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HYPERSPACES OF NONCOMPACT METRIC SPACES

D.W. Curtis

0. Introduction

For a metric space X, the hyperspace 2^X of nonempty compact subsets and the hyperspace $C(X)$ of nonempty compact connected subsets are topologized by the Hausdorff metric, defined by $\rho(A, B) = \inf\{\epsilon : A \subset N_\epsilon(B) \text{ and } B \subset N_\epsilon(A)\}$. It is easily seen that the hyperspace topologies induced by ρ are invariants of the topology on X. It is known that $2^X \approx Q$, the Hilbert cube, if and only if X is a nondegenerate Peano continuum, and $C(X) \approx Q$ if and only if X is a nondegenerate Peano continuum with no free arcs [6]. In this paper we obtain various characterization theorems for hyperspaces of noncompact connected locally connected metric spaces.

THEOREM 1.6: *2^X is an ANR (AR) if and only if X is locally continuum-connected (connected and locally continuum-connected).*

THEOREM 3.3: *$2^X \approx Q\backslash point$ if and only if X ix noncompact, connected, locally connected, and locally compact.*

THEOREM 4.2: *X admits a Peano compactification $\tilde{X}$ such that $(2^{\tilde{X}}, 2^X) \approx (Q, s)$ if and only if X is topologically complete, separable, connected, locally connected, nowhere locally compact, and admits a metric with Property S.*

Analogous results are obtained for $C(X)$. Additionally, we discuss two examples relating to local continuum-connectedness, and an example relating to Property S.

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1. Hyperspaces which are ANR's

A *growth hyperspace* $\mathscr{G}$ of a metric space X is any closed subspace of 2^X satisfying the following condition: if $A \in \mathscr{G}$ and $B \in 2^X$ such that $B \supset A$ and each component of B meets A, then $B \in \mathscr{G}$. Both 2^X and $C(X)$ are growth hyperspaces of X. Another growth hyperspace of particular interest is $\mathscr{G}_A(X)$, the smallest growth hyperspace containing $A \in 2^X$. Thus $\mathscr{G}_A(X) = \{B \in 2^X : B \supset A$ and each component of B meets $A\}$. Growth hyperspaces of Peano continua were studied in [4].

LEMMA 1.1: (*Kelley* [11]). *Let A, $B \in 2^X$ such that $B \in \mathscr{G}_A(X)$ and B has finitely many components. Then there exists a path $\sigma : I \to \mathscr{G}_A(B)$ such that $\sigma(0) = A$ and $\sigma(1) = B$.*

DEFINITION: A metric space X is *continuum-connected* if each pair of points in X is contained in a subcontinuum. X is *locally continuum-connected* if it has an open base of continuum-connected subsets.

Note that in verifying the local property it is sufficient to produce, for each neighborhood U of a point x, a neighborhood $V \subset U$ of x such that each $y \in V$ is connected to x by a subcontinuum in U. For topologically complete metric spaces, the properties of local connectedness, local continuum-connectedness, and local path-connectedness are equivalent, since every complete connected locally connected metric space is path-connected. Examples given later show that in general these properties are not equivalent.

LEMMA 1.2: *Let $A \in 2^X$, with X a locally continuum-connected metric space. Then for arbitrary $\epsilon > 0$ there exists $\tilde{A} \in \mathscr{G}_A(X)$ such that $\rho(A, \tilde{A}) < \epsilon$ and $\tilde{A}$ has finitely many components.*

PROOF: For each $n \geq 1$ choose $\epsilon_n > 0$ such that, whenever $x \in A$ and $y \in X$ with $d(x, y) < \epsilon_n$, there exists a continuum in X connecting x and y with diameter less than $\min\{1/n, \epsilon\}$. For each n let $A_n \subset A$ be a finite ϵ_n-net for A. Then for each $p \in A_{n+1}$ there exists a continuum L_p in X with diameter less than $\min\{1/n, \epsilon\}$, connecting p and some point of A_n. Set $\tilde{A}_1 = A_1$ and $\tilde{A}_{n+1} = \cup\{L_p : p \in A_{n+1}\}$ for each $n \geq 1$. Then $\tilde{A} = cl(\bigcup_1^\infty \tilde{A}_n) = \bigcup_1^\infty \tilde{A}_n \cup A$ has the required properties (note that each component of $\tilde{A}$ meets the finite subset A_1).

LEMMA 1.3: *Let X be a connected and locally continuum-con-*

nected metric space. Then every compact subset is contained in a continuum.

PROOF: Let A be a compact subset of X. There exists by Lemma 1.2 a compact set $\tilde{A} \supset A$ such that $\tilde{A}$ has finitely many components. It is easily seen that X is continuum-conected. Thus the components of $\tilde{A}$ may be connected together by the addition of a finite collection of subcontinua of X, thereby producing a continuum $B \supset \tilde{A} \supset A$.

LEMMA 1.4: *If X is a locally continuum-connected metric space, then every growth hyperspace $\mathscr{G}$ of X is locally path-connected.*

PROOF: Given $A \in \mathscr{G}$ and $\epsilon > 0$, choose $\delta > 0$ such that whenever $x \in A$ and $y \in X$ with $d(x, y) < \delta$, there exists a continuum in X of diameter less than ϵ connecting x and y. We claim that for any $B \in \mathscr{G}$ with $\rho(A, B) < \delta$, there exists a path $\sigma: I \to \mathscr{G}$ between A and B, with $\rho(A, \sigma(t)) < \epsilon$ for each t. We may assume by Lemmas 1.1. and 1.2 that each of A and B has finitely many components. Adding a finite collection of continua to $A \cup B$, which connect each component of A to B and each component of B to A, and all of which have diameter less than ϵ, we obtain an element $C \in \mathscr{G}_A(X) \cap \mathscr{G}_B(X)$ such that $\rho(A, C) < \epsilon$. Then paths between A and C, and B and C, given by Lemma 1.1, will provide the desired path.

LEMMA 1.5: *Let $\mathscr{D} \subset 2^X$* be compact and connected, and let $A \in \mathscr{D}$. *Then $\cup \mathscr{D} \in \mathscr{G}_A(X)$.*

PROOF: Clearly, $\cup \mathscr{D}$ is a compact subset of X and contains A. We show that each component of $\cup \mathscr{D}$ meets A. Let $x \in D \in \mathscr{D}$ be given. For each $\epsilon > 0$ there exists an ϵ-chain $\{D_i\}$ in $\mathscr{D}$ between D and A, and therefore an ϵ-chain $\{q_i\}$ in $\cup \mathscr{D}$ between x and some point of A. Since A is compact, there exists $a \in A$ such that for each $\epsilon > 0$, there is an ϵ-chain in $\cup \mathscr{D}$ between x and a. Then x and a are in the same quasi-component, hence the same component, of $\cup \mathscr{D}$.

THEOREM 1.6: *If X is locally continuum-connected (connected and locally continuum-connected), then every growth hyperspace $\mathscr{G}$ of X is an ANR (AR). Conversely, if there exists a growth hyperspace $\mathscr{G}$ such that $\mathscr{G} \supset C(X)$ and $\mathscr{G}$ is an ANR (AR), then X is locally continuum-connected (connected and locally continuum-connected).*

PROOF: We use the Lefschetz–Dugundji characterization of metric

ANR's [9]: a metric space M is an ANR if and only if, for each open cover α of M, there exists an open refinement β such that every partial β-realization in M of a simplicial polytope K (with the Whitehead topology) extends to a full α-realization of K. Thus, let α be an open cover of $\mathscr{G}$, and assume that the elements of α are open metric balls, with respect to the Hausdorff metric on $\mathscr{G}$. Take an open star-refinement α' of α. By Lemma 1.4 there exists an open refinement β of α' such that each element of β is path-connected. Then every partial β-realization $f: L \to \mathscr{G}$ of a polytope K extends to a partial α-realization $g: L \cup K^1 \to \mathscr{G}$, where K^1 is the 1-skeleton of K. Using Lemma 1.5, we may extend g to a full α-realization $h: K \to \mathscr{G}$ by the following inductive procedure. Consider an n-simplex σ of K, $n \geq 2$, such that h has been defined over $bd\sigma$. Let $r: \sigma \to C(bd\sigma)$ be any extension of the natural injection $bd\sigma \to C(bd\sigma)$. Then define h over σ by setting $h(x) = \cup\{h(p): p \in r(x)\}$. Thus $\mathscr{G}$ is an ANR.

If additionally X is connected, then by Lemma 1.3 every compact subset of X is contained in a continuum. Thus for arbitrary A, $B \in \mathscr{G}$, there exists a continuum C containing $A \cup B$, and $C \in \mathscr{G}$. By Lemma 1.1. there exist paths in $\mathscr{G}_A(C)$ from A to C and in $\mathscr{G}_B(C)$ from B to C, hence a path in $\mathscr{G}$ between A and B. Thus $\mathscr{G}$ is path-connected. Since the argument of the preceding paragraph shows that $\mathscr{G}$ is always n-connected for $n \geq 1$, it follows that $\mathscr{G}$ is an AR.

Conversely, suppose there exists a growth hyperspace $\mathscr{G}$ of X such that $\mathscr{G} \supset C(X)$ and $\mathscr{G}$ is an ANR. Let $x \in X$ and a neighborhood U be given. Since $\mathscr{G}$ is locally path-connected, there exists a neighborhood V of x such that for each $y \in V$, there exists a path $f: I \to \mathscr{G}$ between $\{x\}$ and $\{y\}$ with each $f(t) \subset U$. By Lemma 1.5, $\cup\{f(t): t \in I\} \subset U$ is a continuum. Thus X is locally continuum-connected. And if $\mathscr{G}$ is an AR, and therefore connected, X must also be connected.

The ANR (AR) characterizations for the hyperspaces 2^X and $C(X)$ of a compact metric space X were obtained by Wojdyslawski [15]. These characterizations were extended to complete metric spaces by Tašmetov [13]. Independently, some partial results along these lines were announced by Borges [3].

The following examples show that for noncomplete metric spaces, the property of local continuum-conectedness lies strictly between local connectedness and local path-connectedness.

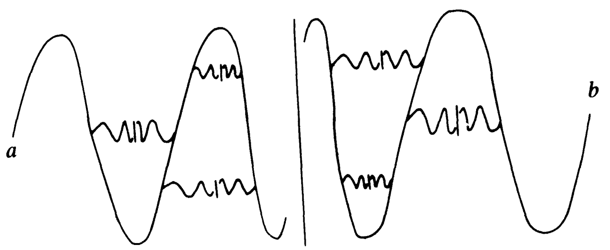
Example 1.7: There exists a connected and locally connected subset of the plane which is not locally continuum-connected.

Proof: There exist disjoint subsets A and B of the plane E^2 such that every nondegenerate continuum in the plane meets both A and B ([10], p. 110). Thus A contains no nondegenerate subcontinuum, and is not locally continuum-connected. However, A is connected and locally connected. Suppose $A = A_1 \cup A_2$ is a separation. Then there exists a closed subset C of the plane separating A_1 and A_2. Since C cannot be 0-dimensional, it contains a nondegenerate subcontinuum D. Then D must meet A, impossible. Thus A is connected, and the same argument applied locally shows that A is locally connected.

EXAMPLE 1.8: There exists a connected and locally continuum-connected subset of the plane which is not locally path-connected.

PROOF: We begin with the continuum

$$S = \{(x, \sin 1/x) : 0 < |x| \leq 1/\pi\} \cup \{(0,t) : |t| \leq 1\}.$$



A countable collection $\{S_i\}$ of progressively smaller copies of S is then fitted inside the individual loops of S as indicated, creating local continuum-connectedness on the limit segment $L = \{(0, t) : |t| \leq 1\} \subset S$. Then for each i, a countable collection $\{S_{ij}\}$ of copies of S is similarly fitted inside the loops of S_i. The infinite iteration of this procedure produces the desired space $X = S \cup (\cup\{S_i : i \geq 1\}) \cup (\cup\{S_{ij} : i, j \geq 1\}) \cup \ldots$.

X is connected and locally continuum-connected. However, X is not locally path-connected at any point on a limit segment such as L. It suffices to show that there exists no path in X between the endpoints $a = (-1/\pi, 0)$ and $b = (1/\pi, 0)$. Suppose there exists such a path σ. Then for some i (in fact, for infinitely many i), σ must contain a subpath σ_i in $S_i \cup (\cup\{S_{ij} : j \geq 1\}) \cup \ldots$ between the corresponding endpoints a_i and b_i of S_i. By the same argument σ_i must contain a subpath σ_{ij} in some $S_{ij} \cup (\cup\{S_{ijk} : k \geq 1\}) \cup \ldots$ between the endpoints a_{ij} and b_{ij} of S_{ij}. Thus the path σ must pass through each member of some nested sequence $(S_i, S_{ij}, S_{ijk}, \ldots)$. But this is impossible, since the limit point of such a sequence is not included in X.

2. Peano compactifications with locally non-separating remainders

Since $2^Y \approx Q$ for every non-degenerate Peano space Y, one way to study the hyperspace of a noncompact space X is to consider, when possible, a Peano compactification $\tilde{X}$ of X, and the corresponding Q-compactification $2^{\tilde{X}}$ of 2^X. The procedure works if the remainder $\tilde{X}\backslash X$ is sufficiently "nice". In this section we specify the desired property of the remainder, and establish the conditions under which such a compactification exists.

DEFINITION: A subset A of X is *locally non-separating in* X if, for each nonempty connected open subset U of X, $U\backslash A$ is nonempty and connected.

Note that if A is locally non-separating, so is every subset of A. It is easily shown that if a locally connected space X has a connected open base $\{U_\alpha\}$ such that each $U_\alpha\backslash A$ is nonempty and connected, then A is locally non-separating.

The motivation for considering locally non-separating subsets comes from the following pair of results on positional properties of intersection hyperspaces. For $A_1, \ldots, A_n \in 2^X$, we define the intersection hyperspaces $2^X(A_1, \ldots, A_n) = \{F \in 2^X : F \cap A_i \neq \emptyset \text{ for each } i\}$ and $C(X; A_1, \ldots, A_n) = \{F \in C(X) : F \cap A_i \neq \emptyset \text{ for each } i\}$. For any nondegenerate Peano space X, $2^X(A_1, \ldots, A_n) \approx Q$, and $C(X; A_1, \ldots, A_n) \approx Q$ if additionally X contains no free arcs [7]. A closed subset F of a metric space Y is a *Z-set* in Y if, for each compact subset K of Y and $\epsilon > 0$, there exists a map $\eta : K \to Y\backslash F$ with $d(\eta, id) < \epsilon$.

PROPOSITION 2.1: *Let A be a closed subset of a Peano continuum X. Then $2^X(A)$ is a Z-set in 2^X if and only if A is locally non-separating in X. More generally, for closed subsets $A, B_1, \ldots, B_n$ of X, $2^X(A, B_1, \ldots, B_n)$ is a Z-set in $2^X(B_1, \ldots, B_n)$ if and only if A is locally non-separating in X and $B_i\backslash A$ is dense in B_i for each i.*

PROOF: Suppose A satisfies the stated conditions, and let $\epsilon > 0$ be given. We must construct a map $\eta : 2^X(B_1, \ldots, B_n) \to 2^X(B_1, \ldots, B_n)\backslash 2^X(A, B_1, \ldots, B_n)$ such that $\rho(\eta, id) < \epsilon$. For each i, there exists a finite $\epsilon/3$-net β_i for B_i such that $\beta_i \subset B_i\backslash A$. By [7], there exists an "expansion" map $h: 2^X(B_1, \ldots, B_n) \to 2^X(\beta_1, \ldots, \beta_n)$ such that $\rho(h, id) \leq \epsilon/3$. And by [8], $2^X(\beta_1, \ldots, \beta_n) \approx \text{inv lim}\,(2^{\Gamma_i}(\beta_1, \ldots, \beta_n), f_i)$, where $\{\Gamma_i\}$ is a sequence of compact connected graphs in X, with each Γ_i containing $\beta_1 \cup \ldots \cup \beta_n$ in its vertex set, and each bonding map $f_i: 2^{\Gamma_{i+1}}(\beta_1, \ldots, \beta_n) \to 2^{\Gamma_i}(\beta_1, \ldots, \beta_n)$ induced by a map $\varphi_i: \Gamma_{i+1} \to C(\Gamma_i)$ such that $\varphi_i(b) = \{b\}$ for each $b \in \beta_1 \cup \ldots \cup \beta_n$. Thus for some i the projection map $p_i: 2^X(\beta_1, \ldots, \beta_n) \to 2^{\Gamma_i}(\beta_1, \ldots, \beta_n)$ satisfies $\rho(p_i, id) < \epsilon/3$.

Let $\mathcal{U}$ be a finite cover of Γ_i by connected open subsets of X with diameters less than $\epsilon/3$. There exists a subdivision $Sd\Gamma_i$ of Γ_i such that each simplex of $Sd\Gamma_i$ is contained in a member of $\mathcal{U}$. To each vertex v of $Sd\Gamma_i$ we assign a point $\kappa(v) \in \cap\{U \in \mathcal{U}: v \in U\}\backslash A$, with $\kappa(b) = b$ if $b \in \beta_1 \cup \ldots \cup \beta_n$. Then κ may be extended to a map $\kappa: Sd\Gamma_i \to X\backslash A$ such that, for each simplex σ of $Sd\Gamma_i$, $\kappa(\sigma) \subset U\backslash A$ for some $U \in \mathcal{U}$ with $\sigma \subset U$ (we use the fact that each $U\backslash A$ is connected, locally connected, and locally compact, therefore path-connected). Thus $d(\kappa, id) < \epsilon/3$, and the induced map $k: 2^{\Gamma_i}(\beta_1, \ldots, \beta_n) \to 2^{X\backslash A}(\beta_1, \ldots, \beta_n)$ satisfies $\rho(k, id) < \epsilon/3$. The composition $kp_ih: 2^X(B_1, \ldots, B_n) \to 2^{X\backslash A}(\beta_1, \ldots, \beta_n) \subset 2^X(B_1, \ldots, B_n)\backslash 2^X(A, B_1, \ldots, B_n)$ satisfies $\rho(kp_ih, id) < \epsilon$, and $2^X(A, B_1, \ldots, B_n)$ is a Z-set in $2^X(B_1, \ldots, B_n)$.

Conversely, suppose the Z-set condition is satisfied. Then each $B_i\backslash A$ must be dense in B_i, otherwise $2^X(A, B_1, \ldots, B_n)$ has a nonempty interior in $2^X(B_1, \ldots, B_n)$. For each i, choose $b_i \in B_i\backslash A$. Given a neighborhood U of a point $y \in A$, let V be a connected open neighborhood of y such that $\bar{V} \subset U\backslash\{b_1, \ldots, b_n\}$. We show that $V\backslash A$ is connected, thus A is locally non-separating. Suppose $V\backslash A = V_0 \cup V_1$ is a separation. There exists a continuum M in V such that $M \cap V_0 \neq \emptyset \neq M \cap V_1$. Let $\mathcal{F} = \{F \in 2^X(M): F\backslash M = \{b_1, \ldots, b_n\}\}$. Then $\mathcal{F}$ is homeomorphic to the connected hyperspace 2^M, and $\mathcal{F} \subset 2^X(B_1, \ldots, B_n)$. For each $\epsilon > 0$ there exists a map $\eta: \mathcal{F} \to 2^X\backslash 2^X(A)$ with $\rho(\eta, id) < \epsilon$. If ϵ is sufficiently small, there exist elements F_0, $F_1 \in \mathcal{F}$ such that $\eta(F_0) \cap V_0 \neq \emptyset$ and $\eta(F_1) \cap V_0 = \emptyset$, and $\eta(F) \cap bdV = \emptyset$ for every $F \in \mathcal{F}$. Then $\eta(\mathcal{F}) = \{\eta(F): \eta(F) \cap V_0 \neq \emptyset\} \cup \{\eta(F): \eta(F) \cap V_0 = \emptyset\}$ is a separation of the connected space $\eta(\mathcal{F})$, impossible.

Proposition 2.2: Let $A, B_1, \ldots, B_n$ be closed subsets of a Peano continuum X. Then $C(X; A, B_1, \ldots, B_n)$ is a Z-set in $C(X; B_1, \ldots, B_n)$ if and only if A is locally non-separating in X and $B_i\backslash A$ is dense in B_i for each i.

Proof: The argument for obtaining the Z-set property is the exact parallel of the corresponding argument in the proof of Proposition 2.1. For the converse, suppose $C(X; A, B_1, \ldots, B_n)$ is a Z-set in $C(X; B_1, \ldots, B_n)$. Actually, we only use the fact that $C(X; A, B_1, \ldots, B_n)$ has empty interior in $C(X; B_1, \ldots, B_n)$. It is immediate that each $B_i \backslash A$ must be dense in B_i, and $X \backslash A$ must be connected. Thus there exists a connected open set G in $X \backslash A$ such that $G \cap B_i \neq \emptyset$ for each i and $\bar{G} \cap A = \emptyset$. Given a neighborhood U of a point $y \in A$, let V be a connected open neighborhood of y such that $\bar{V} \subset U \backslash \bar{G}$, and choose $\epsilon > 0$ such that $N_\epsilon(\bar{V}) \subset U$. Let $\mathscr{W} = \{V \backslash A, G, W_1, W_2, \ldots\}$ be an open cover of $X \backslash A$ such that each W_i is connected and has diameter less than ϵ. By connectedness of $X \backslash A$, we obtain a chain in $\mathscr{W}$ between $V \backslash A$ and G, which in turn leads to connected open sets H and W in $X \backslash A$ such that $H \supset G$, $H \cap V = \emptyset$, $H \cap W \neq \emptyset \neq W \cap V$, and diam $W < \epsilon$. Then $V \cup W \subset U$ is a connected open neighborhood of y, and we claim that $(V \cup W) \backslash A$ is connected. If there exists a separation $(V \cup W) \backslash A = V_0 \cup V_1$, with the connected set W contained in V_1, then $(V \cup W \cup H) \backslash A = V_0 \cup (V_1 \cup H)$ is also a separation. However, there exists a continuum K in the connected open set $V \cup W \cup H$ which meets each B_i, and also meets the open sets V_0 and V_1. Then K is in the interior of $C(X; A, B_1, \ldots, B_n)$ in $C(X; B_1, \ldots, B_n)$, impossible.

DEFINITION: A metric d for a space X has *Property S* if, for each $\epsilon > 0$, there exists a finite connected cover of X with mesh less than ϵ.

If X admits a metric with Property S, then X is locally connected. Without added conditions, the converse is not true (see Lemma 3.2 and Example 4.3).

DEFINITION: A metric d for a connected space X is *strongly connected* if, for each $x, y \in X$, $d(x, y) = \inf\{\text{diam } M : M \text{ is a connected subset containing } x \text{ and } y\}$.

A convex metric on a Peano continuum is an example of a strongly connected metric. If X admits a strongly connected metric, then X is locally connected. Conversely, the proof of the following lemma shows that every connected, locally connected metric space admits a strongly connected metric.

LEMMA 2.3: *Let X be a connected metric space which admits a metric with Property S. Then X admits a strongly connected metric with Property S.*

PROOF: Let d be a metric with Property S. Define a topologically equivalent metric d^* for X by $d^*(x, y) = \inf\{\text{diam } M : M \text{ is a connected subset of } X \text{ containing } x \text{ and } y\}$. It is easily verified that d^* is a metric function. Since $d^*(x, y) \geq d(x, y)$, every open set with respect to d is open with respect to d^*. The converse is easily established, using the local connectedness of X. And since the diameters of connected subsets are the same with respect to d and d^*, d^* is strongly connected and has Property S.

PROPOSITION 2.4: *A connected metric space X has a Peano compactification $\tilde{X}$ with a locally non-separating remainder $\tilde{X}\backslash X$ if and only if X admits a metric with Property S.*

PROOF: Suppose X admits a metric d with Property S. We may assume by Lemma 2.3 that d is also strongly connected. Then the completion $(\tilde{X}, \tilde{d})$ of (X, d) is the desired Peano compactification. That $(\tilde{X}, \tilde{d})$ is connected and has Property S follows from the same properties for (X, d). And since a complete, totally bounded metric space is compact, $(\tilde{X}, \tilde{d})$ is a Peano compactification of (X, d).

Given a nonempty connected open subset U of $\tilde{X}$, we show that the nonempty set $U \cap X$ is connected, thereby verifying that $\tilde{X}\backslash X$ is locally non-separating in $\tilde{X}$. Suppose $U \cap X = H \cup K$ is a separation. Since U is open in $\tilde{X}$, $U \cap X$ is dense in U, and $U \subset \bar{H} \cup \bar{K}$ (the closures are taken in $\tilde{X}$). We must have $\bar{H} \cap \bar{K} \cap U \neq \emptyset$, otherwise $U = (\bar{H} \cap U) \cup (\bar{K} \cap U)$ is a separation. Let $p \in \bar{H} \cap \bar{K} \cap U$. Choose $\delta > 0$ such that the 3δ-neighborhood of p lies in U, and choose points h and k of H and K, respectively, lying in the δ-neighborhood of p. Then $d(h, k) < 2\delta$, and since d is strongly connected there exists a connected subset M of X containing h and k, with $\text{diam } M < 2\delta$. Then M lies in the 3δ-neighborhood of p, therefore in U. Thus $M \subset U \cap X$ is a connected set meeting both H and K, and $H \cup K$ cannot be a separation of $U \cap X$.

Conversely, suppose X has a Peano compactification $\tilde{X}$ such that $\tilde{X}\backslash X$ is locally non-separating. Take any admissible metric $\tilde{d}$ on $\tilde{X}$, and let d be its restriction to X. For every connected open cover $\{U_i\}$ of $\tilde{X}$, $\{U_i \cap X\}$ is a connected cover of X. Since $(\tilde{X}, \tilde{d})$ has finite connected open covers with arbitrarily small mesh, so does (X, d), and d has Property S.

3. Hyperspaces which are homeomorphic to Q\point

LEMMA 3.1: *Let X be a connected, locally connected metric space, with compact subsets A and B such that $A \subset int\ B$. Then only finitely many components of the complement $X \backslash A$ meet $X \backslash B$.*

PROOF: Each component U of $X \backslash A$ must have a limit point in A, otherwise U is both open and closed in X. Thus if $U \backslash B \neq \emptyset$, we must have $U \cap bdB \neq \emptyset$. Suppose there exists an infinite sequence $\{U_i\}$ of distinct components of $X \backslash A$, each extending beyond B. Choose $y_i \in U_i \cap bdB$ for each i. By compactness of bdB, we may assume that $y_i \to y \in bdB$. Since y has a connected neighborhood in $X \backslash A$, the component of $X \backslash A$ containing y meets U_i for almost all i, contradicting our supposition that the U_i are distinct components.

LEMMA 3.2: *Every connected, locally connected, locally compact metric space admits a metric with Property S.*

PROOF: Let $\tilde{X} = X \cup \infty$ be the one-point compactification of such a space X. Then $\tilde{X}$ is metrizable, since X is separable metric. We claim that for any admissible metric d on $\tilde{X}$, the restriction of d to X has Property S (and therefore $\tilde{X}$ is a Peano continuum). Given $\epsilon > 0$, choose a compact subset $A \subset X$ such that the complement $X \backslash A$ lies in the ϵ-neighborhood of ∞, and let $B \subset X$ be a compact neighborhood of A. Then by Lemma 3.1, only finitely many components of $X \backslash A$ extend beyond B. Thus a finite connected cover of B with mesh less than ϵ, together with the finite collection of components of $X \backslash A$ extending beyond B, provides a finite connected cover of X with mesh less than ϵ.

THEOREM 3.3: *$2^X \approx Q \backslash point$ if and only if X is a connected, locally connected, locally compact, noncompact metric space. Similarly, $C(X) \approx Q \backslash point$ if and only if X satisfies the above conditions and contains no free arcs.*

PROOF: Suppose X satisfies the stated conditions. By Lemma 3.2, X admits a metric with Property S, and by Proposition 2.4, X has a Peano compactification $\tilde{X}$ with locally non-separating remainder. Since X is locally compact it must be open in its compactification $\tilde{X}$, and the remainder $\tilde{X} \backslash X$ is closed. By Proposition 2.1, the intersection hyperspace $2^{\tilde{X}}(\tilde{X} \backslash X)$ is a Z-set in $2^{\tilde{X}}$. Thus $(2^{\tilde{X}}, 2^{\tilde{X}}(\tilde{X} \backslash X))$ and $(Q \times [0, 1], Q \times \{0\})$ are homeomorphic as pairs, and $2^X = 2^{\tilde{X}} \backslash 2^{\tilde{X}}(\tilde{X} \backslash X)$ is homeomorphic to $Q \times (0, 1]$, which is homeomorphic to $Q\backslash$point (since Cone $Q \approx Q$).

If in addition X contains no free arcs, then neither does $\tilde{X}$, and the hyperspaces $C(\tilde{X})$ and $C(\tilde{X}; \tilde{X}\backslash X)$ are copies of Q. By Proposition 2.2, $C(\tilde{X}; \tilde{X}\backslash X)$ is a Z-set in $C(\tilde{X})$, and it follows as above that $C(X) \approx Q\backslash$point.

Conversely, if either 2^X or $C(X)$ is homeomorphic to $Q\backslash$point, X must be a connected, locally connected metric space by Theorem 1.6. Since X has a closed imbedding into both 2^X and $C(X)$, X must be locally compact. Obviously, X is noncompact, and if $C(X) \approx Q\backslash$point, X contains no free arcs (otherwise $C(X)$ contains an open 2-cell).

4. Hyperspaces which are homeomorphic to 1^2

With the Hilbert cube Q coordinatized as $\Pi_1^\infty [0, 1]$, let $s = \Pi_1^\infty (0, 1) \subset Q$. Anderson [1] showed that s is homeomorphic to the Hilbert space $1^2 = \{(x_i) \in R^\infty : \Sigma_1^\infty x_i^2 < \infty\}$. Any subspace P of Q such that $(Q, P) \approx (Q, s)$ is called a *pseudo-interior* for Q, and its complement $Q\backslash P$ is a *pseudo-boundary*. A non-trivial example of a pseudo-boundary is the subset $\Sigma = \{(x_i) \in Q : 0 < \inf x_i \text{ and } \sup x_i < 1\}$. Kroonenberg [12] has given the following characterization for pseudo-boundaries, based on the original characterization by Anderson [2].

LEMMA 4.1: *Let $\{K_i\}$ be an increasing sequence of subsets of Q such that*:

i) *each $K_i \approx Q$,*
ii) *each K_i is a Z-set in Q,*
iii) *each K_i is a Z-set in K_{i+1},*
iv) *for each $\epsilon > 0$, there exists a map $f: Q \to K_i$ for some i such that $d(f, id) < \epsilon$.*

Then $\bigcup_1^\infty K_i$ is a pseudo-boundary for Q.

THEOREM 4.2: *The following conditions are equivalent*:

1) *X has a Peano compactification $\tilde{X}$ such that $(2^{\tilde{X}}, 2^X) \approx (Q, s)$,*
2) *X has a Peano compactification $\tilde{X}$ such that $(C(\tilde{X}), C(X)) \approx (Q, s)$,*
3) *X is a topologically complete, separable, connected, locally connected, nowhere locally compact metric space which admits a metric with Property S.*

PROOF: Suppose X satisfies condition 3). Then by Proposition 2.4, X has a Peano compactification $\tilde{X}$ with a locally non-separating remainder. Let $\tilde{d}$ be a convex metric for $\tilde{X}$. Since X is topologically complete and nowhere locally compact, the remainder $\tilde{X}\backslash X$ must be a dense countable union $\bigcup_1^\infty F_i$ of closed, locally non-separating sets in $\tilde{X}$. We may assume that $F_i \subset F_{i+1}$ and F_i has empty interior in F_{i+1}, for each i. This can be arranged inductively as follows. Select a dense sequence $\{x_n\}$ in F_i, a sequence $\{y_n\}$ in $\tilde{X}\backslash F_i$ such that $\tilde{d}(x_n, y_n) < 1/n$ for each n, and a sequence $\{z_n\}$ in $(\tilde{X}\backslash X)\backslash F_i$ such that $\tilde{d}(y_n, z_n) < 1/n$ for each n. Then replace F_{i+1} by the compact set $F_i \cup F_{i+1} \cup \{z_n : n \geq 1\}$.

By Proposition 2.1, each intersection hyperspace $2^{\tilde{X}}(F_i)$ is a Z-set copy of Q in $2^{\tilde{X}}$, and each $2^{\tilde{X}}(F_i) = 2^{\tilde{X}}(F_i, F_{i+1})$ is a Z-set in $2^{\tilde{X}}(F_{i+1})$. Given $\epsilon > 0$, we claim there exists a map $f: 2^{\tilde{X}} \to 2^{\tilde{X}}(F_i)$ for some i, such that $\rho(f, id) \leq \epsilon$. For $D \in 2^{\tilde{X}}$, define $f(D)$ to be the closed ϵ-neighborhood of D in $\tilde{X}$ (with respect to the convex metric $\tilde{d}$). Suppose $f(2^{\tilde{X}})\backslash 2^{\tilde{X}}(F_i) \neq \emptyset$ for each i. Then there exists a convergent sequence $y_i \to y$ in $\tilde{X}$ such that the ϵ-neighborhood of y_i is disjoint from F_i, for each i. It follows that the ϵ-neighborhood of y is disjoint from $\bigcup_1^\infty F_i = \tilde{X}\backslash X$, contrary to the fact that $\tilde{X}\backslash X$ is dense in $\tilde{X}$. Thus by Lemma 4.1, $\bigcup_1^\infty 2^{\tilde{X}}(F_i) = 2^{\tilde{X}}\backslash 2^X$ is a pseudo-boundary for $2^{\tilde{X}}$, and $(2^{\tilde{X}}, 2^X) \approx (Q, s)$.

The proof that $(C(\tilde{X}),\ C(X)) \approx (Q, s)$ is virtually the same as above, using Proposition 2.2.

Conversely, suppose either condition 1) or 2) is satisfied. Since s is a topologically complete, separable, nowhere locally compact metric AR, X must be a topologically complete, separable, connected, locally connected, nowhere locally compact metric space. We show that the remainder $\tilde{X}\backslash X$ is locally non-separating in $\tilde{X}$. For every connected open subset U of $\tilde{X}$, the hyperspace 2^U is a connected open subset of $2^{\tilde{X}}$. Since the pseudo-boundary $Q\backslash s$ is locally non-separating in Q, $2^{\tilde{X}}\backslash 2^X$ is locally non-separating in $2^{\tilde{X}}$. Thus $2^U \cap 2^X = 2^{U \cap X}$ is connected, and $U \cap X$ is connected. It follows from Proposition 2.4 that X admits a metric with Property S.

The first result of this type, $(2^Q, 2^s) \approx (C(Q), C(s)) \approx (Q, s)$, was obtained by Kroonenberg [12].

Using the very powerful Hilbert space characterization theorem of Torunczyk [14], the author has recently shown that $2^X \approx C(X) \approx l^2$ for every topologically complete, separable, connected, locally connected, nowhere locally compact metric space X [5]. The following example illustrates the difference between this result and Theorem 4.2.

EXAMPLE 4.3: There exists a space X such that $2^X \approx C(X) \approx 1^2$, but X does not admit a metric with Property S.

PROOF: The space X is a countable union of copies of 1^2 meeting at a single point θ, and given the uniform topology at θ. X may be realized in 1^2 as follows. Let $N = \bigcup_1^\infty \alpha_i$ be a partition of the positive integers, with each α_i infinite, and for each i set $1_i^2 = \{(x_n) \in 1^2 : x_n = 0$ if $n \notin \alpha_i\}$. Then $X = \bigcup_1^\infty 1_i^2 \subset 1^2$. Clearly, X is a closed, connected, locally connected, nowhere locally compact subset of 1^2, thus $2^X \approx C(X) \approx 1^2$.

The argument that the space X does not admit a metric with Property S is easy. Consider any admissible metric d for X. For some $\delta > 0$, the δ-neighborhood (with respect to d) of θ in X must be contained in the neighborhood $\{x \in X : \|x\| < 1\}$ of θ. Now consider any connected cover of X with mesh less than δ. For each i, any element of the cover intersecting $\{x \in 1_i^2 : \|x\| \geq 1\}$ cannot contain θ, and must therefore lie in $1_i^2 \backslash \theta$. Hence the cover is infinite, and d does not have Property S.

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Added in proof

The condition iv) of the pseudo-boundary characterization Lemma 4.1 is insufficient, and should be replaced by the following condition iv)*: there exists a deformation h: $Q \times [0, 1] \to Q$, with $h(q, 0) = q$ for each $q \in Q$, such that for each $\epsilon > 0$, $h(Q \times [\epsilon, 1]) \subset K_i$ for some i. In the application of Lemma 4.1 contained in the proof of Theorem 4.2, this stronger condition is easily verified (the map f of $2^{\tilde{X}}$ is replaced by the deformation h: $2^{\tilde{X}} \times [0, 1] \to 2^{\tilde{X}}$, where $h(D, t)$ is the closed t-neighborhood of D in $\tilde{X}$).