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by

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Let $V[0, 2\pi]$ denote the class of all complex-valued normalized functions F of bounded variation in $[0, 2\pi]$ extended outside this interval by the rule $F(x+2\pi)-F(x) = F(2\pi)-F(0)$ and let $\sum_{-\infty}^{\infty} C_k e^{ikx}$ be the associated Fourier-Stieltjes series. Let $D(x) = F(x+0)-F(x-0)$ denote the jump of F at x for any F in $V[0, 2\pi]$.

Given any infinite matrix $\Lambda = (\lambda_{n,k})$ of complex numbers, a sequence (s_k) is said to be summable Λ if $\lim_{n\to\infty} \sum_{k=0}^{\infty} \lambda_{n,k} s_k$ exists; it is said to be summable F_Λ if $\lim_{n\to\infty} \sum_{k=0}^{\infty} \lambda_{n,k} s_{k+p}$ exists uniformly in $p = 0, 1, 2, \cdots$.

Fejér [1] (cf. also Zygmund [4] p. 107) had proved that the sequence $\{C_k e^{ikx}+C_{-k}e^{-ikx}-\pi^{-1}D(x)\}$ is summable $(C, 1)$ to zero. In [2], we generalized this theorem of Fejér as follows:

THEOREM A. *Let $\Lambda = (\lambda_{n,k})$ be a matrix such that*

$$\sup_{n \geq 0} \sum_{k=0}^{\infty} |\lambda_{n,k}| = M < \infty.$$

Then for every $F \in V[0, 2\pi]$ and for every $x \in [0, 2\pi]$,

(1) $\{C_k e^{ikx}+C_{-k}e^{-ikx}-\pi^{-1}D(x)\}$ *is summable Λ (or F_Λ) to 0 if and only if $\{\cos kt\}$ is summable Λ (resp. F_Λ) to 0 for all $t \not\equiv 0$ (mod 2π);*

(2) $\{C_k e^{ikx}-C_{-k}e^{-ikx}\}$ *is summable Λ (or F_Λ) to 0 if and only if $\{\sin kt\}$ is summable Λ (resp. F_Λ) to 0 for all $t \not\equiv 0$ (mod 2π);*

(3) $\{C_{\pm k}e^{\pm ikx}-(2\pi)^{-1}D(x)\}$ *is summable Λ (or F_Λ) to 0 if and only if $\{e^{ikt}\}$ is summable Λ to 0 for all $t \not\equiv 0$ (mod 2π).*

From theorem A, we deduced the following generalization of a theorem of Wiener [3] (cf. Zygmund [4] p. 108).

THEOREM B. *Let* $\Lambda = (\lambda_{n,k})$ *be a matrix such that*

$$\sup_{n \geqq 0} \sum_{k=0}^{\infty} |\lambda_{n,k}| = M < \infty.$$

Then for every $F \in V[0, 2\pi]$,

(1) $\{|C_k|^2 + |C_{-k}|^2 - (2\pi^2)^{-1} \sum_{j=0}^{\infty} |D(x_j)|^2\}$ *is summable* Λ *(or* F_Λ*) to* 0 *if and only if* $\{\cos kt\}$ *is summable* Λ *(or* F_Λ*) to* 0 *for all* $t \not\equiv 0$ (mod 2π);

(2) $\{|C_k|^2 - |C_{-k}|^2\}$ *is summable* Λ *(or* F_Λ*) to* 0 *if and only if* $\{\sin kt\}$ *is summable* Λ *(or* F_Λ*) to* 0 *for all* $t \not\equiv 0$ (mod 2π);

(3) $\{|C_{\pm k}|^2 - (2\pi)^{-2} \sum_{j=0}^{\infty} |D(x_j)|^2\}$ *is summable* Λ *(or* F_Λ*) to* 0 *if and only if* $\{e^{ikt}\}$ *is summable* Λ *to* 0 *for all* $t \not\equiv 0$ (mod 2π);

$\{x_j\}$ *being the points of discontinuity of* F *in* $[0, 2\pi)$.

In connection with Fejér's theorem which can be reformulated as follows: if $F \in V[0, 2\pi]$ and $x \in [0, 2\pi]$, the sequence

$$\left\{C_k e^{ikx} + C_{-k} e^{-ikx} - \pi^{-1} \sum_{j=0}^{\infty} D(x_j) \cos k(x-x_j)\right\}$$

is summable $(C, 1)$ to zero, we may ask whether the sequence of moduli is also summable $(C, 1)$ to zero. More generally, we may ask whether the moduli of the sequences defined by

$$A_k(x) = C_k e^{ikx} + C_{-k} e^{-ikx} - \pi^{-1} \sum_{j=0}^{\infty} D(x_j) \cos k(x-x_j)$$

$$iB_k(x) = C_k e^{ikx} - C_{-k} e^{-ikx} - i\pi^{-1} \sum_{j=0}^{\infty} D(x_j) \sin k(x-x_j)$$

$$A_k(x) \pm iB_k(x) = 2C_{\pm k} e^{\pm ikx} - \pi^{-1} \sum_{j=0}^{\infty} D(x_j) e^{\pm ik(x-x_j)}$$

are summable Λ or F_Λ to 0 under different hypotheses on the matrix Λ in theorem A.

The main object of this paper is to show that for an expansive class of matrix methods of summability the answer is in affirmative. In particular, under the hypothesis of Fejér's Theorem not only $\{A_k(x)\}$ but even $\{|A_k(x)|\}$ is summable $(C, 1)$ to 0. The results we establish in this connection enable us to give an alternative proof of Theorem B which does not involve the notion of convolution.

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We first prove the following

THEOREM 1. *Let* $\Lambda = (\lambda_{n,k})$ *be a matrix such that*

$$\sup_{n \geq 0} \sum_{k=0}^{\infty} |\lambda_{n,k}| = M < \infty, \ F \in V[0, 2\pi] \ \text{and} \ x \in [0, 2\pi].$$

Then

(1) *if* $\{\cos kt\}$ *is summable* Λ (*or* F_{Λ}) *to* 0 *for all* $t \not\equiv 0 \pmod{2\pi}$, *the sequences* $\{|A_k(x)|^2\}$ *and* $\{|B_k(x)|^2\}$ *are summable* Λ (*resp.* F_{Λ}) *to* 0;

(2) *if* $\{\sin kt\}$ *is summable* Λ (*or* F_{Λ}) *to* 0 *for all* $t \not\equiv 0 \pmod{2\pi}$, *the sequences* $\{A_k(x)\overline{B_k(x)}\}$ *and* $\{\overline{A_k(x)}B_k(x)\}$ *are summable* Λ (*resp.* F_{Λ}) *to* 0;

(3) *if* $\{e^{ikt}\}$ *is summable* Λ *to* 0 *for all* $t \not\equiv 0 \pmod{2\pi}$, *the sequence* $\{|A(x) \pm iB_k(x)|^2\}$ *is summable* Λ (*or* F_{Λ}) *to* 0.

PROOF. Let g be a function defined in $[0, 2\pi]$ as follows:

$$g(t) = 2^{-1}(\pi - t) \quad \text{for} \quad 0 < t < 2\pi,$$
$$g(0) = g(2\pi) = 0,$$

and outside $[0, 2\pi]$ by periodicity. This function is odd and is continuous except at the origin where it has a jump π. If we put

$$H(t) = F(t) - \pi^{-1} \sum_{j=0}^{\infty} D(x_j) g(t - x_j)$$

where $x_0, x_1, \cdots$ are the points of simple discontinuity of F in $[0, 2\pi)$, then $H \in V[0, 2\pi]$ and is continuous everywhere. It is easily seen that

$$A_k(x) = C_k e^{ikx} + C_{-k} e^{-ikx} - \pi^{-1} \sum_{j=0}^{\infty} D(x_j) \cos k(x - x_j)$$
$$= \pi^{-1} \int_0^{2\pi} \cos k(x - t) \, dH(t)$$

$$iB_k(x) = C_k e^{ikx} - C_{-k} e^{-ikx} - \pi^{-1} i \sum_{j=0}^{\infty} D(x_j) \sin k(x - x_j)$$
$$= i\pi^{-1} \int_0^{2\pi} \sin k(x - t) \, dH(t)$$

and

$$A_k(x) \pm iB_k(x) = 2C_{\pm k} e^{\pm ikx} - \pi^{-1} \sum_{j=0}^{\infty} D(x_j) e^{\pm ik(x - x_j)}$$
$$= \pi^{-1} \int_0^{2\pi} e^{\pm ik(x - t)} \, dH(t).$$

We have

$$\sum_{k=0}^{\infty} \lambda_{n,k} |A_k(x)|^2 = \pi^{-1} \int_0^{2\pi} K_{n1}(t) \overline{dH(t)},$$

$$\sum_{k=0}^{\infty} \lambda_{n,k} |B_k(x)|^2 = \pi^{-1} \int_0^{2\pi} K_{n2}(t) \overline{dH(t)},$$

$$\sum_{k=0}^{\infty} \lambda_{n,k} A_k(x) \overline{B_k(x)} = \pi^{-1} \int_0^{2\pi} K_{n3}(t) \overline{dH(t)},$$

$$\sum_{k=0}^{\infty} \lambda_{n,k} \overline{A_k(x)} B_k(x) = \pi^{-1} \int_0^{2\pi} K_{n4}(t) \overline{dH(t)},$$

and

$$\sum_{k=0}^{\infty} \lambda_{n,k} |A_k(x) \pm i B_k(x)|^2 = \pi^{-1} \int_0^{2\pi} K_{n5}(t) \overline{dH(t)}$$

where

$$K_{n1}(t) = (2\pi)^{-1} \int_0^{2\pi} \sum_{k=0}^{\infty} \lambda_{n,k} \{\cos k(u-t) + \cos k(u+t-2x)\} dH(u),$$

$$K_{n2}(t) = (2\pi)^{-1} \int_0^{2\pi} \sum_{k=0}^{\infty} \lambda_{n,k} \{\cos k(u-t) - \cos k(u+t-2x)\} dH(u),$$

$$K_{n3}(t) = (2\pi)^{-1} \int_0^{2\pi} \sum_{k=0}^{\infty} \lambda_{n,k} \{\sin k(u-t) - \sin k(u+t-2x)\} dH(u),$$

$$K_{n4}(t) = -(2\pi)^{-1} \int_0^{2\pi} \sum_{k=0}^{\infty} \lambda_{n,k} \{\sin k(u-t) + \sin k(u+t-2x)\} dH(u)$$

and

$$K_{n5}(t) = \pi^{-1} \int_0^{2\pi} \sum_{k=0}^{\infty} \lambda_{n,k} e^{\mp ik(u-t)} dH(u).$$

Suppose now that $\{\cos kt\}$ is summable Λ to 0 for all $t \not\equiv 0$ (mod 2π). The integral

$$\int_0^{2\pi} \left\{ \sum_{k=0}^{\infty} \lambda_{n,k} \cos k(u-t) \right\} dH(u)$$

can be written as the sum of two integrals: the first over an interval $[t-\delta, t+\delta]$ where the total variation of H is less than ε and the second over the remainder. The first integral does not exceed $M\varepsilon$ in absolute value whereas the second tends to 0 as $n \to \infty$ by bounded convergence theorem. Thus

$$\lim_{n \to \infty} \int_0^{2\pi} \left\{ \sum_{k=0}^{\infty} \lambda_{n,k} \cos k(u-t) \right\} dH(u) = 0$$

and similarly

$$\lim_{n \to \infty} \int_0^{2\pi} \left\{ \sum_{k=0}^{\infty} \lambda_{n,k} \cos k(u+t-2x) \right\} dH(u) = 0.$$

It follows that

$$\lim_{n\to\infty} K_{n1}(t) = \lim_{n\to\infty} K_{n2}(t) = 0$$

for all t and for all x. Consequently, again, applying bounded convergence theorem, we get

$$\lim_{n\to\infty} \int_0^{2\pi} K_{np}(t)\,d\bar{H}(t) = 0 \quad \text{for} \quad p = 1, 2,$$

so that $\{|A_k(x)|^2\}$ and $\{|B_k(x)|^2\}$ are summable Λ to 0. Similarly we prove that if $\{\cos kt\}$ is summable F_Λ to 0 for all $t \not\equiv 0 \pmod{2\pi}$, these sequences are summable F_Λ to 0. This completes the proof of (1). The proofs of (2) and (3) follow the same line.

If $\Lambda = (\lambda_{n,k})$ is a given matrix, we denote by $|\Lambda| = (|\lambda_{n,k}|)$. From theorem 1, we easily deduce the following

THEOREM 2. *Let* $\Lambda = (\lambda_{n,k})$ *be a matrix such that*

$$\sup_{n\geq 0} \sum_{k=0}^{\infty} |\lambda_{n,k}| = M < \infty, \quad F \in V[0, 2\pi] \quad \text{and} \quad x \in [0, 2\pi].$$

Then

(1) *if* $\{\cos kt\}$ *is summable* $|\Lambda|$ *to* 0 *for all* $t \not\equiv 0 \pmod{2\pi}$, *the sequences* $\{|A_k(x)|\}$ *and* $\{|B_k(x)|\}$ *are summable* $|\Lambda|$ *and consequently summable* Λ *to* 0,

(2) *if* $\{\cos kt\}$ *is summable* $F_{|\Lambda|}$ *to* 0 *for all* $t \not\equiv 0 \pmod{2\pi}$, *the sequences* $\{|A_k(x)|\}$, *and* $\{|B_k(x)|\}$ *are summable* $F_{|\Lambda|}$ *to* 0 *and consequently summable* F_Λ *to* 0.

In fact, applying theorem 1(1) we conclude in the first case that $\{|A_k(x)|^2\}$ and $\{|B_k(x)|^2\}$ are summable $|\Lambda|$ to 0 and by Schwarz inequality we conclude that $\{|A_k(x)|\}$ and $\{|B_k(x)|\}$ are summable $|\Lambda|$ to 0 establishing theorem 2(1). Similarly we prove theorem 2(2).

If we choose $\lambda_{n,k} = 1/(n+1)$ for $k = 0, 1, \cdots, n$ and $\lambda_{n,k} = 0$ for $k > n$ then we obtain the method of arithmetic mean which clearly satisfies the two hypotheses of the above theorem and we obtain the following strengthened version of Fejér's theorem.

"If $f \in V[0, 2\pi]$ and $x \in [0, 2\pi]$, then

$$\lim_{n\to\infty} (n+1)^{-1} \sum_{p}^{n+p} |C_k e^{ikx} + C_{-k} e^{-ikx} - \pi^{-1} \sum_{j=0}^{\infty} D(x_j) \cos k(x-x_j)| = 0$$

uniformly in $p = 0, 1, 2 \cdots$."

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We now prove the following

THEOREM 3. *Let* $\Lambda = (\lambda_{n,k})$ *be a matrix such that*

$$\sup_{n \geq 0} \sum_{k=0}^{\infty} |\lambda_{n,k}| = M < \infty, \; F \in V[0, 2\pi] \quad \text{and} \quad x \in [0, 2\pi].$$

Then

(1) *if* $\{\cos kt\}$ *is summable* Λ (*or* F_Λ) *to* 0 *for all* $t \not\equiv 0$ (mod 2π), *the sequences*

$$\{|C_k e^{ikx} + C_{-k} e^{-ikx}|^2 - \pi^{-2}| \sum_{j=0}^{\infty} D(x_j) \cos k(x - x_j)|^2\}$$

and

$$\{|C_k e^{ikx} - C_{-k} e^{-ikx}|^2 - \pi^{-2}| \sum_{j=0}^{\infty} D(x_j) \sin k(x - x_j)|^2\}$$

are summable Λ (*resp.* F_Λ) *to* 0;

(2) *if* $\{\sin kt\}$ *is summable* Λ (*or* F_Λ) *to* 0 *for all* $t \not\equiv 0$ (mod 2π), *the sequences*

$$\{(C_k e^{ikx} + C_{-k} e^{-ikx})(\bar{C}_k e^{-ikx} - \bar{C}_{-k} e^{ikx})\}$$

and

$$\{(\bar{C}_k e^{-ikx} + \bar{C}_{-k} e^{ikx})(C_k e^{ikx} - C_{-k} e^{-ikx})\}$$

are summable Λ (*resp.* F_Λ) *to* 0;

(3) *if* $\{e^{ikt}\}$ *is summable* Λ *to* 0 *for all* $t \not\equiv 0$ (mod 2π), *the sequences*

$$\{|C_{\pm k}|^2 - (2\pi)^{-2}| \sum_{j=0}^{\infty} D(x_j) e^{\mp ikx_j}|^2\}$$

are summable F_Λ (*and a fortiori summable* Λ) *to* 0.

PROOF: Suppose that $\{\cos kt\}$ is summable Λ to 0 for all $t \not\equiv 0$ (mod 2π). We have

$$\sum_{k=0}^{\infty} \lambda_{n,k} |A_k(x)|^2$$

$$= \sum_{k=0}^{\infty} \lambda_{n,k} \{|C_k e^{ikx} + C_{-k} e^{-ikx}|^2 - \pi^{-2}| \sum_{j=0}^{\infty} D(x_j) \cos k(x - x_j)|^2\}$$

$$- (2\pi^2)^{-1} \sum_{j=0}^{\infty} \overline{D(x_j)} \int_0^{2\pi} \sum_{k=0}^{\infty} \lambda_{n,k} \{\cos k(t - x_j)$$
$$+ \cos k(2x - t - x_j)\} dH(t)$$

$$- (2\pi^2)^{-1} \sum_{j=0}^{\infty} D(x_j) \int_0^{2\pi} \sum_{k=0}^{\infty} \lambda_{n,k} \{\cos k(t - x_j)$$
$$+ \cos k(2x - t - x_j)\} d\bar{H}(t).$$

Since by theorem 1(1), the term on the left tends to zero with $1/n$ and the last two terms tend to zero with $1/n$ by a reasoning used in the proof of theorem 1(1), the first term on the right tends to zero with $1/n$. Similarly using the fact that

$$\sum_{k=0}^{\infty} \lambda_{n,k}|B_k(x)|^2$$

$$= \sum_{k=0}^{\infty} \lambda_{n,k}\{|C_k e^{ikx} - C_{-k} e^{-ikx}|^2 - \pi^{-2}|\sum_{j=0}^{\infty} D(x_j) \sin k(x-x_j)|^2\}$$

$$- (2\pi^2)^{-1} \sum_{j=0}^{\infty} \overline{D(x_j)} \int_0^{2\pi} \sum_{k=0}^{\infty} \lambda_{n,k}\{\cos k(t-x_j)$$
$$- \cos k(2x-t-x_j)\}dH(t)$$

$$- (2\pi^2)^{-1} \sum_{j=0}^{\infty} D(x_j) \int_0^{2\pi} \sum_{k=0}^{\infty} \lambda_{n,k}\{\cos k(t-x_j)$$
$$- \cos k(2x-t-x_j)\}d\overline{H}(t),$$

we prove that the other sequence is also summable Λ to 0. The case of summability F_Λ can be treated analogously. Thus (1) holds.

Suppose that $\{\sin kt\}$ is summable Λ to 0 for all $t \not\equiv 0 \pmod{2\pi}$. We have

$$-i \sum_{k=0}^{\infty} \lambda_{n,k} A_k(x)\overline{B_k(x)}$$

$$= \sum_{k=0}^{\infty} \lambda_{n,k}(C_k e^{ikx} + C_{-k} e^{-ikx})(\bar{C}_k e^{-ikx} - \bar{C}_{-k} e^{ikx})$$

$$+ i(2\pi^2)^{-1} \sum_{j=0}^{\infty} \overline{D(x_j)} \int_0^{2\pi} \sum_{k=0}^{\infty} \lambda_{n,k}\{\sin k(2x-t-x_j)$$
$$+ \sin k(t-x_j)\}dH(t)$$

$$+ i(2\pi^2)^{-1} \sum_{j=0}^{\infty} D(x_j) \int_0^{2\pi} \sum_{k=0}^{\infty} \lambda_{n,k}\{\sin k(2x-t-x_j)$$
$$- \sin k(t-x_j)\}d\overline{H}(t)$$

$$+ i(2\pi^2)^{-1} \sum_{j,l=0}^{\infty} D(x_j)\overline{D(x_l)} \sum_{k=0}^{\infty} \lambda_{n,k}\{\sin k(2x-x_l-x_j)$$
$$- \sin k(x_l-x_j)\}.$$

As in the above case, we see that the term on the left and the last three terms tend to zero with $1/n$ and hence also the first term on the right. Thus the sequence $\{(C_k e^{ikx} + C_{-k} e^{-ikx}) \times (\bar{C}_k e^{-ikx} - \bar{C}_{-k} e^{ikx})\}$ and similarly the sequence $\{(\bar{C}_k e^{-ikx} + \bar{C}_{-k} e^{ikx}) \times (C_k e^{ikx} - C_{-k} e^{-ikx})\}$ are summable Λ to 0. The case of summability F_Λ is analogous. This proves (2).

Finally suppose that $\{e^{ikt}\}$ is summable Λ to 0 for all $t \not\equiv 0 \pmod{2\pi}$. We have

$$\sum_{k=0}^{\infty} \lambda_{n,k}|A_k(x) \pm iB_k(x)|^2 = \sum_{k=0}^{\infty} \lambda_{n,k}\{4|C_{\pm k}|^2 - \pi^{-2}|\sum_{j=0}^{\infty} D(x_j)e^{\mp ikx_j}|^2\}$$

$$- \pi^{-2} \sum_{j=0}^{\infty} \overline{D(x_j)} \int_0^{2\pi} (\sum_{k=0}^{\infty} \lambda_{n,k} e^{\pm ik(x_j-t)})dH(t)$$

$$- \pi^{-2} \sum_{j=0}^{\infty} D(x_j) \int_0^{2\pi} (\sum_{k=0}^{\infty} \lambda_{n,k} e^{\mp ik(x_j-t)})d\overline{H}(t).$$

As in the preceding cases, we see that the term on the left and the last two terms on the right tend to zero with $1/n$ and hence also the first term on the right. This proves the summability $\varLambda$ of the sequence. Since summability $\varLambda$ to zero of $\{e^{ikt}\}$ implies summability $F_\varLambda$ to 0, a repetition of the above argument shows that actually the sequence is summable $F_\varLambda$ to 0. This proves (3).

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We now use theorem 3 to give an alternative proof of theorem B. Suppose that $\{\cos kt\}$ is summable $\varLambda$ to 0 for all $t \not\equiv 0 \pmod{2\pi}$. By theorem 3(1)

$$(1)\quad \lim_{n\to\infty} \sum_{k=0}^{\infty} \lambda_{n,k}\{|C_k e^{ikx}+C_{-k}e^{-ikx}|^2-\pi^{-2}|\sum_{j=0}^{\infty} D(x_j)$$
$$\cos k(x-x_j)|^2\} = 0$$

and

$$(2)\quad \lim_{n\to\infty} \sum_{k=0}^{\infty} \lambda_{n,k}\{|C_k e^{ikx}-C_{-k}e^{-ikx}|^2-\pi^{-2}|\sum_{j=0}^{\infty} D(x_j)$$
$$\sin k(x-x_j)|^2\} = 0.$$

Adding (1) and (2), we get

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} \lambda_{n,k}\{2(|C_k|^2+|C_{-k}|^2)-\pi^{-2}\sum_{j=0}^{\infty} |D(x_j)|^2\} = 0.$$

Suppose that $\{\sin kt\}$ is summable $\varLambda$ to 0 for all $t \not\equiv 0 \pmod{2\pi}$. By theorem 3(2), we have

$$(3)\quad \lim_{n\to\infty} \sum_{k=0}^{\infty} \lambda_{n,k}\{(C_k e^{ikx}+C_{-k}e^{-ikx})(\bar{C}_k e^{-ikx}-\bar{C}_{-k}e^{ikx})\} = 0$$

and

$$(4)\quad \lim_{n\to\infty} \sum_{k=0}^{\infty} \lambda_{n,k}\{(\bar{C}_k e^{-ikx}+\bar{C}_{-k}e^{ikx})(C_k e^{ikx}-C_{-k}e^{-ikx})\} = 0.$$

Adding (3) and (4) we get

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} \lambda_{n,k}(|C_k|^2-|C_{-k}|^2) = 0.$$

Suppose that $\{e^{ikt}\}$ is summable $\varLambda$ to 0 for all $t \not\equiv 0 \pmod{2\pi}$. By theorem 3(3), we have

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} \lambda_{n,k}\{|C_{\pm k}|^2-(2\pi)^{-2}|\sum_{j=0}^{\infty} D(x_j)e^{\mp ikx_j}|^2\} = 0$$

which easily reduces to

$$\lim_{n \to \infty} \sum_{k=0}^{\infty} \lambda_{n,k}\{|C_{\pm k}|^2 - (2\pi)^{-2} \sum_{j=0}^{\infty} |D(x_j)|^2\} = 0.$$

This proves the sufficiency parts of (1), (2), and (3) for summability, Λ the corresponding propositions for summability F_Λ can be proved similarly.

The necessity of (1) and (2) is established by choosing respectively

(1) $F(x) = 0$ for $0 < x < t/2$, $F(x) = 1$ for $t/2 < x < 2\pi - t/2$ and $F(x) = 0$ for $2\pi - t/2 < x < 2\pi$ for which $\pi C_k = -i \sin kt/2$ and $|D(t/2)| = |D(2\pi - t/2)| = 1$ and

(2) $F(x) = f(x) + i2g(x)$ where $f(x) = \pi$ for $t < x < 2\pi - t$ and $f(x) = 0$ for $0 < x < t$ and $2\pi - t < x < 2\pi$ and g as defined above so that $C_k = -i \sin kt + i$. The necessity of (3) follows from the fact that if $\{|C_k|^2 - (2\pi)^{-2} \sum_{j=0}^{\infty} |D(x_j)|^2\}$ is summable Λ or (F_Λ) to 0 for all $F \in V[0, 2\pi]$ then so is $\{|C_{-k}|^2 - (2\pi)^{-2} \sum_{j=0}^{\infty} |D(x_j)|^2\}$. It follows that for each $F \in V[0, 2\pi]$ the sequences

$$\{|C_k|^2 + |C_{-k}|^2 - (2\pi^2)^{-1} \sum_{j=0}^{\infty} |D(x_j)|^2\}$$

and $\{|C_k|^2 - |C_{-k}|^2\}$ are summable Λ (resp. F_Λ) to 0 and consequently by the necessity parts of (1) and (2), $\{\cos kt\}$ and $\{\sin kt\}$ and hence $\{e^{ikt}\}$ are summable Λ to 0 for all $t \not\equiv 0$ (mod 2π). This completes the proof of theorem B.

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