# *Comptes Rendus*

# *Mathématique*

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**Upper bound estimate for the blow-up time of a class of integrodifferential equation of parabolic type involving variable source**

Partial Differential Equations / *Équations aux dérivées partielles*

# Upper bound estimate for the blow-up time of a class of integrodifferential equation of parabolic type involving variable source

## Borne supérieure pour le temps d'explosion d'une classe d'équations intégro-différentielles de type parabolique avec une source variable

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**Abstract.** Consider a class of integrodifferential of parabolic equations involving variable source with Dirichlet boundary condition

$$u_t = \Delta u - \int_0^t g(t-s)\Delta u(x,s)\,\mathrm{d}s + |u|^{p(x)-2}u.$$

By means energy methods, we obtain a lower bound for blow-up time of the solution if blow-up occurs. Furthermore, assuming the initial energy is negative we establish a new blow-up criterion and give an upper bound for blow-up time of the solution.

**Résumé.** Considérons une classe d'équations intégro-différentielles paraboliques comprenant une source variable et avec condition de Dirichlet au bord

$$u_t = \Delta u - \int_0^t g(t-s)\Delta u(x,s)\,\mathrm{d}s + |u|^{p(x)-2}u.$$

À l'aide des méthodes d'énergie nous obtenons une borne inférieure pour le temps où intervient une éventuelle explosion de la solution. De plus, en supposant que l'énergie initiale est négative nous établissons un nouveau critère pour l'explosion et nous donnons une borne supérieure pour le temps d'explosion de la solution.

## 1. Introduction

Let $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$, be a bounded domain with a smooth boundary $\Gamma = \partial\Omega$. The boundary $\Gamma$ of $\Omega$ is assumed to be regular, $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a bounded $C^1(\mathbb{R}^+)$ function and $T$, $p$ are positive constants.

In the recent years, many authors, in the different cases of the values of the memory kernel, specifically when $g = 0$ or $g > 0$, semilinear parabolic problems with a memory term associated with the Laplace operator and source term with Dirichlet type condition has been considered:

$$\begin{cases} u_t - \Delta u + \int_0^t g(t-s)\,\Delta u(x,s)\,\mathrm{d}s = |u|^{p-2}u, & \text{in } \Omega \times (0,T), \\ u(x,t) = 0, & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & \text{in } \Omega, \end{cases} \tag{1}$$

By assuming suitable conditions on $g$, $p$, and $u_0$, using some known theorems in the mathematical literature, the global existence in time, blow-up in finite time, the asymptotic behavior and a lower bound for the blow-up time of the unique weak solution have been discussed.

In this work given a positive measurable function $p(\cdot)$ on $\overline{\Omega}$ satisfying:

$$2 < p_1 = \operatorname*{essinf}_{x\in\Omega} p(x) \leq p(x) \leq p_2 = \operatorname*{esssup}_{x\in\Omega} p(x) \leq p^*(x), \tag{2}$$

and

$$p^*(x) = \begin{cases} \dfrac{np(x)}{\operatorname{esssup}_{x\in\Omega}(n-p(x))} & \text{if } n > p_2 \\ +\infty & \text{if } n \leq p_2 \end{cases}$$

we consider the following semilinear generalized parabolic boundary value problem governed by partial differential equations that describe the evolution of viscoelastic materials with nonlinearities of variable exponent type under Dirichlet type condition:

$$\begin{cases} u_t - \Delta u + \int_0^t g(t-s)\,\Delta u(x,s)\,\mathrm{d}s = |u|^{p(x)-2}u, & \text{in } \Omega \times (0,T), \\ u = 0, & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & \text{in } \Omega. \end{cases} \tag{3}$$

where $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a bounded $C^1(\mathbb{R}^+)$ function, $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ is a bounded domain with smooth boundary, $T \in (0,+\infty]$, and the initial value $u_0 \in H_0^1(\Omega)$.

Problem (3) arises from many important mathematical models in engineering and physical sciences. For example, nuclear science, chemical reactions, heat transfer, population dynamics, biological sciences etc., and have interested a great deal of attention in the research, see [1, 3, 7] and the references therein.

In the following section, we introduce some preliminaries and notations, which will be used throughout this paper.

## 2. Preliminaries

In this section, we list and recall some well-known results and facts from the theory of the Sobolev spaces with a variable exponent (for details, see [4, 5, 6, 8, 9]).

During the rest of this report, $\Omega$ is considered to be a bounded domain of $\mathbb{R}^n$, $n \geq 2$ with a smooth boundary $\Gamma = \partial\Omega$, assuming that $p(\cdot)$ is a measured function on $\overline{\Omega}$ and satisfies the following Zhikov–Fan uniform local continuity condition:

$$\left| p(x) - p(y) \right| \leq \frac{M}{|\log|x - y||}, \quad \text{for all } x, \ y \text{ in } \Omega \text{ with } |x - y| < \frac{1}{2}, \ M > 0. \tag{4}$$

Let $p : \Omega \to [1, \infty]$ be a measurable function. $L^{p(\cdot)}(\Omega)$ denote the set of measurable real-value functions $u$ on $\Omega$ such that

$$A_{p(\cdot)}(u) = \int_{\Omega} |u(x)|^{p(x)} \, \mathrm{d}x < \infty.$$

The variable-exponent space $L^{p(\cdot)}(\Omega)$ equipped with the Luxemburg norm

$$\|u\|_{p(\cdot)} = \inf\left\{ \lambda > 0, \ A_{p(\cdot)}\left( \frac{u}{\lambda} \right) \leq 1 \right\},$$

is a Banach space and it is called variable exponent Lebesgue space.

In general, variable-exponent Lebesgue spaces are similar to classical Lebesgue spaces in many aspects, see the first discussion of $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$ spaces by Kovàcik and Rákosnik in [9].

Here are some properties of the space $L^{p(\cdot)}(\Omega)$, which will be used in the study of the problem (3).

- It results directly from the definition of the norm that

$$\min\left( \|u\|_{p(\cdot)}^{p_1}, \|u\|_{p(\cdot)}^{p_2} \right) \leq A_{p(\cdot)}(u) \leq \max\left( \|u\|_{p(\cdot)}^{p_1}, \|u\|_{p(\cdot)}^{p_2} \right).$$

- The following generalized Hölder inequality

$$\int_{\Omega} |u(x)v(x)| \, \mathrm{d}x \leq \left( \frac{1}{p_1} + \frac{1}{q_1} \right) \|u\|_{p(x)} \|v\|_{q(x)} \leq 2 \|u\|_{p(x)} \|v\|_{q(x)},$$

  holds, for all $u \in L^{p(\cdot)}(\Omega)$, $v \in L^{q(\cdot)}(\Omega)$ with $p(x) \in (1, \infty)$, $q(x) = \frac{p(x)}{p(x)-1}$.

- If condition (4) is fulfilled, and $\Omega$ has a finite measure, $p$ and $q$ are variable exponents such that $p(x) \leq q(x)$ almost everywhere in $\Omega$, then the embedding $L^{q(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$ is continuous.

- If $p \in C(\overline{\Omega})$, $q : \Omega \to [1, +\infty)$ is a measurable function and $\operatorname{essinf}_{x\in\Omega}\left( p^*(x) - q(x) \right) > 0$ with $p^*(x) = \frac{np(x)}{\operatorname{esssup}_{x\in\Omega}(n - p(x))}$, then the embedding $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ is continuous and compact.

**Lemma 1** ([4]). *Let $\Omega$ be a bounded domain of $\mathbb{R}^n$, $p(\cdot)$ and $m(\cdot)$ satisfies (2) and (4), then*

$$\|u\|_{p(\cdot)} \leq B_0 \|\nabla u\|_{m(\cdot)}, \text{ for all } u \in W_0^{1,m(\cdot)}(\Omega). \tag{5}$$

*where the optimal constant of Sobolev embedding $B_0$ is depended on $p_{1,2}$ and $|\Omega|$.*

Throughout the paper, we use $\|\cdot\|_q$ to indicate the $L^q$-norm for $1 \leq q \leq +\infty$. $H_0^1(\Omega)$ is the closure of $C_0^\infty(\Omega)$ with respect to the following equivalent norm:

$$\|u\|_{H_0^1(\Omega)} = \|\nabla u\|_2 = \left( \int_{\Omega} |\nabla u(x)|^2 \, \mathrm{d}x \right)^{\frac{1}{2}}.$$

We assume that:

(A) The memory kernel $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a $C^1(\mathbb{R}^+)$ function satisfying

$$g(t) \geq 0, \quad g'(t) \leq 0, \quad 1 - \int_0^\infty g(s) \, \mathrm{d}s = l > 0.$$

(B) $p(\cdot)$ is a given measurable function on $\overline{\Omega}$ such that:

$$2 < p_1 \leq p(x) \leq p_2 < \frac{2n+4}{n}, \quad n \geq 3; \quad 2 < p_1 \leq p_2 < 4, \quad n = 1, 2.$$

## 3. Blow-up and upper bound of blow-up time

In this section, we derive a lower bound for $T$ if the weak solution $u(x,t)$ of (3) blows up in finite time $T$. We start with a local existence result for the problem (3), which is a direct result of the existence theorem by [2, 10, 11].

**Theorem 2.** *For all $u_0 \in H_0^1(\Omega)$, the problem (3) possesses a weak solution $u$ on $[0, T_0]$ satisfying:*

$$u \in C_w\left([0, T_0]; W^{1,2}(\Omega)\right) \cap C([0, T_0]; L^{p(\cdot)}(\Omega)) \cap W^{1,2}([0, T_0]; L^2(\Omega)).$$

*where $T_0 \in (0, T]$ is a suitable number.*

The energy functional corresponding to problem (3) is

$$E(t) = \frac{1}{2}\left(g \diamond \nabla u\right) + \frac{1}{2}\left(1 - \int_0^t g(s)\,\mathrm{d}s\right)\|\nabla u\|_2^2 - \int_\Omega \frac{1}{p(x)}|u(x,t)|^{p(x)}\,\mathrm{d}x, \tag{6}$$

where

$$\left(g \diamond \nabla u\right)(t) = \int_0^t g(t-s)\int_\Omega |\nabla u(t) - \nabla u(s)|^2\,\mathrm{d}x\mathrm{d}s = \int_0^t g(t-s)\|\nabla u(t) - \nabla u(s)\|_2^2\,\mathrm{d}s.$$

We have

**Lemma 3.** *Let $u(x,t)$ be a weak solution of (3), then $E(t)$ is a nonincreasing function on $[0, T]$.*

**Proof.** Multiplying $u_t(x,t)$ on both sides of (3) and integrating over $\Omega$, we can obtain

$$\frac{\mathrm{d}E(t)}{\mathrm{d}t} = -\frac{1}{2}g(t)|\nabla u(t)|^2 + \frac{1}{2}\left(g' \diamond \nabla u\right) - \int_\Omega u_t^2(x,t)\,\mathrm{d}x \le 0. \tag{7}$$

From the condition (A), we gain $\frac{\mathrm{d}E(t)}{\mathrm{d}t} \le 0$. Thus $E(t)$ is a nonincreasing function on $[0, T]$. □

Our first main result is as follows.

**Theorem 4.** *Under the conditions (A) and (B), assume $u_0 \in H_0^1(\Omega)$ such that $\|u_0\|_2 \ne 0$, and the weak solution $u(x,t)$ of problem (3) blows up in finite time $T$.*

(1) *If $n \ge 3$, then $T$ has a lower bound by*

$$\int_{\|u_0\|_2^2}^\infty \frac{\mathrm{d}y}{C_1 + C_3\,y^{C_2} + C_5\,y^{C_4}},$$

*where*

$$C_1 = \max\left\{\frac{1}{l}E(0), 0\right\} \ge 0, \quad C_2 = \frac{2p_2 - n(p_2 - 2)}{4 - n(p_2 - 2)} > 1, \quad C_4 = \frac{2p_1 - n(p_1 - 2)}{4 - n(p_1 - 2)} > 1, \tag{8}$$

$$0 < C_3 = \frac{2lp_1 + 1}{lp_1}\frac{4 - n(p_2 - 2)}{4}\left(B^2\frac{n(p_2 - 2)\left(2lp_1 + 1\right)}{lp_1\left(4l^2 + 1\right)}\right)^{\frac{n(p_2-2)}{4-n(p_2-2)}}, \tag{9}$$

$$C_5 = \frac{2lp_1 + 1}{lp_1}\frac{4 - n(p_2 - 2)}{4}\left(B^2\frac{n(p_1 - 2)\left(2lp_1 + 1\right)}{lp_1\left(4l^2 + 1\right)}\right)^{\frac{n(p_1-2)}{4-n(p_1-2)}}, \tag{10}$$

*and $B$ be the optimal constant of Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^{\frac{2n}{n-2}}(\Omega)$ which provide the inequality*

$$\|u\|_{\frac{2n}{n-2}} \le B\|\nabla u\|_2.$$

(2) *If $n = 1, 2$, then $T$ has a lower bound by*

$$\int_{\|u_0\|_2^2}^\infty \frac{\mathrm{d}y}{C_1 + C_6\,y^{C_5} + C_8\,y^{C_7}},$$

*where $C_1$ is the constant in* (8),

$$C_5 = \frac{2}{4 - p_2} > 1, \quad C_7 = \frac{2}{4 - p_1} > 1,$$

$$C_6 = \left(\frac{2lp_1 + 1}{2lp_1}\right)\left(\frac{4 - p_2}{2}\right)\left(2B_*^2 \frac{(p_2 - 2)(2lp_1 + 1)}{lp_1(4l^2 + 1)}\right)^{\frac{p_2 - 2}{4 - p_2}} > 0,$$

$$C_8 = \left(\frac{2lp_1 + 1}{2lp_1}\right)\left(\frac{4 - p_1}{2}\right)\left(2B_*^2 \frac{(p_1 - 2)(2lp_1 + 1)}{lp_1(4l^2 + 1)}\right)^{\frac{p_1 - 2}{4 - p_1}} > 0,$$

*and $B_*$ is the best constant of the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^\infty(\Omega)$ which provide the inequality*

$$\|u\|_\infty \le B_* \|\nabla u\|_2.$$

**Proof.**  We define the function

$$\varphi(t) = \int_\Omega u(x, t)^2 \, \mathrm{d}x. \tag{11}$$

Multiplying $u$ on both parties of (3) and integrating by parts, by the condition (A), we have

$$\begin{aligned}
\varphi'(t) &= 2 \int_\Omega u(x, t) u_t(x, t) \, \mathrm{d}x \\
&= 2 \int_\Omega u(x, t) \left(\Delta u - \int_0^t g(t - s) \Delta u(x, s) \mathrm{d}s + |u|^{p(x) - 2} u\right) \mathrm{d}x \\
&= -2\left(1 - \int_0^t g(s) \, \mathrm{d}s\right) \|\nabla u(t)\|_2^2 + 2 \int_\Omega |u|^{p(x)} \mathrm{d}x + I_1 \\
&\le -2l \|\nabla u(t)\|_2^2 + 2 \int_\Omega |u|^{p(x)} \mathrm{d}x + I_1,
\end{aligned} \tag{12}$$

where

$$I_1 = 2 \int_0^t g(t - s) \int_\Omega \nabla u(t) (\nabla u(s) - \nabla u(t)) \, \mathrm{d}x \mathrm{d}s. \tag{13}$$

By Hölder's inequality and the condition (A), we know

$$|I_1| \le 2l \|\nabla u(t)\|_2^2 \int_0^t g(s) \, \mathrm{d}s + \frac{1}{2l} (g \diamond \nabla u) \le 2l(1 - l) \|\nabla u(t)\|_2^2 + \frac{1}{2l} (g \diamond \nabla u), \tag{14}$$

Combining (11) and (14), we gain

$$\varphi'(t) \le -2l^2 \|\nabla u(t)\|_2^2 + 2\left(\int_\Omega |u|^{p_2} \mathrm{d}x + \int_\Omega |u|^{p_1} \mathrm{d}x\right) + \frac{1}{2l} (g \diamond \nabla u). \tag{15}$$

From the condition (A), (6) and Lemma 3, we have

$$\begin{aligned}
\frac{1}{2l} (g \diamond \nabla u) &= \frac{1}{l}\left[E(t) - \frac{1}{2} \|\nabla u(t)\|_2^2 \left(1 - \int_0^t g(s) \, \mathrm{d}s\right) + \int_\Omega \frac{1}{p(x)} |u(x, t)|^{p(x)} \mathrm{d}x\right] \\
&\le \frac{1}{l} E(0) - \frac{1}{2} \|\nabla u(t)\|_2^2 + \frac{1}{lp_1} \int_\Omega |u(x, t)|^{p(x)} \mathrm{d}x.
\end{aligned} \tag{16}$$

Then (15) and (16) show that

$$\varphi'(t) \le -\left(2l^2 + \frac{1}{2}\right) \|\nabla u(t)\|_2^2 + \frac{1}{l} E(0) + \left(2 + \frac{1}{lp_1}\right) \int_\Omega |u|^{p(x)} \mathrm{d}x. \tag{17}$$

Now, we consider the case $n \ge 3$.

Using Hölder's inequality, for any $a > 0$, we see

$$\begin{aligned}
\int_\Omega |u|^{p(x)} \mathrm{d}x &\le \int_\Omega |u|^{p_1} \mathrm{d}x + \int_\Omega |u|^{p_2} \mathrm{d}x = \|u\|_{p_1}^{p_1} + \|u\|_{p_2}^{p_2} \\
&\le \left(a\|u\|_2^{\frac{2n - p_2(n-2)}{2}}\right) \cdot \left(a^{-1} \|u\|_{\frac{2n}{n-2}}^{\frac{n(p_2 - 2)}{2}}\right) + \left(a\|u\|_2^{\frac{2n - p_1(n-2)}{2}}\right) \cdot \left(a^{-1} \|u\|_{\frac{2n}{n-2}}^{\frac{n(p_1 - 2)}{2}}\right).
\end{aligned} \tag{18}$$

On the other hand, we know the following inequality,

$$b^r c^s \le r b + s c, \text{ for all } r, s, b, c > 0, \; r + s = 1. \tag{19}$$

The condition $2 < p_1 \le p_2 < 2 + \frac{4}{n}$ implies $0 < \frac{n(p_1-2)}{4} < \frac{n(p_2-2)}{4} < 1$, then we use (19) and in (18), we have for all $a > 0$,

$$
\begin{aligned}
\|u\|_{p_2}^{p_2} &\le \left( a \|u\|_2^{\frac{2n-p_2(n-2)}{2}} \right)^{\frac{4}{4-n(p_2-2)} \cdot \frac{4-n(p_2-2)}{4}} \cdot \left( a^{-1} \|u\|_{\frac{2n}{n-2}}^{\frac{n(p_2-2)}{2}} \right)^{\frac{n(p_2-2)}{4} \cdot \frac{4}{n(p_2-2)}} \\
&\le \frac{4-n(p_2-2)}{4} \left( a \|u\|_2^{\frac{2n-p_2(n-2)}{2}} \right)^{\frac{4}{4-n(p_2-2)}} + \frac{n(p_2-2)}{4} \left( a^{-1} \|u\|_{\frac{2n}{n-2}}^{\frac{n(p_2-2)}{2}} \right)^{\frac{4}{n(p_2-2)}} \\
&= \frac{4-n(p_2-2)}{4} a^{\frac{4}{4-n(p_2-2)}} \varphi^{\frac{2p_2-n(p_2-2)}{4-n(p_2-2)}}(t) + \frac{n(p_2-2)}{4} a^{\frac{-4}{n(p_2-2)}} \|u\|_{\frac{2n}{n-2}}^2,
\end{aligned}
\tag{20}
$$

similarly, for all $b > 0$,

$$\|u\|_{p_1}^{p_1} \le \frac{4-n(p_1-2)}{4} b^{\frac{4}{4-n(p_1-2)}} \varphi^{\frac{2p_1-n(p_1-2)}{4-n(p_1-2)}}(t) + \frac{n(p_1-2)}{4} b^{\frac{-4}{n(p_1-2)}} \|u\|_{\frac{2n}{n-2}}^2, \tag{21}$$

From (17), (20) and (21), for all $a, b > 0$, we can obtain

$$
\begin{aligned}
\varphi'(t) \le \;& \frac{1}{l} E(0) \\
&+ \left( B^2 \frac{2lp_1+1}{lp_1} \frac{n(p_2-2)}{4} a^{\frac{-4}{n(p_2-2)}} + B^2 \frac{2lp_1+1}{lp_1} \frac{n(p_1-2)}{4} b^{\frac{-4}{n(p_1-2)}} - \left( 2l^2 + \frac{1}{2} \right) \right) \|\nabla u(t)\|_2^2 \\
&+ \frac{2lp_1+1}{lp_1} \frac{4-n(p_2-2)}{4} a^{\frac{4}{4-n(p_2-2)}} \varphi^{\frac{2p_2-n(p_2-2)}{4-n(p_2-2)}}(t) \\
&+ \frac{2lp_1+1}{lp_1} \frac{4-n(p_1-2)}{4} b^{\frac{4}{4-n(p_1-2)}} \varphi^{\frac{2p_1-n(p_1-2)}{4-n(p_1-2)}}(t)
\end{aligned}
\tag{22}
$$

From the condition $p_1 > 2$, taking $\left( B^2 \frac{n(p_2-2)(2lp_1+1)}{lp_1(4l^2+1)} \right)^{\frac{n(p_2-2)}{4}} = a$ and $b = \left( B^2 \frac{n(p_1-2)(2lp_1+1)}{lp_1(4l^2+1)} \right)^{\frac{n(p-2)}{4}}$ in (22), we have

$$\varphi'(t) \le C_1 + C_3 \varphi^{C_2}(t) + C_5 \varphi^{C_4}(t), \tag{23}$$

Integrating (23) from 0 to $t$ and let $y = \varphi(t)$, we obtain the following inequality

$$t \ge \int_{\varphi(0)}^{\varphi(t)} \frac{\mathrm{d}y}{C_1 + C_3 y^{C_2} + C_5 y^{C_4}}. \tag{24}$$

Let $t \to T^-$ in (24), we have the lower bound for blow-up time $T$:

$$T \ge \int_{\|u_0\|_2^2}^{\infty} \frac{\mathrm{d}y}{C_1 + C_3 y^{C_2} + C_5 y^{C_4}}. \tag{25}$$

Next, we consider the case $n = 1, 2$. Using Hölder's inequality and the Sobolev embedding theorem, we gain

$$
\begin{aligned}
\int_\Omega |u|^{p(x)} \mathrm{d}x &\le \|u\|_{p_2}^{p_2} + \|u\|_{p_1}^{p_1} \le \|u\|_2^2 \|u\|_\infty^{p_2-2} + \|u\|_2^2 \|u\|_\infty^{p_1-2} \\
&\le \left( (B_* \|\nabla u(t)\|_2)^{p_2-2} + (B_* \|\nabla u(t)\|_2)^{p_1-2} \right) \|u\|_2^2.
\end{aligned}
$$

The condition $2 < p_1 \le p_2 < 4$ implies $0 < \frac{(p_1-2)}{4} < \frac{(p_2-2)}{4} < 1$, then using (19) to (26), we have for all $d > 0$,

$$
\begin{aligned}
\|u\|_{p_2}^{p_2} &\le \left( d \|u\|_2^2 \right)^{\frac{2}{4-p_2} \cdot \frac{4-p_2}{2}} \cdot \left( d^{-1} \left( B_*^{p_2-2} \|\nabla u(t)\|_2 \right) \right)^{\frac{p_2-2}{2} \cdot \frac{2}{p_2-2}} \\
&\le \frac{4-p_2}{2} d^{\frac{2}{4-p_2}} \varphi^{\frac{2}{4-p_2}}(t) + \frac{(p_2-2)}{2} d^{\frac{-2}{p_2-2}} B_*^2 \|\nabla u(t)\|_2^2,
\end{aligned}
\tag{26}
$$

similarly, for all $e > 0$,

$$\|u\|_{p_1}^{p_1} \le \frac{4-p_1}{2} d^{\frac{2}{4-p_1}} \varphi^{\frac{2}{4-p_1}}(t) + \frac{(p_1-2)}{2} d^{\frac{-2}{p_1-2}} B_*^2 \|\nabla u(t)\|_2^2, \tag{27}$$

Combining (17), (26) and (27), for all $d$, $e > 0$, we know

$$\begin{aligned}
\varphi'(t) \le{} & \frac{1}{l} E(0) \\
& + \left( -\left(2l^2 + \frac{1}{2}\right) + \left(\frac{2lp_1+1}{lp_1}\right) \frac{(p_2-2)}{2} d^{-\frac{2}{p_2-2}} B_*^2 + \left(\frac{2lp_1+1}{lp_1}\right) \frac{(p_1-2)}{2} e^{-\frac{2}{p_1-2}} B_*^2 \right) \|\nabla u(t)\|_2^2 \\
& + \left(\frac{2lp_1+1}{lp_1}\right) \frac{4-p_2}{2} d^{\frac{2}{4-p_2}} \varphi^{\frac{2}{4-p_2}}(t) \\
& + \left(\frac{2lp_1+1}{lp_1}\right) \frac{4-p_1}{2} e^{\frac{2}{4-p_1}} \varphi^{\frac{2}{4-p_1}}(t).
\end{aligned} \tag{28}$$

Also from the condition $p_1 > 2$, taking $\left(2B_*^2 \frac{(p_2-2)(2lp_1+1)}{lp_1(4l^2+1)}\right)^{\frac{p_2-2}{2}} = d$ and $e = \left(2B_*^2 \frac{(p_1-2)(2lp_1+1)}{lp_1(4l^2+1)}\right)^{\frac{p_1-2}{2}}$ in (22), we have

$$\varphi'(t) \le C_1 + C_6 \varphi^{C_5}(t) + C_8 \varphi^{C_7}(t), \tag{29}$$

Then similarly to (24) and (25), we can gain a lower bound for blow-up time $T$:

$$T \ge \int_{\|u_0\|_2^2}^{\infty} \frac{\mathrm{d}y}{C_1 + C_6 y^{C_5} + C_8 y^{C_7}}. \tag{30}$$

Thus Theorem 4 is proved. $\qquad\square$

## 4. Upper bounds for blow-up time

In this section, for the problem (3), we establish a blow-up criterion and obtain an upper bound for the blow-up time of weak solutions by differential inequality technique. We need also the following hypotheses.

(C′) The initial value $u_0 \in H_0^1(\Omega) \cap L^{p(\cdot)}(\Omega)$ satisfies $E(0) < 0$, where $E(t)$ is the energy functional (6) and

$$E(0) = \frac{1}{2} \|\nabla u_0\|_2^2 - \int_\Omega \frac{1}{p(x)} |u_0|^{p(x)} \, \mathrm{d}x.$$

(D′) $g$ satisfies: $1 - \int_0^{+\infty} g(s) \, \mathrm{d}s = l \in \left[ \frac{1}{(p_1-1)^2}, 1 \right]$.

**Theorem 5.** *Under the conditions* (A), (C′) *and* (D′). *Assume,* $p_1 > 2$, $\|u_0\|_2 \ne 0$, $u(x,t)$ *is a weak solution of problem* (3). *Then* $u(x,t)$ *blows up in finite time* $T$. *Furthermore, an upper bound for blow-up time* $T$ *is given by* $\frac{\|u_0\|_2^2}{p_1(2-p_1)E(0)}$.

**Proof.** Let also define

$$\varphi(t) = \int_\Omega |u(x,t)|^2 \, \mathrm{d}x. \tag{31}$$

Then (12) tells us that

$$\varphi'(t) = -2 \|\nabla u(t)\|_2^2 \left(1 - \int_0^t g(t-s) \, \mathrm{d}s\right) + 2 \int_\Omega |u|^{p(x)} \, \mathrm{d}x + I_1, \tag{32}$$

where

$$I_1 = 2 \int_0^t g(t-s) \int_\Omega \nabla u(t) (\nabla u(s) - \nabla u(t)) \, \mathrm{d}x \mathrm{d}s.$$

By Hölder's inequality and the condition (A), we know

$$\begin{aligned}
I_1 \ge{} & \frac{-1}{p_1} \int_0^t g(t-s) \|\nabla u(t)\|_2^2 \, \mathrm{d}s - p_1 \int_0^t g(t-s) \int_\Omega |\nabla u(t) - \nabla u(s)|^2 \, \mathrm{d}x \mathrm{d}s \\
\ge{} & -\frac{1-l}{p_1} \|\nabla u(t)\|_2^2 - p_1 (g \diamond \nabla u).
\end{aligned} \tag{33}$$

Next, we define the function,

$$\psi(t) = -2p_1 E(t) = -p_1 \left(g \diamond \nabla u\right) - p_1 \left(1 - \int_0^t g(s)\,\mathrm{d}s\right) \|\nabla u(t)\|_2^2 + 2p_1 \int_\Omega \frac{1}{p(x)} |u(x,t)|^{p(x)}\,\mathrm{d}x. \quad (34)$$

Using the conditions (A), (D$'$), and (32)–(34), we obtain

$$\begin{aligned}
\varphi'&(t) - \psi(t) \\
&= p_1 \left(g \diamond \nabla u\right) + \int_\Omega |u|^{p(x)}\mathrm{d}x + I_1 \\
&\quad + (p_1 - 2)\left(1 - \int_0^t g(s)\,\mathrm{d}s\right) \|\nabla u(t)\|_2^2 + 2\int_\Omega |u|^{p(x)}\mathrm{d}x - 2p_1 \int_\Omega \frac{1}{p(x)} |u(x,t)|^{p(x)}\,\mathrm{d}x \\
&\geq \left((p_1 - 2)l - \frac{1-l}{p_1}\right) \|\nabla u(t)\|_2^2 + 2\left(1 - \frac{p_1}{p_1}\right) \int_\Omega |u|^{p(x)}\mathrm{d}x \geq 0.
\end{aligned} \quad (35)$$

Also, from the condition (A)and (7), we have

$$\varphi'(t) = -2p_1 E'(t) = p_1 \left(g(t)\,\|\nabla u(t)\|_2^2 - \left(g' \diamond \nabla u\right) + 2\int_\Omega u_t^2(x,t)\,\mathrm{d}x\right) \geq 0. \quad (36)$$

Using (31), (36) and the Schwarz's inequality, we see

$$\begin{aligned}
\varphi(t)\psi'(t) &\geq 2p_1 \int_\Omega u^2(x,t)\,\mathrm{d}x \int_\Omega u_t^2(x,t)\,\mathrm{d}x \geq 2p_1 \left(\int_\Omega u(x,t)\,u_t(x,t)\,\mathrm{d}x\right)^2 \\
&= \frac{p_1}{2} \left(\varphi'(t)\right)^2.
\end{aligned} \quad (37)$$

From the condition (C$'$) and (34), $\psi(0) > 0$. Combining (35) and (36), we get

$$\varphi'(t) \geq \psi(t) > 0, \quad \varphi(t) \geq \varphi(0) > 0, \quad \forall\, t > 0. \quad (38)$$

Hence, (37) and (38) imply

$$\varphi(t)\psi'(t) \geq \frac{p_1}{2}\varphi'(t)\varphi'(t) \geq \frac{p_1}{2}\varphi'(t)\psi(t),$$

which can be written as

$$\frac{\psi'(t)}{\psi(t)} \geq \frac{p_1}{2}\frac{\varphi'(t)}{\varphi(t)}. \quad (39)$$

Integrating (39) from 0 to $t$, then

$$\frac{\psi(t)}{\psi(0)} \geq \left(\frac{\varphi(t)}{\varphi(0)}\right)^{\frac{p_1}{2}}. \quad (40)$$

Using (38) and (39), we have

$$\frac{\varphi'(t)}{\varphi^{\frac{p_1}{2}}(t)} \geq \frac{\psi(0)}{\varphi^{\frac{p_1}{2}}(0)}. \quad (41)$$

Integrating inequality (41) from 0 to $t$, for $p_1 > 2$, one has

$$\frac{1}{\varphi^{\frac{p_1-2}{2}}(t)} \leq \frac{1}{\varphi^{\frac{p_1-2}{2}}(0)} - \frac{p_1 - 2}{2}\frac{\psi(0)}{\varphi^{\frac{p_1}{2}}(0)}\,t. \quad (42)$$

Clearly, (42) cannot hold for all time $t$ and we conclude that $u(x,t)$ blows up in finite time $T$. In fact, let $t \to T^-$, (42) leads to

$$T \leq \frac{\|u_0\|_2^2}{p_1\left(2 - p_1\right)E(0)}.$$

The proof of Theorem 5 is complete. $\qquad\square$

**Remark 6.** Under the conditions of Theorems 4 and 5, we can prove that the lower bounds are smaller than the upper bounds for the blow-up time, i.e.

$$\int_{\|u_0\|_2^2}^{\infty} \frac{\mathrm{d}y}{C_1 + C_3 y^{C_2} + C_5 y^{C_4}} < \frac{\|u_0\|_2^2}{E(0)\,p_1\,(2-p_1)}, \quad n \geq 3;$$

$$\int_{\|u_0\|_2^2}^{\infty} \frac{\mathrm{d}y}{C_1 + C_6 y^{C_5} + C_8 y^{C_7}} < \frac{\|u_0\|_2^2}{E(0)\,p_1\,(2-p_1)}, \quad n = 1,\,2,$$

where $C_1$, $C_2$, $C_3$, $C$, $C_5$, $C_6$, and $C_7$ are the same constants in Theorem 4.

**Proof.** For the case $n \geq 3$, under the conditions of Theorems 4 and 5, we can take $(C_1, C_5) = (0,0)$ and $(C_1, C_3) = (0,0)$, then the lower bound is

$$\min\left( \frac{\|u_0\|_2^{2(1-C_2)}}{C_3(C_2 - 1)}, \frac{\|u_0\|_2^{2(1-C_4)}}{C_5\,(C_4 - 1)} \right) \leq \frac{\min\left( \|u_0\|_2^{2(1-C_2)}, \|u_0\|_2^{2(1-C_4)} \right)}{C_3\,(C_2 - 1)},$$

where $C_2$, $C_3$, $C_4$, and $C_5$ are constants in Theorem 4.

On the other hand, using (20) and (21) for all $\delta_1$, $\delta_2 > 0$, we have

$$\begin{aligned}
E(0) &\geq \frac{1}{2}\|\nabla u_0\|_2^2 - \frac{1}{p_1}\int_\Omega |u_0|^{p(x)}\,\mathrm{d}x \\
&\geq \frac{1}{2}\|\nabla u_0\|_2^2 - \frac{1}{p_1}\left( \|u_0\|_{p_2}^{p_2} + \|u_0\|_{p_1}^{p_1} \right) \\
&\geq \left( \frac{1}{2} - B^2\frac{n(p_2-2)}{4p_1}\delta_1^{\frac{-4}{n(p_2-2)}} - B^2\frac{n(p_1-2)}{4p_1}\delta_2^{\frac{-4}{n(p_1-2)}} \right)\|\nabla u_0\|_2^2 \\
&\quad - \left( \frac{4-n(p_2-2)}{4p_1}\delta_1^{\frac{4}{4-n(p_2-2)}}\|u_0\|_2^{2C_2} + \frac{4-n(p_1-2)}{4p_1}\delta_2^{\frac{4}{4-n(p_1-2)}}\|u_0\|_2^{2C_4} \right).
\end{aligned}$$

Taking $\left( \frac{n(p_2-2)B^2}{p_1} \right)^{\frac{n(p_2-2)}{4}} = \delta_1$ and $\delta_2 = \left( \frac{n(p_1-2)B^2}{p_1} \right)^{\frac{n(p_1-2)}{4}}$ in the above inequality, we know

$$\begin{aligned}
E(0) &\geq -\left( \begin{array}{l} \frac{4-n(p_2-2)}{4p_1}\left( \frac{n(p_2-2)B^2}{p_1} \right)^{\frac{n(p_2-2)}{4-n(p_2-2)}}\|u_0\|_2^{2C_2} \\ + \frac{4-n(p_1-2)}{4p_1}\left( \frac{n(p_1-2)B^2}{p_1} \right)^{\frac{n(p_1-2)}{4-n(p_1-2)}}\|u_0\|_2^{2C_4} \end{array} \right) \\
&\geq -\frac{4-n(p_1-2)}{2p_1}\left( \frac{n(p_2-2)B^2}{p_1} \right)^{\frac{n(p_2-2)}{4-n(p_2-2)}}\max\left( \|u_0\|_2^{2C_2}, \|u_0\|_2^{2C_4} \right).
\end{aligned}$$

Next, we have

$$\begin{aligned}
\frac{\|u_0\|_2^2}{E(0)\,p_1\,(2-p_1)} &\geq \frac{4}{(4-n(p_1-2))\,(p_1-2)}\left( \frac{n(p_2-2)B^2}{p_1} \right)^{\frac{-n(p_2-2)}{4-n(p_2-2)}}\frac{\|u_0\|_2^2}{\max\left( \|u_0\|_2^{2C_2}, \|u_0\|_2^{2C_4} \right)} \\
&= \frac{4}{(4-n(p_1-2))\,(p_1-2)}\left( \frac{n(p_2-2)B^2}{p_1} \right)^{\frac{-n(p_2-2)}{4-n(p_2-2)}}\min\left( \|u_0\|_2^{2-2C_2}, \|u_0\|_2^{2-2C_4} \right).
\end{aligned}$$

From the conditions $2 < p_1 \leq p_2 < 2 + \frac{4}{n}$ and $0 < l < 1$, we obtain

$$\frac{1}{C_3(C_2 - 1)} = \frac{lp_1}{2lp_1 + 1} \frac{2}{(p_2 - 2)} \left(B^2 \frac{n(p_2 - 2)(2lp_1 + 1)}{2lp_1(4l^2 + 1)}\right)^{\frac{-n(p_2 - 2)}{4 - n(p_2 - 2)}}$$

$$\leq \frac{lp_1}{2lp_1 + 1} \frac{2}{(4 - n(p_1 - 2))(p_1 - 2)} \left(B^2 \frac{n(p_2 - 2)(2lp_1 + 1)}{2lp_1(4l^2 + 1)}\right)^{\frac{-n(p_2 - 2)}{4 - n(p_2 - 2)}} (4 - n(p_1 - 2))$$

$$< \frac{4}{(4 - n(p_1 - 2))(p_1 - 2)} \left(B^2 \frac{n(p_2 - 2)}{2p_1}\right)^{\frac{-n(p_2 - 2)}{4 - n(p_2 - 2)}}.$$

Which means

$$\frac{\min\left(\|u_0\|_2^{2 - 2C_2}, \|u_0\|_2^{2 - 2C_4}\right)}{C_3(C_2 - 1)} < \frac{\|u_0\|_2^2}{E(0) p_1 (2 - p_1)}.$$

Also, for the case $n = 1, 2$, can be treated even more similarly, and we have

$$\int_{\|u_0\|_2^2}^\infty \frac{\mathrm{d}y}{C_1 + C_6 y^{C_5} + C_8 y^{C_7}} < \frac{\|u_0\|_2^2}{E(0) p_1 (2 - p_1)},$$

where $C_1$, $C_5$, $C_6$, $C_7$ and $C_8$ are constants in Theorem 4. $\qquad\qquad \square$

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