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THE LOGARITHM OF THE SIEGEL FUNCTION

Daniel S. Kubert*

1. Introduction

Let $a = (a_1, a_2) \in \boldsymbol{Q}^2 - \boldsymbol{Z}^2$. We may define the Siegel function $g_a(\tau)$ as follows. We define the Klein form $k_a(\tau)$, $\operatorname{Im}\tau > 0$ by $k_a(\tau) = e^{-\eta(z,\tau)z/2}\sigma(z, \tau)$ where $z = a_1\tau + a_2$, $\eta(z, \tau)$, $\sigma(z, \tau)$ are respectively the Weierstrass eta-function and the Weierstrass sigma-function for the lattice $[\tau, 1]$. We define $g_a(\tau)$ by

$$g_a(\tau) = k_a(\tau)\eta^2(\tau) \tag{1.1}$$

where η is the Dedekind eta-function. $g_a(\tau)$ has the q-expansion

$$g_a(\tau) = -q_\tau^{1/2B_2(a_1)}e^{2\pi i a_2(a_1-1)/2}(1 - q_z)\prod_{n=1}^{\infty}(1 - q_\tau^n q_z)(1 - q_\tau^n/q_z) \tag{1.2}$$

where $q_\tau = e^{2\pi i\tau}$ and $q_z = e^{2\pi i z}$, $B_2(x) = x^2 - x + \frac{1}{6}$ (see 10).

One may easily see from the q-expansion that if $a - a' \in \boldsymbol{Z}^2$, $g_a(\tau) = \epsilon g_{a'}(\tau)$ where ϵ is a constant depending on a and a'. So up to a constant the function $g_a(\tau)$ depends on the coset of a in $\boldsymbol{Q}^2/\boldsymbol{Z}^2$. Moreover $g_{-a}(\tau) = -g_a(\tau)$. So up to a constant g_a depends on the orbit of a in $\boldsymbol{Q}^2/\boldsymbol{Z}^2/\pm 1$. Let $\Gamma(n)$ denote the congruence subgroup of level n of $\Gamma = SL(2, \boldsymbol{Z})/\pm 1$. From (1.1) or (1.2) it is easy to show that if $\alpha \in \Gamma(n)$ and $na \in \boldsymbol{Z}^2$ then $g_a(\alpha\tau) = \epsilon_a(\alpha)g_a(\tau)$ where $\epsilon_a(\alpha)^{12n} = 1$. In fact, it is easy to explicitly calculate $\epsilon_a(\alpha)$ (see 3, 1–4). ϵ_a will be a character of $\Gamma(n)$.

The functions g_a satisfy the important "distribution relations". These are described as follows: Let $a \in \boldsymbol{Q}^2$. Let $N \in \boldsymbol{Z}^+$.

Let $\{\overline{b_i}\}_i$ be the inverse image of a mod $\boldsymbol{Z}^2$ under multiplication by N. Let $b_i \in \boldsymbol{Q}^2$ be a representative for $\overline{b_i}$. Then

$$\prod_i g_{b_i} = Cg_a \tag{1.3}$$

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where C is a constant. This is proved by showing that the left side and right side have the same divisors on the relevant modular curve which amounts to the fact that the Bernoulli polynomial, $B_2(x)$, satisfies a distribution relation.

We note from (2) that g_a is holomorphic and never 0 on h, the upper-half plane. Thus we may define a logarithm of $g_a(\tau)$ on h which we will denote by $lg_a(\tau)$. Since for $\alpha \in \Gamma(n)$, if $na \in Z^2$, $g_a(\alpha\tau) = \epsilon_a(\alpha)g_a(\tau)$, we have

$$lg_a(\alpha\tau) = lg_a(\tau) + 2\pi i\chi_a(\alpha) \tag{1.4}$$

where $\chi_a(\alpha)$ is a homomorphism from $\Gamma(n)$ to $\boldsymbol{Q}$ and since $\boldsymbol{Q}$ is abelian, χ_a factors through the commutator $\Gamma'(n)$ of $\Gamma(n)$ and we may consider χ_a as a homomorphism from $\Gamma(n)/\Gamma'(n)$ to $\boldsymbol{Q}$. It is clear, moreover, that χ_a only depends on the orbit of a in $\boldsymbol{Q}^2/\boldsymbol{Z}^2/\pm 1$. From the explicit calculation of $\epsilon_a(\alpha)$, we find if $\alpha \in \Gamma(12n^2)$ that $\epsilon_a(\alpha) = 1$. So if $\alpha \in \Gamma(12n^2)$, $\chi_a(\alpha) \in Z$. (See 3, 1-4).

In this paper we wish to develop some properties of χ_a and to explicitly calculate it. Using representation (1.1), we shall see that $\chi_a(\alpha)$ measures the difference between the area of certain parallelograms and the number of lattice points therein. Given this explicit description, we can then fairly easily prove some of the properties of χ_a geometrically. Others, like the distribution property, do not have an obvious geometric proof. This seems due to the fact that the Dedekind eta-function does not have a canonical homogeneous product representation.

We note that representation (1.2) may also be used to calculate χ_a. This has been done by B. Schoeneberg using techniques of Dedekind and Siegel. The expression he obtains has the form of a generalized Dedekind sum ([9], Chapter VIII). Previous to Schoeneberg's work Dieter had calculated the transformation formula for the Klein functions [1].

We note that for the ordinary Dedekind sum, Rademacher has already given an interpretation of it as the number of lattice points in a certain 3-dimensional region [7] (I, p. 318–321). It is therefore not surprising that the generalized Dedekind sums χ_a should have a lattice point interpretation. The techniques used below, in fact, will also yield an expression for the standard Dedekind sum as essentially the number of lattice points in a certain parallelogram minus the area.

Mordell also studied functions of lattice points in certain triangles and proved reciprocity laws for these functions (See 6).

§2. Properties of χ_a

Let $na \in Z^2$. Then for $\alpha, \beta \in \Gamma(n)$.

PROPOSITION 2.1: $\chi_a(\alpha\beta) = \chi_a(\alpha) + \chi_a(\beta)$.

This is just the homomorphism property. One can see from (1.1) that if $\gamma \in \Gamma$, $g_a(\gamma\tau) = C_1 g_{a\gamma}(\tau)$ where we think of γ as a matrix operating on the row vector $a = (a_1, a_2)$. Thus for some constant C

$$lg_a(\gamma\tau) = lg_{a\gamma}(\tau) + C.$$

Suppose $\alpha \in \Gamma(n)$. Then $\gamma\alpha\gamma^{-1} \in \Gamma(n)$ and

$$\begin{aligned} lg_a(\gamma\alpha\gamma^{-1}\tau) - lg_a(\tau) &= lg_a(\gamma\alpha\gamma^{-1}(\gamma\tau)) - lg_a(\gamma\tau) \\ &= lg_a(\gamma\alpha\tau) - lg_a(\gamma\tau) = lg_a\gamma(\alpha\tau) - lg_a\gamma(\tau). \end{aligned}$$

So we have

PROPOSITION 2.2: *If* $\alpha \in \Gamma(n)$, $\gamma \in \Gamma$.

$$\chi_a(\gamma\alpha\gamma^{-1}) = \chi_{a\gamma}(\alpha).$$

Since g_a satisfies a multiplicative distribution relation mod constants, lg_a satisfies an additive distribution relation mod constants. Thus if a and $\{b_i\}_i$ are as in (1.3) we have

PROPOSITION 2.3:

$$\sum_i lg_{b_i}(\tau) = lg_a(\tau) + C, \quad C \text{ constant}$$

Moreover if we take the logarithmic differential of the quantities in Proposition 2.3 we find

PROPOSITION 2.4:

$$\sum_i \frac{\partial g_{b_i}}{g_{b_i}}(\tau) = \frac{\partial g_a}{g_a}(\tau)$$

In particular the coefficients of the power series in q of $\partial g_{b_i}/g_{b_i}$ form additive distributions.

We may form another distribution from (2.3) via integrating (2.4). Let $\gamma \in \Gamma$. For each $a \in \boldsymbol{Q}^2 - \boldsymbol{Z}^2$ we define $\chi_a(\gamma)$ as follows. Choose $r \in \boldsymbol{Z}^+$ such that $\gamma^r \in \Gamma(n)$ where $na \in \boldsymbol{Z}^2$. Define $\chi_a(\gamma)$ as $1/r\chi_a(\gamma^r)$. Then $\chi_a(\gamma)$ is independent of r and, for fixed γ.

PROPOSITION 2.5: *$\chi_a(\gamma)$ forms an additive distribution. If $\gamma = \left[\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right]$ $\chi_a(\gamma)$ is just the distribution $B_2(a_1)/2$.*

§3. Calculations

To calculate χ_a we use formula (1.1). We also will use the distribution identity

$$(3.1) \qquad g(\tfrac{1}{2}, 0)g(\tfrac{1}{2}, \tfrac{1}{2})g(0, \tfrac{1}{2}) = C, \quad C \text{ constant.}$$

Given $a, b \in Q^2 - Z^2$ we may form the function

$$(3.2) \qquad g(a, b) = \frac{g_{a+b}g_{a-b}}{g_a^2} \quad \text{if } a+b, a-b \notin Z^2.$$

From (1.1) we immediately get

$$(3.3) \qquad g(a, b) = \frac{k_{a+b}k_{a-b}}{k_a^2}$$

To begin we must define $lg(a, b)$, the logarithm of $g(a, b)$. Given $r = (r_1, r_2)$ and $s = (s_1, s_2) \in Q^2 - Z^2$, we define

$$(3.4) \quad lk(r, s) = \sum_{\omega \in L-(0)} \left(\log \frac{\omega - r}{\omega - s} + \left(\frac{r}{\omega}\right) + \frac{1}{2}\left(\frac{r}{\omega}\right)^2 - \left(\frac{s}{\omega}\right) - \frac{1}{2}\left(\frac{s}{\omega}\right)^2\right) + \log \frac{r}{s} - \frac{\eta(r)r}{2} + \frac{\eta(s)s}{2}$$

where here, by abuse of notation, r means $r_1\tau + r_2$; s, $s_1\tau + s_2$, and $L = [\tau, 1]$ where log is the principal branch on $\boldsymbol{C} - \boldsymbol{R}^-$ and is given imaginary part πi on the negative real axis.

Then

$$(3.5) \qquad e^{lk(r,s)} = k_r/k_s$$

We define $lg(a, b)$ by

$$(3.6) \qquad lg(a, b) = lk(a+b, a) + lk(a-b, a).$$

Define $[a, h, b, k]$ by $1/(2\pi i)$ $(lg(a+h,\ b+k) - lg(a, b))$ for $a, b \in \boldsymbol{Q}^2 - \boldsymbol{Z}^2$; $h, k \in \boldsymbol{Z}^2$. Then $[a, h, b, k] \in \boldsymbol{Q}$ since g_a differs from g_{a+h} by a root of unity etc. From the definition we have the cocycle relationship.

Proposition 3.8: *If* $h_1, h_2, k_1, k_2 \in \boldsymbol{Z}^2$

$$[a, h_1 + h_2, b, k_1 + k_2] = [a + h_1, h_2, b + k_1, k_2] + [a, h_1, b, k_1].$$

One sees immediately from (3.4) that $lg(a, b)(\gamma\tau) = lg(a\gamma, b\gamma)(\tau)$. From this we see

Proposition 3.9: *If* $\gamma \in \Gamma$, $[a\gamma, h\gamma, b\gamma, k\gamma] = [a, h, b, k]$.

Suppose $na, nb \in \boldsymbol{Z}^2$, $\gamma \in \Gamma(n)$. Then we know $lg(a, b)(\gamma\tau) - lg(a, b)(\tau)$ is a constant which in fact is $2\pi i(\chi_{a+b}(\gamma) + \chi_{a-b}(\gamma) - 2\chi_a(\gamma))$. If we set $h = a\gamma - a$, $k = b\gamma - b$ then since $lg(a, b)(\tau) = lg(a\gamma, b\gamma)(\tau)$, $lg(a, b)(\gamma\tau) - lg(a, b)(\tau) = 2\pi i[a, h, b, h]$. We now calculate $[a, h, b, k]$. Assume in the following that $h \neq 0 \neq k$, and $h \neq \pm k$ so that the elements $0, -h, -k, -h-k$ are distinct. In what follows set $\alpha = a + h$, $\beta = b + k$. Then

$$\begin{aligned} lg(\alpha, \beta) = \sum_{\omega \in L-(0)} & \left(\log\left(\frac{\omega - (\alpha + \beta)}{\omega - (\alpha)}\right) + \log\left(\frac{\omega - (\alpha - \beta)}{\omega - (\alpha)}\right)\right. \\ & + \left(\frac{\alpha + \beta}{\omega}\right) + \frac{1}{2}\left(\frac{\alpha + \beta}{\omega}\right)^2 + \left(\frac{\alpha - \beta}{\omega}\right) \\ & \left. + \frac{1}{2}\left(\frac{\alpha - \beta}{\omega}\right)^2 - 2\left(\frac{\alpha}{\omega}\right) - \left(\frac{\alpha}{\omega}\right)^2\right) + \log\frac{\alpha + \beta}{(\alpha)} \\ & + \log\frac{\alpha - \beta}{\alpha} - \frac{\eta(\alpha + \beta)(\alpha + \beta)}{2} \\ & - \frac{\eta(\alpha - \beta)(\alpha - \beta)}{2} + \eta(\alpha)(\alpha) \end{aligned}$$

$$\begin{aligned} lg(a, b) = \sum_{\omega \in L-(0)} & \left(\log\left(\frac{\omega - (a + b)}{\omega - a}\right) + \log\left(\frac{\omega - (a - b)}{\omega - a}\right)\right. \\ & \left. + \left(\frac{a + b}{\omega}\right) + \frac{1}{2}\left(\frac{a + b}{\omega}\right)^2 + \left(\frac{a - b}{\omega}\right) + \frac{1}{2}\left(\frac{a - b}{\omega}\right)^2 - 2\left(\frac{a}{\omega}\right) - \left(\frac{a}{\omega}\right)^2\right) \\ & + \log\left(\frac{a + b}{a}\right) + \log\left(\frac{a - b}{a}\right) - \frac{\eta(a + b)(a + b)}{2} \\ & - \frac{\eta((a - b))(a - b)}{2} + \eta(a)a. \end{aligned}$$

We calculate the subtraction in two parts, those involving η and the others. Using the fact that η is $\boldsymbol{R}$-linear we find

$$-\frac{\eta(\alpha+\beta)(\alpha+\beta)}{2}-\frac{\eta(\alpha-\beta)(\alpha-\beta)}{2}+\eta(\alpha)(\alpha)$$
$$=-\eta(b+k)(b+k)$$
$$-\frac{\eta(a+b)(a+b)}{2}-\frac{\eta(a-b)(a-b)}{2}+\eta(a)(a)=-\eta(b)b$$

and $-\eta(b+k)(b+k)+\eta(b)b=-\eta(b)k-\eta(k)b-\eta(k)k$. Set

$$*=\sum_{\omega\in L-(0)}\log\left(\frac{\omega-(\alpha+\beta)}{\omega-(\alpha)}+\log\left(\frac{\omega-(\alpha-\beta)}{\omega-(\alpha)}\right)+\left(\frac{\alpha+\beta}{\omega}\right)\right.$$
$$+\frac{1}{2}\left(\frac{\alpha+\beta}{\omega}\right)^2+\left(\frac{\alpha-\beta}{\omega}\right)+\frac{1}{2}\left(\frac{\alpha-\beta}{\omega}\right)^2-2\left(\frac{\alpha}{\omega}\right)-\left(\frac{\alpha}{\omega}\right)^2.$$

Then via $\omega\mapsto\omega+h$

$$*=\sum_{\omega\in L-(-h)}\log\left(\frac{\omega-(a+b+k)}{\omega-a}\right)+\log\left(\frac{\omega-(a-b-k)}{\omega-a}\right)+\left(\frac{a+h+b+k}{\omega+h}\right)$$
$$+\frac{1}{2}\left(\frac{a+h+b+k}{\omega+h}\right)^2+\left(\frac{a+h-b-k}{\omega+h}\right)+\frac{1}{2}\left(\frac{a+h-b-k}{\omega+h}\right)^2-2\left(\frac{a+h}{\omega+h}\right)$$
$$-\left(\frac{a+h}{\omega+h}\right)^2.$$
$$\sum_{\omega\in L-(-h)}\log\left(\frac{\omega-(a+b+k)}{\omega-a}\right)+\left(\frac{a+h+b+k}{\omega+h}\right)$$
$$+\frac{1}{2}\left(\frac{a+h+b+k}{\omega+h}\right)^2-\left(\frac{a+h}{\omega+h}\right)-\frac{1}{2}\left(\frac{a+h}{\omega+h}\right)^2$$

is convergent and equals

$$\sum_{\omega\in L-(-h-k)}\log\left(\frac{\omega-(a+b)}{\omega+k-a}\right)+\left(\frac{a+h+b+k}{\omega+h+k}\right)+\frac{1}{2}\left(\frac{a+h+n+k}{\omega+h+k}\right)^2$$
$$-\left(\frac{a+h}{\omega+h+k}\right)-\frac{1}{2}\left(\frac{a+h}{\omega+h+k}\right)^2.$$

Set

$$t=\sum_{\omega\in L-(0)}\log\left(\frac{\omega-(a+b)}{\omega-a}\right)+\log\left(\frac{\omega-(a-b)}{\omega-a}\right)+\left(\frac{a+b}{\omega}\right)+\frac{1}{2}\left(\frac{a+b}{\omega}\right)^2$$
$$+\left(\frac{a-b}{\omega}\right)+\frac{1}{2}\left(\frac{a-b}{\omega}\right)^2-2\left(\frac{a}{\omega}\right)-\left(\frac{a}{\omega}\right)^2$$
$$\sum_{\omega\in L-(0)}\log\left(\frac{\omega-(a-b)}{\omega-a}\right)+\left(\frac{a-b}{\omega}\right)+\frac{1}{2}\left(\frac{a-b}{\omega}\right)^2-\left(\frac{a}{\omega}\right)-\frac{1}{2}\left(\frac{a}{\omega}\right)^2$$

is convergent and equals

$$\sum_{\omega\in L-(-k)} \log\left(\frac{\omega-(a-b-k)}{\omega+k-a}\right)+\left(\frac{a-b}{\omega+k}\right)+\frac{1}{2}\left(\frac{a-b}{\omega+k}\right)^2-\left(\frac{a}{\omega+k}\right)-\frac{1}{2}\left(\frac{a}{\omega+k}\right)^2.$$

Thus $lg(a+h,b+k)-lg(a,b)$ equals

$$\begin{aligned}
&-\eta(b+k)(b+k)+\eta(b)b+\sum_{\substack{\omega\in L-S\\ S=(0,-h,-k,-h-k)}} \log\left(\frac{\omega-(a+b)}{\omega+k-a}\right)\\
&\quad+\log\left(\frac{\omega-(a-b-k)}{\omega-a}\right)-\log\left(\frac{\omega-(a+b)}{\omega-a}\right)-\log\left(\frac{\omega-(a-b-k)}{\omega+k-a}\right)\\
&\quad+\left(\frac{a+h+b+k}{\omega+h+k}\right)+\frac{1}{2}\left(\frac{a+h+b+k}{\omega+h+k}\right)^2-\left(\frac{a+h}{\omega+h+k}\right)-\frac{1}{2}\left(\frac{a+h}{\omega+h+k}\right)^2\\
&\quad+\left(\frac{a+h-b-k}{\omega+h}\right)+\frac{1}{2}\left(\frac{a+h-b-k}{\omega+h}\right)^2-\left(\frac{a+h}{\omega+h}\right)-\frac{1}{2}\left(\frac{a+h}{\omega+h}\right)^2\\
&\quad-\left(\frac{a+b}{\omega}\right)-\frac{1}{2}\left(\frac{a+b}{\omega}\right)^2+\left(\frac{a}{\omega}\right)+\frac{1}{2}\left(\frac{a}{\omega}\right)^2-\left(\frac{a-b}{\omega+k}\right)-\frac{1}{2}\left(\frac{a-b}{\omega+k}\right)^2\\
&\quad+\frac{a}{\omega+k}+\frac{1}{2}\left(\frac{a}{\omega+k}\right)^2+\sum_{S}\log\left(\frac{\omega-(a+b)}{\omega+k-a}\right)+\log\frac{\omega-(a-b-k)}{\omega-a}\\
&\quad-\log\left(\frac{\omega-(a+b)}{\omega-a}\right)-\log\left(\frac{\omega-(a-b-k)}{\omega+k-a}\right)+\sum_{S-(-h-k)}\left(\frac{a+h+b+k}{\omega+h+k}\right)\\
&\quad+\frac{1}{2}\left(\frac{a+h+b+k}{\omega+h+k}\right)^2-\left(\frac{a+h}{\omega+h+k}\right)-\frac{1}{2}\left(\frac{a+h}{\omega+h+k}\right)^2\\
&\quad+\sum_{S-(-h)}\left(\frac{a+h-b-k}{\omega+h}\right)+\frac{1}{2}\left(\frac{a+h-b-k}{\omega+h}\right)^2-\left(\frac{a+h}{\omega+h}\right)-\frac{1}{2}\left(\frac{a+h}{\omega+h}\right)^2\\
&\quad-\sum_{S-(0)}\left(\frac{a+b}{\omega}\right)+\frac{1}{2}\left(\frac{a+b}{\omega}\right)^2-\left(\frac{a}{\omega}\right)-\frac{1}{2}\left(\frac{a}{\omega}\right)^2-\sum_{S-(-k)}\left(\frac{a-b}{\omega+k}\right)\\
&\quad+\frac{1}{2}\left(\frac{a-b}{\omega+k}\right)^2-\left(\frac{a}{\omega+k}\right)-\frac{1}{2}\left(\frac{a}{\omega+k}\right)^2.
\end{aligned}$$

Now we shall see presently that except for a finite number of ω,

$$\log\left(\frac{\omega-(a+b)}{\omega+k-a}\right)+\log\frac{\omega-(a-b-k)}{\omega-a}-\log\left(\frac{\omega-(a+b)}{\omega-a}\right)-\log\left(\frac{\omega-(a-b-k)}{\omega+k-a}\right)$$

is zero. Thus we may separate this quantity into three convergent sums, $lg(a+h, b+k) - lg(a, b) = (1) + (2) + (3)$ where

$$(1) \qquad = -\eta(b+k)(b+k) + \eta(b)b$$

$$(2) \qquad = \sum_{\omega} \log\left(\frac{\omega-(a+b)}{\omega+k-a}\right) + \log\left(\frac{\omega-(a-b-k)}{\omega-a}\right) - \log\left(\frac{\omega-(a+b)}{\omega-a}\right) - \log\left(\frac{\omega-(a-b-k)}{\omega+k-a}\right)$$

$$\begin{aligned}(3) \quad = & \sum_{S-(-h-k)} \left(\frac{a+h+b+k}{\omega+h+b}\right) + \frac{1}{2}\left(\frac{a+h+b+k}{\omega+h+k}\right)^2 - \left(\frac{a+h}{\omega+h+k}\right) \\ & -\frac{1}{2}\left(\frac{a+h}{\omega+h+k}\right)^2 + \sum_{S-(-h)} \left(\frac{a+h-b-k}{\omega+h}\right) + \frac{1}{2}\left(\frac{a+h-b-k}{\omega+h}\right)^2 \\ & -\left(\frac{a+h}{\omega+h}\right) - \frac{1}{2}\left(\frac{a+h}{\omega+h}\right)^2 - \sum_{S-(0)} \left(\frac{a+b}{\omega}\right) + \frac{1}{2}\left(\frac{a+b}{\omega}\right)^2 - \left(\frac{a}{\omega}\right) \\ & -\frac{1}{2}\left(\frac{a}{\omega}\right)^2 - \sum_{S-(-k)} \left(\frac{a-b}{\omega+k}\right) + \frac{1}{2}\left(\frac{a-b}{\omega+k}\right)^2 - \left(\frac{a}{\omega+k}\right) - \frac{1}{2}\left(\frac{a}{\omega+k}\right)^2 \\ & + \sum_{L-S} \left(\frac{a+h+b+k}{\omega+h+k}\right) + \frac{1}{2}\left(\frac{a+h+b+k}{\omega+h+k}\right)^2 - \left(\frac{a+h}{\omega+h+k}\right) \\ & -\frac{1}{2}\left(\frac{a+h}{\omega+h+k}\right)^2 + \left(\frac{a+h-b-k}{\omega+h}\right) + \frac{1}{2}\left(\frac{a+h-b-k}{\omega+h}\right)^2 \\ & -\left(\frac{a+h}{\omega+h}\right) - \frac{1}{2}\left(\frac{a+h}{\omega+h}\right)^2 - \left(\frac{a+b}{\omega}\right) - \frac{1}{2}\left(\frac{a+b}{\omega}\right)^2 + \left(\frac{a}{\omega}\right) + \frac{1}{2}\left(\frac{a}{\omega}\right)^2 \\ & -\left(\frac{a-b}{\omega+k}\right) - \frac{1}{2}\left(\frac{a-b}{\omega+k}\right)^2 + \left(\frac{a}{\omega+k}\right) + \frac{1}{2}\left(\frac{a}{\omega+k}\right)^2.\end{aligned}$$

We shall first analyze term (3). We recall the definition of $\eta(z, L)$. The Weierstrass zeta function is defined by

$$(3.10) \qquad \zeta(x, L) = \frac{1}{x} + \sum_{\omega \in L-(0)} \frac{1}{x-\omega} + \frac{1}{\omega} + \frac{x}{\omega^2}$$

$\eta(z, L)$ is then defined by

$$(3.11) \qquad \eta(z, L) = \zeta(x+z) - \zeta(x) \quad \text{for } z \in L$$

and is extended by $\boldsymbol{R}$-linearity to $\mathbb{C}$. Evaluating at $x = 0$ we find for $z \in L$, $z \neq 0$

$$(3.12) \qquad \eta(z) = \frac{3}{z} + \sum_{\omega \neq 0, z} \frac{1}{z-\omega} + \frac{1}{\omega} + \frac{z}{\omega^2}.$$

PROPOSITION 3.13: *Expression* (3) *equals* $(2b+k)\eta(k)$.

The calculation will be left to the reader via use of (3.12) with the hint that $(2b+k)\eta(k) = (b+k)\eta(k) + b\eta(k)$. One must also appropriately translate the lattice and one should be guided by the identity that $\frac{1}{2}(a+h+b+k)^2 + \frac{1}{2}(a+h-b-k)^2 - (a+h)^2 - \frac{1}{2}(a+b)^2 - \frac{1}{2}(a-b)^2 + a^2 = (2b+k)k$. The calculation is also valid if $h = 0$ or $k = 0$.

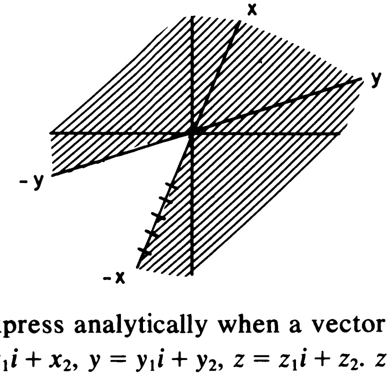
This proposition yields the surprising conclusion that $[a, h, b, k] = lg(a+h,\ b+k) - lg(a, b)$ is actually independent of h. For the proposition shows that expression (3) is independent of h, and it is clear that expressions (1) and (2) are independent of h. Combining (1) and (3) we obtain

$$2\pi i[a, h, b, k] = \eta(k)b - \eta(b)k + (2). \tag{3.14}$$

Set $\langle x, y\rangle = \eta(x)y - \eta(y)x$. By the Legendre relation if $\operatorname{Im} y/x > 0$, $\langle x, y\rangle = 2\pi i$ times the area of the parallelogram generated by the vectors (x_1, x_2), (y_1, y_2). If $\operatorname{Im} y/x < 0$, $\langle x, y\rangle = -2\pi i$ times the area of the parallelogram generated by the vectors (x_1, x_2), (y_1, y_2).

We now evaluate (2). Let $f(z) = \log z/y - \log z/x$ where $\operatorname{Im} x/y \geq 0$. $f(z)$ has two values. On the shaded region $f(z)$ has the value $\log x/y$. On the unshaded region $f(z)$ has the value $\log x/y - 2\pi i$. The ray $\overrightarrow{[0, -x]}$ is in the unshaded region and the ray $\overrightarrow{[0, -y]}$ is in the shaded region. One notes that the half-plane going from y to $-y$ counterclockwise is always in the shaded region.

(3.15)



We wish to express analytically when a vector lies in the unshaded region. Set $x = x_1 i + x_2$, $y = y_1 i + y_2$, $z = z_1 i + z_2$. z lies in the unshaded region iff $\operatorname{Im} x/y > 0$, $\operatorname{Im} x/{-z} \geq 0$, $\operatorname{Im} -z/y > 0$, or $x/y < 0$ and $\operatorname{Im} y/z > 0$ or $y/z > 0$. Now

$$\operatorname{Im} x/y = \frac{x_1 y_2 - y_1 x_2}{y_1^2 + y_2^2}, \quad \operatorname{Im} -z/y = \frac{z_2 b_1 - z_1 b_2}{y_1^2 + y_2^2}.$$

Set $(x, y) = x_1y_2 - x_2y_1$. Then z is in the unshaded region iff

$$(3.16)\qquad (x, y) > 0, (-z, y) > 0, (x, -z) \geq 0 \quad \text{or } x/y < 0 \text{ and } z/y > 0 \quad \text{or } (-z, y) > 0.$$

We now evaluate (2) at the point $\tau = i$. In our application if $\operatorname{Im}(\omega - a)/(\omega + k - a) \geq 0$ we set $y = \omega + k - a$, $x = \omega - a$, $z_1 = \omega - (a + b)$, $z_2 = \omega - (a - b - k)$. Then

$$\log\left(\frac{\omega - (a+b)}{\omega + k - a)}\right) + \log\left(\frac{\omega - (a - b - k)}{\omega - a}\right) - \log\left(\frac{\omega - (a+b)}{\omega - a}\right) - \log\left(\frac{\omega - (a - b - k)}{\omega + k - a}\right) = f(z_1) - f(z_2).$$

This quantity will equal 0 unless z_1 or z_2 is in the unshaded region. If we set $\epsilon(z) = 0$ if z is in the shaded region and $\epsilon(z) = -2\pi i$ if z is in the unshaded region then

$$(3.17)\quad \log\left(\frac{\omega - (a+b)}{\omega + k - a}\right) + \log\left(\frac{\omega - (a - b - k)}{\omega - a}\right) - \log\left(\frac{\omega - (a+b)}{\omega - a}\right) - \log\left(\frac{\omega - (a - b - k)}{\omega + k - a}\right) = \epsilon(z_1) - \epsilon(z_2),\ \operatorname{Im}\frac{\omega - a}{\omega + k - a} \geqslant 0.$$

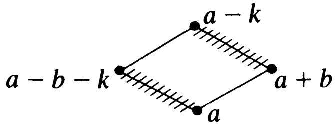
If $\operatorname{Im}(\omega - a/\omega + k - a) \leq 0$ then we set $y = \omega - a$, $x = \omega + k - a$, $z_1 = \omega - (a - b - k)$, $z_2 = \omega - (a + b)$ and with z_1 and z_2 so defined (3.17) again holds. Thus

$$(2)\quad = \sum_{\operatorname{Im}(\omega - a/\omega + k - a) \geq 0} \epsilon(\omega - (a - b - k)) - \epsilon(\omega - (a + b)).$$

$$+ \sum_{\operatorname{Im}(\omega - a/\omega + k - a) < 0} \epsilon(\omega - (a - b - k)) - \epsilon(\omega - (a + b)).$$

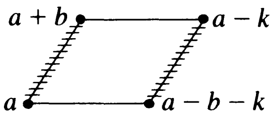
Let us suppose now that b and k are not colinear. Then we may via (3.16) picture the above regions. Suppose first that $\operatorname{Im} b/k > 0$. Then $\sum_{\operatorname{Im}(\omega - a/\omega + k - a) \geq 0} \epsilon(\omega - (a - b - k))$ is 0 and $\sum_{\operatorname{Im}(\omega - a/\omega + k - a) \geq 0} \epsilon(\omega - (a + b))$ equals $-2\pi i$ times the number of lattice points in the triangle with vertices $(a, a + b, a - k)$ with the segment $[a + b, a - k]$ excluded. $\sum_{\operatorname{Im}(\omega - a/\omega + k - a) < 0} \epsilon(\omega - (a + b))$ is 0 and $\sum_{\operatorname{Im}(\omega - a/\omega + k - a) < 0} \epsilon(\omega - (a - b - k))$ equals $-2\pi i$ times the number of lattice points in the triangle with vertices $(a, a - k, a - b - k)$ with the segments $[a, a - k]$, $[a, a - b - k]$ excluded. Thus the total is $-2\pi i$ times the number of points in the parallelogram with vertices $[a, a + b, a - k, a - b - k]$ with the segments $[a + b, a - k]$, $[a, a - b - k]$ excluded.

(3.18)



Now suppose $\operatorname{Im} b/k < 0$. Then $\Sigma_{\operatorname{Im}(\omega - a/\omega + k - a) \geq 0}\, \epsilon(\omega - (a+b))$ is 0 and $\Sigma_{\operatorname{Im}(\omega - a/\omega + k - a) \geq 0} -\epsilon(\omega - (a-b-k))$ equals $2\pi i$ times the number of lattice points in the triangle with vertices $(a, a-k, a-b-k)$ with the segment $[a-k, a-b-k]$ excluded. $\Sigma_{\operatorname{Im}(\omega - a/\omega + k - a) < 0}\, \epsilon(\omega - (a-b-k))$ is 0 and $\Sigma_{\operatorname{Im}(\omega - a/\omega + k - a) < 0} -\epsilon(\omega - (a+b))$ equals $2\pi i$ times the number of lattice points in the triangle with vertices $(a, a+b, a-k)$ with the segment $[a, a+b]$ excluded.

(3.19)



In summary combining (3.14) with (3.18) and (3.19) we find if $(a,\ a+b,\ a-k,\ a-b-k)$ denotes the parallelogram with vertices $a,\ a+b,\ a-k,\ a-b-k$, if

$$\text{(3.20)}\quad \operatorname{Im} b/k > 0, \text{hen } [a, h, b, k] = A(a, a+b, a-k, a-b-k) \\ -I(a, a+b, a-k, a-b-k)$$

where A is the area of $(a, a+b, a-k, a-b-k)$ and I is the number of integral points therein as in (3.18). If

$$\operatorname{Im} b/k < 0, \quad \text{then } [a, h, b, k] = I(a, a+b, a-k, a-b-k) \\ -A(a, a+b, a-k, a-b-k)$$

where I is the number of integral points in $(a, a+b, a-k, a-b-k)$ as in (3.19).

One may also calculate the case when $\operatorname{Im} b/k = 0$, i.e., the case when b and k are colinear but this will not be needed in application.

§4. Determination of χ_a

We would now like to use the results of Section 3 to calculate the homomorphism χ_a. Let a, b belongs to $\boldsymbol{Q}^2 - \boldsymbol{Z}^2$. Suppose n is such that $na,\ nb \in \boldsymbol{Z}^2$. Let $\gamma \in \Gamma(n)$. Then we may use (3.20) to determine

$\chi_{a+b}(\gamma)+\chi_{a-b}(\gamma)-2\chi_a(\gamma)$. From the definitions (1.1), (3.3), we have

$$(4.1)\qquad \chi_{a+b}(\gamma)+\chi_{a-b}(\gamma)-2\chi_a(\gamma)=[a, a\gamma-a, b, b\gamma-b].$$

So $h=a\gamma-a$, $k=b\gamma-b$. Let us first consider the case when k and b are colinear. Suppose $\exists\lambda\in\boldsymbol{R}$ such that $b\gamma=\lambda b$. Then λ must be an eigenvalue of γ and since $\gamma\in SL_2(\boldsymbol{Z})$ and $b\in\boldsymbol{Q}^2$ we must have $\lambda\in\boldsymbol{Q}$. One then sees quickly that $\lambda=\pm1$ and $\mathrm{Tr}(\gamma)=\pm2$, i.e. γ is parabolic. By (2.2) we may assume that $\gamma=[\begin{smallmatrix}1 & mh\\ 0 & 1\end{smallmatrix}]$, $m\in\boldsymbol{Z}$ and $b=(0, b_2)$ $nb_2\in\boldsymbol{Z}$. We show that $[a, a\gamma-a, b, b\gamma-b]$ is zero. This is obvious since $b\gamma-b=0$ and we have shown that $[a,h,b,k]$ does not depend on h. So since $[a,0,b,0]$ is clearly 0, the result follows. Alternatively, γ fixes $i\infty$ and if $b=(0,b_2)$ one checks that $g(a,b)$ has a zero of order 0 at $i\infty$. Thus evaluating $lg(a,b)\gamma(t)-lg(a,b)(t)$ as $t\mapsto i\infty$ we get 0.

Hence we may suppose that b and k are not colinear. We then may use (3.20) to evaluate $\chi_{a+b}(\gamma)+\chi_{a-b}(\gamma)-2\chi_a(\gamma)$ with $h=a\gamma-a$, $k=b\gamma-b$. Seeing that $[a,h,b,k]$ does not depend on h we may change our notation and set

$$(4.2)\qquad [b,k]_a=[a,h,b,k].$$

We have $\chi_{a+b}(\gamma)+\chi_{a-b}(\gamma)-2\chi_b(\gamma)-(\chi_{a+b}(\gamma)+\chi_{a-b}(\gamma)-2\chi_a(\gamma))=2(\chi_a(\gamma)-\chi_b(\gamma)$. So

$$(4.3)\ \chi_a(\gamma)-\chi_b(\gamma)=\tfrac{1}{2}[a, a\gamma-a]_b-\tfrac{1}{2}[b, b\gamma-b]_a\quad a\neq\pm b \bmod \boldsymbol{Z}^2.$$

For $\gamma\in\Gamma(12n^2)$, from the transformation of the Klein function we know that $\chi_a(\gamma)-\chi_b(\gamma)$ must take values in $2\pi i\boldsymbol{Z}$. This is not obvious from (3.20).

Now suppose a is not a 2-point i.e. $2a\notin\boldsymbol{Z}^2$. Let p_1,p_2,p_3 represent the three 2-points in $\boldsymbol{Q}^2/\boldsymbol{Z}^2$. The distribution relation (3.1) tells us that $g_{p_1}g_{p_2}g_{p_3}$ is constant. So for each $\gamma\in\Gamma(2)$

$$(4.4)\qquad \chi_{p_1}(\gamma)+\chi_{p_2}(\gamma)+\chi_{p_3}(\gamma)=0.$$

Then if $na\in Z^2$, $\gamma\in\Gamma(2n)$, n odd or $\gamma\in\Gamma(n)$, n even, $3\chi_a(\gamma)=(\chi_a(\gamma)-\chi_{p_1}(\gamma))+(\chi_a(\gamma)-\chi_{p_2}(\gamma))+\chi_a(\gamma)-\chi_{p_3}(\gamma)$ so

(4.5)

$$\chi_a(\gamma)=\tfrac{1}{6}([a, a\gamma-a]_{p_1}+[a, a\gamma-a]_{p_2}+[a, a\gamma-a]_{p_3})-\tfrac{1}{6}([p_1, p_1\gamma-p_1]_a+[p_2, p_2\gamma-p_2]_a+[p_3, p_3\gamma-p_3]_a).$$

If $\gamma \in \Gamma(n)$ and n is odd we may calculate $\chi_a(\gamma)$ by $\chi_a(\gamma) = \frac{1}{6}\chi_a(\gamma^6)$ and $\gamma^6 \in \Gamma(2n)$.

There is a simpler expression for $\chi_a(\gamma)$ which we may obtain as follows which is, however, not quite as aesthetically pleasing. Consider $g(p_i, a - p_i)$; $g(p_1, a - p_1)g(p_2, a - p_2)g(p_3, a - p_3)$ equals

$$\prod \frac{g_a g_{a+2p_i}}{g_{p_i}^2} = C_1 \prod_i \frac{g_a^2}{g_{p_i}^2} = C_2 g_a^6.$$

So if $na \in \boldsymbol{Z}^2$, $\gamma \in \Gamma(2n)$, n odd, or $\gamma \in \Gamma(n)$, n even

$$(4.6)\quad \chi_a(\gamma) = \tfrac{1}{6}([a - p_1, (a - p_1)(\gamma - 1)]_{p_1} + [a - p_2, (a - p_2)(\gamma - 1)]_{p_2} + [a - p_3, (a - p_3)(\gamma - 1)]_{p_3}).$$

Again, if n is odd and $\gamma \in \Gamma(n)$ one may calculate $\chi_a(\gamma)$ as $\frac{1}{6}\chi_a(\gamma^6)$ and $\gamma^6 \in \Gamma(2n)$.

If p is a 2-point $g(p, a)$ is peculiar in that $g(p, a) = C(g_{a-p}^2/g_a^2)$ so for $\gamma \in \Gamma(8n^2)$, $na \in \boldsymbol{Z}^2$ $[a - p, (a - p)(\gamma - 1)]_p$ must belong to $4\pi i Z$.

We may see this fact geometrically. Set $a' = a - p$ so $2na' \in \boldsymbol{Z}^2$. Set $k = a'(\gamma - 1)$. Then $k \in 4n\boldsymbol{Z}^2$. The parallelogram $(p, p + a', p - k, p - a', -k)$ has area twice an integer since a' has denominator at most $2n$ and $k \in 4n\boldsymbol{Z}^2$. So we must only show that 2 divides $I(p, p + a', p - k, p - a' - k)$. We first consider the case when $\operatorname{Im} b/k > 0$. Consider the transformation $z \mapsto Zp - k - z$ which preserves lattice points since $Zp - k$ belongs to $\boldsymbol{Z}^2$. Then the segment $[p, p - k]$ is taken to itself under this transformation. The triangle $[p, p - k, p + a']$ is taken to the triangle $[p, p - k, p - a' - k]$. The interval $[p, p - a' - k]$ is taken to $[p + a', p - k]$. Moreover the fixed point of the map is $z = p - k/2 \notin \boldsymbol{Z}^2$ since $k \in 2\boldsymbol{Z}^2$. So $I(p, p + a', p - k, p - a' - k)$ equals twice the number of points in the triangle $(p, p + a', p - k)$ excluding the intervals $(p, p - k)$ and $(p, p + a')$ plus the points on $(p, p - k)$ which number is even. The arguments are similar in the case that $\operatorname{Im} b/k < 0$, and are left to the reader. We explore the geometric aspects, more thoroughly in the next section.

We still must show how to calculate χ_a in the case a is a 2-point. Let's say we wish to calculate χ_{p_1}. We have

$$(4.7)\qquad g_{p_1}^{-6} = c\,\frac{g_{p_1+p_2} g_{p_1-p_2}}{g_{p_1}^2} \cdot \frac{g_{p_1-p_3} g_{p_1+p_3}}{g_{p_1^2}},\ c \text{ constant}$$

from the distribution relations. So if $\gamma \in \Gamma(2)$

$$(4.8)\qquad \chi_{p_1}(\gamma) = -\tfrac{1}{6}([p_2, p_2(\gamma - 1)]_{p_1} + [p_3, p_3(\gamma - 1)]_{p_1}).$$

§5. Geometric insights

One might, a priori, define a function $[a, h, b, k]$ as in (3.18) or (3.19) and then try to deduce, geometrically, properties about it. We show here how some of the major properties of $[a, h, b, k]$ can be so deduced without referring to the definition in terms of Siegel functions.

The most obvious property is the independence of $[a, h, b, k]$ on h, since h does not appear in (3.18) or (3.19). So we defined the symbol $[b, k]_a$ as before $b, a \in \boldsymbol{Q}^2 - \boldsymbol{Z}^2$, $k \in \boldsymbol{Z}^2$. We will slightly change the definition in order to make things more symmetric. If $\operatorname{Im} b/k = 0$ we define $[b, k]_a$ to be zero. If $\operatorname{Im} b/k > 0$ we define $[b, k]_a$ to equal the area of the parallelogram $(a, a-k, a+b, a-b-k)$ minus the number of points therein, with each point on the boundary counted with weight one-half. If $\operatorname{Im} b/k < 0$ we define $[b, k]_a$ to equal the number of integral points in the parallelogram $(a, a-k, a+b, a-b-k)$, counted as above, minus the area of the parallelogram. In actual application, this change is unimportant for the following reason. In application, k will equal $b(\gamma-1)$ where $\gamma \in SL(2, \boldsymbol{Z})$ and $a(\gamma-1) \in \boldsymbol{Z}^2$. Under the action of γ, the segment $[a, a+b]$ goes into the segment $[a\gamma, a\gamma + b\gamma]$ which via translation by $-a(\gamma-1)-k$ goes into $[a-k, a+b]$. In particular, since $a(\gamma-1) \in \boldsymbol{Z}^2$ the interval $[a, a+b]$ has the same number of integral points as the interval $[a+b, a-k]$. Likewise the interval $[a, a-b-k]$ has the same number of integral points as the interval $[a-k, a-b-k]$. Thus for $k = b(\gamma-1)$ our revised definition of $[b, k]_a$ agrees with (3.18) and (3.19). We note from the definition that $[b, k]_a$ only depends on the class of a mod $\boldsymbol{Z}^2$ since translation by elements of $\boldsymbol{Z}^2$ preserves lattice points. We now prove directly analogues of propositions (3.8) and (3.9).

PROPOSITION 5.1: *If $na \in \boldsymbol{Z}^2$, $\gamma \in \Gamma(n)$ then*

$$[b\gamma, k\gamma]_a = [b, k]_a.$$

If $\gamma \in \Gamma$

$$[b\gamma, k\gamma]_{a\gamma} = [b, k]_a.$$

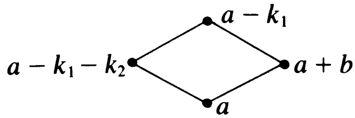
The second statement is immediate since γ preserves orientation, area, and lattice points. If $\gamma \in \Gamma(n)$ then $a - a\gamma$ differs by a lattice point and translation gives the desired first statement.

PROPOSITION 5.2: *Let $k_1, k_2 \in \boldsymbol{Z}^2$. Then*

$$[b, k_1 + k_2]_a = [b + k_1, k_2]_a + [b, k_1]_a.$$

The proof of this is "pictorial". We must distinguish cases corresponding to the relative positions of a, $a+b$, $a-k_1$, $a-k_1-k_2$. One case is when the convex hull of $(a,\ a+b,\ a-k_1,\ a-k_1-k_2)$ is not equal to that of any triangle. We have other cases when it is equal to that of a triangle, and finally we have degenerate cases when some of the subtriangles collapse into lines. We will consider the first case and leave the rest to the reader. Suppose, that the convex hull is not that of any triangle. Let us say that $\operatorname{Im} k_1/b > 0$, $\operatorname{Im}(k_1+k_2)/b > 0$. Then pictorially we have

(5.3)

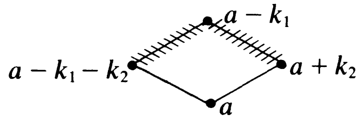


The area of triangle $(a, a+b, a-k_1-k_2)$ plus the area of triangle $(a-k_1-k_2, a+b, a-k)$ equals the area of triangle $(a, a+b, a-k_1)$ plus the area of triangle $(a, a-k_1, a-k_1-k_2)$. We have equivalent statements for numbers of integral points in the respective triangles with the integral points on boundaries counted $\frac{1}{2}$. Since $k_1 \in \boldsymbol{Z}^2$, the number of integral points in $(a-k_1-k_2,\ a+b,\ a-k_1)$ equals the number in $(a-k_2, a+(b+k_1), a)$.

Given a parallelogram $(a, a+b, a-k, a-k-b)$ we may consider it as the sum of 2 triangles, $(a, a+b, a-k)$, $(a, a-k, a-k-b)$. Under the transformation $z \mapsto -k-z$, the latter goes to $(-a, -a-k, -a-b)$. So we may think of $[b, k]_a$ as consisting of the sum of two parts, the a-part, being the difference between the area and number of integral points in $(a, a+b, a-k)$ and the $-a$ part being the difference between the area and number of integral points in $(-a, -a+b, -a-k)$. If we now draw the corresponding "$-a$" diagram of (5.3) and draw the corresponding conclusions we reach the statement that $[b, k_1+k_2]_a - [b+k_1, k_2]_a - [b, k_1]_a$ equals the area of $(a,\ a-k_1,\ a-k_1-k_2)$ minus the number of integral points therein plus the area of $(-a, -a-k_1, -a-k_1-k_2)$ minus the number of points therein. But using the transformation $z \mapsto -k_1-z$, we find this quantity equals the area of $(a, a-k_1, a-k_1-k_2, a+k_2)$ minus the number of integral points therein.

Now $(a, a-k_1, a-k_1-k_2, a+k_2)$ with the sides $(a-k_1, a-k_2)$ and $(a-k_1-k_2, a-k_1)$ excluded form a fundamental domain for the subgroup of $\boldsymbol{Z}^2$ generated by (k_1, k_2). Thus the number of integral points therein equals the area. Here we are giving full value to the integral points on two adjacent sides and excluding the other two.

(5.4)



In the above calculation we counted the integral points on the boundary with value $\frac{1}{2}$. But since opposite sides in this parallelogram differ by translation by an element of $\boldsymbol{Z}^2$, the two calculations are identical and $[b, k_1 + k_2]_a - [b + k_1, k_2]_a - [b, k_1]_a = 0$ which proves the proposition.

We may deduce the following general properties of symbols $[b, k]_a$ satisfying (5.1) and (5.2).

COROLLARY 5.3: *Let $[b, k]_a$ take values in an abelian group and satisfy* (5.1) *and* (5.2). *Suppose $na, nb \in \boldsymbol{Z}^2$.* Then if $\gamma \in \Gamma(n)$, $[b, b(\gamma - 1)]_a$ *depends only on the class of b* mod $\boldsymbol{Z}^2$.

PROOF: We must show if $h \in \boldsymbol{Z}^2$, $[b, b(\gamma - 1)]_a = [b + h, (b + h)(\gamma - 1)]_a$. From (5.2)

$$
\begin{aligned}
&[(b+h), (b+h)(\gamma-1)]_a \\
&\quad = [b+h, b(\gamma-1)]_a + [b\gamma + h, h(\gamma-1)]_a \\
&\quad = [b, b(\gamma-1)+h]_a - [b, h]_a + [b\gamma + h, h(\gamma-1)]_a \\
&\quad = [b, b(\gamma-1)]_a + [b\gamma, h]_a - [b, h]_a + [b\gamma + h, h(\gamma-1)]_a \\
&\quad = [b, b(\gamma-1)]_a + [b\gamma, h\gamma]_a - [b, h]_a \\
&\quad = [b, b(\gamma-1)]_a \text{ by (5.1).}
\end{aligned}
$$

We also note that $[b, 0]_a = 0$ from (5.2).

COROLLARY 5.4: *Let $[b, k]_a$ be as in Corollary 5.3. Then if $na, nb \in \boldsymbol{Z}^2$, $\gamma \mapsto [b, b(\gamma - 1)]_a$ is a homomorphism of $\Gamma(n)$ into the value group.*

PROOF: By the above remark that $[b, 0]_a = 0 \mapsto 0$ under this map. So we must only show given $\gamma_1, \gamma_2 \in \Gamma(n)$, $[b, b(\gamma_1\gamma_2 - 1)]_a = [b, b(\gamma_1 - 1)]_a + [b, b(\gamma_2 - 1)]_a$.

Using (5.1) we have

$$
\begin{aligned}
[b, b(\gamma_1\gamma_2 - 1)]_a &= [b, b(\gamma_1\gamma_2 - \gamma_2 + \gamma_2 - 1)]_a \\
&= [b, b(\gamma_2 - 1)]_a + [b\gamma_2, b\gamma_1\gamma_2 - \gamma_2]_a \\
&= [b, b(\gamma_2 - 1)]_a + [b, b(\gamma_1 - 1)]_a \text{ by (5.2).}
\end{aligned}
$$

We may also prove versions of the distribution relations. Suppose we are given $b \in \boldsymbol{Q}^2 - \boldsymbol{Z}^2$, $c \in \boldsymbol{Q}^2 - \boldsymbol{Z}^2$ and a number $N \in \boldsymbol{Z}^+$ such that if $d = Nb$, $c + d$, $c - d$, $d \notin \boldsymbol{Z}^2$. Then we have

$$\prod_{Na \equiv c(\boldsymbol{Z}^2)} g(a, b) = Cg(c, d) \tag{5.5}$$

by the distribution relations where C is a constant. The group-theoretic analogue of this relation is the following. Let n be such that $na, nb \in \boldsymbol{Z}^2$. Then if $\gamma \in \Gamma(n)$ we have

$$\sum_{Na \equiv c(\boldsymbol{Z}^2)} [b, b(\gamma - 1)]_a = [d, d(\gamma - 1)]_c. \tag{5.6}$$

We would like to prove (5.6) geometrically. This is not hard. Given c, d we consider the parallelogram $(c, c + d, c - k, c - d - k)$ where $k = d(\gamma - 1)$. The integral points of $(c, c + d, c - k, c - d - k)$ may be divided into congruence classes mod N. Let $\{\alpha_i\}$ represent these congruence classes. If we let $I(c, c + d, c - k, c - d - k)$ represent the number of integral points of $(c, c + d, c - k, c - d - k)$ we have

$$I(c, c + d, c - k, c - d - k) = \sum_i I_i(c, c + d, c - k, c - d - k) \tag{5.7}$$

where $I_i(c, c + d, c - k, c - d - k)$ represent the number in congruence class i. We let $I_0(c, c + d, c - k, c - d - k)$ designate those points which are congruent to 0 mod N. Then $I_i(c, c + d, c - k, c - d - k) = I_0(c - \alpha_i, c + d - \alpha_i, c - k - \alpha_i, c - d - k - \alpha_i)$ and $I_0(c - \alpha_i, c + d - \alpha_i, c - k - \alpha_i, c - d - k - \alpha_i)$ equals $I(c - \alpha_i/N, c - \alpha_i/N + d/N, c - \alpha_i/N - k/N, c - \alpha_i/N - d/N - k/N)$ and $d/N = b$, $k/N = b(\gamma - 1)$. Combining this with (5.7) and noticing that the areas of the parallelograms behave as they should we obtain (5.6). If $mb \in \boldsymbol{Z}^2$ and if $\gamma \in \Gamma(m)$ both the left and right side of (5.6) are defined and the proof still holds. If also $mc \in \boldsymbol{Z}^2$ we may interpret the right side as $lg(c, d)\gamma(\tau) - lg(c, d)(\tau)$ and thus also the left side.

We conclude with some observations about congruence properties of integral points in parallelograms. If one tries to get an actual formula for $I(a, a + b, a - k, a - b - k)$ one finds it to be rather horrible, involving greatest integer functions left and right. Given a, b, one may ask if one places a congruence condition on k will $I(a, a + b, a - k, a - b - k)$ satisfy a congruence condition. The answer essentially is no. For if it did, say modulo some number M then $g_{a+b}, g_{a-b}/g_a^2$ would have an Mth-root which was fixed by a congruence subgroup of Γ. But (4, IV) via Shimura's Theorem on the integrality of q-expansion of modular forms says this is impossible except in the obvious cases e.g. a or b is a 2-point, $M = 2$ or a and b are 2-points, $M = 4$. Since Leutbecher and Wohlfahrt obtain similar statements by brute force from the explicit formula for the case of the Dedekind eta function, one surmises that a brute force approach could eventually succeed here also.

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