COMPOSITIO MATHEMATICA

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Compositio Mathematica, tome 40, nº 2 (1980), p. 251-267
<http://www.numdam.org/item?id=CM_1980__40_2_251_0>

COMPOSITIO MATHEMATICA, Vol. 40, Fasc. 2, 1980, pag. 251–267

THE MULTIPLICITY OF BINARY RECURRENCES

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1. Introduction

A linear recurrence of order two is a sequence of rational integers $a_0, a_1, a_2, \ldots$ such that

$$a_{n+2} = Ma_{n+1} - Na_n,\ n \geq 0,\ |a_0| + |a_1| \neq 0,$$

where $M, N \in \mathbb{Z}$ are fixed. Throughout this paper we assume that the sequence is non-degenerate, that is, the polynomial $x^2 - Mx + N$ is irreducible in $\mathbb{Z}[x]$ and the quotient of the roots of $x^2 - Mx + N = 0$ is not a root of unity. We study the number of times that a_n assumes the given integer value λ. We denote this number by $m(\lambda)$. Several estimates for $m(\lambda)$ have been given so far. See for instance [1], [2], [5]. Very recently K.K. Kubota [4] proved that $m(\lambda) \leq 4$. In this paper we improve this result by showing that $m(\lambda) + m(-\lambda) \leq 3$, with finitely many exceptions which can be written down explicitly. Moreover, the upper bound for $m(\lambda) + m(-\lambda)$ is assumed in infinitely many cases, e.g. if $M = 1$, N arbitrary, $a_0 = 1$, $a_1 = -1$ then we see that $a_3 = -1$. In Theorem 2 we prove our assertion for sequences with negative discriminant, that is, $M^2 - 4N < 0$. The proof depends essentially on a p-adic argument given in Theorem 1. No use is made of Strassmann's lemma however. In Theorem 3 we prove that $m(\lambda) + m(-\lambda) \leq 3$ for all non-degenerate sequences with positive discriminant with a single exception. The last part of the paper is devoted to Lucas-sequences of the first kind. These sequences are linear recurrences of order 2 with $a_0 = 0$ and $a_1 = 1$. In Theorem 4 and the corollary to this theorem we prove that $m(\lambda) + m(-\lambda) \leq 2$ if $|\lambda| \geq 2$ and $m(1) + m(-1) \leq 2$ unless $M = \pm 1$, $N = 2, 3, 5$. In [3] K.K. Kubota obtains almost the same result, but the case $M = \pm 1$, $N \equiv 2 \pmod{48}$ remained unproved. The following examples show that there are infinitely many Lucas-sequences such that $m(1) + m(-1) \geq$

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2: if $M = \pm 1$ then $a_2 = M = \pm 1$, if $M^2 = N \pm 1$ then we see that $a_3 = M^2 - N = \pm 1$ and if $M = \pm 12$, $N = 55{,}377$ then $a_5 = \pm 1$. It is not unreasonable to expect that apart from these cases there exist no Lucas-sequences with $m(1) + m(-1) \geq 2$.

2. The general case

LEMMA 1: *Let the non-degenerate recurrent sequence $\{a_n\}_{n=0}^{\infty}$ be defined by $a_n = Ma_{n-1} - Na_{n-2}$ where a_0, a_1, M, $N \in \mathbb{Z}$. Let θ be a root of $x^2 - Mx + N = 0$ and put $\alpha = a_1 - a_0\tilde{\theta}$. Then the recurrent sequence is given by*

$$a_n = \frac{\alpha\theta^n - \tilde{\alpha}\tilde{\theta}^n}{\theta - \tilde{\theta}},$$

where $\tilde{\alpha}$, $\tilde{\theta}$ denote the conjugates of α and θ.

PROOF: Induction.

In studying the multiplicity of the sequence $\{a_n\}_{n=0}^{\infty}$ we may as well consider the multiplicity of the sequence $\alpha\theta^n - \tilde{\alpha}\tilde{\theta}^n$. Suppose we want to study the multiplicity of the value $\alpha\theta^p - \tilde{\alpha}\tilde{\theta}^p$ for some $p \in \mathbb{N}$. By replacing $\alpha\theta^p$ by α we see that without loss of generality it is sufficient to consider the multiplicity of the value $\alpha - \tilde{\alpha}$. Furthermore, we may assume that the algebraic integers α and $\tilde{\alpha}$ have no rational integer factor in common.

LEMMA 2: *Let α, θ be algebraic integers in a quadratic numberfield and denote their conjugates by $\tilde{\alpha}$, $\tilde{\theta}$. Suppose α and $\tilde{\alpha}$ have no common factor in $\mathbb{Z}$. If $\alpha\theta^q - \tilde{\alpha}\tilde{\theta}^q = \epsilon(\alpha - \tilde{\alpha})$ for some $\epsilon \in \{-1, 1\}$ then there exists a rational integer μ such that $\tilde{\theta}^q = \epsilon + \mu\alpha$.*

PROOF: It follows from $\alpha(\theta^q - \epsilon) = \tilde{\alpha}(\tilde{\theta}^q - \epsilon)$ that $\lambda := \alpha(\theta^q - \epsilon)$ is a rational integer. Multiplication by $\tilde{\alpha}$ yields $\lambda\tilde{\alpha} = \alpha\tilde{\alpha}(\theta^q - \epsilon)$. Suppose $\alpha\tilde{\alpha} \nmid \lambda$. Then there exists a prime factor p of $\alpha\tilde{\alpha}$ which divides $\tilde{\alpha}$. Since p is a rational integer we also have $p \mid \alpha$, contradicting our assumption that α and $\tilde{\alpha}$ have no common factor in $\mathbb{Z}$. We therefore conclude that $\alpha\tilde{\alpha} \mid \lambda$ and hence $\theta^q - \epsilon = \mu\tilde{\alpha}$ for some $\mu \in \mathbb{Z}$.

In the following lemmas we assume that the recurrent sequence has negative discriminant, that is $M^2 - 4N < 0$. This implies that α and θ are algebraic integers in an imaginary quadratic field. We may assume that $0 < \arg\theta < \pi/2$ and $0 \leq \arg\alpha < \pi$. This can be achieved by taking the complex conjugate of the equation $\alpha\theta^n - \tilde{\alpha}\tilde{\theta}^n = \pm(\alpha - \tilde{\alpha})$, if necessary, and by the replacements $\theta \to -\theta$ and $\alpha \to -\alpha$. Since the recurrent sequence is non-degenerate, we have $0 < \arg\theta < \pi/2$. Moreover, α cannot be real. For, if $\alpha = \tilde{\alpha}$ then $\alpha\theta^n - \tilde{\alpha}\tilde{\theta}^n = \pm(\alpha - \tilde{\alpha})$ reduces to $\theta^n - \tilde{\theta}^n = 0$ and this implies that $\theta/\tilde{\theta}$ is a root of unity. We therefore assume that $0 < \arg\alpha < \pi$.

THEOREM 1: *Let K be an imaginary quadratic field and $\mathcal{O}_K$ its ring of integers. Let γ, $\eta \in \mathcal{O}_K$ and let $t \in \mathbb{Z}$, $t \neq 0$. Consider the equation $\gamma(1+t\eta)^r - \tilde{\gamma}(1+t\tilde{\eta})^r = \gamma - \tilde{\gamma}$ in the unknown $r \in \mathbb{N}$.*

A) *Suppose that $\gamma\eta - \tilde{\gamma}\tilde{\eta} \neq 0$ and let $g \in \mathcal{O}_K$ divide $\gamma\eta^k - \tilde{\gamma}\tilde{\eta}^k$ for all $k \geq 1$. There are no solutions $r \in \mathbb{N}$ if one of the following conditions is satisfied,*

1) $t \equiv 0 \pmod 2$ *and* $\frac{t}{2} \nmid (\gamma\eta - \tilde{\gamma}\tilde{\eta})/g$,
2) $t \not\equiv 0 \pmod 2$ *and* $t \nmid (\gamma\eta - \tilde{\gamma}\tilde{\eta})/g$.

B) *Suppose that $\gamma\eta - \tilde{\gamma}\tilde{\eta} = 0$ and $\gamma\eta \neq 0$. Let $g \in \mathcal{O}_K$ divide $\eta - \tilde{\eta}$. Then $r = 1$ is the only solution if at least one of the following conditions is satisfied,*

1) $t \equiv 0 \pmod 3$ *and* $\frac{t}{3} \nmid (\eta - \tilde{\eta})/g$,
2) $t \equiv 0 \pmod 3$, $t \nmid (\eta - \tilde{\eta})/g$ *and* $(\eta^2 - \tilde{\eta}^2)/g \equiv 0 \pmod 3$,
3) $t \not\equiv 0 \pmod 3$ *and* $t \nmid (\eta - \tilde{\eta})/g$.

PROOF: The equation $\gamma(1+t\eta)^r - \tilde{\gamma}(1+t\tilde{\eta})^r = \gamma - \tilde{\gamma}$ can be written as

$$\gamma - \tilde{\gamma} + \sum_{k=1}^{r} \binom{r}{k} t^k (\gamma\eta^k - \tilde{\gamma}\tilde{\eta}^k) = \gamma - \tilde{\gamma}$$

and since $\binom{r}{k} = \frac{r}{k}\binom{r-1}{k-1}$, we obtain

$$r \sum_{k=1}^{r} \frac{t^{k-1}}{k} \binom{r-1}{k-1} \frac{\gamma\eta^k - \tilde{\gamma}\tilde{\eta}^k}{g} = 0. \tag{1}$$

Since $r \neq 0$, the sumfactor in (1) must vanish. We first prove part A of our theorem, so we assume that $\gamma\eta - \tilde{\gamma}\tilde{\eta} \neq 0$. Suppose $t \equiv 0 \pmod 2$ and $\frac{t}{2} \nmid (\gamma\eta - \tilde{\gamma}\tilde{\eta})/g$. It is easy to see that $t^{k-1}/k \equiv 0 \pmod{t/2}$ if $k = 2$ and $t^{k-1}/k \equiv 0 \pmod t$ if $k \geq 3$. This implies that $t/2$ divides the first term in the sumfactor, i.e. $\frac{t}{2} \mid (\gamma\eta - \tilde{\gamma}\tilde{\eta})/g$ which is a contradiction. Suppose $t \not\equiv 0 \pmod 2$ and $t \nmid (\gamma\eta - \tilde{\gamma}\tilde{\eta})/g$. It is easy to see that $t^{k-1}/k \equiv 0 \pmod t$ for all $k \geq 2$ and hence $t \mid (\gamma\eta - \tilde{\gamma}\tilde{\eta})/g$, which is a contradiction. We therefore conclude that there exists no solution $r \in \mathbb{N}$ in both cases. We now prove part B. Assume $\gamma\eta = \tilde{\gamma}\tilde{\eta}$ and $r \geq 2$, then equation (1) reduces to

$$r\sum_{k=2}^{r}\frac{t^{k-1}}{k}\binom{r-1}{k-1}\gamma\eta\,\frac{1}{g}(\eta^{k-1}-\tilde{\eta}^{k-1})=0,$$

and hence

$$\binom{r}{2}\sum_{k=2}^{r}\frac{2t^{k-2}}{k(k-1)}\binom{r-2}{k-2}\frac{\eta^{k-1}-\tilde{\eta}^{k-1}}{g}=0.$$

Suppose $r\neq 1$, so that the sumfactor must be zero. Suppose $t\equiv 0(\text{mod } 3)$. Then $2t^{k-2}/k(k-1)\equiv 0(\text{mod } t/3)$ if $k=3$ and $2t^{k-2}/k(k-1)\equiv 0(\text{mod } t)$ if $k\geq 4$. This implies that $t/3$ divides the first term of the sumfactor, that is $(\eta-\tilde{\eta})/g$. This contradicts the assumption we made in B1). Suppose $3\mid t$ and $3\mid(\eta^2-\tilde{\eta}^2)/g$ then the term corresponding to $k=3$ is also zero mod t and hence $t\mid(\eta-\tilde{\eta})/g$ which contradicts the assumption made in B2). Finally, suppose that $t\not\equiv 0(\text{mod } 3)$ and $t\nmid(\eta-\tilde{\eta})/g$. In this case we know that $2t^{k-2}/k(k-1)\equiv 0(\text{mod } t)$ for all $k\geq 3$. Hence $t|(\eta-\tilde{\eta})/g$ which is a contradiction and thus we have proved our theorem.

LEMMA 3: *Let α and θ be integers in an imaginary quadratic field with $0<\arg\alpha<\pi$, $0<\arg\theta<\pi/2$ and $\theta/\tilde{\theta}$ is not a root of unity. Assume that $\theta^p=\epsilon+\mu\tilde{\alpha}$ and $\theta^q=\epsilon'+\mu'\tilde{\alpha}$ for some p, $q\in\mathbb{N}$, μ, $\mu'\in\mathbb{Z}$, ϵ, $\epsilon'\in\{-1,1\}$ with $q>p$ and $|\mu|>1$. Put $q=pr+\delta$, $0\leq\delta<p$. Then $\alpha\theta^\delta-\tilde{\alpha}\tilde{\theta}^\delta=\epsilon'\epsilon^r(\alpha-\tilde{\alpha})$*

PROOF: Observe that

$$\epsilon'(\alpha-\tilde{\alpha})=\alpha\theta^q-\tilde{\alpha}\tilde{\theta}^q=\alpha\theta^\delta(\epsilon+\mu\tilde{\alpha})^r-\tilde{\alpha}\tilde{\theta}^\delta(\epsilon+\mu\alpha)^r.$$

Hence

$$\epsilon^r\epsilon'(\alpha-\tilde{\alpha})=\alpha\theta^\delta-\tilde{\alpha}\tilde{\theta}^\delta+\mu\alpha\tilde{\alpha}\left[\theta^\delta\frac{(1+\epsilon\mu\tilde{\alpha})^r-1}{\mu\tilde{\alpha}}-\tilde{\theta}^\delta\frac{(1+\epsilon\mu\alpha)^r-1}{\mu\alpha}\right].\tag{2}$$

If $p=1$, then $\delta=0$ and the term between square brackets in (2) is divisible by $\mu(\alpha-\tilde{\alpha})$. Hence $\mu^2\alpha\tilde{\alpha}(\alpha-\tilde{\alpha})$ divides $(\alpha-\tilde{\alpha})-\epsilon^r\epsilon'(\alpha-\tilde{\alpha})$. Since $|\mu|>1$ this is only possible if $\epsilon^r\epsilon'=1$ and our lemma is proved in this case. Now assume that $p\geq 2$. The numbers α

and θ belong to a quadratic field, which we denote by $\mathbb{Q}(\sqrt{-d})$, where d is a positive square-free integer. Notice that the term between square brackets in (2) is divisible by $\sqrt{-d}$ if $d \equiv -1 \pmod 4$ and by $2\sqrt{-d}$ if $d \not\equiv -1 \pmod 4$. Put $C(d) = \sqrt{d}$ if $d \equiv -1(4)$ and $C(d) = 2\sqrt{d}$ if $d \not\equiv -1 \pmod 4$. Then (2) implies

$$iC(d)\mu\alpha\tilde{\alpha} \mid (\alpha\theta^{\delta} - \tilde{\alpha}\tilde{\theta}^{\delta} - \epsilon^{r}\epsilon'(\alpha - \tilde{\alpha})).$$

Suppose $\alpha\theta^{\delta} - \tilde{\alpha}\tilde{\theta}^{\delta} \neq \epsilon^{r}\epsilon'(\alpha - \tilde{\alpha})$. This implies

$$C(d)|\mu\alpha\tilde{\alpha}| \leq |\alpha\theta^{\delta} - \tilde{\alpha}\tilde{\theta}^{\delta} - \epsilon^{r}\epsilon'(\alpha - \tilde{\alpha})|.$$

Using $\theta^{p} = \epsilon + \mu\tilde{\alpha}$ and the triangle inequality, we obtain

$$|\alpha|C(d)(|\theta|^{p} - 1) \leq C(d)|\alpha||\theta^{p} - \epsilon| < C(d)|\mu\alpha\tilde{\alpha}| \leq 2|\alpha|(|\theta|^{\delta} + 1),$$

and hence

$$1 < \frac{2}{|\theta|C(d)} + \frac{1 + 2/C(d)}{|\theta|^{p}}. \tag{3}$$

We recall that $0 < \arg\theta < \pi/2$. If $d \not\equiv -1 \pmod 4$, $d \geq 2$ then it follows that

$$\frac{2}{|\theta|C(d)} + \frac{1 + 2/C(d)}{|\theta|^{p}} \leq \frac{1}{\sqrt{1+d}\sqrt{d}} + \frac{1 + 1/\sqrt{d}}{1+d} \leq \frac{1}{\sqrt{6}} + \frac{1 + 1/\sqrt{2}}{3} < 1.$$

If $d \equiv -1 \pmod 4$, $d \geq 11$, then

$$\frac{2}{|\theta|C(d)} + \frac{1 + 2/C(d)}{|\theta|^{p}} \leq \frac{4}{\sqrt{1+d}\sqrt{d}} + \frac{4 + 8/\sqrt{d}}{1+d} \leq \frac{2}{\sqrt{33}} + \frac{4 + 8/\sqrt{11}}{12} < 1.$$

Hence, the only solutions of (3) are those corresponding to $d = 7, 3, 1$. A simple calculation shows that there are no other solutions than $\theta = \frac{1}{2} + \frac{1}{2}\sqrt{-7}$, $1 + \mathrm{i}$, $1 + \sqrt{-3}$, $\frac{3}{2} + \frac{1}{2}\sqrt{-3}$, $\frac{1}{2} + \frac{1}{2}\sqrt{-3}$. The latter four solutions can be ignored since $\theta/\tilde{\theta}$ is a root of unity for these values. We are left with $\theta = \frac{1}{2} + \frac{1}{2}\sqrt{-7}$. From (3) we deduce that $p \leq 3$. The condition $\theta^{p} = \epsilon + \mu\tilde{\alpha}$, $\mu \in \mathbb{Z}$, $|\mu| > 1$ implies that $(\frac{1}{2} + \frac{1}{2}\sqrt{-7})^{p} \pm 1$ is divisible by a rational integer larger than 1 and this is impossible if $p \leq 3$. We therefore conclude that $\alpha\theta^{\delta} - \tilde{\alpha}\tilde{\theta}^{\delta} = \epsilon'\epsilon^{r}(\alpha - \tilde{\alpha})$, as asserted.

LEMMA 4: *Let θ be an integer in an imaginary quadratic field, and assume that $0 < \arg\theta < \pi/2$ and $\theta/\tilde{\theta}$ is not a root of unity. Let p and q be positive integers such that $q > p$.*

a) *If $\theta^q \pm \theta^p = \pm 2$ for some choice of the $\pm$ signs then $(p, q, \theta) = (1, 3, \frac{1}{2}+\frac{1}{2}\sqrt{-7})$ or $(1, 2, \frac{1}{2}+\frac{1}{2}\sqrt{-7})$.*

b) *If $\theta^q \pm 2\theta^p = \pm 3$ then $(p, q, \theta) = (1, 3, \frac{1}{2}+\frac{1}{2}\sqrt{-11})$ or $(1, 2, 1+\sqrt{-2})$.*

c) *If $\theta^q \pm 3\theta^p \in \{\pm 4, \pm 2\}$ then $(p, q, \theta) = (2, 4, \frac{1}{2}+\frac{1}{2}\sqrt{-7})$, $(1, 2, \frac{3}{2}+\frac{1}{2}\sqrt{-7})$, $(1, 4, \frac{1}{2}+\frac{1}{2}\sqrt{-7})$ or $(1, 3, \frac{1}{2}+\frac{1}{2}\sqrt{-15})$.*

PROOF: a) If θ is such that $\theta^q \pm \theta^p = \pm 2$ then $\theta^p \mid 2$ and $p \le 2$. Furthermore, $|\theta|^q \le |\theta|^p + 2 \le |\theta|^2 + 2$, and hence $q \le 4$. If $p = 2$, $q = 4$ then we have a quadratic equation in θ^2. Solving this equation yields $\theta = \pm i, \pm\sqrt{-2}$ which values can be ignored. If $p = 2$, $q = 3$ we can confine ourselves to $\theta^3 \pm \theta^2 - 2 = 0$. If this equation is to have a solution in quadratic integers, then it has also a solution in $\mathbb{Z}$. This actually happens for $\theta^3 + \theta^2 - 2 = 0$ and we find $\theta = 1, -1 \pm i$. In the same way we proceed in the case $p = 1$, $q = 3$. Then we obtain $\theta = \pm\frac{1}{2} \pm \frac{1}{2}\sqrt{-7}$. If $q = 4$, $p = 1$ then $|\theta|^4 \le |\theta| + 2$, contradicting $|\theta| \ge \sqrt{2}$. If $q = 2$, $p = 1$ then we find that $\theta = \pm\frac{1}{2} \pm \frac{1}{2}\sqrt{-7}$.

b) If θ is such that $\theta^q \pm 2\theta^p = \pm 3$, then $\theta^p \mid 3$ and $p \le 2$. Furthermore, $|\theta|^q \le 2|\theta|^p + 3 \le 2|\theta|^2 + 3$ and since $|\theta| \ge \sqrt{3}$ we can conclude that $q \le 4$. we solve the equations $\theta^q \pm 2\theta^p = \pm 3$ in a similar way as in a) and we obtain the solutions given in our lemma.

c) If θ is such that $\theta^q \pm 3\theta^p \in \{\pm 4, \pm 2\}$ then $\theta^p \mid 4$ and $p \le 4$. The collection of values θ satisfying the restrictions $\theta \mid 4$, $0 < \arg\theta < \pi/2$ and such that $\theta/\tilde{\theta}$ is not a root of unity is given by $\{\frac{1}{2}+\frac{1}{2}\sqrt{-7}, \frac{3}{2}+\frac{1}{2}\sqrt{-7}, 1+\sqrt{-7}, \frac{1}{2}+\frac{1}{2}\sqrt{-15}\}$. It is easy to see that for these values $\theta^p \mid 4$ with $p \ge 3$ is impossible. Hence $p \le 2$. If $p = 2$ then $\theta^2 \mid 4$ whence $\theta = \frac{1}{2}+\frac{1}{2}\sqrt{-7}$. We check $\theta^{q-2} \pm 3 \in \{\pm 4/\theta^2, \pm 2/\theta^2\}$ for $\theta = \frac{1}{2}+\frac{1}{2}\sqrt{-7}$ and find that $(p, q, \theta) = (2, 4, \frac{1}{2}+\frac{1}{2}\sqrt{-7})$. If $p = 1$, then we consider $\theta^{q-1} \pm 3 \in \{\pm 4/\theta, \pm 2/\theta\}$ for $\theta \in \{\frac{1}{2}+\frac{1}{2}\sqrt{-7}, \frac{3}{2}+\frac{1}{2}\sqrt{-7}, 1+\sqrt{-7}, \frac{1}{2}+\frac{1}{2}\sqrt{-15}\}$ and we find that $(p, q, \theta) = (1, 3, \frac{1}{2}+\frac{1}{2}\sqrt{-15})$, $(1, 2, \frac{3}{2}+\frac{1}{2}\sqrt{-7})$ or $(1, 4, \frac{1}{2}+\frac{1}{2}\sqrt{-7})$.

LEMMA 5: *Let θ be a complex quadratic integer such that $\theta/\tilde{\theta}$ is not a root of unity and $0 < \arg\theta < \frac{\pi}{2}$. Suppose there exist $p, q \in \mathbb{N}$ with $q > p$ and an algebraic integer $\tilde{\alpha}$ such that $\theta^p = \epsilon' + \mu'\tilde{\alpha}$, $\theta^q = \epsilon + \mu\tilde{\alpha}$ for some $\epsilon, \epsilon' \in \{-1, 1\}$ and $\mu, \mu' \in \mathbb{Z}$. Then $|\mu| \ge |\mu'|$.*

PROOF: Suppose $|\mu'| > |\mu|$. From $|\theta^p - \epsilon'| = |\mu'\tilde{\alpha}|$ and $|\theta^q - \epsilon| = |\mu\tilde{\alpha}|$ we derive $(|\mu| - |\mu'|)|\tilde{\alpha}| = |\theta^q - \epsilon| - |\theta^p - \epsilon'| \ge |\theta|^q - 1 - |\theta|^p - 1$. Hence

$-1 \geq |\mu| - |\mu'| \geq |\theta|^q - |\theta|^p - 2$ or equivalently, $|\theta|^p(|\theta|^{q-p} - 1) \leq 1$. Since $|\theta| \geq \sqrt{2}$, it easily follows that $q - p = 1$. Furthermore, $|\theta| = \sqrt{2}$ and $p \leq 2$. Hence $\theta = \frac{1}{2} + \frac{1}{2}\sqrt{-7}$. Considering $\theta = \epsilon' + \mu'\tilde{\alpha}$ and $\theta^2 = \epsilon' + \mu'\tilde{\alpha}$ for $\theta = \frac{1}{2} + \frac{1}{2}\sqrt{-7}$ we find that $\mu' = \pm 1$. Then $|\mu| < |\mu'|$ implies $\mu = 0$, which is impossible.

LEMMA 6: *Let α and θ be integers in an imaginary quadratic field, and suppose that $\theta/\tilde{\theta}$ is not a root of unity and $0 < \arg\theta < \pi/2$, $0 < \arg\alpha < \pi$. Suppose $\theta^p = \epsilon + \mu\tilde{\alpha}$ with $\mu \in \mathbb{Z}$ and $|\mu| > 3$. Then $\alpha\theta^n - \tilde{\alpha}\tilde{\theta}^n = \pm(\alpha - \tilde{\alpha})$ has no solutions $n = q$ with $q > p$.*

PROOF: Suppose $\alpha\theta^q - \tilde{\alpha}\tilde{\theta}^q = \epsilon'(\alpha - \tilde{\alpha})$. Put $q = pr + \delta$, with $0 \leq \delta < p$, $r > 0$. Then, by Lemmas 2 and 3 we have $\alpha\theta^\delta - \tilde{\alpha}\tilde{\theta}^\delta = \epsilon'\epsilon^r(\alpha - \tilde{\alpha})$ and the equation $\alpha\theta^q - \tilde{\alpha}\tilde{\theta}^q = \epsilon'(\alpha - \tilde{\alpha})$ can be written as

$$\alpha\theta^\delta(\epsilon + \mu\tilde{\alpha})^r - \tilde{\alpha}\tilde{\theta}^\delta(\epsilon + \mu\alpha)^r = \epsilon^r(\alpha\theta^\delta - \tilde{\alpha}\tilde{\theta}^\delta)$$

and hence

$$\alpha\theta^\delta(1 + \epsilon\mu\tilde{\alpha})^r - \tilde{\alpha}\tilde{\theta}^\delta(1 + \epsilon\mu\alpha)^r = \alpha\theta^\delta - \tilde{\alpha}\tilde{\theta}^\delta.$$

In order to apply Theorem 1 we distinguish between two cases.

I) $\theta^\delta - \tilde{\theta}^\delta = 0$. Since $\theta/\tilde{\theta}$ is not a root of unity we see that $\delta = 0$. We apply theorem 1 B) with $\gamma = \alpha$, $\eta = \tilde{\alpha}$, $g = \alpha - \tilde{\alpha}$ and $t = \epsilon\mu$. Then $(\eta - \tilde{\eta})/g = -1$. Since $|\mu| > 3$ the conditions of Theorem 1 B) are fulfilled, whence $r = 1$.

II) $\theta^\delta - \tilde{\theta}^\delta \neq 0$. Since $\alpha\theta^\delta - \tilde{\alpha}\tilde{\theta}^\delta = \epsilon'\epsilon^r(\alpha - \tilde{\alpha})$, Lemma 2 implies that $\theta^\delta = \epsilon'' + \mu'\tilde{\alpha}$ for some $\epsilon'' \in \{-1, 1\}$ and $\mu' \in \mathbb{Z}$. We apply Theorem 1A) with $\gamma = \alpha\theta^\delta$, $\eta = \tilde{\alpha}$, $g = \alpha\tilde{\alpha}(\alpha\theta^\delta - \tilde{\alpha}\tilde{\theta}^\delta)$ and $t = \epsilon\mu$. Then $(\gamma\eta - \tilde{\gamma}\tilde{\eta})/g = \alpha\tilde{\alpha}(\theta^\delta - \tilde{\theta}^\delta)/\alpha\tilde{\alpha}(\alpha\theta^\delta - \tilde{\alpha}\tilde{\theta}^\delta) = \mu'(\tilde{\alpha} - \alpha)/\epsilon''(\alpha - \tilde{\alpha}) = -\epsilon''\mu'$. Notice that g divides $\gamma\eta^k - \tilde{\gamma}\tilde{\eta}^k$ for all $k \geq 1$. It now follows from Theorem 1A) that either $r = 0$, contradicting $r > 0$, or $\mu \mid \mu'$ if $\mu \equiv 1 \pmod 2$ and $(\mu/2) \mid \mu'$ if $\mu \equiv 0 \pmod 2$. By Lemma 5 we know that $|\mu'| \leq |\mu|$ and hence $|\mu| = |\mu'|$ or $|\mu| = 2|\mu'|$. Suppose $|\mu| = |\mu'|$. Then $\theta^p = \epsilon \pm \mu'\tilde{\alpha}$ and $\theta^\delta = \epsilon'' + \mu'\tilde{\alpha}$. Hence $\theta^p \pm \theta^\delta = \pm 2$ or 0. Since θ is not a root of unity we have $\theta^p \pm \theta^\delta = \pm 2$. We observe that $\theta^\delta \mid 2$ and hence $|\theta^\delta| \leq 2$. This contradicts $\theta^\delta = \epsilon'' + \mu'\tilde{\alpha}$ because $|\mu'| = |\mu| > 3$. Suppose $|\mu| = 2|\mu'|$ then $\theta^p = \epsilon \pm 2\mu'\tilde{\alpha}$ and $\theta^\delta = \epsilon'' + \mu'\tilde{\alpha}$. Hence $\theta^p \pm 2\theta^\delta = \pm 3, \pm 1$. Since θ is not a root of unity we have $\theta^p \pm 2\theta^\delta = \pm 3$. By Lemma 4 we find that $(\theta, p, \delta) = (\frac{1}{2} + \frac{1}{2}\sqrt{-11}, 3, 1)$ or $(1 + \sqrt{-2}, 2, 1)$. Since $\theta^\delta = \epsilon'' + \mu'\tilde{\alpha}$, where $|\mu'| \geq 2$, one of the numbers $\frac{1}{2} + \frac{1}{2}\sqrt{-11} \pm 1$ and one of the numbers $1 + \sqrt{-2} \pm 1$ must be divisible by a rational integer larger than 1, which is not the case. We therefore conclude that there exists no solution $n = q$ with $q > p$.

LEMMA 7: *For given α and θ, all solutions $n \geq 0$ of the equations $\alpha\theta^n - \tilde{\alpha}\tilde{\theta}^n = \pm(\alpha - \tilde{\alpha})$ are given:*

i) $\theta = \frac{1}{2} + \frac{1}{2}\sqrt{-7}$, $\alpha = \frac{1}{2} + \frac{1}{2}\sqrt{-7}$ *then* $n = 0, 1, 2, 4, 12$
ii) $\theta = 1 + \sqrt{-2}$, $\alpha = \sqrt{-2}$ *then* $n = 0, 1, 2, 5$
iii) $\theta = \frac{1}{2} + \frac{1}{2}\sqrt{-11}$, $\alpha = \frac{1}{2} + \frac{1}{2}\sqrt{-11}$ *then* $n = 0, 1, 4$
iv) $\theta = \frac{1}{2} + \frac{1}{2}\sqrt{-11}$, $\alpha = -\frac{3}{2} + \frac{1}{2}\sqrt{-11}$ *then* $n = 0, 1, 3$
v) $\theta = \frac{1}{2} + \frac{1}{2}\sqrt{-15}$, $\alpha = -\frac{3}{2} + \frac{1}{2}\sqrt{-15}$ *then* $n = 0, 1, 3$
vi) $\theta = \frac{1}{2} + \frac{1}{2}\sqrt{-19}$, $\alpha = \frac{1}{2} + \frac{1}{2}\sqrt{-19}$ *then* $n = 0, 1, 6$.

PROOF: The cases i), ii), iii) and vi) can be dealt with immediately. Notice that $(\frac{1}{2} + \frac{1}{2}\sqrt{-7})^{12} = -1 - 45(\frac{1}{2} - \frac{1}{2}\sqrt{-7})$, $(1 + \sqrt{-2})^5 = 1 - 11\sqrt{-2}$, $(\frac{1}{2} + \frac{1}{2}\sqrt{-11})^4 = 1 + 5(\frac{1}{2} - \frac{1}{2}\sqrt{-11})$ and $(\frac{1}{2} + \frac{1}{2}\sqrt{-19})^6 = 1 - 56(\frac{1}{2} - \frac{1}{2}\sqrt{-19})$ respectively. Hence Lemma 6 implies that $n \leq 12$, $n \leq 5$, $n \leq 4$ and $n \leq 6$ respectively, and this finite collection of possibilities can easily be checked by considering the corresponding recurrent sequences.

In case iv) we notice that $\theta^4 = 1 + 5\tilde{\theta}$. Put $n = 4k + \delta$, with $0 \leq \delta \leq 3$, then the equation reads

$$\alpha\theta^\delta(1 + 5\tilde{\theta})^k - \tilde{\alpha}\tilde{\theta}^\delta(1 + 5\theta)^k = \pm(\alpha - \tilde{\alpha}).$$

Considering this equation mod 5 if $\delta = 0$ and mod $5\theta\tilde{\theta}$ if $\delta > 0$ we see that $\alpha\theta^\delta - \tilde{\alpha}\tilde{\theta}^\delta \equiv \pm(\alpha - \tilde{\alpha}) \pmod 5$ if $\delta = 0$ and $\alpha\theta^\delta - \tilde{\alpha}\tilde{\theta}^\delta \equiv \pm(\alpha - \tilde{\alpha}) \pmod{15}$ if $\delta > 0$. This implies $\alpha\theta^\delta - \tilde{\alpha}\tilde{\theta}^\delta = \pm(\alpha - \tilde{\alpha})$ and hence

$$\alpha\theta^\delta(1 + 5\tilde{\theta})^k - \tilde{\alpha}\tilde{\theta}^\delta(1 + 5\theta)^k = \alpha\theta^\delta - \tilde{\alpha}\tilde{\theta}^\delta.$$

We can now apply Theorem 1A) with $\gamma = \alpha\theta^\delta$, $\eta = \tilde{\theta}$, $g = \sqrt{-11}$ and $t = 5$. We find that $k = 0$ and hence $\eta = \delta \leq 3$ and our solutions follow. In case v) we notice that $\theta^3 = -1 + 3\alpha$. Put $q = 3k + \delta$, $0 \leq \delta \leq 2$ and suppose that $\alpha\theta^q - \tilde{\alpha}\tilde{\theta}^q = \epsilon(\alpha - \tilde{\alpha})$ for some $\epsilon \in \{-1, 1\}$. Then Lemma 3 implies that $\alpha\theta^\delta - \tilde{\alpha}\tilde{\theta}^\delta = (-1)^k\epsilon(\alpha - \tilde{\alpha})$ and the equation $\alpha\theta^q - \tilde{\alpha}\tilde{\theta}^q = \epsilon(\alpha - \tilde{\alpha})$ can now be written as $\alpha\theta^\delta(1 - 3\tilde{\alpha})^k - \tilde{\alpha}\tilde{\theta}^\delta(1 - 3\alpha)^k = \alpha\theta^\delta - \tilde{\alpha}\tilde{\theta}^\delta$. If $\delta = 0$, we can apply Theorem 1 B2) with $\gamma = \alpha\theta^\delta = \alpha$, $\eta = \tilde{\alpha}$, $g = \alpha - \tilde{\alpha}$ and $t = -3$, so that $(\eta - \tilde{\eta})/g = -1$, $(\eta^2 - \tilde{\eta}^2)/g = -(\alpha + \tilde{\alpha}) = 3$. We conclude that $k = 0, 1$. If $\delta \neq 0$, application of Theorem 1A) with $\gamma = \alpha\theta^\delta$, $\eta = \tilde{\alpha}$, $g = 6\sqrt{-15}$ and $t = -3$ gives us $k = 0$ as the only solution. Hence $q = 3k + \delta \leq 3$ and our solutions follow.

THEOREM 2: *Let the non-degenerate recurrent sequence of rational integers* $\{a_m\}_{m=0}^{\infty}$ *be given by* $a_0 \geq 0$, $(a_1, a_0) = 1$, $a_m = Ma_{m-1} - Na_{m-2}$ *with* $M, N \in \mathbb{Z}$ *and* $M \geq 0$. *Assume that* $M^2 - 4N < 0$. *If* $a_m = \pm a_0$ *has more than three solutions* m, *then one of the following cases holds*:

$M = 1$, $N = 2$, $a_0 = a_1 = 1$ *then* $m = 0, 1, 2, 4, 12$.
$M = 1$, $N = 2$, $a_0 = 1, a_1 = -1$ *then* $m = 0, 1, 3, 11$.
$M = 3$, $N = 4$, $a_0 = a_1 = 1$ *then* $m = 0, 1, 2, 6$.
$M = 2$, $N = 3$, $a_0 = a_1 = 1$ *then* $m = 0, 1, 2, 5$.

REMARK 1: The restriction $M \geq 0$ is not essential, for negative M we may consider the sequence $a'_m = (-1)^m a_m$ which satisfies the recurrence relation $a'_m = (-M)a'_{m-1} - Na'_{m-2}$. Furthermore, if $(a_0, a_1) = g > 1$ then we may consider the sequence a_m/g and therefore the condition $(a_0, a_1) = 1$ is not restrictive.

REMARK 2: If a recurrent sequence assumes the value λ more than three times in absolute value, then it can be transformed into one of the cases of theorem 2. Let m_0 be smallest solution of $|a_m| = \lambda$ and consider the recurrent sequence starting with $a_{m_0}, a_{m_0+1}, \ldots$. Divide the terms of this sequence by (a_{m_0}, a_{m_0+1}) and, if necessary, change the negative M into $(-M)$ as we have done in Remark 1. The recurrence that we have thus obtained satisfies the conditions of Theorem 2, and it assumes its starting value more than three times in absolute value. Conversely by reversing the process we can construct from the cases mentioned in Theorem 2 all sequences with $m(\lambda) + m(-\lambda) \geq 3$ for some $\lambda \in \mathbb{N}$.

PROOF: of Theorem 2. According to Lemma 1 the sequence is given by $a_m = (\alpha\theta^m - \tilde{\alpha}\tilde{\theta}^m)/(\theta - \tilde{\theta})$, where θ is a root of $x^2 - Mx + N = 0$ and $\alpha = a_1 - a_0\tilde{\theta}$. For θ we choose the root with positive imaginary part. Furthermore, $M = \theta + \tilde{\theta}$ and $N = \theta\tilde{\theta}$. Since $M \geq 0$, The real part of θ is non-negative, and hence $0 < \arg\theta \leq \pi/2$. Since $a_0 = (\alpha - \tilde{\alpha})/(\theta - \tilde{\theta})$ ≥ 0 we see that $0 \leq \arg\alpha \leq \pi$. Moreover, $\theta/\tilde{\theta}$ is not a root of unity and α is not real and therefore $0 < \arg\theta < \pi/2$, $0 < \arg\alpha < \pi$. The equation $\alpha_n = \pm a_0$ can be rewritten as

$$\alpha\theta^n - \tilde{\alpha}\tilde{\theta}^n = \pm(\alpha - \tilde{\alpha}). \tag{4}$$

We assume that α and $\tilde{\alpha}$ have no common factor. This is no restriction because we can divide (4) by such a factor.

Suppose that (4) has at least four solutions, which we denote by

$n = 0$, p, q, r with $r > q > p > 0$. According to Lemma 2 we have $\theta^p = \epsilon' + \mu'\tilde{\alpha}$ and $\theta^q = \epsilon + \mu\tilde{\alpha}$, where ϵ', $\epsilon \in \{-1, 1\}$, μ, $\mu' \in \mathbb{Z}$. Because there exists a larger solution r, Lemma 6 together with Lemma 5 implies that $|\mu'| \leq |\mu| \leq 3$. We consider the following possibilities:

I) $|\mu| = |\mu'|$. Then $\theta^q \pm \theta^p \in \{-2, 0, 2\}$. Since θ is not a root of unity we have $\theta^q \pm \theta^p = \pm 2$. By Lemma 4 we see that $(p, q, \theta) = (1, 3, \frac{1}{2} + \frac{1}{2}\sqrt{-7})$ or $(1, 2, \frac{1}{2} + \frac{1}{2}\sqrt{-7})$. The conditions $\theta^p = \pm 1 + \mu'\alpha$ and $\theta^q = \pm 1 + \mu\alpha$ then imply $\alpha = -\frac{3}{2} + \frac{1}{2}\sqrt{-7}$ or $\alpha = \frac{1}{2} + \frac{1}{2}\sqrt{-7}$ respectively.

II) $|\mu| = 3$, $|\mu'| = 2$. Then $2\theta^q \pm 3\theta^p \in \{-5, -1, 1, 5\}$ and since θ is not a root of unity we have $2\theta^q \pm 3\theta^p = \pm 5$. This implies $|\theta|^p = \sqrt{5}$ or 5 and $p \leq 2$. Furthermore $|\theta|^q \leq \frac{3}{2}|\theta|^p + \frac{5}{2} \leq 10$ and hence $q \leq 2$. Solving $2\theta^2 \pm 3\theta = \pm 5$ yields no relevant solutions.

III) $|\mu| = 3, |\mu'| = 1$. Then $\theta^q \pm 3\theta^p \in \{-4, -2, 2, 2, 4\}$. By Lemma 4 we see that $(p, q, \theta) = (2, 4, \frac{1}{2} + \frac{1}{2}\sqrt{-7})$, $(1, 2, \frac{3}{2} + \frac{1}{2}\sqrt{-7})$, $(1, 4, \frac{1}{2} + \frac{1}{2}\sqrt{-7})$ or $(1, 3, \frac{1}{2} + \frac{1}{2}\sqrt{-15})$. The conditions $\theta^p = \pm 1 + \mu'\tilde{\alpha}$ and $\theta^q = \pm 1 + \mu\tilde{\alpha}$ then imply $\alpha = \frac{1}{2} + \frac{1}{2}\sqrt{-7}$, $-\frac{1}{2} + \frac{1}{2}\sqrt{-7}$, $\frac{1}{2} + \frac{1}{2}\sqrt{-7}$ or $-\frac{3}{2} + \frac{1}{2}\sqrt{-15}$ respectively.

iv) $|\mu| = 2, |\mu'| = 1$. Then $\theta^q \pm 2\theta^p = \pm 3, \pm 1$ and since θ is not a root of unity we have $\theta^q \pm 2\theta^p = \pm 3$. Lemma 4 implies that $(p, q, \theta) = (1, 3, \frac{1}{2} + \frac{1}{2}\sqrt{-11})$ or $(1, 2, 1 + \sqrt{-2})$. The conditions $\theta^p = \pm 1 + \mu'\tilde{\alpha}$ and $\theta^q = \pm 1 + \mu\tilde{\alpha}$ then imply $\alpha = -\frac{3}{2} + \frac{1}{2}\sqrt{-11}$ or $\alpha = \sqrt{-2}$ respectively.

Summarizing, if equation (4) has at least four solutions then (α, θ) assumes one of the following values, $(\frac{1}{2} + \frac{1}{2}\sqrt{-7}, \frac{1}{2} + \frac{1}{2}\sqrt{-7})$, $(-\frac{1}{2} + \frac{1}{2}\sqrt{-7}, \frac{3}{2} + \frac{1}{2}\sqrt{-7})$, $(-\frac{3}{2} + \frac{1}{2}\sqrt{-7}, \frac{1}{2} + \frac{1}{2}\sqrt{-7})$, $(-\frac{3}{2} + \frac{1}{2}\sqrt{-11}, \frac{1}{2} + \frac{1}{2}\sqrt{-11})$, $(-\frac{3}{2} + \frac{1}{2}\sqrt{-15}, \frac{1}{2} + \frac{1}{2}\sqrt{-15})$ or $(\sqrt{-2}, 1 + \sqrt{-2})$. In the first case it follows from Lemma 7 that (4) has the solutions $n = 0, 1, 2, 4, 12$. In the second case equation (4) reads $(-\frac{1}{2} + \frac{1}{2}\sqrt{-7})(\frac{3}{2} + \frac{1}{2}\sqrt{-7})^n - (-\frac{1}{2} - \frac{1}{2}\sqrt{-7})(\frac{3}{2} - \frac{1}{2}\sqrt{-7})^n = \pm\sqrt{-7}$ and since $\frac{3}{2} - \frac{1}{2}\sqrt{-7} = -(\frac{1}{2} + \frac{1}{2}\sqrt{-7})^2$ we see that $n = 0, 1, 2, 6$ are the only solutions. Since $-\frac{3}{2} + \frac{1}{2}\sqrt{-7} = (\frac{1}{2} + \frac{1}{2} - 7)^2$ we see that $n = 0, 1, 3, 11$ are the only solutions in the third case. In the fourth, fifth and sixth case the solutions of (4) are given by Lemma 7 and only in the sixth case there exist more than three solutions namely $n = 0, 1, 2, 5$. It is easy to see that we obtain precisely the sequences mentioned in our theorem.

THEOREM 3: *Let $\{a_n\}_{n=0}^{\infty}$ be a non-degenerate recurrent sequence of rational integers given by $a_0 \geq 0$, $(a_0, a_1) = 1$, $a_n = Ma_{n-1} - Na_{n-2}$ with $M, N \in \mathbb{Z}$ and $M > 0$ and $M^2 - 4N > 0$. The equation $a_n = \pm a_0$ has at most three solutions n, unless $M = 1$, $N = -1$, $a_0 = 1$, $a_1 = -1$. Then $a_n = \pm 1$ has the solutions $n = 0, 1, 3, 4$.*

REMARK: The remarks we made after the statement of Theorem 2 are also valid for Theorem 3.

PROOF: Just as in the proof of Theorem 2 we consider the equation

$$\alpha\theta^n - \tilde{\alpha}\tilde{\theta}^n = \pm(\alpha - \tilde{\alpha}), \tag{4a}$$

where α and θ are integers in a real quadratic numberfield $\mathbb{Q}(\sqrt{d})$. Without loss of generality we may assume that $\theta \geq |\tilde{\theta}|$. Since $\theta + \tilde{\theta}$, $(\theta - \tilde{\theta})/\sqrt{d} \in \mathbb{Z}$ and $\theta/\tilde{\theta} \neq \pm 1$ we have $\theta \neq |\tilde{\theta}|$ and $\theta \geq |\tilde{\theta}| + 1$. Suppose that (4a) has four solutions given by $n = 0, k, l, m$. Eliminating α and $\tilde{\alpha}$ from $\alpha\theta^k - \tilde{\alpha}\tilde{\theta}^k = \pm(\alpha - \tilde{\alpha})$, $\alpha\theta^l - \tilde{\alpha}\tilde{\theta}^l = \pm(\alpha - \tilde{\alpha})$ and $\alpha\theta^m - \tilde{\alpha}\tilde{\theta}^m = \pm(\alpha - \tilde{\alpha})$ we obtain

$$\frac{\theta^k - \epsilon}{\tilde{\theta}^k - \epsilon} = \frac{\theta^l - \epsilon'}{\tilde{\theta}^l - \epsilon'} = \frac{\theta^m - \epsilon''}{\tilde{\theta}^m - \epsilon''} = \frac{\tilde{\alpha}}{\alpha} \text{ for some } \epsilon, \epsilon', \epsilon'' \in \{-1, 1\}. \tag{5}$$

At least two epsilons must be equal to either $+1$ or -1. We distinguish two cases according to whether the first possibility occurs, or the second.

I) Suppose that $l > k$ and

$$\frac{\theta^k - 1}{\tilde{\theta}^k - 1} = \frac{\theta^l - 1}{\tilde{\theta}^l - 1}. \tag{6}$$

If $\tilde{\theta} > 0$ then $|(\theta^r - 1)/(\tilde{\theta}^r - 1)|$ increases monotonically with $r \in \mathbb{N}$ as can be seen from

$$\left|\frac{\theta^r - 1}{\tilde{\theta}^r - 1}\right| \left|\frac{\theta^{r+1} - 1}{\tilde{\theta}^{r+1} - 1}\right|^{-1} = \left|\frac{\theta^r - 1}{\theta^{r+1} - 1}\right| \left|\tilde{\theta} + \frac{\tilde{\theta} - 1}{\tilde{\theta}^r - 1}\right| < \frac{1}{\theta}\left|\tilde{\theta} + \frac{1}{\tilde{\theta}^{r-1} + \cdots + 1}\right|$$
$$\leq \frac{1}{\theta}(|\tilde{\theta}| + 1) \leq 1,$$

and (6) cannot occur. If $\tilde{\theta} < 0$ we distinguish three cases,

i) k and l even. This is equivalent to considering (6) with θ^2 instead of θ. Because $\tilde{\theta}^2 > 0$ we have already dealt with this case.

ii) k odd. Then

$$\frac{\theta^k - 1}{|\tilde{\theta}|^k + 1} = \left|\frac{\theta^k - 1}{\tilde{\theta}^k - 1}\right| = \left|\frac{\theta^l - 1}{\tilde{\theta}^l - 1}\right| \geq \frac{\theta^l - 1}{|\tilde{\theta}|^l + 1}.$$

However, $(\theta^r - 1)/(|\tilde{\theta}|^r + 1)$ increases with $r \in \mathbb{N}$, as can be seen from

$$\frac{\theta^r-1}{|\theta^r+1}\left(\frac{\theta^{r+1}-1}{|\tilde{\theta}|^{r+1}+1}\right)^{-1}=\frac{\theta^r-1}{\theta^{r+1}-1}\frac{|\theta|^{r+1}+1}{|\tilde{\theta}|^r+1}<\frac{1}{\theta}\left(|\tilde{\theta}|+\frac{1-|\tilde{\theta}|}{|\tilde{\theta}|^r+1}\right)$$
$$<\frac{1}{\theta}(|\tilde{\theta}|+1)\leq 1.$$

iii) k even, l odd. Then comparison of the signs in (6) shows that $-1<\tilde{\theta}<0$. Notice that

$$\left|\frac{\theta^k-1}{\tilde{\theta}^k-1}\right|=\frac{1}{|\tilde{\theta}+1|}\cdot\frac{\theta^k-1}{|\tilde{\theta}^{k-1}-\tilde{\theta}^{k-2}+\cdots 1|}\leq\frac{1}{|\tilde{\theta}+1|}\cdot\frac{\theta^k-1}{1+|\tilde{\theta}|}$$
$$\leq(\theta+1)\frac{\theta^k-1}{|\tilde{\theta}|+1}$$

and

$$\left|\frac{\theta^l-1}{\tilde{\theta}^l-1}\right|>\frac{\theta^l-1}{2}.$$

Hence (6) implies

$$\theta^l-1<2\frac{\theta+1}{|\tilde{\theta}|+1}(\theta^k-1).$$

Suppose that $l\geq k+3$, then the latter inequality implies $\theta<2$, hence $\theta=\frac{1}{2}+\frac{1}{2}\sqrt{5}$, which is impossible. We therefore conclude that $l=k+1$. There exists a third solution m however and (5) implies that

$$\frac{\theta^m-\epsilon}{\tilde{\theta}^m-\epsilon}=\frac{\theta^k-1}{\tilde{\theta}^k-1}\text{ for some }\epsilon\in\{-1,1\}.$$

If $\epsilon=1$ then we know that $|m-k|=|m-l|=|l-k|=1$ which is impossible. If $\epsilon=-1$ then $(\theta^m+1)/(\tilde{\theta}^m+1)$ and $(\theta^k-1)/(\tilde{\theta}^k-1)$ have opposite signs. Thus we have shown in i, ii, iii) that in (5) at most one epsilon equals $+1$.

II) Suppose that $l>k$ and

$$\frac{\theta^k+1}{\tilde{\theta}^k+1}=\frac{\theta^l+1}{\tilde{\theta}^l+1}. \tag{7}$$

We observe that $(\theta^x+1)/(\tilde{\theta}^x+1)$ is an increasing function of $x>0$ if $\tilde{\theta}>0$ and thus (7) cannot be satisfied. If $\tilde{\theta}<0$, then we distinguish four cases,

i) k, l even. This is equivalent to considering (7) with θ^2 instead of θ.

ii) k even and l odd. By consideration of the signs in (7) we see that $-1 < \tilde{\theta} < 0$. Hence

$$\theta^k + 1 > \frac{\theta^k + 1}{\tilde{\theta}^k + 1} = \frac{\theta^l + 1}{\tilde{\theta}^l + 1} > \theta^l + 1,$$

which is impossible.

iii) k and l odd. Then (7) can be written as

$$\frac{\theta^k + 1}{|\tilde{\theta}|^k - 1} = \frac{\theta^l + 1}{|\tilde{\theta}|^l - 1}.$$

The sequence $(\theta^{2r-1} + 1)/(|\tilde{\theta}|^{2r-1} - 1)$ increases with $r \in \mathbb{N}$, as can be seen from

$$\frac{\theta^{2r-1} + 1}{\theta^{2r+1} + 1} \frac{|\tilde{\theta}|^{2r+1} - 1}{|\tilde{\theta}|^{2r-1} - 1} \leq \frac{\theta + 1}{\theta^3 + 1} \frac{|\tilde{\theta}|^3 - 1}{|\tilde{\theta}| - 1}$$

$$= \frac{|\tilde{\theta}|^2 + |\tilde{\theta}| + 1}{\theta^2 - \theta + 1} \leq \frac{|\tilde{\theta}|^2 + |\tilde{\theta}| + 1}{(|\tilde{\theta}| + 1)^2 - |\tilde{\theta}|} = 1.$$

Note that both equality signs hold if and only if $r = 1$ and $\theta = 1 - \tilde{\theta}$. Hence $k = 1$, $l = 3$ and $\theta = 1 - \tilde{\theta}$. We know that there exists a third solution m and (5) implies

$$\frac{\theta^m - \epsilon}{\tilde{\theta}^m - \epsilon} = \frac{\theta + 1}{\tilde{\theta} + 1} \text{ for some } \epsilon \in \{-1, 1\}.$$

Suppose $\theta > 2$, then $\tilde{\theta} = 1 - \theta < -1$ and m must be odd. Hence

$$\frac{\theta^5 - 1}{(\theta - 1)^5 + 1} \leq \frac{\theta^m - \epsilon}{(\theta - 1)^m + \epsilon} = -\frac{\theta^m - \epsilon}{\tilde{\theta}^m - \epsilon} = -\frac{\theta + 1}{\tilde{\theta} + 1} = \frac{\theta + 1}{\theta - 2}$$

which implies $2\theta^5 - 5\theta^4 + 5\theta^2 - 6\theta + 2 \leq 0$ and we may conclude that $\theta < \frac{1}{2} + \frac{1}{2}\sqrt{13}$. There exists no algebraic integer of the shape $\theta = \frac{1}{2} + \frac{1}{2}q\sqrt{d}$ such that $2 < \theta < \frac{1}{2} + \frac{1}{2}\sqrt{13}$. Thus $\theta < 2$ and hence $\theta = \frac{1}{2} + \frac{1}{2}\sqrt{5}$. Then it follows that

$$\frac{\theta^m - 1}{\frac{1}{2} + \frac{1}{2}\sqrt{5}} \leq \left|\frac{\theta^m - \epsilon}{\tilde{\theta}^m - \epsilon}\right| = \left|\frac{\theta + 1}{\tilde{\theta} + 1}\right| = \theta^4$$

and hence $m \leq 5$. Checking $(\theta^m - \epsilon)/(\tilde{\theta}^m - \epsilon) = (\theta + 1)/(\tilde{\theta} + 1)$for $m \leq 5$ we find that $\epsilon = -1$ and $m = 3, 4$.

iv) k odd l even. Comparison of signs in (7) shows that $-1 < \tilde{\theta} < 0$. Hence

$$(\theta + 1)(\theta^k + 1) \geq \frac{\theta^k + 1}{\tilde{\theta} + 1} \cdot \frac{1}{1 + |\tilde{\theta}| + \cdots + |\tilde{\theta}|^{k-1}} = \frac{\theta^k + 1}{\tilde{\theta}^k + 1}$$
$$= \frac{\theta^l + 1}{\tilde{\theta}^l + 1} > \frac{\theta^l + 1}{2},$$

and this implies $\theta^l + 1 < 2(\theta + 1)\ (\theta^k + 1)$. Suppose $l \geq k + 3$ then $\theta^{k+3} + 1 < 2(\theta + 1)\ (\theta^k + 1)$. One easily confirms that $\theta < 2.2$ and hence $\theta = \frac{1}{2} + \frac{1}{2}\sqrt{5}$. On using $\theta\tilde{\theta} = -1$ we see that $(\theta^l + 1)/(\tilde{\theta}^l + 1) = \theta^l$ and (7) now implies $\theta^l = (\theta^k + 1)/(\tilde{\theta}^k + 1) \leq (\theta^k + 1)/(\tilde{\theta} + 1) = \theta^2(\theta^k + 1)$ and consequently $k = 1, l = 4$. Suppose $l = k + 1$ or $k = 1, l = 4$. We know that there exists a third solution m and (5) implies that

$$\frac{\theta^m - \epsilon}{\tilde{\theta}^m - \epsilon} = \frac{\theta^l + 1}{\tilde{\theta}^l + 1} \text{ for some } \epsilon \in \{-1, 1\}.$$

If $\epsilon = 1$ then the terms have opposite sign, which is impossible. If $\epsilon = -1$ and $m > l$ then we are in case i) or ii) since l is even. If $\epsilon = -1$ and $m < l$ then it follows that either m $= 1$, $l = 4$, $k = 3$ or $m = 3$, $l = 4$, $k = 1$ and also that $\theta = \frac{1}{2} + \frac{1}{2}\sqrt{5}$.

We can conclude that the solution of (5) is given by $\theta = \frac{1}{2} + \frac{1}{2}\sqrt{5}$ and $k = 1$, $l = 3$, $m = 4$. It also follows from (5) that $\tilde{\alpha}/\alpha = (\frac{1}{2} + \frac{1}{2}\sqrt{5})^4$ and hence $\alpha = \pm(\frac{3}{2} - \frac{1}{2}\sqrt{5})$. The corresponding recurrent sequence is given by $M = 1$, $N = -1$, $a_0 = 1$, $a_1 = -1$. Thus our assertion follows.

3. Lucas-sequences of the first kind

A Lucas-sequence of the first kind is a recurrent sequence defined by

$$a_{m+2} = Ma_{m+1} - Na_m,\ a_0 = 0,\ a_1 = 1;\ M, N \in \mathbb{Z}.$$

LEMMA 8: *Let* $\{a_m\}_{m=0}^{\infty}$ *be a sequence given by* $a_{m+2} = Ma_{m+1} - Na_m$ $a_0 = 0$, $a_1 = 1$. *Let* θ *be a solution of* $x^2 - Mx + N = 0$. *Then*

$$a_m = \frac{\theta^m - \tilde{\theta}^m}{\theta - \tilde{\theta}}.$$

PROOF: This is an immediate consequence of Lemma 1.

First, we consider the number of solutions m of the equation $a_m = \pm 1$. This is equivalent to $\theta^m - \tilde{\theta}^m = \pm(\theta - \tilde{\theta})$. For reasons of computational smoothness we shall consider the equation $\theta^{n+1} - \tilde{\theta}^{n+1} = \pm(\theta - \tilde{\theta})$ in n. In the following lemma we consider only quadratic integers in an imaginary quadratic field, merely because sequences with negative discriminant, i.e. $M^2 - 4N < 0$, are more difficult to deal with than sequences with positive discriminant.

LEMMA 9: *Let θ be an integer in a quadratic imaginary field and $\theta \notin \mathbb{Z}$. If $\theta^{n+1} - \tilde{\theta}^{n+1} = \pm(\theta - \tilde{\theta})$ has three or more solutions then either $\theta/\tilde{\theta}$ is a root of unity or $\theta + \tilde{\theta} = \pm 1$.*

PROOF: Suppose that $\theta/\tilde{\theta}$ is not a root of unity and suppose that there exist three solutions which we denote by $n = 0$, p, q with $q > p$. If q is odd then $(\theta^2 - \tilde{\theta}^2) \mid (\theta^{q+1} - \tilde{\theta}^{q+1}) = \pm(\theta - \tilde{\theta})$ and hence $\theta^2 - \tilde{\theta}^2 = \pm(\theta - \tilde{\theta})$. After division by $\theta - \theta$ we obtain $\theta + \tilde{\theta} = \pm 1$ and our lemma is proved. In the remaining lines we therefore assume that q is even. It suffices to prove the lemma in case $0 \le \arg\theta \le \pi/2$. We also notice that θ and $\tilde{\theta}$ have no factors in common. This follows from $\theta^{p+1} - \tilde{\theta}^{p+1} = \pm(\theta - \tilde{\theta})$, which can be written as $\theta^p + \theta^{p-1}\tilde{\theta} + \cdots + \tilde{\theta}^p = \pm 1$. Hence $0 < \arg\theta < \pi/2$. By Lemma 2 we know that $\theta^p = \epsilon(1 + \mu\tilde{\theta})$ for some $\mu \in \mathbb{Z}$ and $\epsilon \in \{-1, 1\}$. According to Lemma 6 we have $|\mu| \le 3$, since there exists a larger solution q. Then $\theta^p = \epsilon(1 + \mu\tilde{\theta})$ implies $|\theta|^p - 1 \le 3|\theta|$. Suppose $p \ge 3$, then $|\theta| < 2$. If $|\theta| = \sqrt{2}$ then $\theta \in \{\frac{1}{2} + \frac{1}{2}\sqrt{-7}, \sqrt{-2}, 1 + i\}$. In the first case our lemma is proved and the two latter cases can be ignored because θ and $\tilde{\theta}$ cannot have a common factor. If $|\theta| = \sqrt{3}$ then $\theta \in \{\frac{1}{2} + \frac{1}{2}\sqrt{-11},\ 1 + \sqrt{-2},\ \sqrt{-3},\ \frac{3}{2} + \frac{1}{2}\sqrt{-3}\}$. In the first case our lemma is proved. If $\theta = 1 + \sqrt{-2}$ then we deduce from $|\theta|^p - 1 \le 3|\theta|$ that $p \le 3$ and hence $p = 3$. However $(1 + \sqrt{-2})^3 = -5 + \sqrt{-2}$ which contradicts $\theta^3 = \pm(1 + \mu\tilde{\theta})$, $\mu \in \mathbb{Z}$. The cases $\theta = \sqrt{-3}, \frac{3}{2} + \frac{1}{2}\sqrt{-3}$ can be ignored. Assume that $p \le 2$. If $p = 1$, then $\theta = \pm(1 + \mu\tilde{\theta})$ and by comparison of the imaginary parts we see that $\mu = \mp 1$, hence $\theta + \tilde{\theta} = \pm 1$. Suppose $p = 2$. Then $\theta^2 = \epsilon(1 + \mu\tilde{\theta})$ and this implies $\theta^2 - \tilde{\theta}^2 = -\epsilon\mu(\theta - \tilde{\theta})$, Hence $\theta + \tilde{\theta} = -\epsilon\mu$. Suppose that $|\mu| \ge 2$. Then, according to Lemmas 2 and 3 we see that $q = pr = 2r$ for some $r \in \mathbb{N}$ and $\theta^{q+1} - \tilde{\theta}^{q+1} = \epsilon^r(\theta - \tilde{\theta})$. On substituting $q = 2r$ and $\theta^2 = \epsilon(1 + \mu\tilde{\theta})$ in $\theta^{q+1} - \tilde{\theta}^{q+1} = \epsilon^r(\theta - \tilde{\theta})$, we obtain

$$\theta(1 + \mu\tilde{\theta})^r - \tilde{\theta}(1 + \mu\theta)^r = \theta - \tilde{\theta}.$$

We now apply Theorem 1 with $\gamma = \theta$, $\eta = \tilde{\theta}$, $t = \mu$ and $g = \theta - \tilde{\theta}$.

Notice that $\gamma\eta - \tilde{\gamma}\tilde{\eta} = 0$. Since $(\eta - \tilde{\eta})/g = -1$ and $(\eta^2 - \tilde{\eta}^2)/g = -(\theta + \tilde{\theta}) = \epsilon\mu$, Theorem 1B implies that $r = 0$, 1, corresponding to the known solutions 0 and p. We therefore conclude that $|\mu| = 1$, hence $\theta + \tilde{\theta} = \pm 1$ and our lemma is proved.

THEOREM 4: *Let* $a_{m+2} = Ma_{m+1} - Na_m$, $a_0 = 0$, $a_1 = 1$ *be a non-degenerate Lucas-sequence. Then the equation* $|a_m| = 1$ *has at most two solutions, unless*

a) $|M| = 1$, $N = 2$, *then* $|a_m| = 1$ *for* $m = 1, 2, 3, 5, 13$ *only.*
b) $|M| = 1$, $N = 3$, *then* $|a_m| = 1$ *for* $m = 1, 2, 5$ *only.*
c) $|M| = 1$, $N = 5$, *then* $|a_m| = 1$ *for* $m = 1, 2, 7$ *only.*

PROOF: Without loss of generality we may assume that $M \geq 1$. We separate the proof into two parts.

I) $M^2 - 4N < 0$. Suppose that $|a_m| = 1$ or, equivalently, $\theta^m - \tilde{\theta}^m = \pm(\theta - \tilde{\theta})$, has three or more solutions. Then Lemma 9 implies that $\theta + \tilde{\theta} = \pm 1$ and since $\theta + \tilde{\theta} = M \geq 1$ we have $M = \theta + \tilde{\theta} = 1$. If $N = 2$, 3 or 5 then $\theta = \frac{1}{2} + \frac{1}{2}\sqrt{-7}$, $\frac{1}{2} + \frac{1}{2}\sqrt{-11}$ or $\frac{1}{2} + \frac{1}{2}\sqrt{-19}$ respectively and Lemma 7 yields all solutions given in the exceptional cases a), b) and c) in our theorem. Assume $N \neq 2, 3, 5$. Notice that $\{a_m\}_{m=0}^{\infty} \equiv 0, 1, 1, 1, \ldots \pmod{N}$ and thus $|a_m| = 1$ implies $a_m = 1$. Notice that $\{a_m\}_{m=0}^{\infty} \equiv 0, 1, 1, 0, -1, -1, 0, 1, 1, \ldots (\mathrm{mod}(N-1))$ and thus $a_m = 1$ implies $m \equiv 1, 2 \pmod{6}$. Hence we must solve the equation $\theta^{n+1} - \tilde{\theta}^{n+1} = \theta - \tilde{\theta}$ with $n \equiv 0, 1 \pmod{6}$. Suppose $n \equiv 1 \pmod{6}$ then

$$(\theta^{(n+1)/2} + \tilde{\theta}^{(n+1)/2}) \frac{\theta^{(n+1)/2} - \theta^{(n+1)/2}}{\theta - \tilde{\theta}} = 1.$$

Since both factors are rational integers it follows that $\theta^{(n+1)/2} + \tilde{\theta}^{(n+1)/2} = (\theta^{(n+1)/2} - \tilde{\theta}^{(n+1)/2})/(\theta - \tilde{\theta}) = \pm 1$. Thus we find $2\theta^{(n+1)/2} = \pm(\theta - \tilde{\theta} + 1) = \pm 2\theta$ which implies $n = 1$. We now assume that $n \equiv 0 \pmod{6}$. Notice that $\theta^3 + 1 = (\theta + 1)(\theta^2 - \theta + 1) = -(\theta + 1)(\theta\tilde{\theta} - 1) = -(\theta + 1)(N - 1)$. Put $n = 3r$ then $\theta^{n+1} - \theta^{n+1} = \theta - \tilde{\theta}$ implies

$$\theta(-1 - (N-1)(\theta + 1))^r - \tilde{\theta}(-1 - (N-1)(\tilde{\theta} + 1))^r = \theta - \tilde{\theta}.$$

We know that r is even, by $6 \mid n$, and we may as well consider

$$\theta(1 + (N-1)(\theta + 1))^r - \tilde{\theta}(1 + (N-1)(\tilde{\theta} + 1))^r = \theta - \tilde{\theta}.$$

We apply Theorem 1 with $\gamma = \theta$, $\eta = \theta + 1$, $t = N - 1$, $g = \theta - \tilde{\theta}$. Note that $(\gamma\eta - \tilde{\gamma}\tilde{\eta})/g = 2$. Since $t = N - 1 \neq 1, 2, 4$ Theorem 1A implies that $r = 0$. Hence there exists no third solution if $N \neq 2, 3, 5$, as asserted.

II) $M^2 - 4N > 0$. In this case the solutions $\theta, \tilde{\theta}$ of $x^2 - Mx + N = 0$ are real. Suppose that $\theta\tilde{\theta} = N < 0$ then it is easy to see that $a_{m+1} > a_m > 0$ if $m > 1$ and the theorem is proved. Suppose that $\theta\tilde{\theta} = N > 0$. Then without loss of generality we may assume that $\theta > \tilde{\theta} > 0$. The function $\theta^x - \tilde{\theta}^x$ is strictly increasing in $x > 0$ and there cannot exist three solutions for the equation $\theta^m - \tilde{\theta}^m = \pm(\theta - \tilde{\theta})$.

COROLLARY: *Let* $\{a_m\}_{m=0}^{\infty}$ *be a non-degenerate Lucas-sequence of the first kind. Let* $\lambda \geq 2$. *Then* $|a_m| = \lambda$ *has at most two solutions* m.

PROOF: Let d be the smallest integer such that $|a_d| = \lambda$ or, equivalently, $\theta^d - \tilde{\theta}^d = \pm\lambda(\theta - \tilde{\theta})$. If $\theta^q - \tilde{\theta}^q = \pm\lambda(\theta - \tilde{\theta})$ for some $q \in \mathbb{N}$ then $\theta^{(q,d)} \equiv \tilde{\theta}^{(q,d)} \pmod{\lambda(\theta - \tilde{\theta})}$ where (q, d) is the largest common divisor of q and d. Hence $\theta^{(q,d)} - \tilde{\theta}^{(q,d)} = \mu\lambda(\theta - \tilde{\theta})$ for some $\mu \in \mathbb{Z}$. Furthermore, $\theta^{(q,d)} - \tilde{\theta}^{(q,d)}$ divides $\theta^d - \tilde{\theta}^d$ and hence $\mu \mid 1$. Thus $\theta^{(q,d)} - \tilde{\theta}^{(q,d)} = \pm\lambda(\theta - \tilde{\theta})$ and since d is minimal and $\lambda \geq 2$ this implies $d = (q, d)$, i.e. $d \mid q$. Put $q = rd$ then we find that $(\theta^d)^r - (\tilde{\theta}^d)^r = \pm(\theta^d - \tilde{\theta}^d)$. By Lemma 8 this implies that $a_r^* = \pm 1$ for the Lucas-sequence given by $a_{m+2}^* = M^* a_{m+1}^* - N^* a_m^*$ with $M^* = \theta^d + \tilde{\theta}^d$ and $N^* = (\theta\tilde{\theta})^d$. According to Theorem 4 there exist at most two solutions for $|a_r^*| = 1$, unless $N^* = 2, 3, 5$. Since $N^* = (\theta\tilde{\theta})^d$ is a perfect power, the latter cases cannot occur and our corollary is proved.

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(Oblatum 21-VIII-1978 & 9-I-1979)

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