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Davit Harutyunyan

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Metric Space Analysis / *Analyse sur les espaces métriques*

On the Geometric Rigidity interpolation estimate in thin bi-Lipschitz domains

Davit Harutyunyan [a]

[a] Department of Mathematics, University of California Santa Barbara, USA
E-mail: harutyunyan@ucsb.edu

Abstract. This work is concerned with developing asymptotically sharp geometric rigidity estimates in thin domains. A thin domain Ω in space is roughly speaking a shell with non-constant thickness around a regular enough two dimensional compact surface. We prove a sharp geometric rigidity interpolation inequality that permits one to bound the L^p distance of the gradient of a $\boldsymbol{u} \in W^{1,p}$ field from any constant proper rotation $\boldsymbol{R}$, in terms of the average L^p distance (nonlinear strain) of the gradient from the rotation group, and the average L^p distance of the field itself from the set of rigid motions corresponding to the rotation $\boldsymbol{R}$. The constants in the estimate are sharp in terms of the domain thickness scaling. If the domain mid-surface has a constant sign Gaussian curvature then the inequality reduces the problem of estimating the gradient $\nabla\boldsymbol{u}$ in terms of the nonlinear strain $\int_\Omega \operatorname{dist}^p(\nabla\boldsymbol{u}(x), SO(3))\mathrm{d}x$ to the easier problem of estimating only the vector field $\boldsymbol{u}$ in terms of the nonlinear strain with no asymptotic loss in the constants. This being said, the new interpolation inequality reduces the problem of proving "any" geometric one well rigidity problem in thin domains to estimating the vector field itself instead of the gradient, thus reducing the complexity of the problem.

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1. Introduction

Let $S \subset \mathbb{R}^3$ be a bounded connected regular enough[1] surface (while the standard regularity that we will impose on S will be bi-Lipschitz, we will also impose a stronger piecewise C^2 regularity on S for the purpose of construction of an Ansatz). Given a small parameter $h > 0$, a shell S_h of thickness h is the $h/2$ neighborhood of S in the normal direction, i.e., $S^h = \{x + t\boldsymbol{n}(x) \ : \ x \in S, \ t \in (-h/2, h/2)\}$, where for any point $x \in S$, the vector $\boldsymbol{n}(x)$ is the unit normal to S at x. Thin spatial domains are roughly shells with non-constant thickness. Namely, assume again $h > 0$ is a small parameter and the functions $g_1^h(x), g_2^h(x) \colon S \to (0,\infty)$ are order h Lipschitz functions, i.e., they fulfill the uniform conditions

$$h \le g_1^h(x), g_2^h(x) \le c_1 h, \quad \text{and} \quad |\nabla g_1^h(x)| + |\nabla g_2^h(x)| \le c_2 h, \quad \text{for a.e.} \quad x \in S, \tag{1}$$

[1] So that it has a unit normal a.e.

for some fixed constants $c_1, c_2 > 0$. Then the set Ω^h given by

$$\Omega^h = \{x + t\boldsymbol{n}(x) \ : \ x \in S, \ t \in (-g_1^h(x), g_2^h(x))\}, \tag{2}$$

is a thin domain with (possibly varying) thickness of order h. Determining the rigidity of thin domains is a central task in nonlinear elasticity. The problem has been solved for plates only[2] by Friesecke, James and Müller [3]. The term "rigidity" here refers to the geometric rigidity of a thin domain, which despite the physical intuition, seems to have no mathematical definition at present. A natural way of defining it would be through the celebrated geometric rigidity estimate of Friesecke, James and Müller, which reads as follows: *Assume* $\Omega \subset \mathbb{R}^3$ *is open bounded connected and Lipschitz. Then there exists a constant* $C_I = C_I(\Omega)$, *such that for every vector field* $\boldsymbol{u} \in H^1(\Omega)$, *there exists a constant proper rotation* $\boldsymbol{R} \in SO(3)$, *such that*

$$\|\nabla \boldsymbol{u} - \boldsymbol{R}\|^2_{L^2(\Omega)} \leq C_I \int_\Omega \operatorname{dist}^2(\nabla \boldsymbol{u}(x), SO(3)) \mathrm{d}x. \tag{3}$$

It is known that for thin domains Ω, the constant C_I in (3) blows up as $h \to 0$ and it typically has the asymptotic form $C_I = ch^\alpha$ for some $c > 0$ and $\alpha < 0$. In particular, in the case when Ω is a plate, i.e., $\Omega = \omega \times (-h/2, h/2)$ for some open bounded connected set $\omega \subset \mathbb{R}^2$ with Lipschitz boundary, then $C_I = ch^{-2}$ in (3), see [3]. Given now any thin domain $\Omega \subset \mathbb{R}^n$, it would be mathematically adequate to say that the geometric rigidity of Ω is the exponent α in $C = ch^\alpha$ provided it exists as $h \to 0$. There is a large amount of evidence in the literature pointing to the yet unproven fact that α should depend only on the domain mid-surface S, while the constant c will depends also on the constants c_1 and c_2 in (1), we refer the reader to [1–5] for details. The present work is concerned with studying (3) for any thin domains $\Omega \subset \mathbb{R}^3$. Namely, we prove an interpolation version of (3), which reduces the problem to the estimation of the deviation of the vector field $\boldsymbol{u}$ itself (not the gradient) from the group of rigid motions. This is apparently a significant reduction of the complexity of the problem, taking into account the fact, that in the case of uniformly positive or negative Gaussian curvature thin domains (this refers to the Gaussian curvature of the mid-surface S), no asymptotic loss of the constant is expected as the lower bounds and the Ansätze in [5] suggest. A somewhat more detailed observation appears in the next section.

2. Main result

Theorem 1. *Let* $S \subset \mathbb{R}^3$ *be a connected compact bi-Lipschitz surface with nonempty relative interior, and let* $h \in (0,1)$. *Assume the family of Lipschitz functions* $g_1^h, g_2^h \colon S \to (0,\infty)$ *fulfills the uniform conditions*

$$h \leq g_1^h(x), g_2^h(x) \leq c_1 h, \quad \text{and} \quad |\nabla g_1^h(x)| + |\nabla g_2^h(x)| \leq c_2 h, \quad \text{for a.e.} \quad x \in S, \tag{4}$$

for some constants $c_1, c_2 > 0$, *and denote the family of thin domains*

$$\Omega^h = \{x + t\boldsymbol{n}(x) \ : \ x \in S, \ t \in (-g_1^h(x), g_2^h(x))\}.$$

Let $1 < p < \infty$ *and let* $\|\cdot\|_p$ *denote the* $L^p(\Omega^h)$ *norm. Then there exists constants* $C, h_0 > 0$, *depending only on* p, S *and the constants* $c_1, c_2 > 0$ *such that for any vector field* $\boldsymbol{u} \in W^{1,p}(\Omega)$, *any proper rotation* $\boldsymbol{R} \in SO(3)$, *and any constant vector* $\boldsymbol{b} \in \mathbb{R}^3$ *one has the estimate*

$$\|\nabla \boldsymbol{u} - \boldsymbol{R}\|_p^2 \leq C\left(\frac{\|\boldsymbol{u} - \boldsymbol{R}x - \boldsymbol{b}\|_p \|\operatorname{dist}(\nabla \boldsymbol{u}, SO(3))\|_p}{h} + \|\boldsymbol{u} - \boldsymbol{R}x - \boldsymbol{b}\|_p^2 + \|\operatorname{dist}(\nabla \boldsymbol{u}, SO(3))\|_p^2\right), \tag{5}$$

for all $h \in (0, h_0)$. *Moreover, if in addition* S *has a patch that is* C^2*–regular, then the exponent of* h *in the inequality* (5) *is optimal for* Ω^h, *i.e., there exists a family of deformations* $\boldsymbol{u}^h \in W^{1,p}(\Omega, \mathbb{R}^3)$ *realizing the asymptotics of* h *in* (5).

[2] Or for shells that have a flat part

Some remarks are in order. Note first that for any given displacement $\boldsymbol{v}\in C^2(\Omega^h,\mathbb{R}^3)$, taking the sequence of deformations $\boldsymbol{u}^\epsilon=\boldsymbol{x}+\epsilon\boldsymbol{v}$, we have for small enough ϵ by linearizing at the identity matrix the point-wise approximate formula:

$$\operatorname{dist}(\nabla\boldsymbol{u}^\epsilon(x),SO(3))=\operatorname{dist}(\boldsymbol{I}+\epsilon\nabla\boldsymbol{v}(x),SO(3))\approx\frac{\epsilon}{2}|\nabla\boldsymbol{v}(x)+\nabla\boldsymbol{v}^T(x)|=\epsilon|e(\boldsymbol{v}(x))|,\quad x\in\Omega^h,$$

thus taking $\boldsymbol{R}=\boldsymbol{I}$ and $\boldsymbol{b}=0$ in (5) and then letting ϵ go to zero we derive from (5) the estimate

$$\|\nabla\boldsymbol{v}\|_p^2\leq C\left(\frac{\|\boldsymbol{u}\|_p\|e(\boldsymbol{v})\|_p}{h}+\|\boldsymbol{v}\|_p^2+\|e(\boldsymbol{v})\|_p^2\right)\quad\text{for all }\ v\in C^2(\Omega^h,\mathbb{R}^3),\tag{6}$$

where $e(\boldsymbol{v})=\frac{1}{2}(\nabla\boldsymbol{v}+\nabla\boldsymbol{v}^T)$ is the linear strain. Of course (6) holds for all $v\in W^{1,p}(\Omega^h,\mathbb{R}^3)$ too by density. The estimate (6) is a Korn interpolation inequality and is the linear version of (5). A slightly stronger version of (6) has been proven in [6, 7] for the L^2 norm, where in place of the product $\|\boldsymbol{u}\|_2\|e(\boldsymbol{v})\|_2$ one has $\|\boldsymbol{n}\cdot\boldsymbol{u}\|_2\|e(\boldsymbol{v})\|_2$, i.e., only the normal component of the field enters the estimate.

Let now K denote the Gaussian curvature of S. Tovstik and Smirnov have constructed an Ansatz [8] that realizes the asymptotics $\alpha=-4/3$ in the constant $C=ch^\alpha$ in (3) in the case $K<0$. Also the author has constructed an Ansatz [7] that gives the asymptotics $\alpha=-1$ in (3) in the case $K>0$. Furthermore, it has been proven in [7] that if zero boundary condition is imposed on the vector field $\boldsymbol{u}$ on the thin face of the thin domain Ω^h, then in the linear version of (3), i.e., in Korn's first inequality

$$\|\nabla\boldsymbol{u}\|^2_{L^2(\Omega^h)}\leq ch^\alpha\|e(\boldsymbol{u})\|^2_{L^2(\Omega^h)}$$

one indeed has $\alpha=-4/3$ and $\alpha=-1$ in the cases $K<0$ and $K>0$ respectively, (see [7] for details). Also, it has been shown [7] that

$$\|\boldsymbol{u}\|^2_{L^2(\Omega^h)}\leq Ch^\beta\|e(\boldsymbol{u})\|^2_{L^2(\Omega^h)},$$

where $\beta=-1/3$ for $K<0$ and $\beta=0$ for $K>0$. This implies that the estimate (5) indeed does not suffer an asymptotic loss of constants (this means that when one utilizes the interpolation inequality to prove that regular geometric rigidity estimate, the constant obtained in the latter one is still asymptotically optimal) at least in the cases $K><0$, and thus from this point on, the interpolation estimate (5) (and the linear analog (6)) can be utilized for the purpose of proving asymptotically optimal rigidity estimates on gradient fields, where one needs to only estimate the vector field in terms of the energy, instead of the gradient.

3. Proof of the main result

Proof of Theorem 1. We divide the proof into several steps for the convenience of the reader.

Step 1. We first somewhat simplify the estimate (5). Namely, first of all a translation by a fixed vector $\boldsymbol{u}=\boldsymbol{v}+\boldsymbol{b}$ does not change the gradient, thus we can assume without loss of generality that $\boldsymbol{b}=0$. Next, denoting $\boldsymbol{u}=\boldsymbol{R}\boldsymbol{v}$, the left hand side of (5) will become $\|\boldsymbol{v}-\boldsymbol{I}\|_p^2$, and the right hand side of (5) will remain the same expression written out for $\boldsymbol{v}$ in place of $\boldsymbol{u}$. This being said we can assume without loss of generality that $\boldsymbol{R}=\boldsymbol{I}$ and $\boldsymbol{b}=0$ in (5). Finally, making a change of variables $\boldsymbol{u}=\boldsymbol{v}+\boldsymbol{x}$ will transform the new form of (5) to the estimate

$$\|\nabla\boldsymbol{v}\|_p^2\leq C\left(\frac{\|\boldsymbol{v}\|_p\|\operatorname{dist}(\nabla\boldsymbol{v}+\boldsymbol{I},SO(3))\|_p}{h}+\|\boldsymbol{v}\|_p^2+\|\operatorname{dist}(\nabla\boldsymbol{v}+\boldsymbol{I},SO(3))\|_p^2\right)\tag{7}$$

to be now proven.

Step 2. In the second step we prove the following statement: *Under the conditions of Theorem 1, the estimate* (7) *holds if and inly if one has for any field* $\boldsymbol{v}\in W^{1,p}(\Omega^h,\mathbb{R}^3)$ *the estimate*

$$\|\nabla\boldsymbol{v}\|_p^2\le C_1\left(\frac{\|\boldsymbol{v}\|_p^2}{h^t}+\frac{\|\mathrm{dist}(\nabla\boldsymbol{v}+\boldsymbol{I},SO(3))\|_p^2}{h^{2-t}}\right),\tag{8}$$

for any $t\in[0,2]$. *Here* $C_1>0$, *and* C *and* C_1 *in* (7) *and* (8) *are comparable, namely,* $1/2\le\frac{C_1}{C}\le 2$. Evidently as $h>0$ is small, we have $h^t,h^{2-t}\le 1$ for $t\in[0,2]$ and (7) implies (8) by the arithmetic-geometric mean inequality with $C_1=\frac{3}{2}C$. Assume now (8) holds. Given the fixed vector field $\boldsymbol{v}\in W^{1,p}(\Omega^h,\mathbb{R}^3)$, if $\frac{\|\boldsymbol{v}\|_p^2}{h^{t_0}}=\frac{\|\mathrm{dist}(\nabla\boldsymbol{v}+\boldsymbol{I},SO(3))\|_p^2}{h^{2-t_0}}$ for some $t_0\in[0,2]$, then we choose $t=t_0$ in (8) and get the estimate

$$\|\nabla\boldsymbol{v}\|_p^2\le 2C_1\frac{\|\boldsymbol{v}\|_p\|\mathrm{dist}(\nabla\boldsymbol{v}+\boldsymbol{I},SO(3))\|_p}{h}.$$

If $\frac{\|\boldsymbol{v}\|_p^2}{h^t}<\frac{\|\mathrm{dist}(\nabla\boldsymbol{v}+\boldsymbol{I},SO(3))\|_p^2}{h^{2-t}}$ for any $t\in[0,2]$ then (8) implies by choosing $t=2$ in (8) the bound

$$\|\nabla\boldsymbol{v}\|_p^2\le 2C_1\|\mathrm{dist}(\nabla\boldsymbol{v}+\boldsymbol{I},SO(3))\|_p^2,$$

which in tern yields (7) with $C=2C_1$. The case $\frac{\|\boldsymbol{v}\|_p^2}{h^t}>\frac{\|\mathrm{dist}(\nabla\boldsymbol{v}+\boldsymbol{I},SO(3))\|_p^2}{h^{2-t}}$ for any $t\in[0,2]$ is analogous.

Step 3. Now we focus our attention to the simplified estimate (8) with no product terms. Fix $\gamma\in[0,1]$. We divide the shell Ω^h into small compact shells with in-plane size of order h^γ. Denoting $m=[1/h^\gamma]+1$, we have $N=O(m^2)$ shells $\Omega_1^h,\Omega_2^h,\dots,\Omega_N^h$, with thickness of order h and in plane size roughly h^γ. The constant C_I in the estimate (3) is in fact invariant under change of variables $x'=\lambda x$, with $\lambda>0$, and it depends only on the Lipschitz constant of the boundary $\partial\Omega$. Thus by choosing the partition of Ω^h such that all composing parts Ω_i^h have a uniform Lipschitz constant, we get by the estimate (3) for the vector field $\boldsymbol{v}+\boldsymbol{x}$,

$$\|\nabla\boldsymbol{v}+\boldsymbol{I}-\boldsymbol{R}_i\|_{L^p(\Omega_i^h)}\le\frac{C}{h^{1-\gamma}}\|\mathrm{dist}(\nabla\boldsymbol{v}+\boldsymbol{I},SO(3))\|_{L^p(\Omega_i^h)},\quad i=1,2,\dots,N,\tag{9}$$

for some rotations $\boldsymbol{R}_i\in SO(3)$ and some uniform constant $C>0$ that depends only on S and the constants c_1 and c_2 in (4). Consequently we obtain from (9) the bound

$$\|\nabla\boldsymbol{v}\|_{L^p(\Omega_i^h)}\le\|\boldsymbol{I}-\boldsymbol{R}_i\|_{L^p(\Omega_i^h)}+\frac{C}{h^{1-\gamma}}\|\mathrm{dist}(\nabla\boldsymbol{v}+\boldsymbol{I},SO(3))\|_{L^p(\Omega_i^h)},\quad i=1,2,\dots,N.\tag{10}$$

Denoting $\boldsymbol{b}_i=\fint_{\omega_i^h}(\boldsymbol{v}(x)+(\boldsymbol{I}-\boldsymbol{R}_i)x)\mathrm{d}x$, we have by the Poincaré inequality and (9),

$$\begin{aligned}\|\boldsymbol{v}+(\boldsymbol{I}-\boldsymbol{R}_i)x-\boldsymbol{b}_i\|_{L^p(\Omega_i^h)}&\le Ch^\gamma\|\nabla\boldsymbol{v}+\boldsymbol{I}-\boldsymbol{R}_i\|_{L^p(\Omega_i^h)}\\&\le Ch^{2\gamma-1}\|\mathrm{dist}(\nabla\boldsymbol{v}+\boldsymbol{I},SO(3))\|_{L^p(\Omega_i^h)},\end{aligned}\tag{11}$$

which gives the bound

$$\|(\boldsymbol{I}-\boldsymbol{R}_i)x-\boldsymbol{b}_i\|_{L^p(\Omega_i^h)}\le\|\boldsymbol{v}\|_{L^p(\Omega_i^h)}+Ch^{2\gamma-1}\|\mathrm{dist}(\nabla\boldsymbol{v}+\boldsymbol{I},SO(3))\|_{L^p(\Omega_i^h)},\quad i=1,2,\dots,N.\tag{12}$$

Next we claim that

$$\|(\boldsymbol{I}-\boldsymbol{R}_i)x-\boldsymbol{b}_i\|_{L^p(\Omega_i^h)}\ge Ch^\gamma\|\boldsymbol{I}-\boldsymbol{R}_i\|_{L^p(\Omega_i^h)},\tag{13}$$

for some $C>0$ uniformly in $i=1,2,\dots,N$. Indeed we have the obvious formula

$$\|\boldsymbol{I}-\boldsymbol{R}_i\|^p_{L^p(\Omega_i^h)}=|\boldsymbol{I}-\boldsymbol{R}_i|^p|\Omega_i^h|,\quad i=1,2,\dots,N.\tag{14}$$

To estimate the other term we interpret it geometrically. First of all the fact that S is bi-Lipschitz means that it has a finite atlas with bi-Lipschitz patches, thus S is locally the graph of a Lipschitz map upon a rotation of coordinates. Next, as S is compact, it satisfies the following geometric condition: *There exist* $\sigma,\delta\in(0,1)$ *such that for every* $r\in(0,\delta)$ *and every* $x\in S$ *one has*

$$\frac{\mathcal{H}^2(B_r(x)\cap S)}{\mathcal{H}^2(B_{2r}(x)\cap S)}\leq\sigma, \tag{15}$$

where $\mathcal{H}^2$ *is the two-dimensional Hausdorff measure (surface measure in this case).* Condition (4) roughly means that there can be no infinitesimal local concentrations of the surface S. Note next that each local rotation $\boldsymbol{R}_i$ rotates around a unit vector $\boldsymbol{n}_i\in\mathbb{R}^3$, which means that the operator $\boldsymbol{R}_ix-\boldsymbol{b}_i\colon\mathbb{R}^3\to\mathbb{R}^3$ projects onto the plane π_i orthogonal to $\boldsymbol{n}_i$, then rotates by $\boldsymbol{R}_i$ inside π_i, and then translates by the vector $-\boldsymbol{b}_i$. Assume the plane π_i is applied at the tip of the vector $\boldsymbol{b}_i$. Note that as each shell S_i^h has in-plane size of order h^γ, then keeping in mind condition (15) (first uncsaling each piece by h^γ) at most half of S_i^h (in terms of the measure) gets projected into the disc $D_i(\boldsymbol{b}_i,\frac{\sigma h^\gamma}{2})$, which is centered at $\boldsymbol{b}_i$, lies in π_i and has radius $\frac{\sigma h^\gamma}{2}$. As in two dimensions one always has $|(\boldsymbol{I}-\boldsymbol{R})x|=\frac{1}{\sqrt{2}}|\boldsymbol{I}-\boldsymbol{R}||x|$ for any rotation $\boldsymbol{R}\in SO(2)$ and any vector $x\in\mathbb{R}^2$, then taking into account the above observation, we have for small enough h the obvious estimate

$$\|(\boldsymbol{I}-\boldsymbol{R}_i)x-\boldsymbol{b}_i\|^p_{L^p(S_i^h)}\geq\frac{(\sigma h^\gamma)^p|\boldsymbol{I}-\boldsymbol{R}_i|^p|S_i^h|}{2^{2p}},\quad i=1,2,\dots,N. \tag{16}$$

The estimate (13) immediately follows from (14) and (16). Finally putting together (10), (12) and (13) and summing up the obtained estimates in i, we discover the bound

$$\|\nabla\boldsymbol{v}\|_{L^p(\Omega^h)}\leq C\left(\frac{1}{h^\gamma}\|\boldsymbol{v}\|_{L^p(\Omega^h)}+\frac{1}{h^{1-\gamma}}\|\mathrm{dist}(\nabla\boldsymbol{v}+\boldsymbol{I},SO(3))\|_{L^p(\Omega^h)}\right), \tag{17}$$

which is equivalent to (8). In the case when S has patch that is C^2, then an Ansatz realizing the asymptotics of h in Korn's first inequality for shells with positive Gaussian curvature has been constructed in [7]. It turns out that the same Ansatz works for (5) too. For convenience of the reader we present the Ansatz here. We choose a patch $S_0\subset S$ that is given by $S_0=\{\boldsymbol{r}(\theta,z)\ :\ \theta\in[0,s],\ z\in[0,s]\}$ in the principal coordinate (θ,z) parametrization $\boldsymbol{r}=\boldsymbol{r}(\theta,z)$, where $s>0$ is fixed. Denote next $A_z=\left|\frac{\partial\boldsymbol{r}}{\partial z}\right|$, $A_\theta=\left|\frac{\partial\boldsymbol{r}}{\partial\theta}\right|$, the two nonzero components of the metric tensor of S_0 and the two principal curvatures by κ_z and κ_θ. The signs of κ_z and κ_θ are chosen such that κ_z and κ_θ are positive for a sphere. Denoting the normal to S_0 coordinate variable by t, we have that the gradient of a vector field $\boldsymbol{u}=(u_t,u_\theta,u_z)\in W^{1,p}(S_0^h,\mathbb{R}^3)$ has the form

$$\nabla\boldsymbol{y}=\begin{bmatrix} u_{t,t} & \dfrac{y_{t,\theta}-A_\theta\kappa_\theta u_\theta}{A_\theta(1+t\kappa_\theta)} & \dfrac{u_{t,z}-A_z\kappa_z u_z}{A_z(1+t\kappa_z)}\\[2ex] u_{\theta,t} & \dfrac{A_zu_{\theta,\theta}+A_zA_\theta\kappa_\theta u_t+A_{\theta,z}u_z}{A_zA_\theta(1+t\kappa_\theta)} & \dfrac{A_\theta u_{\theta,z}-A_{z,\theta}u_z}{A_zA_\theta(1+t\kappa_z)}\\[2ex] u_{z,t} & \dfrac{A_zu_{z,\theta}-A_{\theta,z}u_\theta}{A_zA_\theta(1+t\kappa_\theta)} & \dfrac{A_\theta u_{z,z}+A_zA_\theta\kappa_zu_t+A_{z,\theta}u_\theta}{A_zA_\theta(1+t\kappa_z)}\end{bmatrix} \tag{18}$$

in the orthonormal basis $\boldsymbol{e}_t$, $\boldsymbol{e}_\theta$, $\boldsymbol{e}_z$, where $f,x=\partial_xf$. We finally choose $\boldsymbol{u}=\boldsymbol{I}+\epsilon\boldsymbol{v}$, where

$$\begin{cases} v_t=W(\frac{\theta}{\sqrt{h}},z)\\ v_\theta=-\frac{t\cdot W_{,\theta}\left(\frac{\theta}{\sqrt{h}},z\right)}{A_\theta\sqrt{h}}\\ v_z=-\frac{t\cdot W_{,z}\left(\frac{\theta}{\sqrt{h}},z\right)}{A_z},\end{cases} \tag{19}$$

where w is a smooth function compactly supported on the mid-surface S_0. For the rotation $\boldsymbol{R}=\boldsymbol{I}$ and the vector field $\boldsymbol{b}=0$, this choice will give equality in (5) for every fixed small enough $h>0$ by choosing $\epsilon>0$ small enough. $\square$

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