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Number theory / *Théorie des nombres*

A counterexample of two Romanov type conjectures

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Abstract. In this note, we disprove two Romanov type conjectures posed by Chen.

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For any subset A of natural numbers set $\mathbb{N}$, let $A(x) = |A \cap [1, x]|$. In 2008, Chen communicated to the authors of [3] and made the following two conjectures therein.

Conjecture 1. *Let $\mathscr{A}$ and $\mathscr{B}$ be two sets of positive integers. If there exists a constant $c > 0$ such that $\mathscr{A}(\log x/\log 2)\mathscr{B}(x) > cx$ for all sufficiently large x, then the set $\{2^a + b : a \in \mathscr{A},\ b \in \mathscr{B}\}$ has positive lower asymptotic density.*

Conjecture 2. *Let $\mathscr{A}$ and $\mathscr{B}$ be two sets of positive integers. If there exists a constant $c > 0$ such that $\mathscr{A}(\log x/\log 2)\mathscr{B}(x) > cx$ for infinitely many x, then the set $\{2^a + b : a \in \mathscr{A},\ b \in \mathscr{B}\}$ has positive upper asymptotic density.*

Romanov's theorem [6] offers a positive answer to Chen's conjectures when $\mathscr{B}$ is the set of primes. However, it will not be the case in general. We construct a counterexample which disproves simultaneously Chen's conjectures.

Let p_i be the i^{th} odd prime and $d_t = p_1 p_2 \cdots p_t$ for $t \in \mathbb{Z}_+$. For any $t \in \mathbb{Z}_+$, define

$$\mathscr{B}_t = \{n : n \in \mathbb{N},\ d_t \mid n\} \cap \left[2^{2^{t^2}}, 2^{2^{(t+1)^2}}\right), \quad \mathscr{B} = \bigcup_{t=1}^{\infty} \mathscr{B}_t$$

and

$$\mathscr{A} = \mathbb{N}, \quad \mathscr{C} = 2^{\mathscr{A}} + \mathscr{B} = \left\{2^a + b : a \in \mathscr{A},\ b \in \mathscr{B}\right\}.$$

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Let x be a large number and j be the number such that

$$2^{2^{j^2}} \le x < 2^{2^{(j+1)^2}}, \tag{1}$$

which means that

$$\sqrt{\log\log x} < j \le \frac{2}{\sqrt{\log 2}}\sqrt{\log\log x}. \tag{2}$$

From the Chebyshev estimate, we have

$$\begin{aligned} d_j &= \exp\left(\sum_{2<p\le p_j}\log p\right) \\ &\le \exp\left(2j\log j\right) \\ &\le \exp\left(3\sqrt{\log\log x}\log\log\log x\right). \end{aligned} \tag{3}$$

In view of equation (1), we know $2^{2^{(j-1)^2}} < \sqrt{x}$. So by the construction of $\mathscr{B}$ and equation (3), we have

$$\begin{aligned} \mathscr{B}(x) &\ge \frac{x-2^{2^{j^2}}}{d_j} + \frac{2^{2^{j^2}} - 2^{2^{(j-1)^2}}}{d_{j-1}} - 2 \\ &\ge \frac{x-2^{2^{(j-1)^2}}}{d_j} - 2 \\ &\gg \frac{x}{\exp\left(3\sqrt{\log\log x}\log\log\log x\right)}. \end{aligned} \tag{4}$$

It follows that

$$\mathscr{A}\left(\log x/\log 2\right)\mathscr{B}(x) \gg \frac{x\log x}{\exp\left(3\sqrt{\log\log x}\log\log\log x\right)} > x.$$

It remains to prove $\mathscr{C}(x) = o(x)$ as $x \to \infty$. It is clear that

$$\mathscr{C}(x) = \#\left\{c \le x : c = 2^a + b, a \in \mathscr{A}, b \in \mathscr{B}\right\} \le S_1(x) + S_2(x), \tag{5}$$

where

$$S_1(x) = \#\left\{c \le x : c = 2^a + b, a \in \mathscr{A}, b \in \mathscr{B}_j\right\}$$

and

$$S_2(x) = \#\left\{c \le x : c = 2^a + b, a \in \mathscr{A}, b \notin \mathscr{B}_j\right\}.$$

Note that if $c = 2^a + b$ for some $b \in \mathscr{B}_j$, then $p_i \nmid c$ for any $1 \le i \le j$. This fact leads to the following bound

$$\begin{aligned} S_1(x) &\le \sum_{\substack{c\le x \\ \left(c,\prod_{p\le p_j} p\right)=1}} 1 \\ &= \sum_{\ell \mid \prod_{p\le p_j} p} \mu(\ell)\left\lfloor \frac{x}{\ell}\right\rfloor \\ &\le x\prod_{p\le p_j}\left(1-\frac{1}{p}\right) + 2^j. \end{aligned} \tag{6}$$

The same observation yields the following estimate

$$\begin{aligned} S_2(x) &\le \#\left\{c \le x : c = 2^a + b, a \in \mathscr{A}, b \in \mathscr{B}_{j-1}\right\} + 2^{2^{(j-1)^2}}\frac{\log x}{\log 2} \\ &\ll x\prod_{p\le p_{j-1}}\left(1-\frac{1}{p}\right) + \sqrt{x}\log x + 2^j. \end{aligned} \tag{7}$$

It can be seen that
$$2^j \ll \exp\left(3\sqrt{\log\log x}\right) \ll \sqrt{x}$$
from equation (2). Therefore, combing equations (5), (6) and (7) we have
$$\begin{aligned}\mathscr{C}(x) &\ll x \prod_{p \le p_{j-1}} \left(1 - \frac{1}{p}\right) + \sqrt{x}\log x \\ &\ll x\left(\log p_{j-1}\right)^{-1} + \sqrt{x}\log x \\ &\ll x\left(\log\log\log x\right)^{-1},\end{aligned}$$
where the last but one step follows from the Mertens estimate, which is surely more to expectation than our requirement.

We remark that the Romanov type problems start from the remarkable paper [6], where it is proved that there is a positive lower asymptotic density of odd numbers which can be represented by the sum of a prime and a power of 2. In the opposite direction, van der Corput [1] showed that there is a positive lower asymptotic density of odd numbers none of whose members can be represented by the sum of a prime and a power of 2. Subsequently, Erdős [2] constructed an arithmetic progression of odd numbers having the same property required in the paper of van der Corput. The results of van der Corput and Erdős give a negative answer to an old conjecture of de Polignac [4, 5].

At present, the author of this note has no answer to the following question. Let $\mathscr{B}$ be a set of positive integers satisfying $\mathscr{B}(x) = O(x/\log x)$, then is it true that
$$\limsup_{x\to\infty} \frac{\mathscr{C}(x)}{\mathscr{B}(x)} = \infty,$$
where
$$\mathscr{C} = 2^{\mathbb{N}} + \mathscr{B} = \left\{2^k + b : k \in \mathbb{N},\ b \in \mathscr{B}\right\}.$$

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