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Number Theory and Reductive Group Theory / *Théorie des nombres et théorie des groupes réductifs*

On non-admissible irreducible modulo p representations of $\mathrm{GL}_2(\mathbb{Q}_{p^2})$

Sur les représentations irréductibles non-admissibles modulo p de $\mathrm{GL}_2(\mathbb{Q}_{p^2})$

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Abstract. We use a Diamond diagram attached to a 2-dimensional reducible split mod p Galois representation of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_{p^2})$ to construct a non-admissible smooth irreducible mod p representation of $\mathrm{GL}_2(\mathbb{Q}_{p^2})$ following the approach of Daniel Le.

Résumé. Nous utilisons un diagramme de Diamond attaché à une représentation galoisienne mod p semi-simple réductible de dimension 2 de $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_{p^2})$ pour construire une représentation mod p non-admissible irréductible lisse de $\mathrm{GL}_2(\mathbb{Q}_{p^2})$ en suivant l'approche de Daniel Le.

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1. Introduction

Let p be a prime number, $\mathbb{Q}_p$ be the field of p-adic numbers, and $\overline{\mathbb{F}_p}$ be an algebraic closure of the finite field $\mathbb{F}_p$ of cardinality p. The study of the admissibility of smooth irreducible representations of connected reductive p-adic groups goes back to Harish–Chandra (see [6]). Building upon his work, Jacquet proved that every such representation over the field of complex numbers is admissible (see [8], see also [3]). This result was extended by Vignéras to smooth irreducible representations over any algebraically closed field of characteristic not equal to p (cf. [12]). In

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the note [1], the authors ask whether this is true for smooth irreducible representations over algebraically closed fields of characteristic p. It is known that every smooth irreducible representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ over $\overline{\mathbb{F}_p}$ is admissible (see [2]). However, Daniel Le recently constructed *non-admissible* smooth irreducible $\overline{\mathbb{F}_p}$-linear representations of $\mathrm{GL}_2(F)$, for F a finite unramified extension of $\mathbb{Q}_p$ of degree at least 3 and for $p > 2$, providing a negative answer to the question raised above (see [9]). In this paper, we follow Le's approach and construct non-admissible irreducible representations of $\mathrm{GL}_2(\mathbb{Q}_{p^2})$ where $\mathbb{Q}_{p^2}$ is the unramified extension of $\mathbb{Q}_p$ of degree 2. These results support the viewpoint of Breuil and Paškūnas that the mod p (and p-adic) representation theory of $\mathrm{GL}_2(F)$ becomes more complicated as soon as $F \neq \mathbb{Q}_p$ (see [5], see also [11]).

Let $G = \mathrm{GL}_2(\mathbb{Q}_{p^2})$, $K = \mathrm{GL}_2(\mathbb{Z}_{p^2})$, and $\Gamma = \mathrm{GL}_2(\mathbb{F}_{p^2})$, where $\mathbb{Z}_{p^2}$ is the ring of integers of $\mathbb{Q}_{p^2}$ with residue field $\mathbb{F}_{p^2}$. Fix an embedding $\mathbb{F}_{p^2} \hookrightarrow \overline{\mathbb{F}_p}$. Let I and I_1 denote the Iwahori and the pro-p Iwahori subgroups of K respectively, and K_1 denote the first principal congruence subgroup of K. Write N for the normalizer of I (and of I_1) in G. As a group, N is generated by I, the center Z of G, and by the element $\Pi = \left(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\right)$. All representations considered in this paper from now on are over $\overline{\mathbb{F}_p}$-vector spaces. For a character χ of I, χ^s denotes its Π-conjugate sending g in I to $\chi(\Pi g \Pi^{-1})$.

A weight is a smooth irreducible representation of K. The K-action on such a representation factors through Γ and thus any weight is described by a 2-tuple $(r_0, r_1) \otimes \det^m := \mathrm{Sym}^{r_0} \overline{\mathbb{F}_p}^2 \otimes (\mathrm{Sym}^{r_1} \overline{\mathbb{F}_p}^2)^{\mathrm{Frob}} \otimes \det^m$ of integers with $0 \leq r_0, r_1 \leq p-1$ together with a determinant twist for some $0 \leq m < p^2 - 1$ (see [4, Lemma 2.16 and Proposition 2.17]). Given a weight σ, its subspace σ^{I_1} of I_1-invariants has dimension 1. If χ_σ denotes the corresponding smooth character of I and $\chi_\sigma \neq \chi_\sigma^s$, then there exists a unique weight σ^s such that $\chi_{\sigma^s} = \chi_\sigma^s$ (see [10, Theorem 3.1.1]).

A *basic 0-diagram* is a triplet (D_0, D_1, r) consisting of a smooth KZ-representation D_0, a smooth N-representation D_1 and an IZ-equivariant isomorphism $r : D_1 \xrightarrow{\sim} D_0^{I_1}$ with the trivial action of p on D_0 and D_1. Given such a diagram such that $D_0^{K_1}$ has finite dimension, the smooth injective K-envelope $\mathrm{inj}_K D_0$ admits a non-canonical N-action which glues together with the K-action to give a smooth G-action on $\mathrm{inj}_K D_0$ (see [5, Theorem 9.8]). The G-subrepresentation of $\mathrm{inj}_K D_0$ generated by D_0 is smooth admissible and its K-socle equals the K-socle $\mathrm{soc}_K D_0$ of D_0.

From now on, assume that p is odd. Let $\rho : \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_{p^2}) \to \mathrm{GL}_2(\overline{\mathbb{F}_p})$ be a continuous generic Galois representation such that p acts trivially on its determinant and $\mathscr{D}(\rho)$ be the set of weights, called *Diamond weights*, associated to ρ as described in [5, Section 11]. Breuil and Paškūnas attach a family of basic 0-diagrams $(D_0(\rho), D_1(\rho), r)$, called *Diamond diagrams*, to ρ such that $\mathrm{soc}_K D_0(\rho) = \bigoplus_{\sigma \in \mathscr{D}(\rho)} \sigma$ (see [5, Theorem 13.8]).

For a finite unramified extension F of $\mathbb{Q}_p$ of degree at least 3, Le uses a Diamond diagram attached to an *irreducible* $\rho : \mathrm{Gal}(\overline{\mathbb{Q}_p}/F) \to \mathrm{GL}_2(\overline{\mathbb{F}_p})$ to construct an infinite dimensional diagram which gives rise to a non-admissible smooth irreducible representation of $\mathrm{GL}_2(F)$ (see [9]). His strategy does not work for a Diamond diagram attached to an irreducible Galois representation of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_{p^2})$ because such a diagram does not have suitable Π-action dynamics. However, for $F = \mathbb{Q}_{p^2}$, we observe that a Diamond diagram attached to a *reducible split* ρ has an indecomposable subdiagram with suitable Π-action dynamics so that Le's method can be used to obtain a non-admissible irreducible representation of $G = \mathrm{GL}_2(\mathbb{Q}_{p^2})$.

2. Reducible Diamond diagram

Let ω_2 be Serre's fundamental character of level 2 for the fixed embedding $\mathbb{F}_{p^2} \hookrightarrow \overline{\mathbb{F}_p}$, and let $\rho : \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_{p^2}) \to \mathrm{GL}_2(\overline{\mathbb{F}_p})$ be a continuous reducible split generic Galois representation. The restriction of ρ to the inertia subgroup is, up to a twist by some character, isomorphic to

$$\begin{pmatrix} \omega_2^{r_0+1+(r_1+1)p} & 0 \\ 0 & 1 \end{pmatrix}$$

for some $0 \le r_0, r_1 \le p-3$, not both equal to 0 or equal to $p-3$ (see [4, Corollary 2.9 (i)] and [5, Definition 11.7 (i)]). Define the weight

$$\sigma := (r_0+1, p-2-r_1) \otimes \det^{p-1+r_1 p}.$$

Then the set of Diamond weights for ρ is given by

$$\mathscr{D}(\rho) = \left\{(r_0, r_1), \sigma, \sigma^s, (p-3-r_0, p-3-r_1) \otimes \det^{r_0+1+(r_1+1)p}\right\}$$

(see [5, Lemma 11.2 or Section 16, Example (ii)]). Fix a Diamond diagram $(D_0(\rho), D_1(\rho), r)$ attached to ρ, and identify $D_1(\rho)$ with $D_0(\rho)^{I_1}$ as IZ-representations via r. There is a direct sum decomposition $D_0(\rho) = \bigoplus_{\nu \in \mathscr{D}(\rho)} D_{0,\nu}(\rho)$ of K-representations with $\mathrm{soc}_K D_{0,\nu}(\rho) = \nu$ (see [5, Proposition 13.4]).

Now define

$$D_0 := D_{0,\sigma}(\rho) \oplus D_{0,\sigma^s}(\rho) \qquad \text{and} \qquad D_1 := D_0^{I_1}.$$

It follows from [5, Theorem 15.4 (ii)] that (D_0, D_1, r) is an indecomposable subdiagram of $(D_0(\rho), D_1(\rho), r)$. Set

$$\tau := (r_0+2, r_1) \otimes \det^{p-2+(p-1)p} \qquad \text{and} \qquad \tau' := (p-1-r_0, p-3-r_1) \otimes \det^{r_0+(r_1+1)p}.$$

The graded pieces of the socle filtrations of $D_{0,\sigma}(\rho)$ and $D_{0,\sigma^s}(\rho)$, with the convention that we ignore a weight if a negative entry appears, are as follows (see [5, Theorem 14.8 or Section 16, Example (ii)]):

$$D_{0,\sigma}(\rho):\ \sigma \;\text{———}\; \tau \oplus \tau^s \;\text{———}\; (p-4-r_0, r_1-1) \otimes \det^{r_0+2}$$

$$D_{0,\sigma^s}(\rho):\ \sigma^s \;\text{———}\; \tau' \oplus \tau'^s \;\text{———}\; (r_0-1, p-4-r_1) \otimes \det^{(r_1+2)p}.$$

We have from [5, Corollary 14.10] that

$$D_1 = \chi_\sigma \oplus \chi_\tau \oplus \chi_\tau^s \oplus \chi_\sigma^s \oplus \chi_{\tau'} \oplus \chi_{\tau'}^s. \tag{1}$$

For an IZ-representation V and an IZ-character χ, we write V^χ for the χ-isotypic part of V.

3. An infinite dimensional diagram and the construction

Let $D_0(\infty) := \bigoplus_{i\in\mathbb{Z}} D_0(i)$ be the smooth KZ-representation with component-wise KZ-action, where there is a fixed isomorphism $D_0(i) \cong D_0$ of KZ-representations for every $i \in \mathbb{Z}$. Following [9], we denote the natural inclusion $D_0 \xrightarrow{\sim} D_0(i) \hookrightarrow D_0(\infty)$ by ι_i, and write $v_i := \iota_i(v)$ for $v \in D_0$ for every $i \in \mathbb{Z}$. Let $D_1(\infty) := D_0(\infty)^{I_1}$. We define a Π-action on $D_1(\infty)$ as follows. Let $\lambda = (\lambda_i) \in \prod_{i\in\mathbb{Z}} \overline{\mathbb{F}_p}^\times$. For all integers $i \in \mathbb{Z}$, define

$$\Pi v_i := \begin{cases} (\Pi v)_i & \text{if } v \in D_1^{\chi_\sigma}, \\ (\Pi v)_{i+1} & \text{if } v \in D_1^{\chi_\tau}, \\ \lambda_i (\Pi v)_i & \text{if } v \in D_1^{\chi_{\tau'}}. \end{cases}$$

This uniquely determines a smooth N-action on $D_1(\infty)$ such that $p = \Pi^2$ acts trivially on it. Thus we get a basic 0-diagram $D(\lambda) := (D_0(\infty), D_1(\infty), \mathrm{can})$ with the above actions where can is the canonical inclusion $D_1(\infty) \hookrightarrow D_0(\infty)$.

Theorem 1. *There exists a smooth representation π of G such that*

(i) $(\pi|_{KZ}, \pi|_N, \mathrm{id})$ *contains* $D(\lambda)$,
(ii) π *is generated by* $D_0(\infty)$ *as a* G*-representation, and*
(iii) $\mathrm{soc}_K \pi = \mathrm{soc}_K D_0(\infty)$.

Proof. Let Ω be the smooth injective K-envelope of D_0 equipped with the KZ-action such that p acts trivially. The smooth injective I-envelope $\mathrm{inj}_I D_1$ of D_1 appears as an I-direct summand of Ω. Let e denote the projection of Ω onto $\mathrm{inj}_I D_1$. There is a unique N-action on $\mathrm{inj}_I D_1$ compatible with that of I and compatible with the action of N on D_1. By [5, Lemma 9.6], there is a non-canonical N-action on $(1-e)(\Omega)$ extending the given I-action. This gives an N-action on Ω whose restriction to IZ is compatible with the action coming from KZ on Ω.

Now let $\Omega(\infty) := \bigoplus_{i\in\mathbb{Z}}\Omega(i)$ with component-wise KZ-action where there is a fixed isomorphism $\Omega(i)\cong\Omega$ of KZ-representations for every $i\in\mathbb{Z}$. We wish to define a compatible N-action on $\Omega(\infty)$. As before, denote the natural inclusion $\Omega \xrightarrow{\sim} \Omega(i) \hookrightarrow \Omega(\infty)$ by ι_i, and write $v_i := \iota_i(v)$ for $v\in\Omega$. Let Ω_χ denote the smooth injective I-envelope of an I-character χ. Thus, from (1), we have $e(\Omega) = \mathrm{inj}_I D_1 = \Omega_{\chi_\sigma}\oplus\Omega_{\chi_\tau}\oplus\Omega_{\chi_\tau^s}\oplus\Omega_{\chi_\sigma^s}\oplus\Omega_{\chi_{\tau'}}\oplus\Omega_{\chi_{\tau'}^s}$. If $v\in(1-e)(\Omega)$, we define $\Pi v_i := (\Pi v)_i$ for all integers i. Otherwise, we define $\Pi v_i := (\Pi v)_i$ if $v\in\Omega_{\chi_\sigma}$, $\Pi v_i := (\Pi v)_{i+1}$ if $v\in\Omega_{\chi_\tau}$, and $\Pi v_i := \lambda_i(\Pi v)_i$ if $v\in\Omega_{\chi_{\tau'}}$. By demanding that Π^2 acts trivially, this defines a smooth N-action on $\Omega(\infty)$ which is compatible with the N-action on $D_1(\infty)$, and whose restriction to IZ is compatible with the action coming from KZ on $\Omega(\infty)$. By [10, Corollary 5.5.5], we have a smooth G-action on $\Omega(\infty)$. We then take π to be the G-representation generated by $D_0(\infty)$ inside $\Omega(\infty)$. If follows easily from the construction that π satisfies the properties (i), (ii) and (iii). □

Theorem 2. *If $\lambda_i\neq\lambda_0$ for all $i\neq 0$, then any smooth representation π of G satisfying the properties* (i), (ii), *and* (iii) *of Theorem 1 is irreducible and non-admissible.*

Proof. Let $\pi'\subseteq\pi$ be a non-zero subrepresentation of G. By property (iii), we have either $\mathrm{Hom}_K(\sigma,\pi')\neq 0$ or $\mathrm{Hom}_K(\sigma^s,\pi')\neq 0$. We consider the case $\mathrm{Hom}_K(\sigma,\pi')\neq 0$; the other case is treated analogously. There exists a non-zero $(c_i)\in\bigoplus_{i\in\mathbb{Z}}\overline{\mathbb{F}_p}$ such that

$$\left(\sum_i c_i\iota_i\right)(D_{0,\sigma}(\rho))\cap\pi'\neq 0.$$

We claim that

$$\left(\sum_i c_i\iota_{i+j}\right)(D_0)\subset\pi' \qquad \text{for all } j\in\mathbb{Z}. \tag{2}$$

We first show that $\left(\sum_i c_i\iota_i\right)(D_{0,\sigma^s}(\rho))\subset\pi'$. Note that $\left(\sum_i c_i\iota_i\right)(D_{0,\sigma}(\rho))\cap\pi'\neq 0$ is equivalent to $\left(\sum_i c_i\iota_i\right)(\sigma)\subset\pi'$. Since $\left(\sum_i c_i\iota_i\right)(D_1^{\chi_\sigma})\subset\pi'$ and π' is stable under the Π-action, we have $\left(\sum_i c_i\iota_i\right)(D_1^{\chi_\sigma^s})\subset\pi'$. By Frobenius reciprocity, we have a non-zero K-equivariant map

$$\mathrm{Ind}_I^K\left(\left(\sum_i c_i\iota_i\right)\left(D_1^{\chi_\sigma^s}\right)\right)\to\pi' \tag{3}$$

whose image is $\left(\sum_i c_i\iota_i\right)(I(\delta(\sigma),\sigma^s))$, where δ is the bijection on the set of Diamond weights $\mathscr{D}(\rho)$ defined in [5, Section 15], and $I(\delta(\sigma),\sigma^s)$ is the K-subrepresentation of $D_{0,\delta(\sigma)}(\rho)$ with cosocle σ^s (and socle $\delta(\sigma)$). In our setting, δ maps σ to σ^s and vice versa (see [5, Lemma 15.2]). Thus $I(\delta(\sigma),\sigma^s)=\sigma^s$ and so $\left(\sum_i c_i\iota_i\right)(\sigma^s)\subset\pi'$. Let $R\big((\sum_i c_i\iota_i)(\sigma)\big)$ be the K-subrepresentation of the compact induction $\text{c-Ind}_{KZ}^G\big((\sum_i c_i\iota_i)(\sigma)\big)$ defined in [5, Section 17]. By [5, Lemmas 17.1, 17.4 and 17.8], we have

$$\mathrm{Ind}_I^K\left(\left(\sum_i c_i\iota_i\right)\left(D_1^{\chi_\sigma^s}\right)\right)\subset R\left(\left(\sum_i c_i\iota_i\right)(\sigma)\right),$$

and by Frobenius reciprocity, there is a non-zero map

$$\text{c-Ind}_{KZ}^G\left(\left(\sum_i c_i\iota_i\right)(\sigma)\right)\to\pi' \tag{4}$$

which restricts to the map (3). So the image Q of $R\big((\sum_i c_i\iota_i)(\sigma)\big)$ in π' under the map (4) contains $\left(\sum_i c_i\iota_i\right)(\sigma^s)$. Since $\mathrm{soc}_K Q\subset\mathrm{soc}_K\pi=\mathrm{soc}_K D_0(\infty)$ and the Jordan–Hölder factors of

$R\big((\sum_i c_i\iota_i)(\sigma)\big)$ are multiplicity free (see [5, Lemma 17.11]), $\mathrm{soc}_K Q$ is isomorphic to a subrepresentation of the direct sum of the weights in $\mathscr{D}(\rho)$. Therefore by [5, Lemma 19.5], $\mathrm{soc}_K Q = \big(\sum_i c_i\iota_i\big)(\sigma^s)$, and by [5, Lemma 19.7], Q contains a copy of the K-representation $D_{0,\sigma^s}(\rho)$. But $\big(\sum_i c_i\iota_i\big)(D_{0,\sigma^s}(\rho))$ is the unique K-subrepresentation of π isomorphic to $D_{0,\sigma^s}(\rho)$ and with K-socle $\big(\sum_i c_i\iota_i\big)(\sigma^s)$. Thus $\big(\sum_i c_i\iota_i\big)(D_{0,\sigma^s}(\rho)) = Q \subset \pi'$.

Now, since $\big(\sum_i c_i\iota_i\big)(\sigma^s) \subset \pi'$, a symmetric argument shows that $\big(\sum_i c_i\iota_i\big)(D_{0,\sigma}(\rho)) \subset \pi'$. Thus

$$\left(\sum_i c_i\iota_i\right)(D_0) \subset \pi'.$$

Therefore

$$\left(\sum_i c_i\iota_i\right)\left(D_1^{\chi_\tau}\right) \subset \pi' \qquad \text{and} \qquad \left(\sum_i c_i\iota_i\right)\left(D_1^{\chi_\tau^s}\right) \subset \pi'.$$

Since π' is stable under the Π-action, we have

$$\left(\sum_i c_i\iota_{i+1}\right)\left(D_1^{\chi_\tau^s}\right) \subset \pi' \qquad \text{and} \qquad \left(\sum_i c_i\iota_{i-1}\right)\left(D_1^{\chi_\tau}\right) \subset \pi'.$$

In particular,

$$\left(\sum_i c_i\iota_{i+1}\right)(D_{0,\sigma}(\rho)) \cap \pi' \neq 0 \qquad \text{and} \qquad \left(\sum_i c_i\iota_{i-1}\right)(D_{0,\sigma}(\rho)) \cap \pi' \neq 0.$$

By the same arguments as above, we find that

$$\left(\sum_i c_i\iota_{i+1}\right)(D_0) \subset \pi' \qquad \text{and} \qquad \left(\sum_i c_i\iota_{i-1}\right)(D_0) \subset \pi'.$$

The claim (2) is now proved by repeatedly using the Π-action.

For $(d_i) \in \bigoplus_{i\in\mathbb{Z}} \overline{\mathbb{F}_p}$, let $\#(d_i)$ denote the number of non-zero d_i's. Among all the non-zero elements (c_i) of $\bigoplus_{i\in\mathbb{Z}} \overline{\mathbb{F}_p}$ for which $\big(\sum_i c_i\iota_i\big)(D_0) \subset \pi'$, we pick one with $\#(c_i)$ minimal. We may also assume that $c_0 \neq 0$ using (2). We now show that $\#(c_i) = 1$. Assume to the contrary that $\#(c_i) > 1$. Since $\big(\sum_i c_i\iota_i\big)(D_1^{\chi_{\tau'}}) \subset \pi'$ and π' is stable under the Π-action, we have

$$\left(\sum_i \lambda_i c_i\iota_i\right)\left(D_1^{\chi_{\tau'}^s}\right) \subset \pi'.$$

Since $\big(\sum_i \lambda_0 c_i\iota_i\big)(D_1^{\chi_{\tau'}^s})$ is also clearly in π', subtracting it from the above, we get

$$\left(\sum_i (\lambda_i - \lambda_0) c_i\iota_i\right)\left(D_1^{\chi_{\tau'}^s}\right) \subset \pi'.$$

Writing $(c_i') := ((\lambda_i - \lambda_0)c_i)$, we see that

$$\left(\sum_i c_i'\iota_i\right)(D_{0,\sigma^s}(\rho)) \cap \pi' \neq 0.$$

Following the same arguments as in the previous paragraphs, we get that $\big(\sum_i c_i'\iota_i\big)(D_0) \subset \pi'$. However, the hypothesis $\lambda_i \neq \lambda_0$ for all $i \neq 0$, and the assumption $\#(c_i) > 1$ imply that (c_i') is non-zero and $\#(c_i') = \#(c_i) - 1$ contradicting the minimality of $\#(c_i)$. Therefore, we have $c_0\iota_0(D_0) \subset \pi'$. So $\iota_0(D_0) \subset \pi'$. Using (2) again, we get that $\bigoplus_{j\in\mathbb{Z}} \iota_j(D_0) = D_0(\infty) \subset \pi'$. By property (ii), we have $\pi' = \pi$.

The non-admissibility of π is clear because $\pi^{K_1} \supseteq \mathrm{soc}_K \pi$ and $\mathrm{soc}_K \pi$ is not finite dimensional by property (iii). □

Remark 3. If the diagram $(D_0(\rho), D_1(\rho), r)$ is defined over $\mathbb{F}_{p^2}$ and $(\lambda_i) \in \prod_{i\in\mathbb{Z}} \mathbb{F}_{p^2}^\times$, then the representation π in Theorem 1 has a model π_0 over $\mathbb{F}_{p^2}$. Furthermore, π_0 is absolutely irreducible and non-admissible if the (λ_i) satisfy the hypothesis of Theorem 2. In fact, for any field C containing $\mathbb{F}_{p^2}$, the methods of this paper produce an absolutely irreducible non-admissible smooth C-representation $C \otimes_{\mathbb{F}_{p^2}} \pi_0$ of G.

Now let C be an arbitrary field of characteristic p with algebraic closure $\overline{C}$. From the discussion in the previous paragraph, the representation $\overline{C} \otimes_{\mathbb{F}_{p^2}} \pi_0$ is a smooth irreducible $\overline{C}$-representation which has a model $C' \otimes_{\mathbb{F}_{p^2}} \pi_0$ over C', where $C' = C\mathbb{F}_{p^2} \subset \overline{C}$. By [7, Lemma II.5], there exists a smooth irreducible C-representation π_C such that $\overline{C} \otimes_{\mathbb{F}_{p^2}} \pi_0$ is a $\overline{C}$-subrepresentation of $\overline{C} \otimes_C \pi_C$. Since $\overline{C} \otimes_{\mathbb{F}_{p^2}} \pi_0$ is non-admissible, $\overline{C} \otimes_C \pi_C$ is non-admissible and hence π_C is non-admissible by [7, Lemma III.1(ii)]. Thus we obtain a smooth irreducible non-admissible representation of G over any field C of characteristic p.

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