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Inequalities Involving q-Analogue of Multiple Psi Functions

Inégalités impliquant des q-analogues des fonction psi multiples

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Abstract. Logarithmic derivative of the multiple gamma function is known as the multiple psi function. In this work q-analogue of multiple psi functions of order n have been considered. Subadditive, superadditive and convexity properties of higher order derivatives of these functions are derived. Some related inequalities for these functions and their ratios are also obtained.

Résumé. La dérivée logarithmique de la fonction gamma multiple est connue comme la fonction psi multiple. Dans ce travail, des q-analogues de fonctions psi multiples d'ordre n ont été considérés. Des propriétés de sous-additivité, superadditivité et convexité des dérivées d'ordre supérieur de ces fonctions en découlent. Certaines inégalités apparentées sont également obtenues pour ces fonctions et leur rapports.

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1. Introduction

The multiple gamma functions of Barnes [4, 5] introduced more than a century ago have been taken up during the last decades because they enter in several different areas of modern mathematics represented by e.g., S. N. M. Ruijsenaars [13] and M. F. Vignéras [16]. These functions are useful in summation of series and infinite products and in the theory of elliptic functions and theta functions [7]. The inverse of the double gamma function is the well known G-function [7]. Multiple gamma functions Γ_n are useful to study the determinant of Laplacians on the n-dimensional unit sphere S^n [7]. The multiple gamma functions Γ_n ($n \in \mathbb{N}$) which are also defined and denoted by $G_n := \Gamma_n^{(-1)^{n-1}}$ satisfy the following recurrence relation

$$\Gamma_{n+1}(z+1)\Gamma_n(z) = \Gamma_{n+1}(z), \quad \Gamma_1(z) = \Gamma(z), \quad \Gamma(1) = 1, \quad (n \in \mathbb{N}, z \in \mathbb{C}), \tag{1}$$

where Γ denotes the familiar Euler's gamma function (see, e.g., [14, Sections 1.1 and 1.4]). Here and throughout, let $\mathbb{N}$ and $\mathbb{C}$ be sets of positive integers and complex numbers, respectively. The above relations (1) with the condition $(-1)^n \frac{d^{n+1}}{dx^{n+1}} \log\Gamma_n(x) \geq 0$, $x > 0$ are known as the conditions of generalized Bohr–Mollerup theorem. Logarithmic derivative of the multiple gamma function is known as the multiple psi function and is denoted by $\Psi_n(z) = \frac{\Gamma_n'(z)}{\Gamma_n(z)}$. The poly multiple gamma function $\Psi_n^{(m)}$ is the m-th order derivative of Ψ_n.

Quantum calculus (or q-calculus), sometimes called calculus without limits plays important role in mathematical physics, quantum physics, theoretical physics, approximation theory and in many branches of applied science and engineering. Because of the nonlinear structure of the quantum calculus, several problems remain unsolved. Nowadays, most of the researchers and mathematicians [6, 8, 12, 14, 15] are interested in q-calculus due to its applications in various fields of science and engineering.

Throughout this paper we suppose $0 < q < 1$. The q-deformed gamma function was first introduced by F. Jackson in [10, 11] and is defined as [8]

$$\Gamma_q(x) = \lim_{n\to\infty} \int_0^{[n]_q} \left(1 - \frac{t}{[n]_q}\right)_q^n t^{x-1} \mathrm{d}_q t = \int_0^{[\infty]_q} E_q(-t)\, t^{x-1} \mathrm{d}_q t, \tag{2}$$

where $[x]_q$, $[\infty]_q$ and $[n]_q!$ are defined as

$$[x]_q = \frac{1-q^x}{1-q}, [\infty]_q = \frac{1}{1-q} \text{ and } [n]_q! = 1(1+q)(1+q+q^2)\cdots(1+q+q^2+\cdots+q^{n-1})$$

and q-analogue of exponential function is defined as

$$E_q(x) = \sum_{k=0}^{\infty} \frac{q^{k(k-1)/2}}{[k]_q!} x^k.$$

The q-binomial (Gaussian binomial) is defined as

$$(x+y)_q^n = \sum_{k=0}^{n} \binom{n}{k}_q x^k y^{n-k}$$

with

$$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

It is well known that

$$\lim_{q\to 1} \Gamma_q(x) = \Gamma(x) = \int_0^{\infty} e^{-t} t^{x-1} \mathrm{d}t.$$

The q-analogue of multiple gamma functions of order n are denoted by $\Gamma_n(z;q) = (G_n(z;q))^{(-1)^{n-1}}$ and satisfies a q-analogue of generalized Bohr–Mollerup theorem:

Theorem 1 ([15]). *There exists a unique hierarchy of functions which satisfy*

(1) $G_n(z+1;q) = G_{n-1}(z;q)G_n(z;q)$,
(2) $G_n(1,q) = 1$,
(3) $\frac{\mathrm{d}^{n+1}}{\mathrm{d}x^{n+1}} \log G_n(x+1;q) \geq 0$ *for* $x \geq 0$,
(4) $G_0(z;q) = [z]_q$.

A function f is called subadditive on a set I of real numbers if $f(x+y) \leq f(x) + f(y)$ for all $x, y \in I$ such that $x+y \in I$. If the inequality reverses, then f is called superadditive on I. If $f(xy) \leq f(x)f(y)$ holds for all $x, y \in I$ such that $xy \in I$, then f is known as submultiplicative. If the inequality reverses then f is called supermultiplicative. These functions play vital role in number theory, in the theory of differential equations and also in the theory of convex bodies.

H. Alzer and S. Ruscheweyh [3] proved that $x \mapsto (\Gamma(x))^\alpha$ is subadditive on $(0,\infty)$ if and only if $\alpha^* \leq \alpha \leq 0$, where $\alpha^* \approx -0.946850\ldots$. In [1], H. Alzer derived that $\Psi(e^x)$ is strictly concave on $\mathbb{R}$, where $\Psi(x) = \frac{\mathrm{d}}{\mathrm{d}x}\log\Gamma(x)$ is known as the psi (digamma) function. Recently in 2007, H. Alzer [2] proved the subadditive and superadditive properties of Euler's gamma function and obtained the following interesting inequality

$$\left(\frac{\Gamma(x+y+c)}{\Gamma(x+y)}\right)^{1/\alpha} < \left(\frac{\Gamma(x+c)}{\Gamma(x)}\right)^{1/\alpha} + \left(\frac{\Gamma(y+c)}{\Gamma(y)}\right)^{1/\alpha}.$$

The above inequality holds for all $x, y > 0$ if and only if $\alpha \le \max(1, c)$, where $0 < c \neq 1$. The reverse inequality is valid for all positive x and y if and only if $\alpha \le \min(1, c)$. In [9], B.-N. Guo, F. Qi and Q.-M. Luo discussed the additivity of polygamma functions. In [12] T. Mansour and A. Sh. Shabani derived several inequalities for the q-digamma function. Recently, N. Batir [6] obtained monotonicity properties of q-digamma and q-trigamma functions.

The above results motivate us to derive subadditive, superadditive and convexity properties of functions involving $\Psi_n(x;q) = \frac{\Gamma'_n(x;q)}{\Gamma_n(x;q)}$ and its higher order derivatives ($\Psi_n^{(m)}(x;q) = \frac{\mathrm{d}^m}{\mathrm{d}x^m}\Psi_n(x;q)$). Further we obtain some related inequalities for these functions and their ratios.

2. Inequalities for q-Poly Multiple Gamma Function

The q-analogue of multiple gamma functions of order n can also be defined as [15, (6)]

$$\Gamma_n(z;q) = (G_n(z;q))^{(-1)^{n-1}}, \tag{3}$$

where

$$\begin{aligned}\log G_n(z+1;q) = &-\binom{z}{n}\log(1-q) - \sum_{k=1}^{\infty}\binom{-k}{n-1}\log(1-q^{z+k}) \\ &+ \sum_{j=0}^{n-1} G_{n,j}(z)\left\{\sum_{k=1}^{\infty} k^j \log(1-q^k)\right\},\end{aligned} \tag{4}$$

where

$$\binom{z-u}{n-1} = \sum_{j=0}^{n-1} G_{n,j}(z)u^j.$$

It can be noted that

$$\binom{-k}{n-1} = (-1)^{n-1}\binom{n+k-2}{n-1}. \tag{5}$$

Let $f(z) = \log(1-q^{z+k})$. Then

$$\begin{aligned} f'(z) &= -\ln q \sum_{k=0}^{\infty} \frac{q^{z+k}}{1-q^{z+k}} = -\ln q \sum_{k=1}^{\infty}\frac{q^{kz}}{1-q^k} \\ \implies f^{(m+1)}(z) &= -(\ln q)^{m+1}\sum_{k=1}^{\infty}\frac{k^m q^{kz}}{1-q^k}. \end{aligned} \tag{6}$$

From (3), we have

$$\log G_n(z;q) = (-1)^{n-1}\log\Gamma_n(z;q)). \tag{7}$$

Using (7), (3), (5) and (6) and differentiating (4) with respect to z, $m+1$ times ($m \ge n$) we have

$$\Psi_n^{(m)}(z;q) = (\ln q)^{m+1}\sum_{k=1}^{\infty}\binom{n+k-2}{n-1}\frac{k^m q^{kz}}{1-q^k}, \quad 0<q<1, \quad m \ge n.$$

Hence, clearly we have the following conclusions

(i) If m is odd (even), then $\Psi_n^{(m)}(x;q) > (<) 0$ for $0<q<1$ and $x>0$.
(ii) If m is odd (even), then $\Psi_n^{(m)}(x;q)$ is decreasing (increasing) for $0<q<1$ and $x>0$.

Thus we have the following results.

Theorem 2. *$\Psi_n^{(m)}(x;q)$ is convex (concave) on $(0,\infty)$ if m is odd (even) for $0<q<1$.*

Corollary 3. *$\Psi_n^{(m)}(e^x;q)$ is convex (concave) on $\mathbb{R}$ if m is odd (even) for $0<q<1$.*

Now we will discuss about subadditive and superadditive properties.

Theorem 4. *For $0<q<1$, $a\geq 0$, $x,y>0$ and $m\geq n$, the following inequalities hold:*

$$\begin{aligned}\Psi_n^{(m)}(a+x+y;q) &\leq \Psi_n^{(m)}(a+x;q)+\Psi_n^{(m)}(a+y;q), \quad n \text{ is odd;}\\ \Psi_n^{(m)}(a+x+y;q) &\geq \Psi_n^{(m)}(a+x;q)+\Psi_n^{(m)}(a+y;q), \quad n \text{ is even.}\end{aligned}$$

Proof. Let $f(x)=\Psi_n^{(m)}(a+x+y;q)-\Psi_n^{(m)}(a+x;q)+\Psi_n^{(m)}(a+y;q)$. Let m be odd. Then keeping y fixed, we have

$$\begin{aligned}f'(x) &= \Psi_n^{(m+1)}(a+x+y;q)-\Psi_n^{(m+1)}(a+x;q)\\ &= (\ln q)^{m+1}\sum_{k=1}^{\infty}\binom{n+k-2}{n-1}\frac{k^m}{1-q^k}\left(q^{k(a+x+y)}-q^{k(a+x)}\right)\end{aligned}$$

Since $u\geq v\Rightarrow q^u\leq q^v$ for $0<q<1$. Therefore, $f'(x)<0$. Hence, f is decreasing for $x,y\in(0,\infty)$ and $0<q<1$.

Clearly, $\lim_{x\to\infty}f(x)>0$.

Consequently, $f(x)>0$ for $x,y>0$ and $0<q<1$.

Similarly, if m is even we can show that $f(x)<0$ for $x,y>0$ and $0<q<1$. which proves the theorem. □

Now we have the following corollary explaining the additivity of $\Psi_n^{(m)}(x;q)$ for $x>0$ and $0<q<1$.

Corollary 5. *$\Psi_n^{(m)}(x;q)$ is subadditive (superadditive) if m is odd (even) for $x>0$ and $0<q<1$.*

Proof. $a=0$ in Theorem 4 gives the proof of the corollary. □

Theorem 6. *For $0<q<1$, $a\geq 0$, $x,y>0$, the following inequalities hold*

$$\left[\Psi_n^{(m)}(a+x+y;q)\right]^2 \leq \Psi_n^{(m)}(a+x;q)\Psi_n^{(m)}(a+y;q), \quad \text{if } m \text{ is odd;} \tag{8}$$

$$\left[\Psi_n^{(m)}(a+x+y;q)\right]^2 \geq \Psi_n^{(m)}(a+x;q)\Psi_n^{(m)}(a+y;q), \quad \text{if } m \text{ is even,} \tag{9}$$

for all $m\geq n$.

Proof. Let $m\geq n$. We will use the fact that $u\geq v\geq 0\Rightarrow q^u\leq q^v$ for $0<q<1$ to prove the theorem.

Case I. Let m be odd. Then $(\ln q)^{m+1}>0$. Therefore,

$$\begin{aligned}\Psi_n^{(m)}(a+x;q) &= (\ln q)^{m+1}\sum_{k=1}^{\infty}\binom{n+k-2}{n-1}\frac{k^m q^{k(a+x)}}{1-q^k}\\ &\geq (\ln q)^{m+1}\sum_{k=1}^{\infty}\binom{n+k-2}{n-1}\frac{k^m q^{k(a+x+y)}}{1-q^k}\\ &\geq \Psi_n^{(m)}(a+x+y;q)>0.\end{aligned} \tag{10}$$

Similarly,

$$\Psi_n^{(m)}(a+y;q)\geq \Psi_n^{(m)}(a+x+y;q)>0. \tag{11}$$

Now combining (10) and (11), we have

$$\Psi_n^{(m)}(a+x;q)\Psi_n^{(m)}(a+y;q)\geq\left[\Psi_n^{(m)}(a+x+y;q)\right]^2.$$

Case II. Let m be even. Then $(\ln q)^{m+1} < 0$ for $0 < q < 1$. Then

$$\begin{aligned}
\Psi_n^{(m)}(a+x;q) &= (\ln q)^{m+1}\sum_{k=1}^{\infty}\binom{n+k-2}{n-1}\frac{k^m q^{k(a+x)}}{1-q^k}\\
\implies 0 &< -\Psi_n^{(m)}(a+x;q) \le -\Psi_n^{(m)}(a+x+y;q)\\
\text{and } 0 &< -\Psi_n^{(m)}(a+y;q) \le -\Psi_n^{(m)}(a+x+y;q).
\end{aligned}$$

Combining above inequalities, we have

$$\Psi_n^{(m)}(a+x;q)\Psi_n^{(m)}(a+y;q) \le \left[\Psi_n^{(m)}(a+x+y;q)\right]^2,$$

which proves the theorem. □

Theorem 7. *Let $0 < q < 1$, $a \ge 0$ and $0 < x, y < 1$. Then*

$$\left[\Psi_n^{(m)}(a+xy;q)\right]^2 > \Psi_n^{(m)}(a+x;q)\Psi_n^{(m)}(a+y;q).$$

Proof. Let $f(x) = \left[\Psi_n^{(m)}(a+x;q)\right]^2$. Then

$$f'(x) = 2\Psi_n^{(m)}(a+x;q)\Psi_n^{(m+1)}(a+x;q) < 0.$$

which implies that $f(x)$ is a decreasing on $(0,1)$. Therefore,

$$\begin{aligned}
\left[\Psi_n^{(m)}(a+xy;q)\right]^2 &> \left[\Psi_n^{(m)}(a+x;q)\right]^2;\\
\left[\Psi_n^{(m)}(a+xy;q)\right]^2 &> \left[\Psi_n^{(m)}(a+y;q)\right]^2.
\end{aligned}$$

Combining the above inequalities, we have

$$\left[\Psi_n^{(m)}(a+xy;q)\right]^4 > \left[\Psi_n^{(m)}(a+x;q)\Psi_n^{(m)}(a+y;q)\right]^2.$$

Since, $\Psi_n^{(m)}(a+x;q)\Psi_n^{(m)}(a+y;q) > 0$. Therefore,

$$\left[\Psi_n^{(m)}(a+xy;q)\right]^2 > \Psi_n^{(m)}(a+x;q)\Psi_n^{(m)}(a+y;q).$$

Hence the theorem is proved. □

Now we will find the bounds for the ratio of q-analogue of multiple psi functions.

Theorem 8. *Let a, b, c, d, e, f be real numbers and $f(x)$ be a function defined as*

$$f(x) = \frac{\Psi_n^{(m)}(a+bx;q)^c}{\Psi_n^{(m)}(d+ex;q)^f}, \quad x \ge 0, \quad m \ge n \ge 1.$$

(a) *If $a, b, d, e > 0$, $c \le 0$ and $f \ge 0$, then $f(x)$ is increasing in $[0,\infty)$ and for all $x \in [0,1]$ the following inequality holds*

$$\frac{\Psi_n^{(m)}(a;q)^c}{\Psi_n^{(m)}(d;q)^f} \le \frac{\Psi_n^{(m)}(a+bx;q)^c}{\Psi_n^{(m)}(d+ex;q)^f} \le \frac{\Psi_n^{(m)}(a+b;q)^c}{\Psi_n^{(m)}(d+e;q)^f}.$$

(b) *If $a, b, d, e > 0$, $c \ge 0$ and $f \le 0$, then $f(x)$ is decreasing in $[0,\infty)$ and for all $x \in [0,1]$ the following inequality holds*

$$\frac{\Psi_n^{(m)}(a+b;q)^c}{\Psi_n^{(m)}(d+e;q)^f} \le \frac{\Psi_n^{(m)}(a+bx;q)^c}{\Psi_n^{(m)}(d+ex;q)^f} \le \frac{\Psi_n^{(m)}(a;q)^c}{\Psi_n^{(m)}(d;q)^f}.$$

Proof. Let $g(x) = \ln f(x)$. Then

$$g'(x) = \frac{bc\Psi_n^{(m)}(d+ex;q)\Psi_n^{(m+1)}(a+bx;q) - ef\Psi_n^{(m)}(a+bx;q)\Psi_n^{(m+1)}(d+ex;q)}{\Psi_n^{(m)}(a+bx;q)\Psi_n^{(m)}(d+ex;q)}. \tag{12}$$

Now,

$$\Psi_n^{(m)}(d+ex;q)\Psi_n^{(m+1)}(a+bx;q)<0,\ \Psi_n^{(m+1)}(d+ex;q)\Psi_n^{(m)}(a+bx;q)<0$$
$$\text{and } \Psi_n^{(m)}(a+bx;q)\Psi_n^{(m)}(d+ex;q)>0.$$

Hence, using the conditions of part (a), we have $g'(x)\geq 0$, which implies that $g(x)$ is increasing in $[0,\infty)$. Consequently, $f(x)$ is increasing in $[0,\infty)$ and for $0\leq x\leq 1$, $f(0)\leq f(x)\leq f(1)$, which proves part (a) of the theorem.

Note that the condition (b) reverses the inequalities given in part (a). Hence, proceeding similarly like part (a), part (b) of the theorem can be established. □

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