme of $Q^p_n$.

$n = 0$. We set $BL^p_0 = Q^p_0 = \mathbb{M}_{p,1}$. $Z_0$ is the subscheme of $BL^p_0$ defined by the equation $\eta^0 = 0$, so

$$(Z_0 \subset BL^p_0) \simeq (\{0\} \subset \mathbb{A}^p).$$

$n = 1$. $BL^p_1$ is the closed subscheme of $Q^p_1$ defined by the equation

$$E_{01} = \eta^0 - \eta^{01} \cdot \eta^1 = 0. \tag{3.5}$$

Clearly projection to the $(\eta^{01}, \eta^1)$ factor of $P^p_1$ defines an open immersion of $BL^p_1$ into the affine space $\mathbb{M}_{p,p} \times \mathbb{M}_{p,1}$, hence the map $d_0 : BL^p_1 \to BL^p_0$ is smooth and the entries of $\eta^1$ form a regular sequence on $BL^p_1$. Turning to $d_1$, consider the diagram:

$$\begin{array}{ccl} BL^p_1 & \xrightarrow{\ j\ } & Q^p_1 \subset \mathbb{M}_{p,1} \times \mathbb{M}_{p,1} \times \mathbb{M}_{p,p} \\ & {\scriptstyle d_1} \searrow & \downarrow {\scriptstyle (\eta^0)_*} \\ & & BL^p_0 = \mathbb{M}_{p,1}. \end{array}$$

The immersion $j$ is defined by the equation $E_{01} = 0$ (3.5). Now in $M_{p,1}(\Omega^1_{Q^p_1/BL^p_0})$

$$dE_{01} = -d\eta^{01} \cdot \eta^1 + \eta^{01} \cdot d\eta^1.$$

If we look at the matrix describing $dE_{01}$ in terms of $d\eta^{01}$ and $d\eta^1$ we see that it has maximal rank at a point $x$ of $Q^p_1$ if either $\eta^{01}$ is invertible at $x$, or $\eta^1$ is not identically zero at $x$. However on $Q^p_1$ one or the other of these properties must hold at each point as a consequence of the discussion following the proof of Lemma 3.1, since if $\eta^1$ is zero at $x$ then all the differentials in $K(\eta^1)_*$ will be zero there too. Hence the face map $d_1 : BL^p_1 \to BL^p_0$ is smooth.

Now we can observe since $d_1$ is smooth, the entries of $\eta^0$ form a regular sequence on $BL^p_1$, and we have verified parts (iii) and (iv) of the theorem for $BL^p_1$. Furthermore, we know that both the Koszul complexes $K(\eta^i)_*$ for $i = 0$ and 1 are resolutions of $\mathcal{O}_{Z^p_1}$ and that

$$\Lambda^* \eta^{01} : K(\eta^0)_* \to K(\eta^1)_*$$

is a quasi-isomorphism.

Finally, we must check that our definition of $BL^p_1$ is compatible with the single degeneracy $s_0 : Q_0^p \to Q_1^p$; i.e., that

$$s_0(BL_0^p = Q_0^p) \subset BL_1^p \subset Q_1^p;$$

but this is obvious, for $s_0(BL_0^p)$ is the subscheme of $Q_1^p$ where $\eta^{01} = I_{p,p}$ and $\eta^0 = \eta^1$ and this is certainly a subscheme of $BL_1^p$. Note that $\Lambda^*\eta^{01}$ is the identity on $s_0(BL_1^p)$.

$n \geq 2$: Before starting the general inductive step, we need some notation. For all $n \geq 0$ we shall write $S_n^p$ for the direct factor

$$\mathrm{Spec}(\mathbb{Z}[\eta^n, \{\eta^{\alpha_0,\ldots,k}\}_{k \geq 1, \alpha_k - \alpha_{k-1} = 1}])$$

of $P_n^p$.

(3.6) Our induction hypothesis is that for $k = 1, \ldots, n-1$ we have constructed closed subschemes $BL_k^p \subset Q_k^p$ with the following eight properties (which are easily checked for $BL_1^p$).

(a) For all $i = 0, \ldots, k$ $\quad BL_k^p \subset d_i^{-1}(BL_{k-1}^p)$ and for all

$i = 0, \ldots, k-1 \quad BL_{k-1}^p \subset s_i^{-1}(BL_k^p)$.

(b) The natural map $BL_k^p \to S_k^p$ is an open immersion.

(c) The entries of $(\eta^j)$ for $j = 0, \ldots, k$ form regular sequences on $BL_k^p$ and the Koszul complexes $K(\eta^j)_*$ are all resolutions of $\mathcal{O}_{Z_k^p}$ where $Z_k \subset BL_k^p$ is defined by the equation $\eta^j = 0$ for any $j$.

(d) For each multi-index $0 \leq \alpha_0 < \alpha_1 \leq k$

$$\eta^{\alpha_0} = \eta^{\alpha_0\alpha_1}\eta^{\alpha_1}$$

and $\Lambda^*\eta^{\alpha_0\alpha_1} : K(\eta^{\alpha_0})_* \to K(\eta^{\alpha_1})_*$ is a quasi-isomorphism.

(e) For each $k \leq n-1$ we have a map of complexes

$$\Lambda^*\eta^{0,\ldots,k} : K(\eta^0)_* \to K(\eta^k)_*$$

of degree $k-1$ on $BL_k^p$, such that $\Lambda^1\eta^{0,\ldots,k} = \eta^{0,\ldots,k}$. Recall that for all $j \leq k$ and $\alpha : [j] \to [k]$, we define

$$\Lambda^*(\eta^\alpha) = BL_\cdot^p(\alpha)^*(\Lambda^*\eta^{0,\ldots,j})$$

which is a map of degree $j-1$

$$K(\eta^{\alpha_0})_* \to K(\eta^{\alpha_j})_*.$$

(f) For all $k \leq n-1$, all degeneracies $s_i : BL_{k-1}^p \to BL_k^p$ $(i = 0, \ldots, k-1)$ and all multi-indexes $0 \leq \alpha_0 < \ldots < \alpha_j \leq k$, $s_i^*\Lambda^*\eta^\alpha$ vanishes identically on $BL_{k-1}^p$.

(g) Again for all $k \leq n-1$ and all multi-indices $0 \leq \alpha_0 < \ldots < \alpha_j \leq k$, in the complex of $\mathcal{O}_{BL_k^p}$ modules

$$\mathrm{Hom}(K(\eta^{\alpha_0})_*, K(\eta^{\alpha_j})_*)_*$$

we have the equality (where $D$ is the standard differential, see the proof of theorem 2.2)

$$D(\Lambda^*\eta^\alpha) = \sum_{k=1}^{j-1} \{(-1)^{(1-k)(j-k)}\Lambda^*\eta^{\alpha_0,\ldots,\alpha_k}\Lambda^*\eta^{\alpha_k,\ldots,\alpha_j} + (-1)^k \Lambda^*\eta^{\alpha_0,\ldots,\hat{\alpha}_k,\ldots,\alpha_j}\}.$$

(h) For all $k \leq n-1$, all $j \leq k$ and all multi-indices $\alpha:[j] \to [k]$, the matrices of functions $\Lambda^i\eta^\alpha$ extend, for all $i \leq p$, to $S_k^p$ via the open immersion $BL_k^p \to S_k^p$; i.e., the entries of the $\Lambda^i\eta^\alpha$ may be expressed as polynomials in the entries $\eta^\alpha$ for $0 \leq \alpha_0 < \ldots < \alpha_j \leq k$ with either $\alpha_0 = k$ or $\alpha_k - \alpha_{k-1} = 1$.

We now set $BL_n^p$ equal to the closed subscheme of

$$R_n^p = \bigcap_{\substack{0 \leq i < n \\ \alpha:[i]\to[n]}} Q(\alpha)^{-1}(BL_i^p) \subset Q_n^p$$

defined by the matrix of equations (where we set $\eta^{\alpha_0,\ldots,\alpha_i} = 0$ if $i > p$):

$$\eta^{0,\ldots,n} d_n(\eta^n) - \sum_{k=1}^{n-1} \{(-1)^{(1-k)(n-k)}\eta^{0,\ldots,k}\Lambda^k\eta^{k,\ldots,n} + (-1)^k \eta^{0,\ldots,\hat{k},\ldots,n}\} = 0. \tag{3.7}$$

We shall write the left-hand side of this equation as $E_{0,\ldots,n}$. Note that $\Lambda^k\eta^{k,\ldots,n}$ makes sense on $R_n^p$ by virtue of the existence of $\Lambda^k\eta^{0,\ldots,n-k}$ on $BL_{n-k}^p$, so that the construction of the twisting cocyle $\Lambda^*\eta^{0,\ldots,n}$ is an integral part of the construction of $BL_\cdot^p$.

We must now check that $BL_n^p$ satisfies conditions (a)–(h).

(a) $BL_n^p$ has been constructed as a subscheme of $\bigcap_{i=0}^{n} d_i^{-1}(BL_n^p)$ so the first part is tautologous. To check compatibility with degeneracies we first remark that for any $i = 0,\ldots,n-1$ and any $j = 0,\ldots,n$, we have

$$d_j \cdot s_i = s_k \cdot d_l : Q_{n-1}^p \to Q_{n-1}^p$$

where

$$\begin{Bmatrix} k = i-1 \\ j = l \end{Bmatrix} \text{ if } j < i \text{ and } \begin{Bmatrix} k = i \\ j = l+1 \end{Bmatrix} \text{ if } j > i+1$$

while

$$d_i s_i = Id : Q^p_{n-1} \to Q^p_{n-1}$$

hence

$$s_i(BL^p_{n-1}) \subset \bigcap_{0 \le j \le n} d_j^{-1}(BL_{n-1}).$$

Therefore it suffices to show that (see 3.7) $s_i^*(E_{0,\dots,n})$ vanishes on $BL^p_{n-1}$. However by the induction hypothesis $\Lambda^*\eta^{\alpha_0,\dots,\alpha_j} = 0$ for $2 \le j \le n-1$ if $\alpha_k = \alpha_{k+1}$ for any $k \le j-1$ and $\Lambda^*\eta^{k,k} = I$ for all $k \le n-1$, so

$$\begin{aligned}
&s_i^*(E_{0,\dots,n}) = \eta^{0,\dots,i,i,\dots,n-1}\\
&- \sum_{k=1}^{i} \{(-1)^{(1-k)(n-k)}\eta^{0,\dots,k}\Lambda^k\eta^{k,\dots,i,i,\dots,n-1}\\
&+ (-1)^k\eta^{0,\dots,\hat{k},\dots,i,i,\dots,n-1}\}\\
&- \sum_{k=i}^{n-1} (-1)^{(1-k-1)(n-k-1)}\eta^{0,\dots,i,i,\dots,k}\Lambda^{k+1}\eta^{k+1,\dots,n-1}\\
&+ (-1)^{k+1}\eta^{0,\dots,i,i,\dots,\hat{k},\dots,n-1}\}\\
&= (-1)^i\eta^{0,\dots,\hat{i},i,\dots,n-1} + (-1)^{i+1}\eta^{0,\dots,i,\hat{i},\dots,n-1}\\
&= 0.
\end{aligned}$$

(b) Turning to the natural map $BL^p_n \to S^p_n$, first observe that it factors through a map

$$h : BL^p_n \to BL^p_{n-1} \times \mathrm{Spec}(\mathbb{Z}[\{\eta^{0,\alpha_1,\dots,\alpha_k}\}_{1 \le \alpha_1 < \dots < \alpha_{k-1} = \alpha_k - 1 \le n-1}])$$

$$\stackrel{\mathrm{def}}{=} BL^p_{n-1} \times V^p_n$$

which is the product of the face map

$$d_0 : BL^p_n \to BL^p_{n-1}$$

and the natural map

$$BL^p_n \to V^p_n.$$

Given the induction hypothesis and that $S^p_n = S^p_{n-1} \times V^p_n$, in order to show that $BL^p_n \to S^p_n$ is an open immersion it is sufficient to show that $h$

is an open immersion. Next observe that $h$ is the composition of the closed immersion

$$j_n : BL_n^p \to BL_{n-1}^p \times_{Q_{n-1}^p} Q_n^p$$

deduced from the diagram

$$\begin{array}{ccccc} BL_n^p & \longrightarrow & Q_n^p & \longrightarrow & P_n^p \\ \downarrow d_0 & & \downarrow d_0 & & \downarrow d_0 \\ BL_{n-1}^p & \to & Q_{n-1}^p & \to & P_{n-1}^p \end{array}$$

(where the horizontal maps are the natural inclusions) with the projection

$$BL_{n-1}^p \times_{Q_{n-1}^p} Q_n^p \to BL_{n-1}^p \times V_n^p$$

deduced from the projection from

$$P_n^p = P_{n-1}^p \times \operatorname{Spec} \mathbb{Z}[\{\eta^{0,\alpha_1,\ldots,\alpha_k}\}_{1 \le \alpha_1 < \ldots < \alpha_k \le n}] \overset{\text{def}}{=} P_{n-1}^p \times W_n^p \text{ to}$$

$$P_{n-1}^p \times V_n^p,$$

and the inclusion $BL_{n-1}^p \to P_{n-1}^p$. Our first step is therefore to study the closed immersion $j_n$, which is defined by the ideal $I_n^p$ generated by the entries of the matrices

$$E_{\alpha_0,\ldots,\alpha_k} = \eta^{\alpha_0,\ldots,\alpha_k} d_k(\eta^{\alpha_k}) - \sum_{r=1}^{k-1} \{(-1)^{(1-r)(k-r)} \eta^{\alpha_0,\ldots,\alpha_r} \Lambda^r \eta^{\alpha_r,\ldots,\alpha_k} + (-1)^r \eta^{\alpha_0,\ldots,\hat{\alpha}_r,\ldots,\alpha_k}\}$$

as $\alpha$ runs through all multiindices $0 = \alpha_0 < \alpha_1 < \ldots < \alpha_k \le n$ for $k = 1,\ldots,n$. Note that the use of $\Lambda^r \eta^{\alpha_r,\ldots,\alpha_k}$ in those equations is allowable since it is defined on $BL_{n-1}^p$.

LEMMA 3.8: *For all $n \ge 1$, the ideal $I_n^p$ defining the closed immersion $j_n$ is in fact equal to its subideal $J_n^p$ generated by the entries of the $E_\alpha$ as $\alpha$ runs through only those indices with $\alpha_k - \alpha_{k-1} = 1$.*

PROOF: Consider $E_\alpha$ with $0 = \alpha_0 < \ldots < \alpha_k \le n$ and $\alpha_k - \alpha_{k-1} \ge 1$. We use induction first on $n \ge 1$ and then on $d(\alpha) = \alpha_k - \alpha_0 - k$ to prove that the entries of $E_\alpha$ lie in $J_n^p$. The case $n = 1$ we have seen already, while for any $n \ge 1$ if $d(\alpha) = 0$ we must have $\alpha_i - \alpha_{i-1} = 1$ for all

$i = 1,\dots,k$ and in particular $\alpha_k - \alpha_{k-1} = 1$, hence we can start the induction off. Suppose now that $J_m^p = I_m^p$ for all $m = 1,\dots,n-1$ and that for all $\beta = (0 = \beta_0 < \beta_1 < \dots < \beta_l \leq n)$ with $d(\beta) < d(\alpha)$ for a given $\alpha = (0 = \alpha_0 < \alpha_1 < \dots < \alpha_k \leq n)$ with $\alpha_k - \alpha_{k-1} > 1$ we know that the entries of $E_\beta$ lie in $J_n^p$. Since $d_{n-1}d_0 = d_0 d_n$ we have a commutative diagram:

$$\begin{array}{ccc} BL_n^p & \xrightarrow{j_n} & BL_{n-1}^p \times_{Q_{n-1}^p} Q_n^p \\ \big\downarrow{\scriptstyle d_n} & & \big\downarrow{\scriptstyle d_{n-1}\times d_n} \\ BL_{n-1}^p & \xrightarrow{j_{n-1}} & BL_{n-2}^p \times_{Q_{n-2}^p} Q_{n-1}^p \end{array}$$

and $(d_{n-1} \times d_n)^* J_{n-1}^p \subset J_n^p$ is the subideal generated by the entries of those $E_\beta$ with $0 = \beta_0 < \beta_1 < \dots < \beta_{l-1} = \beta_l - 1 \leq n-1$. We may therefore suppose that $\alpha_k = n$. Let us write $\alpha'$ for the multiindex $0 = \alpha_0 < \dots < \alpha_{k-1} < \alpha_k - 1 < \alpha_k = n$; then the entries of $E_{\alpha'}$ lie in $J_n^p$ and if we can show that the entries of

$$-(E_{\alpha'}d_k(\eta^n) + (-1)^k E_\alpha) \tag{3.9}$$

lie in $J_n^p$ we shall be done. Expanding out (3.9) we get, since $d_{k+1}(\eta^n)d_k(\eta^n) = 0$:

$$\begin{aligned} &\sum_{r=1}^{k-1}\{(-1)^{(1-r)(k+1-r)}\eta^{\alpha_0,\dots,\alpha_r}\Lambda^r\eta^{\alpha_r,\dots,\alpha_{k-1},n-1,n} \\ &+ (-1)^r\eta^{\alpha_0,\dots,\hat{\alpha}_r,\dots,\alpha_{k-1},n-1,n} \\ &+ (-1)^{k-1}\eta^{\alpha_0,\dots,\alpha_{k-1},n-1}\Lambda^k\eta^{n-1,n}\}d_k(\eta^n) \\ &+ (-1)^k\sum_{r=1}^{k-1}\{(-1)^{(1-r)(k-r)}\eta^{\alpha_0,\dots,\alpha_r}\Lambda^r\eta^{\alpha_r,\dots,\alpha_{k-1},n} \\ &+ (-1)^r\eta^{\alpha_0,\dots,\hat{\alpha}_r,\dots,\alpha_{k-1},n}\}. \end{aligned} \tag{3.10}$$

We want to show that formula (3.10) vanishes mod $J_n^p$. By part $g$ of the induction hypothesis (3.6) we know that for $0 < r \leq k-1$,

$$\begin{aligned} &\Lambda^r\eta^{\alpha_r,\dots,\alpha_{k-1},n-1,n}d_k(\eta^n) \\ &= (-1)^{(k-r)}d_{r-1}(\eta^{\alpha_r})\Lambda^{r-1}\eta^{\alpha_r,\dots,\alpha_{k-1},n-1,n} \\ &+ \sum_{s=1}^{k-1-r}\{(-1)^{(1-s)(k+1-r-s)}\Lambda^r\eta^{\alpha_r,\dots,\alpha_{r+s}}\Lambda^{r+s-1}\eta^{\alpha_{r+s},\dots,\alpha_{k-1},n-1,n} \\ &+ (-1)^s\Lambda^r\eta^{\alpha_r,\dots,\hat{\alpha}_s,\dots,\alpha_{k-1},n-1,n}\} \\ &+ (-1)^{1-k+r}\Lambda^r\eta^{\alpha_r,\dots,\alpha_{k-1},n-1}\Lambda^{k-1}\eta^{n-1,n} \\ &+ (-1)^{(k-r)}\Lambda^r\eta^{\alpha_r,\dots,\alpha_{k-1},n}. \end{aligned} \tag{3.11}$$

Furthermore, by the induction hypothesis (since $d(\alpha_0,\ldots,\alpha_r) < d(\alpha)$):

$$\eta^{\alpha_0,\ldots,\alpha_r} d_r(\eta^{\alpha_r}) = \sum_{i=1}^{r-1} \{(-1)^{(1-i)(r-i)} \eta^{\alpha_0,\ldots,\alpha_i} \Lambda^i \eta^{\alpha_i,\ldots,\alpha_r} + (-1)^i \eta^{\alpha_0,\ldots,\hat{\alpha}_i,\ldots,\alpha_r}\} \tag{3.12}$$

mod $J_n^p$, and since by definition, the entries of $E_{\alpha_0,\ldots,\hat{\alpha}_r,\ldots,\alpha_{k-1},n-1,n}$ lie in $J_{p,n}$ we have:

$$\begin{aligned} &\eta^{\alpha_0,\ldots,\hat{\alpha}_r,\ldots,\alpha_{k-1},n-1,n} d_k(\eta^n) \\ &= \sum_{t=1}^{r-1} \{(-1)^{(1-t)(k-t)} \eta^{\alpha_0,\ldots,\alpha_t} \Lambda^t \eta^{\alpha_t,\ldots,\hat{\alpha}_r,\ldots,\alpha_{k-1},n-1,n} \\ &+ (-1)^t \eta^{\alpha_0,\ldots,\hat{\alpha}_t,\ldots,\hat{\alpha}_r,\ldots,\alpha_{k-1},\ldots,n-1,n}\} \\ &+ \sum_{t=r+1}^{k-1} \{(-1)^{(1-t+1)(k-t+1)} \eta^{\alpha_0,\ldots,\hat{\alpha}_r,\ldots,\alpha_t} \Lambda^{t-1} \eta^{\alpha_t,\ldots,\alpha_{k-1},n-1,n} \\ &+ (-1)^{(t-1)} \eta^{\alpha_0,\ldots,\hat{\alpha}_r,\ldots,\hat{\alpha}_t,\ldots,\alpha_{k-1},n-1,n}\} \end{aligned} \tag{3.13}$$

mod $J_n^p$. We may now use formulae 3.11, 3.12, 3.13 in succession to rewrite 3.10 as

$$\begin{aligned} &\sum_{r=1}^{k-1} (-1)^{(1-r)(k+1-r)} (-1)^{k-r} \cdot \\ &\cdot \sum_{u=1}^{r-1} \{(-1)^{(1-u)(r-u)} \eta^{\alpha_0,\ldots,\alpha_u} \Lambda^u \eta^{\alpha_u,\ldots,\alpha_r} \\ &+ (-1)^u \eta^{\alpha_0,\ldots,\hat{\alpha}_u,\ldots,\alpha_r}\} \Lambda^{r-1} \eta^{\alpha_r,\ldots,\alpha_k - 1,\alpha_k} \\ &+ \sum_{r=1}^{k-1} (-1)^{(1-r)(k+1-r)} \eta^{\alpha_0,\ldots,\alpha_r} \\ &+ \sum_{v=1}^{k-1-r} \{(-1)^{(1-v)(k+1-r-v)} \Lambda^r \eta^{\alpha_r,\ldots,\alpha_{r+v}} \Lambda^{r+v-1} \eta^{\alpha_{r+v},\ldots,\alpha_{k-1},\alpha_k - 1,\alpha_k} \\ &+ (-1)^v \Lambda^r \eta^{\alpha_r,\ldots,\hat{\alpha}_{r+v},\ldots,\alpha_k}\} \\ &+ \sum_{r=1}^{k-1} (-1)^{(1-r)(k+1-r)} \eta^{\alpha_0,\ldots,\alpha_r} \cdot \\ &\cdot \{(-1)^{(1-(k-r))} \Lambda^r \eta^{\alpha_r,\ldots,\alpha_{k-1},\alpha_k - 1} \Lambda^{k-1} \eta^{\alpha_k - 1,\alpha_k} \\ &+ (-1)^r \Lambda^r \eta^{\alpha_r,\ldots,\alpha_{k-1},\alpha_k}\} \\ &+ \sum_{r=1}^{k-1} (-1)^r \sum_{t=1}^{r-1} \{(-1)^{(1-t)(k-t)} \eta^{\alpha_0,\ldots,\alpha_t} \Lambda^t \eta^{\alpha_t,\ldots,\hat{\alpha}_r,\ldots,\alpha_{k-1},\alpha_k - 1,\alpha_k} \\ &+ (-1)^t \eta^{\alpha_0,\ldots,\hat{\alpha}_t,\ldots,\hat{\alpha}_r,\ldots,\alpha_k - 1,\alpha_k}\} \end{aligned}$$

$$+\sum_{r=1}^{k-1}(-1)^r\sum_{t=r+1}^{k-1}\{(-1)^{(1-t+l)(k-t+l)}\eta^{\alpha_0,\dots,\hat{\alpha}_r,\dots,\alpha_t}\Lambda^{t-1}\eta^{\alpha_t,\dots,\alpha_{k-1},\alpha_k}$$
$$+(-1)^{t-1}\eta^{\alpha_0,\dots,\hat{\alpha}_r,\dots,\hat{\alpha}_t,\dots,\alpha_{k-1}\alpha_k-1,\alpha_k}\}$$
$$+\sum_{r=1}^{k-1}(-1)^r\{(-1)^k\eta^{\alpha_0,\dots,\hat{\alpha}_r,\dots,\alpha_k-1}\Lambda^{k-1}\eta^{\alpha_k-1,\alpha_k}$$
$$+(-1)^{(k-1)}\eta^{\alpha_0,\dots,\hat{\alpha}_r,\dots,\alpha_{k-1},\alpha_k}\}$$
$$+(-1)^{k-1}\sum_{r=1}^{k-1}\{(-1)^{(1-r)(k-r)}\eta^{\alpha_0,\dots,\alpha_r}$$
$$\times\Lambda^r\eta^{\alpha_r,\dots,\alpha_k-1}\Lambda^{k-1}\eta^{\alpha_k-1,\alpha_k}$$
$$+(-1)^r\eta^{\alpha_0,\dots,\hat{\alpha}_r,\dots,\alpha_k-1}\Lambda^{k-1}\eta^{\alpha_k-1,\alpha_k}\}$$
$$+(-1)^k\sum_{r=1}^{k-1}\{(-1)^{(1-r)(k-r)}\eta^{\alpha_0,\dots,\alpha_r}\Lambda^r\eta^{\alpha_r,\dots,\alpha_{k-1},\alpha_k}$$
$$+(-1)^k\eta^{\alpha_0,\dots,\hat{\alpha}_r,\dots,\alpha_k}\}$$

It is now a straightforward, but unfortunately extremely tedious exercise for the reader to check that this monstrous formula is in fact identically zero, completing the proof of Lemma 3.8.

Given that $j_n$ is defined by the ideal $J_n^p$, we now observe that the matrices of coordinates $\{\eta^\alpha\}_{1\le\alpha_1<\dots<\alpha_k\le n}$ on $W_n^p$ can be divided into two classes; those for which $\alpha_k-\alpha_{k-1}>1$ and those for which $\alpha_k-\alpha_k=1$, and that the correspondence $\alpha\to\alpha'$ used in the proof of Lemma 3.8 defines a bijection between these two classes. Each matrix of equations $E_{\alpha'}=0$ for $\alpha'=(0=\alpha_0<\alpha_1<\dots<\alpha_{k-1}<\alpha_k-1<\alpha_k)$ expresses ($\alpha=(\alpha_0=0<\alpha_1<\dots<\alpha_{k-1}<\alpha_k=n)$) $\eta^\alpha$ as a function of $\eta^{\beta_0,\dots,\beta_l}$ with either $l<k$ or $\beta_l-\beta_{l-1}=1$. (Note that by the induction hypothesis $\Lambda^r\eta^{\alpha_r,\dots,\alpha_{k-1},\alpha_k-1,\alpha_k}$ is a polynomial function of the $\eta^\beta$ for $\{\beta_0,\dots,\beta_l\}\subset\{\alpha_r,\dots,\alpha_{k-1},\alpha_k-1,\alpha_k\}$.) That the map $h:BL_n^p\to BL_{n-1}^p\times V_n^p$ is an open immersion now follows from:

LEMMA 3.15: *Let S be a scheme, $A_1,\dots,A_n,B$ disjoint finite sets of independent variables, $Q\subset S\times\operatorname{Spec}\mathbb{Z}[A_1,\dots,A_n,B]$ a Zariski open subset. Suppose $Y\subset Q$ is a closed subscheme defined by polynomial equations, one for each $a\in A_i$, as i runs from 1 to n:*

$$a=f_a(A_1,\dots,A_{i-1},B)$$

*(if $a\in A_1$ then we suppose $f_a$ is a function of the variables in B alone). Then the natural map*

$$Y\to S\times\operatorname{Spec}\mathbb{Z}[B]$$

*is an open immersion.*

PROOF: This is essentially obvious. Clearly it is enough to show that if $Q = S \times \operatorname{Spec} \mathbb{Z}[A_1, \ldots, A_n, B]$ then $Y \simeq S \times \operatorname{Spec} \mathbb{Z}[B]$. We shall procede by induction on $n$. The case $n = 0$ is trivial. Let us write $I_n$ for the sheaf of ideals in $\mathcal{O}_S[A_1, \ldots, A_n, B]$ generated by the $f_a$ for $a \in A_i$ with $i \leq m$. By the induction hypothesis we may suppose that the natural map

$$\mathcal{O}_S[B] \to \mathcal{O}_S[A_1, \ldots, A_{n-1}, B]/I_{n-1}$$

is an isomorphism. Hence the map

$$\mathcal{O}_S[A_n, B]/(\{f_a\}_{a \in A_n}) \to \mathcal{O}_S[A_1, \ldots, A_n, B]/I_n$$

is an isomorphism. It is therefore sufficient to prove the lemma when $n = 1$; but then it becomes entirely obvious.

This completes the proof of part (b) of the induction step.

(c) Turning to $\eta^j$ for $j = 0, \ldots, n - 1$ we observe that by the equations defining $BL_n^p$ we have, on $BL_n^p$:

$$\eta^j = \eta^{j,j+1} \eta^{j+1,j+2}, \ldots, \eta^{n-1,n},$$

hence $\eta^j$ extends as a matrix $\tilde{\eta}_j$ of functions on $S_n^p$. By the construction of $BL_n^p$ we know that it is the open subset of $S_n^p$ on which the ideals generated by the entries of $\tilde{\eta}^j$ and $\eta^n$ coincide for all $j = 0, \ldots, n - 1$. On $S_n^p$ the entries of $\eta^n$ automatically form a regular sequence, and so by restriction they form a regular sequence on $BL_n^p$. For $j < n$, we first observe that we know already that on $BL_2^p$:

$$\Lambda^* \eta^{01} : K(\eta^0)_* \to K(\eta^1)_*$$

is a weak equivalence. Pulling back via the map:

$$BL^p((j, n)) : BL_n^p \to BL_2^p$$

we see that

$$\Lambda^* \eta^{j,n} : K(\eta^j)_* \to K(\eta^n)_*$$

is a weak equivalence, hence firstly, $K(\eta^j)_*$ is a resolution of $\mathcal{O}_{Z_n^p}$ where $Z_n^p$ is the closed subscheme of $BL_n^p$ defined equally by the ideals generated by the entries of $\eta^j$ for any $j = 0, \ldots, n$; and secondly, the entries of $\eta^j$ form a regular sequence on $BL_n^p$ for all $j = 0, \ldots, n$.

(d) If $0 \leq \alpha_0 < \alpha_1 \leq n$, by part (c) we know that $\Lambda^*\eta^{\alpha_0 n}$ and $\Lambda^*\eta^{\alpha_1 n}$ are weak equivalences. Since $\Lambda^*\eta^{\alpha_0,n}$ is homotopic to $\Lambda^*\eta^{\alpha_0\alpha_1}\Lambda^*\eta^{\alpha_1,n}$ it follows that the latter map is a weak equivalence and hence $\Lambda^*\eta^{\alpha_0,\alpha_1}$ is also a weak equivalence.

Before turning to (e) we need a lemma (which gives us part (h) of the inductive process).

LEMMA 3.16: *For each multiindex* $0 \leq \alpha_0 < \ldots < \alpha_k \leq n$ *with* $k \leq n - 1$ *the map*

$$\Lambda^*\eta^{\alpha_0,\ldots,\alpha_k}: K(\eta^{\alpha_0})_* \to K(\eta^{\alpha_k})_*$$

*extends to a map of complexes on* $S_n^p$

$$\tilde{\Lambda}^*\eta^{\alpha_0,\ldots,\alpha_k}: K(\tilde{\eta}^{\alpha_0})_* \to K(\tilde{\eta}^{\alpha_k})_*.$$

PROOF: If $\alpha_0 > 0$ this follows by induction on $n$, since $S_n^p = S_{n-1}^p \times V_n^p$ and $\Lambda^*\eta^{\alpha_0,\ldots,\alpha_k}$ is pulled back from $BL_{n-1}^p \subset S_{n-1}^p$ via $d_0: BL_n^p \to BL_{n-1}^p$. If $\alpha_0 = 0$, then $\Lambda^*\eta^{\alpha_0,\ldots,\alpha_k}$ is induced from $BL_{n-1}^p$ via $d_i$ for some $i > 0$; therefore we want to extend $d_i: BL_n^p \to BL_{n-1}^p$ to a map $\tilde{d}_i: S_n^p \to S_{n-1}^p$, since we know by the induction hypothesis that $\Lambda^*\eta^{\beta_0,\ldots,\beta_l}$ extends across $S_{n-1}^p$ for all $0 \leq \beta_0 < \ldots < \beta_l \leq n-1$. In order to construct $\tilde{d}_i$ it suffices to show that for all $k < n$ and $0 \leq \alpha_0 < \ldots < \alpha_k \leq n$, $\eta^{\alpha_0,\ldots,\alpha_k}$ extends to a global section $\tilde{\eta}^{\alpha_0,\ldots,\alpha_k}$ of $\mathcal{O}_{S_n^p}$. By the proof of Lemma 3.8 we see that any $\eta^{\alpha_0,\ldots,\alpha_k}$ can be expressed as a function of $\eta^{\beta_0,\ldots,\beta_l}$ with either $l < k$ or $\beta_l - \beta_{l-1} = 1$ (i.e. $\eta^{\beta_0,\ldots,\beta_l}$ is one of the coordinate functions on $S_n^p$). Therefore by induction we are reduced to the case of $l = 0$, which we have already seen in part (c) above.

(e) We now wish to construct $\Lambda^*\eta^{0,\ldots,n}$. Consider the map of complexes $G_*: K(\eta^0)_* \to K(\eta^n)_*$ defined on $BL_n^p$ by:

$$G_* = \sum_{k=1}^{n-1} \{(-1)^{(1-k)(n-k)}\Lambda^*\eta^{0,\ldots,k}\Lambda^*\eta^{k,\ldots,n} + (-1)^k\Lambda^*\eta^{0,\ldots,\hat{k},\ldots,n}\}.$$

By Lemma 3.16 this extends to a map

$$\tilde{G}_*: K(\tilde{\eta}^0)_* \to K(\tilde{\eta}^n_*)$$

of complexes on $S_n^p$. By part (g) of the induction hypothesis we know that for all $0 \leq \alpha_0 < \ldots < \alpha_k \leq n$ with $k < n$

$$D(\Lambda^*\eta^{\alpha_0,\ldots,\alpha_k}) = \sum_{j=1}^{k-1} \{(-1)^{(1-j)(k-j)}\Lambda^*\eta^{\alpha_0,\ldots,\alpha_j}\Lambda^*\eta^{\alpha_j,\ldots,\alpha_k} + (-1)^j\Lambda^*\eta^{\alpha_0,\ldots,\hat{\alpha}_j,\ldots,\alpha_k}\};$$

a straightforward but tedious calculation shows that $D(G_*) = 0$ on $BL^p_n$, hence $D(\tilde{G}_*) = 0$ on $S^p_n$. However, the complex

$$\mathrm{Hom}_{\mathcal{O}_{S^p_n}}(K(\tilde{\eta}^0)_*, K(\eta^n)_*)$$

is acyclic in positive (homological) degrees since it is quasi-isomorphic to the complex

$$\mathrm{Hom}_{\mathcal{O}_{S^p_n}}(K(\tilde{\eta}^0)_*, \mathcal{O}_{S^p_n}/(\eta^n) = \mathcal{O}_{\tilde{Z}^p_n})$$

which is concentrated in non-positive (homological) degrees, because $K(\eta^n)_*$ is a resolution of $\mathcal{O}_{\tilde{Z}^p_n}$; therefore there exists a map of complexes $\Lambda^*\eta^{0,\dots,n}: K(\tilde{\eta}^0)_* \to K(\tilde{\eta}^n)_*$ such that

$$D(\Lambda^*\eta^{0,\dots,n}) = \tilde{G}_*.$$

This completes the proof of (e).

(f) We now want to show that we can in fact make a more restrictive choice of $\Lambda^*\eta^{0,\dots,n}$. First we observe that by the proof of Lemma 3.16 each degeneracy for $i = 0,\dots,n-1$, $s_i: BL^p_{n-1} \to BL^p_n$ extends to a map $\tilde{s}_i: S^p_{n-1} \to S^p_n$ which is the inclusion of the affine subspace $\Delta_i \subset S^p_n$ defined by the equations $\eta^{i-1,i} = I$, $\eta^{i,i+1} = I$ and $\eta^\alpha = 0$ if $|\alpha| > 1$ with $(i, i+1)$ a subsequence of $(\alpha_0,\dots,\alpha_k)$. Let us write $\Delta = \bigcup_i \Delta_i$ and if $i = \{0 \le i_1 < \dots < i_j \le n-1\}$ is a multiindex we write

$$\Delta_i = \bigcap_{k=1}^{j} \Delta_{i_k},$$

finally we denote the ideals defining $\Delta$ and $\Delta_{\underline{i}}$ in $\mathcal{O}_{S^p_n}$ as $\mathcal{I}_\Delta$ and $\mathcal{I}_{\Delta_i}$ respectively. Examining the equations defining $\Delta_i \simeq S^p_{n-1}$ in $S^p_n$ we see that $\Delta_{\underline{i}}(\underline{i} = (i_1 < \dots < i_k))$ is isomorphic to $S^p_{n-k}$ via the map

$$\tilde{s}_{\underline{i}} = \tilde{s}_{i_k} \cdot \tilde{s}_{i_{k-1}} \cdot \dots \cdot \tilde{s}_{i_1} : S^p_{n-k} \to S^p_n.$$

LEMMA 3.17: *The natural map $(d_0)^n: \Delta \to BL^p_0 \simeq \mathbb{A}^p_{\mathbb{Z}}$ defined by the matrix of functions $\tilde{\eta}^n$ on $S^p_n$ is a flat morphism.*

PROOF: For each $j \ge 1$ and each $j$-tuple $\underline{i} = (i_1,\dots,i_j)$ the affine space $\Delta_i$ is flat over $BL^p_0$ since $d^n_0 \cdot s_{\underline{i}} = d_0^{n-k}: S^p_{n-k} \to BL^p_0$ is flat by the definition of $S^p_{n-k}$. Now by construction $\mathcal{O}_\Delta$ has a resolution $\mathcal{O}_\Delta \to R^*_\Delta$ where for $j \geq 0$ ($\underline{i}$ running through all $0 \leq i_1 < \ldots < i_j \leq n-1$):

$$R^j_\Delta = \oplus_{\underline{i}} \mathcal{O}_{\Delta_{\underline{i}}}.$$

Since $R^*_\Delta$ is a resolution by flat $\mathcal{O}_{BL^p_0}$ modules, $\mathcal{O}_\Delta$ is itself a flat $\mathcal{O}_{BL^p_0}$ module, and hence the entries of $\eta^n$ form a regular sequence in $\mathcal{O}_\Delta$.

It follows that $K(\eta^n)_* \otimes_{\mathcal{O}_{S^p_n}} \mathcal{I}_\Delta$ is a resolution of the module $\mathcal{O}_{\tilde{Z}^p_n} \otimes_{\mathcal{O}_{S^p_n}} \mathcal{I}_\Delta$. By the induction hypothesis $\tilde{G}_*$ vanishes on $\Delta$, and so may be viewed as a map

$$K(\tilde{\eta}^0)_* \to K(\eta^n)_* \otimes_{\mathcal{O}_{S^p_n}} \mathcal{I}_\Delta$$

such that $D(\tilde{G}_*) = 0$; since $K(\eta^n)_* \otimes_{\mathcal{O}_{S^p_n}} \mathcal{I}_\Delta$ is acyclic in positive degrees and $K(\tilde{\eta}^0)_*$ is a complex of free modules, there is a map

$$\Lambda^* \eta^{0,\ldots,n} : K(\tilde{\eta}^0)_* \to K(\eta^n)_* \otimes \mathcal{I}_\Delta \subset K(\eta^n)_*$$

such that $D(\Lambda^* \eta^{0,\ldots,n}) = \tilde{G}_*$.

Having completed part (f) of the induction process we see that parts (g) and (h) have already been covered, completing the construction of $BL^p_\bullet$.

Turning to the proof of parts (ii), (iii) and (iv) of the theorem, we see first that as a Zariski open subset of $S^p_n$ which is an affine space over $\mathbb{Z}$, $BL^p_n$ is automatically smooth over $\mathbb{Z}$ for all $n \geq 0$. Similarly for all $n \geq 0$, $Z^p_n \subset BL^p_n$ is defined by the regular sequence $\eta^n$, i.e., it is the inverse image under $(d_0)^n$ of $Z^p_0 \subset BL^p_0$, and since $S^p_n$ is smooth over $BL^p_0$, $Z^p_n$ is smooth over $Z^p_0 = \mathrm{Spec}(\mathbb{Z})$, thus proving (iii). For part (iv) we observe that for $i < n$, $d^*_i(\eta^{n-1}) = \eta^n$ and so $d_i^{-1}(Z^p_{n-1}) = Z^p_n$ both in the sense of schemes and of algebraic cycles. If $i = n$, $d^*_n(\eta^{n-1}) = \eta^{n-1}$; however by construction $\eta^{n-1}$ is again a regular sequence defining the subscheme $Z^p_n$, and so $d_n^{-1}(Z^p_{n-1}) = Z^p_n$ again, both as schemes and cycles.

Finally (v); we have already made the observation that the condition on the $\Lambda^* \eta^{0,\ldots,n}$ making them a twisting cocycle is expressed by equation (3.4), which is equivalent to part (g) of the induction hypothesis. Hence to complete the proof of the theorem we need only observe that for all $n \geq 0$ and all $i = 0, \ldots, n$, $K(\eta^i)_*$ is a resolution of $\mathcal{O}_{Z^p_n}$.

We now show that $BL^p_\bullet$ does indeed classify local complete intersections.

THEOREM 3.18: *Let $Y$ be a codimension $p$ local complete intersection subscheme of a scheme $X$. Then there exists an open cover $\{U_\alpha\}$ of $X$ and a morphism of simplicial schemes*:

$$\chi_Y : N_\bullet\{U_\alpha\} \to BL^p_\bullet$$

(*which we shall just write* $\chi$ *when there is no chance for confusion*) *such that*

(i) $\chi_Y^{-1}(Z^p_\cdot) = Y \cap N_\cdot\{U_\alpha\}$

(ii) $\forall n \geq 0,\ i \geq 1,\ \mathrm{Tor}_i^{\mathcal{O}_{BL^p_n}}(\mathcal{O}_{Z^p_n}, \mathcal{O}_{N_n\{U_\alpha\}}) = 0$;

*i.e.*, $\chi_Y$ *and* $Z^p_\cdot$ *are transverse, and by classical intersection theory* [25] *we have*

$$[Z_n]\cdot_{\chi_Y}[N_n\{U_\alpha\}] = [Y \cap N_n\{U_\alpha\}].$$

(iii) $\chi_Y^*(\Lambda^*\eta)$ *is a twisted resolution of* $\mathcal{O}_Y$ *on* $N_\cdot\{U_\alpha\}$.

(*Any* $\chi$ *satisfying* (i), (ii), (iii) *will be said to "classify* $Y$".)

PROOF: For $\{U_\alpha\}$ we may choose an affine open cover of $X$ such that for each $\alpha$, $Y \cap U_\alpha$ is generated by a regular sequence $(f_1^\alpha, \ldots, f_p^\alpha)$ in $\Gamma(U_\alpha, \mathcal{O}_X)$. Hence we have for each $\alpha$, a map $f^\alpha : U_\alpha \to \mathbb{A}^p$ defined by the column vector $(f_1^\alpha, \ldots, f_p^\alpha)^t$, and hence a map

$$\chi_0 = \coprod f^\alpha : \coprod U_\alpha = N_0\{U_\alpha\} \to BL_0^p = \mathbb{A}^p$$

satisfying (i) and (ii) of the theorem. On each $U_\alpha \cap U_\beta$ $f^\alpha$ and $f^\beta$ are related by a matrix $f^{\alpha\beta}$ such that $f^\alpha = f^{\alpha\beta} \cdot f^\beta$. By Lemma (3.1) the triple $(f^\alpha, f^{\alpha\beta}, f^\beta)$ defines a map $U_\alpha \cap U_\beta \to BL_1^p$, and so we may define:

$$\chi_1 = \coprod_{\alpha,\beta} (f^\alpha, f^{\alpha,\beta}, f^\beta).$$

Note that $\chi_0^*(\eta^0) = \{f^\alpha\}$ while $\chi_1^*(\eta^0, \eta^{01}, \eta^1) = (f^\alpha, f^{\alpha\beta}, f^\beta)$, and that the diagram

$$\begin{array}{ccc}
U_\alpha & \xrightarrow{f^\alpha} & BL_0^p = \mathbb{A}^p \\
\uparrow & & \uparrow d_1 \\
U_\alpha \cap U_\beta & \xrightarrow{f^{\alpha\beta}} & BL_1^p \\
\downarrow & & \downarrow d_0 \\
U_\beta & \xrightarrow{f^\beta} & BL_0^p
\end{array}$$

commutes, so that $\chi_0 d_i = d_i \chi_1$ for $i = 0, 1$. In order to define $\chi_n$ for $n \geq 2$ we procede by induction on $n$. Suppose that we have defined $\chi_m$ for $m < n$. Then on each component $U_{\alpha_0} \cap \ldots \cap U_{\alpha_n}$ of $N_n\{U_\alpha\}$ we have for each $k < n$ and each multi-index $0 \leq i_0, \ldots, i_k \leq n$, maps

$$\Lambda^* f^{\alpha_{i_0}, \ldots, \alpha_{i_k}}: K(f^{\alpha_{i_0}})_* \to K(f^{\alpha_{i_k}})_*$$

of degree $(k-1)$, where $\Delta^* f^{\alpha_{i_0}, \ldots, \alpha_{i_k}}$ is the pull back via the composition

$$U_{\alpha_0} \cap \ldots \cap U_{\alpha_n} \to U_{\alpha_{i_0}} \cap \ldots \cap U_{\alpha_{i_k}} \xrightarrow{\chi_k} BL_k^p$$

of $\Lambda^* \eta^{0, \ldots, k}$. Since $X$ is separated $U_{\alpha_0} \cap \ldots \cap U_{\alpha_n}$ is affine and $\mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Y) = 0$ for $i < 0$. Now if

$$G_n = \sum_{j=1}^{n-1} \{(-1)^{(1-j)(n-j)} f^{\alpha_0, \ldots, \alpha_j} \Lambda^j f^{\alpha_j, \ldots, \alpha_n} + (-1)^j f^{\alpha_0, \ldots, \hat{\alpha}_j, \ldots, \alpha_n}\} \tag{3.19}$$

we have $G_n d^{n-1}_{f_n^\alpha} = 0$ since this is true for the $\eta$'s on $BL_n^p$; hence there exists

$$f^{\alpha_0, \ldots, \alpha_n}: K(f^{\alpha_0})_1 \to K(f^{\alpha_n})_n$$

such that

$$f^{\alpha_0, \ldots, \alpha_n} d^n_{f_n^\alpha} = G_n. \tag{3.20}$$

We may now define

$$\chi_n \Big|_{U_{\alpha_0} \cap \ldots \cap U_{\alpha_n}} = (\{f^{\alpha_{i_0}, \ldots, \alpha_{i_k}}\}_{0 \leq i_0 < \ldots < i_k \leq n}) \to BL_n^p.$$

It is straightforward to check that $\chi_n$ is compatible with the face and degeneracy maps

$$BL_n^p \rightleftarrows BL_{n-1}^p$$

and that we have therefore defined a morphism of simplicial schemes. Statement (ii) of the theorem follows from the fact that for each $n \geq 0$ the inverse image under $\chi_n$ of each of the regular sequences generating $\mathcal{I}_{Z_n^p}$ is a regular sequence generating the ideal of $Y$ in $N_n\{U_\alpha\}$. Statement (iii) is an immediate consequence of (ii) together with the fact that $\Lambda^* \eta$ is a twisted resolution of $\mathcal{O}_{Z_\bullet^p}$.

In order to justify completely the assertion that $BL_\bullet^p$ classifies codimension $p$ local complete intersections, we need to examine what happens if we make a different choice of open cover and local equations for $Y$.

PROPOSITION 3.21: *Let $X, Y$ be as in* (3.18). *Suppose that there exist two open covers $\{U_\alpha\}$, $\{V_\beta\}$ and morphisms*

$$\chi^0 : N_.\{U_\alpha\} \to BL^p_.$$
$$\chi^1 : N_.\{V_\beta\} \to BL^p_.$$

*classifying $Y$. Then $\chi^0$ and $\chi^1$ are homotopic in the sense that there exists a common refinement $\{W_\gamma\}$ of $\{U_\alpha\}$ and $\{V_\beta\}$, with refinement maps*

$$\rho^0 : N_.\{W_\gamma\} \to N_.\{U_\alpha\}$$
$$\rho^1 : N_.\{W_\gamma\} \to N_.\{V_\beta\},$$

*and a map*

$$H : N_.\{W_\gamma\} \times I_. \to BL^p_.$$

*such that for $i = 0, 1$ the restriction of $H$ to $N_.\{W_\gamma\} \times \{i\}$ coincides with $\chi^i \cdot \rho^i$ (where $I_.$ is the simplicial unit interval;* [*M*]).

PROOF: We may assume $\{W_\gamma\} = \{U_\alpha\} = \{V_\beta\}$; i.e., it is sufficient to compare the two different classifying maps $\chi^i \cdot \rho^i$ $(i = 0, 1)$ from $N_.\{W_\gamma\}$ to $BL^p_.$. The simplicial scheme (which is a hypercovering of $X$):

$$D_. = N_.\{W_\gamma\} \times I_.$$

has the following simple description. If we write $W = \coprod_\gamma W_\gamma$, viewing $W$ as a scheme over $X$ we can, for each $k \geq 0$, form the $(k+1)$-fold fibre product (over $X$) $W^{k+1} = W \times \ldots \times W = N_k\{W_\gamma\}$. $I_k$ may be identified with the set of increasing sequences $(i_0 \leq i_1 \leq \ldots \leq i_k)$ such that $i_j = 0$ or 1 for all $j$; we may also identify $I_k$ with the set $\{-1, \ldots, k\}$ by the rule $(i_0 \leq \ldots \leq i_k) \mapsto j$ such that $i_j < i_{j+1}$. Now observe that we may write

$$D_k = W^{k+1} \times I_k$$

as

$$\coprod_j W^{k+1}_{(j)} \quad \text{or} \quad \coprod_{i_0 \leq \ldots \leq i_k} W^{k+1}_{i_0, \ldots, i_k}.$$

So

$$D_0 = W_0 \amalg W_1 = W_{(-1)} \amalg W_{(0)}$$

and

$$\begin{aligned} D_1 &= (W \times W)_{0,0} \amalg (W \times W)_{0,1} \amalg (W \times W)_{1,1} \\ &= W_{(-1)} \amalg W_{(0)} \amalg W_{(1)} \end{aligned}$$

with the face maps

$$d_i : D_1 \to D_0$$

being, for $i = 1, 0$:

$$(W \times W)_{j,k} \xrightarrow{d_1} W_j$$

and

$$(W \times W)_{j,k} \xrightarrow{d_0} W_k$$

respectively. $Y \cap D_0$ has local equations $f_0^\alpha$ on $W_{\alpha,0}$ (where $W_0 = \amalg W_{\alpha,0}$) and $f_1^\alpha$ on $W_{\alpha,1}$ (in general we shall write $\Lambda^* f_i^{0,\ldots,n} = \chi_n^{i*}(\Lambda^* \eta^{0,\ldots,n})$); on $D_1$ these local equations are related by the transition matrices $f_i^{\alpha\beta}$ on $(W_\alpha \cap W_\beta)_{i,i}$ already defined by the $\chi^i$ for $i = 0, 1$, however on each $(W_\alpha \cap W_\beta)_{01}$ we must choose a new transition matrix $f_{01}^{\alpha\beta}$

$$f_0^\alpha = f_{01}^{\alpha\beta} f_1^\beta.$$

Proceeding in this fashion, we may suppose that for $k = 0, \ldots, n-1$ and all $(\alpha_0, \ldots, \alpha_k)$ we have defined maps $(-1 \le j \le k)$

$$f_{(j)}^{\alpha_0, \ldots, \alpha_k} : K(f_j^{\alpha_0})_1 \to K(f_k^{\alpha_j})_k$$

where

$$f_{(j)}^{\alpha_i} = \begin{cases} f_0^{\alpha_i} & \text{if } j \ge i \\ f_1^{\alpha_i} & \text{if } j < i \end{cases}$$

such that

$$f_{(j)}^{\alpha_0,\ldots,\alpha_k} = \begin{cases} f_0^{\alpha_0,\ldots,\alpha_k} & \text{if } j = k \\ f_1^{\alpha_0,\ldots,\alpha_k} & \text{if } j = -1 \end{cases}$$

Then the $f_{(j)}^{\alpha_0,\ldots,\alpha_k}$ satisfy equation 3.19 on each $(W_{\alpha_0} \cap \ldots \cap W_{\alpha_k})_{(j)}$; hence we may choose $h_{(j)}^{\alpha_0,\ldots,\alpha_n}$ to satisfy (3.20). Furthermore, if $j = -1$ (or $n$, respectively) the $h_{(j)}^{\alpha_0,\ldots,\alpha_k}$ version of equation (3.19) involves only the $f_0^{\alpha_0,\ldots,\alpha_k}$ ($f_1^{\alpha_0,\ldots,\alpha_k}$ respectively), hence we may choose $h_{(j)}^{\alpha_0,\ldots,\alpha_n}$ to equal $f_0^{\alpha_0,\ldots,\alpha_n}$ if $j = -1$ ($f_1^{\alpha_0,\ldots,\alpha_n}$ if $j = n$). Clearly the $h_j^{\alpha_0,\ldots,\alpha_n}$ now define our map

$$H : N_{.}\{W_\gamma\} \times I_{.} \to BL_{.}^{p}.$$

Following [20] we know $H$ defines a homotopy between $\chi^1$ and $\chi^0$.

## §4. Universal cycle classes

First, we shall construct universal cycle classes in the Chow ring for local complete intersections in the category of varieties over a fixed field $k$. Until further notice we shall abuse notation and for each $p \geq 0$ denote the simplicial varieties $Z_{.}^{p} \otimes_{\mathbb{Z}} k$ and $BL_{.}^{p} \otimes_{\mathbb{Z}} k$ as simply $Z_{.}^{p}$ and $BL_{.}^{p}$. Our object is to construct for all $p \geq 0$, classes:

$$\gamma[Z_{.}^{p}] \in H_{Z_{.p}}^{p}(BL_{.}^{p}, \mathbf{K}_p).$$

We begin by recalling various properties of the sheaves $\mathbf{K}_p$, $p \geq 0$.

THEOREM 4.1 (Quillen): *Let $X$ be a scheme, regular and of finite type over a field. Then for each $p \geq 0$, the sheaf $\mathbf{K}_p$ associated to the presheaf*

$$U \mapsto K_p(U)$$

*on $X$ defined by the Quillen K-functors ($K_*(U)$ is the K-theory of locally free $\mathcal{O}_U$-modules [Q]), has a flasque resolution:*

$$\mathbf{K}_p \to \mathbf{R}_p^*$$

*where $\mathbf{R}_p^i(U) = \bigoplus_{x \in U \cap X^{(i)}} K_{p-i}(\mathbb{K}(x))$, $X^{(i)}$ being the set of points of codimension $i$ in $X$.*

PROOF: [21] The complex $\mathbf{R}_p^*$ forms part of the $E_1$ term of a spectral sequence

$$E_1^{i,j}(X) = \bigoplus_{x \in X^{(i)}} K_{-i-j}\mathbb{K}(x) \Rightarrow K'_{-i-j}(X),$$

where $K'_*(X)$ is the Quillen $K$-theory of the category of coherent sheaves on $X$. This spectral sequence is contravariant for flat morphisms.

PROPOSITION 4.3: (i) *If $X$ is a noetherian excellent scheme then for all $i \geq 0$:*

$$E_2^{p,-p}(X) = CH^p(X) \tag{4.4}$$

*where $CH^p(X)$ is the Chow homology group of codimension $p$ cycles on $X$ modulo rational equivalence* [*F*].

(ii) *The isomorphism* (4.4) *is compatible with the contravariance with respect to flat maps of both domain and codomain.*

PROOF: The proof of (i) is in [21]; though one needs to observe that the definition of the Chow ring Quillen uses coincides with Fulton's definition of the Chow groups. The proof of (ii) is by inspection of the definitions in [21] and [10] of the two pull back maps concerned.

We must also observe that this proposition may be generalized as follows:

COROLLARY 4.5: *Let $Y$, $X$ be schemes smooth over a field $k$, $Y$ a codimension $p$ subscheme of $X$. Then we have isomorphisms*:

(i) $H_Y^i(X, \mathbf{K}_p) = \begin{cases} 0 & i < p \\ \mathbb{Z}\gamma[Y] & i = p \end{cases}$ *where $\gamma[Y]$ corresponds to the generator $\mathbb{K}[Y] \in K_0(\mathbb{K}(Y)) = \Gamma_Y(X, \mathbf{R}_p^p) = \mathbb{Z}$*

(ii) *If $i,j > p$: $H_Y^i(X, \mathbf{K}_j) = H^{i-p}(Y, \mathbf{K}_{j-p})(\simeq CH^{i-p}(Y)$ if $i = j$).*

(iii) *If $f: Z \to X$ is smooth, we have a commutative diagram for all $i \geq p$:*

$$\begin{array}{ccccc} H_Y^i(X, \mathbf{K}_i) & \longrightarrow & CH^{i-p}(Y) & \longrightarrow & CH^i(X) \\ \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f|_{f^{-1}(Y)}^*} & & \downarrow{\scriptstyle f^*} \\ H_{f^{-1}(Y)}^i(Z, \mathbf{K}_i) & \longrightarrow & CH^{i-p}(f^{-1}(Y)) & \longrightarrow & CH^i(Z) \end{array}$$

(*Note that $f|_{f^{-1}(Y)}$ is also smooth*).

PROOF: (i) is immediate from the existence of the resolution $\mathbf{K}_p \to \mathbf{R}_p^*$, since:

$$\Gamma_Y(X, \mathbf{R}_p^i) = \begin{cases} 0 & i < p \\ K_0(\mathbb{K}(Y)) & i = p \end{cases}.$$

(ii) follows from the observation that for $i \geq p$:

$$\Gamma_Y(X, \mathbf{R}_i^*) = g_* \mathbf{R}_{i-p}^*[p]$$

where $g: Y \to X$ is the inclusion of $Y$ in $X$ together with the isomorphism (4.4) for $Y$ instead of $X$. (iii) follows from the fact that not only is the spectral sequence (4.2) contravariant for flat (and therefore smooth) maps, but that from its construction [21] it is clear that for all $i \geq p$ the diagram:

$$\begin{array}{ccccc} \Gamma(X, \mathbf{R}_i^*) & \longleftarrow & \Gamma_Y(X, \mathbf{R}_i^*) & \longrightarrow & \Gamma(Y, \mathbf{R}_{i-p}^*)[p] \\ \downarrow & & \downarrow & & \downarrow \\ \Gamma(Z, \mathbf{R}_i^*) & \longleftarrow & \Gamma_{f^{-1}(Y)}(Z, \mathbf{R}_i^*) & \longrightarrow & \Gamma(f^{-1}(Y), \mathbf{R}_{i-p}^*)[p]. \end{array}$$

commutes.

Returning to the construction of $\gamma[Z_{\cdot}^p/k]$, we first observe that there is a spectral sequence:

$$E_1^{i,j} = H_{Z_i^p}^j(BL_i^p, \mathbf{K}_p) \Rightarrow H_{Z_{\cdot}^p}^{i+j}(BL_{\cdot}^p, \mathbf{K}_p). \tag{4.6}$$

From Corollary (4.5):

$$H_{Z_i^p}^j(BL_i^p, \mathbf{K}_p) = \begin{cases} 0 & j < p \\ \mathbb{Z}\gamma[Z_i^p] & j = p \end{cases}$$

together with (4.6) this implies:

$$H_{Z_{\cdot}^p}^p(BL_{\cdot}^p, \mathbf{K}_p) = H^0(i \mapsto H_{Z_i^p}^p(BL_i^p, \mathbf{K}_p)),$$

where the differentials in the complex

$$i \mapsto H_{Z_i^p}^p(BL_i^p, \mathbf{K}_p) \tag{4.7}$$

are induced by the face maps of the simplicial scheme $BL_{\cdot}^p$.

PROPOSITION 4.8: *There is an isomorphism* $H^p_{Z^p_\cdot}(BL^p_\cdot, \mathbf{K}_p) = \mathbb{Z}$ *and* $H^p_{Z^p_\cdot}(BL^p_\cdot, \mathbf{K}_p)$ *has a canonical generator* $\gamma[Z^p_\cdot]$ *which restricts to* $\gamma[Z^p_0]$ *on* $BL^p_0$.

PROOF: It suffices to show that $\gamma[Z^p_0]$ is a cocycle in the complex (4.7), that is:

$$d_0^*(\gamma[Z^p_0]) = d_1^*(\gamma[Z^p_0]) \in H^p_{Z^p_1}(BL^p_1, \mathbf{K}_p).$$

However, since the face maps $d_0$ and $d_1$ are flat, and the isomorphism (4.4) is compatible with pullback along flat maps it is enough to show that

$$d_0^*([Z^p_0]) = d_1^*([Z^p_0]) \in CH^p_{Z^p_1}(BL^p_1)$$

however, by (ii) of Theorem 3.3 and (iii) of Corollary 4.5, we have:

$$d_0^*([Z^p_0]) = d_1^*([Z^p_0]) = [Z^p_1] \in CH^p_{Z^p_1}(BL^p_1).$$

DEFINITION 4.9: (i) The cycle class of $Z^p_\cdot$, $\gamma[Z^p_\cdot]$ is the canonical generator of $H^p_{Z^p_\cdot}(BL^p_\cdot, \mathbf{K}_p)$ found in the preceding proposition.

(ii) Let $Y$ be a codimension $p$ subscheme, locally a complete intersection, of the variety $X$ defined over the field $k$. Then if $\{U_\alpha\}$ is an open cover of $X$ such that there exists a classifying map

$$\chi_y : N_\cdot\{U_\alpha\} \to BL^p_\cdot$$

*with* $\chi_Y^{-1}(Z^p_\cdot) = Y \cap N_\cdot\{U_\alpha\}$, then we define the *cycle class of* $Y$:

$$\gamma[Y] = \chi_Y^*(\gamma[Z^p_\cdot]) \in H^p_{Y \cap N_\cdot\{U_\alpha\}}(N_\cdot\{U_\alpha\}, \mathbf{K}_p) \simeq H^p_Y(X, \mathbf{K}_p).$$

PROPOSITION 4.10: *The cycle class* $\gamma[Y]$ *of definition* (4.9) *part* ii), *is independent of the classifying map* $\chi_Y$.

PROOF: By Proposition (3.21) we know any two classifying maps are homotopic; but it is a standard fact that homotopic maps between simplicial schemes induce the same map on cohomology:

LEMMA 4.11: *Let* $f, g : X_\cdot \to Y_\cdot$ *be maps of simplicial schemes over a fixed base* $S$. *Let* $\mathscr{F}$ *be a sheaf (or complex of sheaves) on the big Zariski site over* $S$, *and*

$$f^*, g^* : H^*(Y_\cdot, \mathscr{F}_{Y_\cdot}) \to H^*(X_\cdot, \mathscr{F}_{X_\cdot})$$

*the induced maps. Then if* $f$ *and* $g$ *are homotopic* $f^* = g^*$.

PROOF OF LEMMA: By construction $X_\bullet \times I_\bullet$ is a (Zariski) hypercovering of $X_\bullet$ via the natural projection map, and so we have a commutative diagram of isomorphisms

$$\begin{array}{ccccc} H^*(X_\bullet \times \{0\}, \mathscr{F}_{X_\bullet}) & \xleftarrow{i_0^*} & H^*(X_\bullet \times I_\bullet, \mathscr{F}_{X_\bullet \times I_\bullet}) & \xrightarrow{i_1^*} & H^*(X_\bullet \times \{1\}, \mathscr{F}_{X_\bullet}) \\ & \diagdown\!\!\diagdown & \uparrow{\scriptstyle p_{X_\bullet}^*} & \diagup\!\!\diagup & \\ & & H^*(X_\bullet, \mathscr{F}_{X_\bullet}) & & \end{array}$$

By assumption, there exists a map

$$H: X_\bullet \times I_\bullet \to Y.$$

such that $i_0 \cdot H = f$ and $i_1 \cdot H = g$; then

$$f^* = H^* \cdot i_0^* = H^* \cdot p_{X_\bullet}^{*^{-1}} = H^* \cdot i_1^* = g^*,$$

and the lemma is proved.

The proposition now follows immediately, taking $\mathscr{F} = \mathbf{K}_p$.

We now wish to verify that this definition of the cycle class has the right geometric properties.

THEOREM 4.13: *Suppose $Y \subset X$ are as in* (ii) *of Definition* (4.9).

(i) *If $f: Z \to X$ is a flat morphism, then $\gamma[f^{-1}Y] = f^*\gamma[Y]$.*

(ii) *If $T \subset X$ is a codimension $q$ subscheme, locally a complete intersection with $\mathrm{Tor}_i^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_T) = 0$ for $i > 0$, so that $T \cap Y$ is a local complete intersection in $Y$, then $\gamma(T) \cup \gamma(Y) = \gamma(T \cap Y)(-1)^{pq}$.*

(iii) *If $U \subset X$ is an affine open set in which the ideal of $Y$ is generated by a regular sequence $(f_1, \ldots, f_p)$, then $\gamma[Y]|_U$ is represented by the Cech cocycle $(-1)^{p(p-1)/2}\gamma(f_1, \ldots, f_p)$ where*

$$\gamma(f_1, \ldots, f_p) \in C_Y^p(\{U, U_{f_1}, \ldots, U_{f_p}\}, \mathbf{K}_p)$$

*is the cocycle whose value on $U \cap U_{f_1} \cap \ldots \cap U_{f_p}$ is the symbol $\{f_1, \ldots, f_p\} \in K_p(X)$ defined by the product* [24]

$$K_1(U_{f_1}) \otimes \ldots \otimes K_1(U_{f_p}) \to K_p(U_{f_1} \cap \ldots \cap U_{f_p}).$$

(iv) *If $X$ is smooth over $k$ then $\gamma[Y]$ coincides with the class defined by the isomorphism in* (ii) *of Corollary* 4.5.

PROOF: (i) If $f: Z \to X$ is flat and $\{U_\alpha\}$ is an open cover of $X$ and

$$\chi_Y : N.\{U_\alpha\} \to BL_0^p$$

a map classifying $Y$, then if we define $f \circ \chi_Y$ as the composition

$$\begin{array}{ccc} N.\{f^{-1}U_\alpha\} & & \\ \Big\downarrow f & \searrow^{f \circ \chi_Y} & \\ N.\{U_\alpha\} & \xrightarrow{\chi_Y} & BL_.^p/k \end{array}$$

$f \circ \chi_Y$ classifies $\chi_{f^{-1}(Y)}$, and hence

$$\begin{aligned} \gamma[f^{-1}(Y)] &= \chi_{f^{-1}(Y)}^* \gamma[Z_.^p] \\ &= f^*(\chi_Y^* \gamma[Z_.^p]) = f^*(\gamma[Y]). \end{aligned}$$

To prove (ii) we first observe that we may suppose that the classifying maps $\chi_Y, \chi_T$ are both defined relative to the same open cover of $X$. Then $Y \cap T$ is classified by the map $\chi_{Y \cap T}$ that represents the tensor product of the twisted resolutions of $\mathcal{O}_Y$ and $\mathcal{O}_T$, i.e. $\chi_{Y \cap T}$ factors through the natural map:

$$\mu_{p,q} : BL_.^p \times BL_.^q \to BL_.^{p+q}$$

such that

$$\mu_{p,q}^*(\eta^{0,\ldots,n}) = (\eta^{0,\ldots,n} \otimes I) + (I \otimes \eta^{0,\ldots,n}).$$

Hence it is sufficient to show that

$$(-1)^{pq} \mu_{p,q}^*(\gamma[Z_.^{p+q}]) = \gamma[Z_.^p] \cup \gamma[Z_.^q].$$

Using the spectral sequence (4.6) it is sufficient to check the result in degree zero, i.e.:

$$(-1)^{pq} \mu_{p,q}^*(\gamma[Z_0^{p+q}]) = \gamma[Z_0^p] \cup \gamma[Z_0^q].$$

Since the map $BL_0^p \times BL_0^q \to BL_0^{p+q}$ is just the product

$$\mathbb{A}_k^p \times \mathbb{A}_k^p \to \mathbb{A}_k^{p+q}$$

and $\mu^*_{p,q}(Z_0^{p,q}) = (Z_0^p \times \mathbb{A}_k^q) \cap (\mathbb{A}^p \times Z_0^p)$. The equality (4.4) now follows from the compatibility of the product on $K$-theory with intersection theory ([12], [15]). (iii) is an immediate consequence of (i) and (ii) together with the fact that the cycle class of a Cartier divisor $V(f) \subset U$ is the element of $H^1_{V(f)}(U, \mathbf{K}_1)$ coming from the section $f \in \Gamma(U_f, \mathbf{K}_1)$ via the boundary map in the long exact cohomology sequence for the pair $(V(f) \subset U)$. For part (iv), we know by (ii) of Corollary 4.5 and the standard local to global spectral sequence that we need only check the equality of the two classes locally. The result now follows by (iii) together with the compatability of the $K$-theory product with intersections, using induction on $p$ together with the fact that the classes obviously coincide if $p = 1$ in which case $\gamma[Y]$ is the cycle class in $H^1_Y(X, \mathbf{K}_1 \simeq \mathcal{O}_X^*)$ of the Cartier divisor $Y$.

One can define for all schemes $X$ and all codimension $p$ subschemes $j: Y \to X$ a cap product:

$$\cap : H^i_Y(X, \mathbf{K}_i) \otimes CH^j(X) \to CH^{j+i-p}(Y)$$

which is induced by the product ([12], [15])

$$\mathbf{K}_i \otimes \mathbf{R}^*_j \to \mathbf{R}^*_{i+j}$$

together with the isomorphism

$$\mathbb{H}^{j+i}_Y(X, \mathbf{R}^*_{i+j}) = CH^{i+j-p}(Y).$$

In particular cap product with $\gamma[Y]$, if $Y$ is a local complete intersection subscheme of a variety $X$ over $k$, defines a homomorphism

$$j^* : CH^i(X) \to CH^i(Y).$$

Such a Gysin homomorphism has already been defined by Verdier [23] by geometrical methods. We wish to compare the two maps, using a slight reinterpretation of Verdier's construction.

First we need a special case:

LEMMA 4.15: *Let $j: Y \to X$ be a codimension $p$ subscheme, locally a complete intersection. Then*

$$\gamma[Y] \cap [X] = (-1)^{p(p-1)/2}[Y] \in CH^0(Y)$$

PROOF: Since $CH^0(Y) \simeq \bigoplus_{x \in Y \cap X^{(p)}} K_0(\mathbb{K}(x))$ we need only check that the two classes agree in some affine neighborhood of each generic point of $Y$. Using (iii) of Theorem 4.13 and induction on $p$, we may suppose that $p = 1$, $X$ is affine, $Y \cap X^{(p)}$ consists of a single point $y$ and that the ideal of $Y$ in $\mathcal{O}_X(X)$ is generated by a single element $f$. Then $\gamma[Y] \cap [X]$ is the image under the boundary map

$$\partial : K_1\mathbb{K}(X) \to \bigoplus_{x \in X^{(1)}} K_0(\mathbb{K}(x))$$

of the element $\{f\} \in K_1(\mathbb{K}(x))$. By ([14],[21]) $\partial\{f\} = [\mathcal{O}_Y]$ and the lemma is proved.

Given $Y \xrightarrow{j} X$ a codimension $p$ regular immersion of varieties over $k$, there is a flat family $D_{X/Y} \xrightarrow{\pi} \mathbb{A}^1 = \mathrm{Spec}(k[t])$ together with an immersion $Y \times \mathbb{A}^1 \xrightarrow{i} D_{X/Y}$ such that $\pi \circ j$ is the natural projection $Y \times \mathbb{A}^1 \to \mathbb{A}^1$, and there are $\mathbb{A}^1$-isomorphisms

$$(\pi^{-1}(\mathbb{A}^1 - \{0\}), Y \times (\mathbb{A}^1 - \{0\})) \to X \times (\mathbb{A}^1 - \{0\})$$
$$(\pi^{-1}\{0\}, Y \times \{0\}) \to (N_{X/Y}, Y).$$

We now define Verdier's Gysin homomorphism $j^* : CH^*(X) \to CH^*(Y)$ as follows. For an integral subscheme $Z \subset X$, $i^*([Z])$ is defined as $\pi^{*-1}(\overline{[Z \times (\mathbb{A}^1 - \{0\})]} \cap \gamma[D_0])$ where $\gamma[D_0] \in H^1_{D_0}(D_{X/Y}, \mathbf{K}_1)$ is the special fibre $\pi^{-1}\{0\}$ regarded as a Cartier divisor, $\overline{Z \times (\mathbb{A}^1 - \{0\})}$ is the closure of $Z \times (\mathbb{A}^1 - \{0\})$ in $D_{X/Y}$ and $\pi^{*-1} : CH^*(N_{X/Y}) \to CH^*(Y)$ is the isomorphism induced by the flat projection $\pi : N_{X/Y} \to Y$. This last isomorphism is cap product with $(-1)^{p(p-1)/2}$ times the cycle class $\gamma_N[Y]$ of the zero section of $N_{X/Y}$. This is because if $S \subset Y$ is an integral subscheme, then

$$\pi^*[S] \cap \gamma_N[Y] = (-1)^{p(p-1)/2}[S] \in CH^*(Y).$$

To see this we observe that $\pi^*[S] = [N_{X/Y}|_S]$ and that if $\{U_\alpha\}$ is an open cover of $Y$ such that $N_{X/Y}|_{U_\alpha}$ is trivial for all $\alpha$, $Y \hookrightarrow N_{X/Y}$ may be classified by a map $N_.\{\pi^{-1}(u_\alpha)\} \to BL^p_.$. Now $\{U_\alpha \cap S\}$ is a trivializing open cover for $N_{X/Y}|_S$ and $S \hookrightarrow N_{X/Y}|_S$ may be classified by the composition

$$N_.\{\pi^{-1}(u_\alpha \cap S)\} \to N_.\{\pi^{-1}(u_\alpha)\} \to BL^p_.$$

so if $\sigma : S \hookrightarrow Y$ is the natural inclusion, $\sigma^*\gamma[Y] = \gamma[S] \in H^p_S(N_{X/Y}|_S, \mathbf{K}_p)$.

But we have

$$\begin{aligned}\pi^*[S] \cap \gamma_N[Y] &= \pi^*[S] \cap \tilde{\sigma}^*\gamma_N[Y] \\ &= \pi^*[S] \cap \gamma_N[S] = (-1)^{p(p-1)/2}[S] \qquad \text{(by Lemma 4.15)},\end{aligned}$$

where $\tilde{\sigma}: N_{X/Y}|_S \hookrightarrow N_{X/Y}$ is the natural inclusion.

It remains to show that

$$\overline{([Z \times \mathbb{A}^1 - \{0\}]} \cap \gamma[D_0]) \cap \gamma_N[Y] = Z \cap \gamma[Y].$$

Examining the local equations for $j(Y \times \mathbb{A}^1)$, $D_0$, and $Y \times \{0\} \subset D_0$ we find that for any cycle $[S] \in CH^*(D)$:

$$([S] \cap \gamma[D_0]) \cap \gamma_N[Y] = ([S] \cap \gamma[j(Y \times \mathbb{A}^1)]) \cap \gamma[D_0].$$

Since $\overline{[Z \times (\mathbb{A}^1 - \{0\})]} \cap \gamma[j(Y \times \mathbb{A}^1)]$ is a cycle on $Y \times \mathbb{A}^1 \subset D$ we may regard $\overline{([Z \times (\mathbb{A}^1 - \{0\}]} \cap \gamma[Y \times \mathbb{A}^1]) \cap \gamma[D_0]$ as the specialization of the cycle

$$[Z \times (\mathbb{A}^1 - \{0\})] \cap \gamma[Y \times (\mathbb{A}^1 - \{0\})] = ([Z] \times \gamma(Y)) \times (\mathbb{A}^1 - \{0\})$$

to $Y \times \{0\}$, which is clearly $[Z] \cap \gamma[Y]$. Summarizing:

PROPOSITION 4.16: *Given $Y \overset{j}{\hookrightarrow} X$ a codimension $p$ regular immersion of varieties, the cycle class $\gamma[Y] \in H^p_Y(X, \mathbf{K}_p)$ defines a Gysin homomorphism*

$$\begin{aligned}&CH^*(X) \to CH^*(Y) \\ &x \mapsto x \cap \gamma[Y]\end{aligned}$$

*which coincides up to a factor of $(-1)^{p(p-1)/2}$ with the map defined by Verdier in* [23].

A corollary of this proposition is that if $Z \subset X$ is an integral subscheme which intersects $Y$ properly (i.e. codim($S = Y \cap Z$) = codim $Y$ + codim $X$) then

$$(-1)^{p(p-1)/2}[Z] \cap \gamma[Y] = \sum_{\substack{T \text{ irreducible} \\ \text{components of } S}} \mu_T(Y, Z)[T]$$

(the "classical" intersection product of $Y$ and $Z$). We can now see this in two ways; first of all by classical methods it is true for Verdier's definition of $[Z] \cap \gamma[Y]$, alternatively one can use the methods of the proof of the compatibility of the $K$-theoretic product on the Chow Ring with the classical intersection product given in ([12], [14]).

The construction of the universal cycle class in étale cohomology is somewhat easier than in the Chow ring. We fix an integer $n \neq 0$, and consider the category of schemes over $\mathbb{Z}[1/n]$; for each $p \geq 1$ $Z^p_\cdot$ and $BL^p_\cdot$ will denote the pullbacks of the simplicial schemes defined in §3 over $\mathrm{Spec}(\mathbb{Z}[1/n])$. We want to construct a cycle class

$$\gamma([Z^p_\cdot] \in H^{2p}_{Z^p_\cdot}(BL^p_\cdot, \mu_n^{\otimes p})$$

lying in the relative étale cohomology (as defined in [9]) of the pair of simplicial schemes $(BL^p_\cdot, BL^p_\cdot - Z^p_\cdot)$. There is a natural spectral sequence ([9]):

$$E_1^{i,j} = H^j_{Z^p_i}(BL^p_i, \mu_n^{\otimes p}) \Rightarrow H^{i+j}_{Z^p_\cdot}(BL^p_\cdot, \mu_n^{\otimes p}).$$

However by ([7] 2.2.8) we know that

$$H^j_{Z^p_i}(BL^p_i, \mu_n^{\otimes p}) = 0 \text{ for } j < 2p.$$

Therefore there is an isomorphism:

$$H^{2p}_{Z^p_\cdot}(BL^p_\cdot, \mu_n^{\otimes p}) \simeq \mathrm{Ker}(H^{2p}_{Z^p_0}(BL^p_0, \mu_n^{\otimes p}) \xrightarrow{d_0^* - d_1^*} H^{2p}_{Z^p_1}(BL^p_1, \mu_n^{\otimes p})). \tag{4.17}$$

By [7] (2.3.8) we know

$$d_i^*(\gamma[Z^p_\cdot]) = \gamma[Z^p_1] \in H^{2p}_{Z^p_\cdot}(BL^p_\cdot, \mu_n^{\otimes p}),$$

for $i = 0$ and 1, hence we can make the following:

DEFINITION 4.18: (i) The universal cycle class

$$\gamma[Z^p_\cdot] \in H^{2p}_{Z^p_\cdot}(BL^p_\cdot, \mu_n^{\otimes p})$$

is the class defined by the element $\gamma[Z^p_0]$ via the isomorphism (4.17).

(ii) Let $X$ be a scheme of finite type over $\mathbb{Z}[1/n]$, and $Y \subset X$ a codimension $p$ subscheme, locally a complete intersection. Then there exists an open cover $\{U_\alpha\}$ of $X$, and a map

$$\chi_Y : N_.\{U_\alpha\} \to BL^p_.$$

classifying $Y$ in the sense of (3.7). We define the cycle class of $Y$ to be:

$$\gamma[Y] = \chi_Y^*(\gamma[Z^p_.]) \in H_Y^{2p}(X, \boldsymbol{\mu}_n^{\otimes p}).$$

Using the method of (4.10) it is clear that $\gamma[Y]$ is well-defined, i.e. independent of $\{U_\alpha\}$ and $\chi_Y$.

The following proposition may be proved using the same methods used to prove Theorem 4.13.

PROPOSITION 4.19: *Suppose $i: Y \to X$ is a regular codimension $p$ embedding of schemes over $\mathbb{Z}[1/n]$. Then:*

(i) *If $f: Z \to X$ is a flat morphism, then*

$$\gamma[f^{-1}(Y)] = f^*\gamma[Y] \in H^{2p}_{f^{-1}(Y)}(X, \boldsymbol{\mu}_n^{\otimes p}).$$

(ii) *If $j: T \to X$ is a codimension $q$ regular embedding such that $j \times i: T \cap Y \to X$ is also regular then*

$$\gamma(T \cap Y) = \gamma(T) \cup \gamma(Y).$$

(iii) *If $X$ is smooth over $\mathbb{Z}[1/n]$ then $\gamma[Y]$ coincides with the class defined in* ([7] 2.2).

Finally we construct cycle classes for local complete intersections in crystalline cohomology. We refer to [4] for the results on crystalline cohomology that we need. For simplicity we restrict our attention to schemes of finite type over a fixed perfect field $k$ of characteristic $p > 0$. We wish to construct for each $p \geq 1$ and for every codimension $p$ subscheme $Y$ locally a complete intersection in a scheme $X$ of finite type over $k$ a class:

$$\gamma[Y] \in H^{2p}_{\text{crys}}(X/W) = \varprojlim H^{2p}_{\text{crys}}(X/W_n)$$

(where $W_n$ is the usual ring of truncated Witt vectors). By cohomological descent ([4] §7.8) there is an isomorphism

$$H^{2p}_{\text{crys}}(X/W_n) \xrightarrow{\sim} H^{2p}_{\text{crys}}(N_.\{U_\alpha\}/W_n)$$

for each open cover $\{U_\alpha\}$ of $X$. (The crystalline cohomology of a simplicial scheme $X_.$ over $k$ is the cohomology of the sheaf $\mathcal{O}_{X./W_n}$ in the crystallic topos of $X./W_n$ (op. cit.)). Hence to construct $\gamma[Y]$ it is enough to construct a universal class:

$$\gamma_{\text{crys}}[Z^p_.] \in H^{2p}_{\text{crys}}(BL^p_./W) = \varprojlim H^{2p}_{\text{crys}}(BL^p_./W_n).$$

Here we view $Z^p_.$ and $BL^p_.$ as schemes over Spec($k$) by pulling back from spec($\mathbb{Z}$). By (op. cit., §74)

$$H^{2p}_{\text{crys}}(BL^p_./W_n) \simeq \mathbb{H}^{2p}_{\text{Zar}}(BL^p_./W_n, \Omega^{\cdot}_{BL^p_./W_n})$$

where in the right hand side of this equation $BL^p_./W_n$ is the simplicial scheme smooth over $W_n$ obtained by base change from $BL^p_./\mathbb{Z}$. Hence we get maps:

$$\begin{aligned} &H^{2p}_{DR}(BL^p_./W) \\ &\quad = \mathbb{H}^p(BL^p_./W, \Omega^{\cdot}_{BL^p_./W}) \longrightarrow \varprojlim \mathbb{H}^{2p}(BL^p_./W_n, \Omega^{\cdot}_{BL^p_./W_n}) \\ &\quad \xrightarrow{\sim} \varprojlim H^{2p}_{\text{crys}}(BL^p_./W_n) \\ &\quad = H^{2p}_{\text{crys}}(BL^p_./W), \end{aligned} \tag{4.21}$$

so in order to construct our universal class $\gamma_{\text{crys}}[Z^p_.]$, it is enough to construct a class

$$\gamma_{DR}[Z^p_.] \in H^{2p}_{DR}(BL^p_./W).$$

This is done in both [1] and [18] and we shall sketch the construction here. If $X$ is a smooth scheme over $W$, and $Y \subset X$ is a codimension $p$ subscheme, also smooth over $W$, there is a natural morphism of complexes of $\mathcal{O}_X$ modules:

$$\phi^{\cdot} : \Omega^{\cdot}_{Y/W} \to \mathcal{H}^p_Y(\Omega^{\cdot}_{X/W})[p].$$

which is made up from maps

$$\phi^i : \Omega^i_{Y/W} \to \mathcal{H}^p_Y(\Omega^{p+i}_{X/W}).$$

For each $y \in Y$, the map on stalks, $\phi^i_y$ is constructed as follows; we choose local equations $x_1, \ldots, x_p$ for $Y$ in a neighborhood $U$ of $y$ in $X$, these define a local cycle class $\gamma_{x_1, \ldots, x_p} \in H^p_Y(U, \Omega^p_{X/W})$. given by the Cech cocycle for the cover $(U, U_{x_1}, \ldots, U_{x_p})$ of $U$ defined by the section

$$\frac{dx_1}{x_1} \wedge \ldots \wedge \frac{dx_p}{x_p} \in \Gamma(U_{x_1} \cap \ldots \cap U_{x_p}, \Omega^p_{X/W}).$$

Given any $\omega \in \Omega^i_{Y/W}(U)$, we can lift it to an element $\tilde{\omega} \in \Omega^i_{X/W}(U)$, and we set:

$$\phi^i(\omega) = \tilde{\omega} \wedge \gamma_{x_1, \ldots, x_p}.$$

One may easily show that $\phi^i(\omega)$ is independent of all the choices made, and that $\phi^{\cdot}$ is a map of complexes. Since $\phi^{\cdot}$ is defined naturally it is also well-defined if $X$ and $Y$ are both simplicial schemes. In particular there is a map of complexes of sheaves on $BL^p_{\cdot}/W$:

$$\phi^{\cdot}: \Omega^{\cdot}_{Z^p_{\cdot}/W} \to H^p_{Z^p_{\cdot}}(BL^p_{\cdot}\Omega^{\cdot}_{BL^p_{\cdot}/W})[p].$$

$\phi^{\cdot}$ induces a map

$$\phi^*: H^0_{DR}(Z^p_{\cdot}/W) \to \mathbb{H}^0(BL^p_{\cdot}, \mathscr{H}^p_{Z^p_{\cdot}}(\Omega^{\cdot}_{BL^p_{\cdot}/W})[p]).$$

By [19] III §8.7, the sheaves $\mathscr{H}^i_{Z^p}(\Omega^{i+p}_{BL^p/W})$ are zero for $i \neq p$, hence

$$H^0(BL^p_{\cdot}, \mathscr{H}^p_{Z^p_{\cdot}}(\Omega^{\cdot}_{BL^p_{\cdot}/W})[p]) \simeq \mathbb{H}^{2p}_{Z^p_{\cdot}}(BL^p_{\cdot}, \Omega^{\cdot}_{BL^p_{\cdot}/W}),$$

and composing $\phi^*$ with the map from cohomology with supports on $Z^p_{\cdot}$ to cohomology with supports on $BL^p$, we get a homomorphism:

$$\phi: H^0_{DR}(Z^p_{\cdot}/W) \to H^{2p}_{DR}(BL^p_{\cdot}/W).$$

DEFINITION 4.22: (i) The cycle class $\gamma[Z^p_{\cdot}] \in H^{2p}_{\mathrm{crys}}(BL^p_{\cdot}/W)$ is the image of the canonical generator $[Z^p_{\cdot}] \in H^0_{DR}(Z^p_{\cdot}/W)$ under the map $\phi$ composed with the homomorphism (4.21).

(ii) If $Y \subset X$ is codimension $p$ subscheme, locally a complete intersection in a scheme of finite type over $W$, the cycle class

$$\gamma[Y] \in H^{2p}_{\mathrm{crys}}(X/W)$$

is defined as the inverse image of $\gamma[Z^p_{\cdot}]$ under any map

$$\chi: X \to BL^p_{\cdot}$$

classifying $Y$. Using the method of (4.10) one sees that this class is independent of the choices made.

In view of the complexity of the construction of cycle classes for smooth subschemes of smooth varieties over $k$ in [1], we shall not compare this definition with that of (op. cit.).

## §5. Determinental subschemes

Clearly there must be elements in the cohomology groups $H^p(X, \mathbf{K}_p)$ of an algebraic variety which do not arise from local complete intersections in the manner §4. I am unable to give a geometric description of all the elements of $H^p(X, \mathbf{K}_p)$, but it is possible to identify some of the "codimension two Cartier cycles" not coming from local complete intersections.

THEOREM 5.1: *Let $X$ be an algebraic variety and $Y \subset X$ a codimension two subscheme the structure sheaf of which locally has projective resolutions of length two (such a $Y$ may be called "perfect"). Then $Y$ has a cycle class $\gamma[Y] \in H^2_Y(X, \mathbf{K}_p)$.*

PROOF: $Y$ is in fact locally determinental (see [5] for a proof of this result, which goes back to Hilbert) and so there is an open cover $\{U_\alpha\}$ of $X$ such that on each $U_\alpha$ there is a resolution

$$0 \to \mathcal{O}^n_{U_\alpha} \xrightarrow{\phi^\alpha} \mathcal{O}^{n+1}_{U_\alpha} \to \mathcal{O}_{U_\alpha} \to \mathcal{O}_{Y \cap U_\alpha} \to 0. \tag{5.2}$$

Note that we can make $m$ independent of $\alpha$, since $X$ is quasi-compact, by adding superfluous generators as necessary without affecting the locally determinental nature of $Y$. In $U_\alpha$ *the ideal* $\mathcal{I}_{Y \cap U_\alpha}$ of $Y$ in $\mathcal{O}_{U_\alpha}$ is generated by the maximal minors of the matrix of the differential $\phi^\alpha$, and is the inverse image (both scheme and cycle-theoretic) of the standard determinental subscheme of $\mathbb{M}_{n,(n+1)}$ by the obvious map, also denoted $\phi^\alpha$, which classifies the differential between $\mathcal{O}^n_{U_\alpha}$ and $\mathcal{O}^{n+1}_{U_\alpha}$.

We want to do for locally determinental subschemes such as $Y$ what we did for local complete intersections in §3. That is construct a smooth simplicial scheme $BD_\bullet$ such that every perfect codimension two subscheme $Y$ is the inverse image of a universal subscheme $Z_\bullet \subset BD_\bullet$ by a suitable classifying map. Clearly $BD_0 = \mathbb{M}_{n,n+1}$. However since the resolution (*) cannot be explicitly reconstituted from a knowledge the generators of the ideal $\mathcal{I}_Y$ alone, if we start with one resolution "$\phi^\alpha$" and want to obtain a second resolution "$\phi^{\beta}$" we need more information than just the transition matrix relating the two sets of generators of the ideal. By Lemma (3.1) there is an open subset $\Gamma \subset \mathbb{M}_{n,n+1} \times \mathbb{M}_{n+1,n+1}$ consisting of all points $(X, Y)$ such that the ideals $(Y.\Delta(X))$ and $(\Delta(X))$ coincide. Now even though the generators $\{Y.\Delta(X)\} = \{\sum_j y_{i,j}\Delta_j(X)\}$ do not determine a single resolution of the ideal $(Y.\Delta(X)) = (\Delta(X))$ we do know that this ideal has projective dimension one and so the kernel $\mathscr{E}$ of the map

$$\mathcal{O}_{\Gamma}^{n+1} \xrightarrow{(Y.\Delta(X))} \mathcal{O}_{\Gamma}$$

is projective and so locally free. Hence there is a $GL_n$ — torsor $F(\mathscr{E})$ over $\Gamma$, the frame bundle of $\mathscr{E}$, the sections of which correspond to isomorphisms $\mathscr{E} = \mathcal{O}_{\Gamma}^{n}$. I claim that $F(\mathscr{E})$ is the right choice for $BD_1$. If $S$ is a perfect codimension two subscheme of an algebraic variety $T$, determinental on the elements of an open cover $\{U_{\alpha}\}$ of $T$, then on the overlaps $U_{\alpha} \cap U_{\beta}$ the triples $(\phi^{\alpha}, \phi^{\beta}, Y^{\alpha\beta})$ consisting of resolutions

$$0 \to \mathcal{O}_{U_\alpha \cap U_\beta}^{n} \xrightarrow{\phi^{\alpha}} \mathcal{O}_{U_\alpha \cap U_\beta}^{n+1} \to \mathcal{O}_{U_\alpha \cap U_\beta} \to \mathcal{O}_{S \cap U_\alpha \cap U_\beta} \to 0$$

$$0 \to \mathcal{O}_{U_\alpha \cap U_\beta}^{n} \xrightarrow{\phi^{\beta}} \mathcal{O}_{U_\alpha \cap U_\beta}^{n+1} \to \mathcal{O}_{U_\alpha \cap U_\beta} \to \mathcal{O}_{S \cap U_\alpha \cap U_\beta} \to 0$$

and transition matrices

$$\begin{array}{ccc} \mathcal{O}_{U_\alpha \cap U_\beta}^{n+1} & \xrightarrow{Y^{\alpha\beta}} & \mathcal{O}_{U_\alpha \cap U_\beta}^{n+1} \\ & {\scriptstyle \Delta(\phi^{\beta})} \searrow \quad \swarrow {\scriptstyle \Delta(\phi^{\alpha})} & \\ & \mathcal{O}_{U_\alpha \cap U_\beta} & \end{array}$$

are clearly classified by morphisms

$$U_{\alpha} \cap U_{\beta} \xrightarrow{\eta} F(\mathscr{E}) = BD_1$$

which are transverse to the standard determinental subscheme of $F(\mathscr{E})$. "Transverse" here means that if $Z \subset F(\mathscr{E})$ is the inverse image of the standard determinental subscheme $D$ of $\mathbb{M}_{n,n+1}$ then

$$\mathrm{Tor}_i^{\mathcal{O}_{F(\mathscr{E})}}(\mathcal{O}_{U_\alpha \cap U_\beta}, \mathcal{O}_Z) = 0 \text{ for } i > 0.$$

In order to construct $BD_{\cdot}$ starting from $\mathrm{BD}_0 = \mathbb{M}_{n,n+1}$, $BD_1 = F(\mathscr{E})$ we can adapt the method used in §3. There are two face maps $d_0, d_1$ and a degenerary $s_0$ between $BD_1$ and $BD_0$:

$$BD_0 \underset{\xrightarrow{s_0}}{\overset{\xrightarrow{d_0}}{\xleftarrow{d_1}}} BD_1.$$

$d_0$ is the composition of the structural map $F(\mathscr{E}) \to \Gamma$, together with projection $\Gamma \to \mathbb{M}_{n,n+1}$. $d_1$ is the map which classifies the "second" map

$$\mathcal{O}^n_{BD_1} \to \mathcal{O}^{n+1}_{BD_1}.$$

$s_0$ classifies the pair of resolutions consisting of the standard resolution of the determinental subscheme of $BD_0$ repeated twice, together with the identity map. Now define $\Lambda_k$ for each $k \geq 2$ as the pull back of the diagram

$$\begin{array}{ccccccc} BD_1 & & BD_1 \ldots\ldots BD_1 & & BD_1 \\ {}_{d_1}\searrow & \swarrow_{d_0} \quad {}_{d_1}\searrow & & {}_{d_1}\searrow & \swarrow_{d_0} \\ & BD_0 & BD_0 & BD_0 & \end{array} \qquad (k \text{ copies of } BD_1)$$

On $\Lambda_k$ we have $(k+1)$ different resolutions $\phi_\alpha$ $(\alpha = 0, \ldots, k)$ of the ideal of the same perfect codimension two subscheme $Z_k \subset \Lambda_k$:

$$0 \to \mathcal{O}^n_{\Lambda_k} \overset{\phi_\alpha}{\to} \mathcal{O}^{n+1}_{\Lambda_k} \to \mathcal{O}_{\Lambda_k} \to \mathcal{O}_{Z_k} \to 0$$

together with transition matrices $\eta^{\alpha,\alpha+1}$ between $\phi_\alpha$ and $\phi_{\alpha+1}$. The scheme $BD_k$ classifying all the possible transition matrices $\phi_{\alpha_0,\alpha_1}$ for $0 \leq \alpha_0 \leq \alpha_1 \leq k$ of $(k+1)$ different resolutions of the same determinental ideal, together with all homotopies between them is the smooth subvariety of

$$P_k = \Lambda_k \times \prod_{0 \leq \alpha_0 \leq \alpha_1 \leq \alpha_2 \leq k} \mathbb{M}_{n+1,n},$$

defined by the equations

$$E_{\alpha_0,\alpha_1,\alpha_2} = \eta^{\alpha_0\alpha_1\alpha_2} \cdot \phi_{\alpha_2} - \eta^{\alpha_0\alpha_1}\eta^{\alpha_1\alpha_2} + \eta^{\alpha_0\alpha_2} = 0$$

$$E_{\alpha_0,\alpha_1,\alpha_2,\alpha_3} = \eta^{\alpha_0\alpha_1}\eta^{\alpha_1\alpha_2\alpha_3} - \eta^{\alpha_0\alpha_2\alpha_3} + \eta^{\alpha_0\alpha_1\alpha_3} - \eta^{\alpha_0\alpha_1\alpha_2}(\eta^{\alpha_2\alpha_3}|_{Ker\,\phi_{\alpha_2}})$$

where for $i = 2$ or 3, $(\alpha_0, \ldots, \alpha_i)$ ranges over all $i$-tuples $\alpha: [i] \to [k]$ and where we write the coordinates on $P_k$ as

$$(\phi_k, \phi_{k-1}, \ldots, \phi_0, \{\eta^{\alpha_0, \alpha_1}\}_{0 \le \alpha_0 < \alpha_1 \le k}, \{\eta^{\alpha_0, \alpha_1, \alpha_2}\}_{0 \le \alpha_0 < \alpha_1 < \alpha_2 \le 3}).$$

(If $\alpha_1 > \alpha_0 + 1$, $\eta^{\alpha_0 \alpha_1}$ is defined inductively using the relationship

$$\eta^{\alpha_0 \alpha_1} = \eta^{\alpha_0, \alpha_1 - 1}\eta^{\alpha_1 - 1, \alpha_1} - \eta^{\alpha_0, \alpha_1 - 1, \alpha_1}\phi_{\alpha_1}.$$

The construction of $BD_.$ together with the perfect codimension two subscheme $Z_. \subset BD_.$ and the universal cycle class

$$\gamma[Z_0] \in H^2_{Z_0}(BD_0, \mathbf{K})$$

now follows just as in §§3 and 4.

The obstacle to extending these ideas to the more general determinental ideals generated by the maximal minors of $r \times s$ matrices for $r < s - 1$ (see [8] for some of the properties of these subschemes) is that not every point $(X, Y)$ in the open set $\Gamma$ of Lemma (3.1) gives rise to an $\binom{s}{r}$-tuple $\{Y, \Delta(X)\}$ of functions which are the maximal minors of some matrix. In fact $\{Y, \Delta(X)\}$ must satisfy the Plucker relations (see for example [17] V.2). It follows that there is a closed subscheme $P \subset \Gamma$ such that if $(X, Y) \in P$, there is a coherent subsheaf $\mathscr{E} \subset \mathcal{O}^s_{P,(X,Y)}$ which is generically a rank $r$ direct summand. Therefore if we restrict $\mathscr{E}$ to the open set $Q \subset P$ on which it is locally free we find that $Q$ classifies determinental ideals coming from two different matrices together with the transition matrices relating the two sets of generators of the ideals. Unfortunately $P$ is singular and there seems no reason to believe that its singular locus misses $Q$. One might still however be able to construct a universal cycle class for subschemes whose ideals are locally generated by the maximal minors of $r \times s$ matrices if the following "Purity Theorem" for algebraic $K$-theory were known to be true:

QUESTION 5.2: Let $X$ be an algebraic variety and suppose its singular locus $\Sigma$ has codimension at least $p + 1$. Then is $H^1_\Sigma(X, \mathbf{K}_p) = 0$?

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