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On the two-dimensional singular stochastic viscous nonlinear wave equations

Ruoyuan Liu [a] **and Tadahiro Oh** [a]

[a] School of Mathematics, The University of Edinburgh, and The Maxwell Institute for the Mathematical Sciences, James Clerk Maxwell Building, The King's Buildings, Peter Guthrie Tait Road, Edinburgh, EH9 3FD, United Kingdom

E-mails: ruoyuan.liu@ed.ac.uk, hiro.oh@ed.ac.uk

Abstract. We study the stochastic viscous nonlinear wave equations (SvNLW) on $\mathbb{T}^2$, forced by a fractional derivative of the space-time white noise ξ. In particular, we consider SvNLW with the singular additive forcing $D^{\frac{1}{2}}\xi$ such that solutions are expected to be merely distributions. By introducing an appropriate renormalization, we prove local well-posedness of SvNLW. By establishing an energy bound via a Yudovich-type argument, we also prove pathwise global well-posedness of the defocusing cubic SvNLW. Lastly, in the defocusing case, we prove almost sure global well-posedness of SvNLW with respect to certain Gaussian random initial data.

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1. Introduction

1.1. *Stochastic viscous nonlinear wave equations*

In [22], Kuan and Čanić proposed the following wave equation on $\mathbb{R}^2$ augmented by the viscous effect:

$$\partial_t^2 u - \Delta u + 2\mu D \partial_t u = F_{\text{ext}}(u), \tag{1}$$

where $\mu > 0$ is a constant, $D = |\nabla| = \sqrt{-\Delta}$, and $F_{\text{ext}}(u)$ denotes an external forcing, which may in general depend on the unknown u. The equation (1) appears in the study of fluid-structure interaction in the three-dimensional space where the Dirichlet–Neumann operator models the coupling between a viscous, incompressible fluid and an elastic structure. Here, the viscosity term $2\mu D\partial_t u$ in (1) represents the effect of the Cauchy stress tensor of Newtonian fluid in the vertical direction (namely, in z-direction). See [22] for the derivation of (1).

The general solution to the homogeneous linear viscous wave equation:

$$\partial_t^2 u - \Delta u + 2\mu D \partial_t u = 0 \tag{2}$$

is given by

$$u(t) = e^{(-\mu|\nabla| + \sqrt{(\mu^2-1)|\nabla|^2})t} f_1 + e^{(-\mu|\nabla| - \sqrt{(\mu^2-1)|\nabla|^2})t} f_2.$$

When $\mu \geq 1$, we have $-\mu|\xi| + \sqrt{(\mu^2-1)|\xi|^2} \sim -\mu^{-1}|\xi|$ and thus the equation (1) is purely of parabolic type. In this case, we can study well-posedness of (1), simply by using the Schauder estimate for the Poisson kernel (see Lemma 10 below). On the other hand, when $0 < \mu < 1$, the solution to (2) with initial data $(u, \partial_t u)|_{t=0} = (u_0, u_1)$ is given by

$$u = e^{-\mu D t}\left(\cos\left(\sqrt{1-\mu^2} D t\right) + \frac{\mu}{\sqrt{1-\mu^2}} \sin\left(\sqrt{1-\mu^2} D t\right)\right) u_0 + e^{-\mu D t} \frac{\sin(\sqrt{1-\mu^2} D t)}{\sqrt{1-\mu^2} D} u_1, \tag{3}$$

and thus we see that the equation exhibits an interesting mixture of the wave dispersion and the parabolic regularization by the fluid viscosity. For this reason, we will restrict our attention to $0 < \mu < 1$. Note that, when $0 < \mu < 1$, a precise value of μ (and $\sqrt{1-\mu^2}$) in (3) does not play any important role in terms of the well-posedness theory, and thus without loss of generality, we set $\mu = \frac{1}{2}$ in the remaining part of the paper. See also Footnote 3 below.

In [22], Kuan and Čanić studied well-posedness and ill-posedness of the following viscous nonlinear wave equation (vNLW) on $\mathbb{R}^2$:

$$\partial_t^2 u - \Delta u + D\partial_t u + u^k = 0$$

in both the deterministic and probabilistic settings (in particular with random initial data). See also [24,27]. In a recent preprint [23], Kuan and Čanić also studied the following stochastic viscous wave equation with a multiplicative noise on $\mathbb{R}^d$, $d = 1, 2$:

$$\partial_t^2 u - \Delta u + D\partial_t u = f(u)\xi,$$

where f is a Lipschitz function and ξ denotes the (Gaussian) space-time white noise on $\mathbb{R}_+ \times \mathbb{R}^2$.

In this paper, we consider the following stochastic vNLW (SvNLW) with an additive stochastic forcing on the two-dimensional torus $\mathbb{T}^2 = (\mathbb{R}/\mathbb{Z})^2$:

$$\partial_t^2 u - \Delta u + D\partial_t u + u^k = D^\alpha \xi \tag{4}$$

where $\alpha \geq 0$ and ξ denotes the (Gaussian) space-time white noise on $\mathbb{R}_+ \times \mathbb{T}^2$. By a standard argument (see, for example, Lemma 8 below), we see that the stochastic convolution Ψ, satisfying

$$\partial_t^2 \Psi - \Delta\Psi + D\partial_t \Psi = D^\alpha \xi$$

(say, with the zero initial data), is almost surely a continuous function on $\mathbb{R}_+ \times \mathbb{T}^2$, when $\alpha < \frac{1}{2}$. It is worthwhile to note that a combination of the wave dispersion and the dissipation by the fluid viscosity yields $\frac{3}{2}$-smoothing on the noise (rather than the usual one degree of smoothing for stochastic heat equations [12,29] and stochastic wave equations [18,31]). For this reason, we set $\alpha = \frac{1}{2}$ in this paper and study the following Cauchy problem for SvNLW on $\mathbb{T}^2$:

$$\begin{cases} \partial_t^2 u + (1-\Delta) u + D\partial_t u + u^k = \sqrt{2} D^{\frac{1}{2}} \xi \\ (u, \partial_t u)|_{t=0} = (u_0, u_1). \end{cases} \tag{5}$$

In this case, the corresponding stochastic convolution is merely a distribution and thus we need to introduce a proper renormalization to give a precise meaning to the equation.

Remark 1. In (5), we replaced $-\Delta$ by $1-\Delta$. This modifications simplifies part of the argument (so that we do not need to make a separate analysis at the zeroth frequency). Furthermore, this modification, along with the extra factor $\sqrt{2}$, is necessary for the almost sure global well-posedness result (Theorem 4). Note that Theorems 2 and 3 apply to (4) with $\alpha = \frac{1}{2}$ with essentially identical proofs.

1.2. *Renormalized SvNLW*

In this subsection, we briefly go over the renormalization procedure for (5), following the discussion in [18, 19, 31]. Let Ψ be the solution to the following linear stochastic viscous wave equation:

$$\begin{cases} \partial_t^2 \Psi + (1-\Delta)\Psi + D\partial_t \Psi = \sqrt{2} D^{\frac{1}{2}} \xi \\ (\Psi, \partial_t \Psi)|_{t=0} = (0,0). \end{cases} \tag{6}$$

By writing in the Duhamel formulation (= mild formulation), the stochastic convolution Ψ can be expressed as

$$\Psi(t) = \sqrt{2} \int_0^t S(t-t') D^{\frac{1}{2}} \mathrm{d}W(t'), \tag{7}$$

where the linear propagator $S(t)$ is defined by

$$S(t) = e^{-\frac{D}{2}t} \frac{\sin(t[\![D]\!])}{[\![D]\!]} \qquad \text{with } [\![D]\!] = \sqrt{1 - \tfrac{3}{4}\Delta} \tag{8}$$

and W denotes a cylindrical Wiener process on $L^2(\mathbb{T}^2)$:[1]

$$W(t) = \sum_{n\in\mathbb{Z}^2} B_n(t) e_n.$$

Here, $e_n(x) = e^{2\pi i n\cdot x}$ and $\{B_n\}_{n\in\mathbb{Z}^2}$ is defined by $B_n(t) = \langle \xi, \mathbb{1}_{[0,t]} \cdot e_n \rangle_{t,x}$, where $\langle \cdot, \cdot \rangle_{t,x}$ denotes the duality pairing on $\mathbb{R}_+ \times \mathbb{T}^2$. As a result, we see that $\{B_n\}_{n\in\mathbb{Z}^2}$ is a family of mutually independent complex-valued[2] Brownian motions conditioned so that $B_{-n} = \overline{B_n}$, $n \in \mathbb{Z}^2$.

Given $N \in \mathbb{N}$, we define the truncated stochastic convolution $\Psi_N = \mathbf{P}_N \Psi$, where $\mathbf{P}_N$ denotes the frequency cutoff onto the spatial frequencies $\{|n| \le N\}$. Then, for each fixed $t \ge 0$ and $x \in \mathbb{T}^2$, a direct computation shows that $\Psi_N(t,x)$ is a mean-zero real-valued Gaussian random variable with variance

$$\begin{aligned} \sigma_N(t) &\stackrel{\text{def}}{=} \mathbb{E}\big[\Psi_N(t,x)^2\big] = 2 \sum_{\substack{n\in\mathbb{Z}^2 \\ |n|\le N}} \int_0^t e^{-(t-t')|n|} \bigg[\frac{\sin((t-t')[\![n]\!])}{[\![n]\!]} \bigg]^2 |n| \mathrm{d}t' \\ &= \sum_{\substack{n\in\mathbb{Z}^2 \\ |n|\le N}} \frac{1}{\langle n\rangle^2} - \frac{e^{-t|n|}}{[\![n]\!]^2} \bigg(1 - \frac{|n|^2}{4\langle n\rangle^2} \cos(2[\![n]\!]t) + \frac{|n|[\![n]\!]}{2\langle n\rangle^2} \sin(2[\![n]\!]t) \bigg) \\ &\sim \sum_{\substack{n\in\mathbb{Z}^2 \\ |n|\le N}} \frac{1}{\langle n\rangle^2} \sim \log N \longrightarrow \infty, \end{aligned} \tag{9}$$

as $N \to \infty$, where

$$\langle n\rangle = (1+|n|^2)^{\frac{1}{2}} \qquad \text{and} \qquad [\![n]\!] = \sqrt{1 + \tfrac{3}{4}|n|^2}.$$

From this computation, we see that $\{\Psi_N(t)\}_{N\in\mathbb{N}}$ is almost surely unbounded in $W^{0,p}(\mathbb{T}^2)$ for any $1 \le p \le \infty$.

Let us now consider the truncated SvNLW with the regularized noise:

$$\begin{cases} \partial_t^2 u_N + (1-\Delta) u_N + D\partial_t u_N + u_N^k = \sqrt{2} D^{\frac{1}{2}} \mathbf{P}_N \xi \\ (u_N, \partial_t u_N)|_{t=0} = (u_0, u_1). \end{cases} \tag{10}$$

[1] Hereafter, we drop the harmless factor 2π.

[2] In particular, B_0 is a standard real-valued Brownian motion. Note that we have, for any $n \in \mathbb{Z}^2$,

$$\mathrm{Var}(B_n(t)) = \mathbb{E}\big[\langle \xi, \mathbb{1}_{[0,t]} \cdot e_n \rangle_{t,x} \overline{\langle \xi, \mathbb{1}_{[0,t]} \cdot e_n \rangle_{t,x}}\big] = \|\mathbb{1}_{[0,t]} \cdot e_n\|^2_{L^2_{t,x}} = t.$$

Proceeding with the first order expansion ([5, 12, 28]):

$$u_N = \Psi_N + v_N, \tag{11}$$

we see that the residual term v_N satisfies the following equation:

$$\partial_t^2 v_N + (1-\Delta) v_N + D\partial_t v_N + \sum_{\ell=0}^{k} \binom{k}{\ell} \Psi_N^\ell v_N^{k-\ell} = 0. \tag{12}$$

Note that the power Ψ_N^ℓ does not converge to any limit as $N \to \infty$. This is where we introduce the Wick renormalization:

$$:\!\Psi_N^\ell(t,x)\!: \stackrel{\text{def}}{=} H_\ell(\Psi_N(t,x); \sigma_N(t)), \tag{13}$$

where $H_\ell(x,\sigma)$ is the Hermite polynomial of degree ℓ with variance parameter σ. See Subsection 2.1. This yields the renormalized version of (12):

$$\partial_t^2 v_N + (1-\Delta) v_N + D\partial_t v_N + \sum_{\ell=0}^{k} \binom{k}{\ell} :\!\Psi_N^\ell\!: v_N^{k-\ell} = 0. \tag{14}$$

In Lemma 8, we show that the Wick power $:\!\Psi_N^\ell\!:$ converges to a limit $:\!\Psi^\ell\!:$ in $C([0,T]; W^{-\varepsilon,\infty}(\mathbb{T}^2))$ for any $\varepsilon > 0$ and $T > 0$, almost surely. Then, by taking $N \to \infty$, we obtain the limiting equation:

$$\partial_t^2 v + (1-\Delta) v + D\partial_t v + \sum_{\ell=0}^{k} \binom{k}{\ell} :\!\Psi^\ell\!: v^{k-\ell} = 0. \tag{15}$$

At the level of u_N, in view of (11), we define the renormalized nonlinearity $:\!u_N^k\!:$ by

$$:\!u_N^k\!: = :\!(\Psi_N + v_N)^k\!: = \sum_{\ell=0}^{k} \binom{k}{\ell} :\!\Psi_N^\ell\!: v_N^{k-\ell}. \tag{16}$$

Then, if v_N solves (14), then $u_N = \Psi_N + v_N$ satisfies the following truncated renormalized SvNLW:

$$\partial_t^2 u_N + (1-\Delta) u_N + D\partial_t u_N + :\!u_N^k\!: = \sqrt{2} D^{\frac{1}{2}} \mathbf{P}_N \xi. \tag{17}$$

Similarly, if v solves (15), then $u = \Psi + v$ satisfies the following renormalized SvNLW:

$$\partial_t^2 u + (1-\Delta) u + D\partial_t u + :\!u^k\!: = \sqrt{2} D^{\frac{1}{2}} \xi, \tag{18}$$

where the renormalized nonlinearity $:\!u^k\!:$ is defined as in (16) (by dropping the subscript N).

1.3. *Main results*

Our main goal is to study well-posedness of the renormalized SvNLW (18). More precisely, we study the following Duhamel formulation of (15) endowed with initial data $(v, \partial_t v)|_{t=0} = (u_0, u_1)$:

$$v(t) = V(t)(u_0, u_1) - \sum_{\ell=0}^{k} \binom{k}{\ell} \int_0^t S(t-t') :\!\Psi^\ell\!: v^{k-\ell}(t')\, \mathrm{d}t', \tag{19}$$

where the linear propagator $V(t)$ is defined by

$$V(t)(u_0,u_1) = e^{-\frac{D}{2}t}\left(\cos(t[\![D]\!]) + \frac{D}{2[\![D]\!]}\sin(t[\![D]\!])\right)u_0 + e^{-\frac{D}{2}t}\frac{\sin(t[\![D]\!])}{[\![D]\!]}u_1. \tag{20}$$

Then, given the almost sure regularity of the Wick powers $:\!\Psi^\ell\!:$, standard deterministic analysis yields the following local well-posedness result.

Theorem 2. *Let $k \geq 2$ be an integer and $s \geq 1$. Then, the renormalized SvNLW* (18) *is locally well-posed in $\mathscr{H}^s(\mathbb{T}^2) = H^s(\mathbb{T}^2) \times H^{s-1}(\mathbb{T}^2)$ in the sense that the following statement holds true almost surely; given $(u_0, u_1) \in \mathscr{H}^s(\mathbb{T}^2)$, there exists a unique local-in-time solution v to* (15) *with initial data $(v, \partial_t v)|_{t=0} = (u_0, u_1)$, where $\{:\Psi^\ell:\}_{\ell=1}^k$ denotes the stochastic convolution Ψ defined in* (7) *and its Wick powers $:\Psi^\ell:$, $\ell = 2, \ldots, k$, defined in Lemma 8 below.*

Furthermore, there exists an almost surely positive stopping time $T = T(\omega)$ such that the solution $u_N = \Psi_N + v_N$ to the truncated renormalized SvNLW (17) (*and its time derivative $\partial_t u_N$*) *converges to the solution $u = \Psi + v$ to* (18) (*constructed above*) *in $C([0,T]; H^{-\varepsilon}(\mathbb{T}^2))$* (*and to $\partial_t u$ in $C([0,T]; H^{-1-\varepsilon}(\mathbb{T}^2))$, respectively*), *$\varepsilon > 0$, almost surely, as $N \to \infty$. Here, v_N denotes the solution to* (14) *with $(v_N, \partial_t v_N)|_{t=0} = (u_0, u_1)$.*

See Proposition 11 below for the local well-posedness statement at the level of the residual term $v = u - \Psi$, satisfying (19). We point out that the regularity of initial data can be lowered but we do not pursue this issue here. See Remark 12.

Next, we turn our attention to the global well-posedness problem. In the cubic case ($k = 3$), we have the following pathwise global well-posedness result.

Theorem 3. *Let $k = 3$ and $s \geq 1$. Then, the renormalized cubic SvNLW* (18) *is globally well-posed in $\mathscr{H}^s(\mathbb{T}^2)$ in the sense that the following statement holds true almost surely; given $(u_0, u_1) \in \mathscr{H}^s(\mathbb{T}^2)$, there exists a unique global-in-time solution v to* (15) *with initial data $(v, \partial_t v)|_{t=0} = (u_0, u_1)$, where Ψ, $:\Psi^2:$, and $:\Psi^3:$ are the stochastic convolution Ψ defined in* (7) *and its Wick powers defined in Lemma 8 below.*

In proving Theorem 3, we study (15) with $k = 3$:

$$\partial_t^2 v + (1 - \Delta) v + D\partial_t v + v^3 + 3v^2\Psi + 3v{:}\Psi^2{:} + {:}\Psi^3{:} = 0. \tag{21}$$

From the proof of Theorem 2, we see that it suffices to control the $\mathscr{H}^1$-norm of $\vec{v}(t) \overset{\text{def}}{=} (v(t), \partial_t v(t))$. For this purpose, we study the evolution of the energy (with $k = 3$)

$$E(\vec{v}) = \frac{1}{2}\int_{\mathbb{T}^2} \big(v^2 + |\nabla v|^2\big) dx + \frac{1}{2}\int_{\mathbb{T}^2} (\partial_t v)^2 dx + \frac{1}{k+1}\int_{\mathbb{T}^2} v^{k+1} dx \tag{22}$$

for the standard nonlinear wave equation (NLW):

$$\partial_t^2 u + (1 - \Delta) u + u^k = 0. \tag{23}$$

As in the case of the stochastic NLW studied in [19], the energy $E(\vec{v})$ is not conserved under (21) due to the singular perturbative term $3v^2\Psi + 3v{:}\Psi^2{:} + {:}\Psi^3{:}$. For our problem, the dissipation by the viscous term comes in rescue and allows us to establish a double exponential growth bound on $E(\vec{v})$ via a Yudovich-type argument [7, 45]. See Section 4 for details. In [10], Burq and Tzvetkov used an analogous Yudovich-type argument and proved probabilistic global well-posedness of the defocusing cubic NLW, (23) with $k = 3$, on the three-dimensional torus $\mathbb{T}^3$ with randomized initial data in $L^2(\mathbb{T}^3)$. A key difference between Theorem 3 and [10] is that, thanks to the dissipative smoothing effect, we can handle data (namely, the stochastic convolution and its Wick powers) of slightly negative regularity.

Lastly, we consider global well-posedness of (18) with random initial data. More precisely, consider a pair (u_0^ω, u_1^ω) of random functions defined by

$$u_0^\omega = \sum_{n \in \mathbb{Z}^2} \frac{g_n(\omega)}{\langle n \rangle} e_n \qquad \text{and} \qquad u_1^\omega = \sum_{n \in \mathbb{Z}^2} h_n(\omega) e_n. \tag{24}$$

Here, $\{g_n, h_n\}_{n \in \mathbb{Z}^2}$ is a family of independent standard complex-valued Gaussian random variables such that $\overline{g_n} = g_{-n}$ and $\overline{h_n} = h_{-n}$, $n \in \mathbb{Z}^2$. We assume that $\{g_n, h_n\}_{n \in \mathbb{Z}^2}$ is independent from the space-time white noise ξ in (18). A standard computation shows that $(u_0^\omega, u_1^\omega) \in \mathscr{H}^s(\mathbb{T}^2) \setminus \mathscr{H}^0(\mathbb{T}^2)$ for any $s < 0$, almost surely. In particular, (u_0^ω, u_1^ω) in (24) is much rougher than the

$\mathscr{H}^1$-initial data considered in Theorems 2 and 3. Our goal is to prove almost sure global well-posedness of (18) with respect to the Gaussian random initial data (u_0^ω, u_1^ω) in (24).

For this purpose, let us first define the stochastic convolution Φ with the Gaussian random initial data (u_0^ω, u_1^ω) in (24):

$$\begin{cases} \partial_t^2 \Phi + (1-\Delta)\Phi + D\partial_t \Phi = \sqrt{2} D^{\frac{1}{2}} \xi \\ (\Phi, \partial_t \Phi)|_{t=0} = (u_0^\omega, u_1^\omega). \end{cases} \tag{25}$$

By writing (25) in the Duhamel formulation, we have

$$\Phi(t) = V(t)(u_0^\omega, u_1^\omega) + \sqrt{2} \int_0^t S(t-t') D^{\frac{1}{2}} \mathrm{d}W(t'), \tag{26}$$

where $V(t)$ is as in (20). A direct computation with (9) and (24) shows that $\Phi_N(t,x) = \mathbf{P}_N \Phi(t,x)$ is a mean-zero real-valued Gaussian random variable with variance

$$\begin{aligned} \alpha_N &\stackrel{\text{def}}{=} \mathbb{E}\big[\Phi_N(t,x)^2\big] = \mathbb{E}\big[\big(\mathbf{P}_N V(t)(u_0^\omega, u_1^\omega)(x)\big)^2\big] + \mathbb{E}\big[\big(\mathbf{P}_N \Psi(t,x)\big)^2\big] \\ &= \sum_{\substack{n \in \mathbb{Z}^2 \\ |n| \le N}} \frac{1}{\langle n \rangle^2} \sim \log N \end{aligned} \tag{27}$$

for any $t \ge 0$, $x \in \mathbb{T}^2$, and $N \ge 1$. Note that, unlike $\sigma_N(t)$ in (9), the variance α_N is time independent. This is due to the fact that the distribution of $(\Phi_N(t), \partial_t \Phi_N(t))$ is invariant under the linear dynamics (25). As in (13), we then define the Wick power by

$$:\!\Phi_N^\ell(t,x)\!: \stackrel{\text{def}}{=} H_\ell(\Phi_N(t,x); \alpha_N) \tag{28}$$

for $k \in \mathbb{N}$. As before, it follows that the Wick power $:\!\Phi_N^\ell\!:$ converges to a limit $:\!\Phi^\ell\!:$ in $C([0,T]; W^{-\varepsilon,\infty}(\mathbb{T}^2))$ for any $\varepsilon > 0$ and $T > 0$, almost surely; see Lemma 8. Then, by proceeding as in Subsection 1.2, namely, by (i) first considering the truncated equation (10) with the random initial data (u_0^ω, u_1^ω) in (24), (ii) using the first order expansion $u_N = \Phi_N + v_N$ and introducing Wick renormalizations, and (iii) taking a limit $N \to \infty$, we arrive at the following (renormalized) reformulation of (18) in this setting:

$$\begin{cases} \partial_t^2 v + (1-\Delta) v + D\partial_t v + \sum_{\ell=0}^k \binom{k}{\ell} :\!\Phi^\ell\!: v^{k-\ell} = 0 \\ (v, \partial_t v)|_{t=0} = (0,0). \end{cases} \tag{29}$$

We now state an almost sure global well-posedness result of (18) with the random initial data (u_0^ω, u_1^ω) in (24).

Theorem 4. *Let $k \in 2\mathbb{N}+1$. Then, the renormalized SvNLW* (18) *is almost surely globally well-posed with the random initial data (u_0^ω, u_1^ω) defined in* (24) *in the sense that the following statement holds true almost surely; there exists a unique global-in-time solution v to* (29) *with the zero initial data, where $\{:\!\Phi^\ell\!:\}_{\ell=1}^k$ denotes the stochastic convolution Φ defined in* (26) *and its Wick powers $:\!\Phi^\ell\!:$, $\ell = 2, \dots, k$, defined in Lemma 8 below.*

The proof of Theorem 4 is based on Bourgain's invariant measure argument [4, 5]. By viewing the SvNLW dynamics (18) as the "superposition" of the (renormalized) NLW dynamics (23) and the Ornstein–Uhlenbeck dynamics (for $\partial_t u$):

$$\partial_t(\partial_t u) = -D\partial_t u + \sqrt{2} D^{\frac{1}{2}} \xi,$$

we expect the Gibbs measure (for the standard NLW (23)), formally given by

$$\text{“}\mathrm{d}\vec{\rho}(u, \partial_t u) = Z^{-1} e^{-E(u, \partial_t u)} \mathrm{d}u \mathrm{d}(\partial_t u)\text{”}, \tag{30}$$

to be invariant under the SvNLW dynamics (18).

Let $\vec{\mu}_1$ be the induced probability measure under the map: $\omega \in \Omega \longmapsto (u_0^\omega, u_1^\omega)$, where (u_0^ω, u_1^ω) is as in (24). Then, we can write $\vec{\mu}_1$ as $\vec{\mu}_1 = \mu_1 \otimes \mu_0$, where μ_s denote a Gaussian measure on periodic distributions, formally defined by

$$d\mu_s = Z_s^{-1} e^{-\frac{1}{2}\|u\|_{H^s}^2} du = Z_s^{-1} \prod_{n \in \mathbb{Z}^2} e^{-\frac{1}{2}\langle n \rangle^{2s} |\hat{u}(n)|^2} d\hat{u}(n). \tag{31}$$

Note that μ_1 corresponds to the massive Gaussian free field, while μ_0 corresponds to the white noise. Then, by renormalizing the potential part of the energy $E(\vec{u})$ in (22), we can indeed construct the Gibbs measure $\vec{\rho}$ as a probability measure such that $\vec{\rho}$ and $\vec{\mu}_1$ are mutually absolutely continuous. By exploiting the formal invariance of the Gibbs measure $\vec{\rho}$ under (18), Bourgain's invariant measure argument yields almost sure global well-posedness of (18) with respect to the Gibbs measure $\vec{\rho}$; see Theorem 14 below. By invoking the mutual absolute continuity of $\vec{\rho}$ and $\vec{\mu}_1$, we then conclude Theorem 4. See Subsection 5.1 for details.

As a corollary to Theorem 4 and the Cameron–Martin theorem [11], we obtain the following almost sure global well-posedness of (18) with deterministic $\mathscr{H}^1$-initial data (v_0, v_1) perturbed by the random functions (u_0^ω, u_1^ω) in (24). Let us introduce some notations. *Fix* $\vec{v}_0 = (v_0, v_1) \in \mathscr{H}^1(\mathbb{T}^2)$. With $\vec{u}_0^\omega = (u_0^\omega, u_1^\omega)$, define the stochastic convolution $\Phi[\vec{v}_0 + \vec{u}_0^\omega]$ with the shifted initial data $\vec{v}_0 + \vec{u}_0^\omega$:

$$\begin{aligned} \Phi[\vec{v}_0 + \vec{u}_0^\omega](t) &= V(t)(\vec{v}_0 + \vec{u}_0^\omega) + \sqrt{2}\int_0^t S(t-t')D^{\frac{1}{2}} dW(t') \\ &= V(t)\vec{v}_0 + \Phi(t), \end{aligned} \tag{32}$$

where Φ is as in (26). Given $N \in \mathbb{N}$, set $\Phi_N[\vec{v}_0 + \vec{u}_0^\omega] = \mathbf{P}_N \Phi[\vec{v}_0 + \vec{u}_0^\omega]$. In view of

$$H_\ell(x+y;\sigma) = \sum_{j=0}^{\ell} \binom{\ell}{j} x^{\ell-j} H_j(y;\sigma)$$

and (28), we define the Wick power $:\!(\Phi_N[\vec{v}_0 + \vec{u}_0^\omega])^\ell\!:$ by

$$\begin{aligned} :\!(\Phi_N[\vec{v}_0 + \vec{u}_0^\omega])^\ell(t,x)\!: &\stackrel{\text{def}}{=} H_\ell(\Phi_N[\vec{v}_0 + \vec{u}_0^\omega](t,x); \alpha_N) \\ &= \sum_{j=0}^{\ell} \binom{\ell}{j} (V(t)\vec{v}_0)^j :\!\Phi_N^{\ell-j}(t,x)\!:. \end{aligned} \tag{33}$$

where α_N is as in (27) and $:\!\Phi_N^{\ell-j}\!: \; = \; :\!(\Phi_N[\vec{u}_0^\omega])^{\ell-j}\!:$ is as in (28). Thanks to the H^1-regularity of $V(t)\vec{v}_0$ and the almost sure convergence of $:\!\Phi_N^{\ell-j}\!:$, we see that the Wick power $:\!(\Phi_N[\vec{v}_0 + \vec{u}_0^\omega])^\ell\!:$ converges to a limit $:\!(\Phi[\vec{v}_0 + \vec{u}_0^\omega])^\ell\!:$ in $C([0,T]; W^{-\varepsilon,\infty}(\mathbb{T}^2))$ for any $\varepsilon > 0$ and $T > 0$, almost surely; see Subsection 5.2.

Proceeding as in Subsection 1.2 with the first order expansion $u_N = \Phi_N[\vec{v}_0 + \vec{u}_0^\omega] + v_N$ and taking $N \to \infty$, we can reformulate the renormalized SvNLW (18) with the shifted initial data $(u, \partial_t u)|_{t=0} = \vec{v}_0 + \vec{u}_0^\omega$ as

$$\begin{cases} \partial_t^2 v + (1-\Delta)v + D\partial_t v + \sum_{\ell=0}^k \binom{k}{\ell} :\!(\Phi[\vec{v}_0 + \vec{u}_0^\omega])^\ell\!: v^{k-\ell} = 0 \\ (v, \partial_t v)|_{t=0} = (0,0). \end{cases} \tag{34}$$

Corollary 5. *Let $k \in 2\mathbb{N}+1$ and fix $(v_0, v_1) \in \mathscr{H}^1(\mathbb{T}^2)$. Then, the renormalized SvNLW* (18) *is almost surely globally well-posed with respect to the shifted initial data $(v_0, v_1) + (u_0^\omega, u_1^\omega)$, where (u_0^ω, u_1^ω) is as in* (24)*, in the sense that there exists almost surely a unique global-in-time solution v to* (34) *with the zero initial data.*

In Subsection 5.2, we sketch the argument. See [36] for a further discussion on probabilistic well-posedness and other aspects (such as the large deviation principle) for random initial data of the form "a smooth deterministic function + a rough random perturbation".

We conclude this introduction by stating several remarks.

Remark 6.

(i) The pathwise global well-posedness result (Theorem 3) relies on a dispersive PDE argument. As such, the coefficient $\sqrt{2}$ on the noise $D^{\frac{1}{2}}\xi$ in the equation (18) plays no role and thus Theorem 3 also applies to the cubic SvNLW (18) (with $k = 3$) with a general coefficient on the noise. On the other hand, Theorem 4 relies on the invariant measure argument and thus the coefficient on the noise in (18) must be $\sqrt{2}$.[3]

At this point, we do not know how to extend the pathwise global well-posedness (Theorem 3) to the (super-)quintic case. Even with a smoother noise, one would need to use a trick introduced in [26, 35] to handle the higher homogeneity. See [27]. It may also be of interest to investigate if a parabolic PDE approach such as those in [29, 44] can be applied to handle the (super-)quintic case.

(ii) As mentioned in Remark 1, Theorems 2 and 3 apply to (4) with $\alpha = \frac{1}{2}$ with essentially identical proofs. When $\alpha < \frac{1}{2}$, the equation (4) is no longer singular (namely, a solution is a function and there is no need for a renormalization). See [27] for pathwise global well-posedness results for higher values of $k \in 2\mathbb{N}+1$, when $\alpha < \frac{1}{2}$.

(iii) When $k = 2$, the equation (18) is no longer defocusing. Even in this case, however, the Gibbs measure can be constructed; see [6, 39] and thus an analogue of almost sure global well-posedness (Theorem 4) holds when $k = 2$. In the non-defocusing case with $k \geq 3$, namely, either (i) even $k \geq 4$ or (ii) with the nonlinearity $-{:}u^k{:}$, $k \in 2\mathbb{N}+1$, (i.e. with the negative sign) in (18), it is known that the Gibbs measure is not constructible [9, 39] and hence Bourgain's invariant measure argument is not applicable in this case.

(iv) For the physical reason, it is of interest to investigate the well-posedness issue of (18) on $\mathbb{R}^2$. In this case, due to the unboundedness of the domain, the integrability becomes an issue. In view of the pathwise global well-posedness (Theorem 3), we expect that the dispersive techniques as in [43] may be applied to treat the cubic case. As for the higher order nonlinearity, it may be possible to adapt the parabolic approach as in [29]. We plan to address this issue in a forthcoming work.

(v) As for the model (5) without renormalization, we expect a *triviality* result to hold. Roughly speaking, extreme oscillations make solutions u_N to (10) with regularized noises tend to a solution to the linear stochastic viscous wave equation (6) (or the trivial solution) as the regularization is removed. Such a triviality result (in the absence of renormalization) is known for stochastic NLW and stochastic nonlinear heat equations; see [1, 21, 32, 37].

Remark 7. From its derivation, the viscous wave equation is most relevant physically in two spatial dimensions; see [22, 24]. At the same time, it is of interest to study the equation in other spatial dimensions.

(i) Let us first consider the following equation on $\mathbb{T}^d$:

$$\partial_t^2 u + (1-\Delta)u + D\partial_t u + u^k = D^\alpha \xi. \tag{35}$$

When $\alpha < \frac{3-d}{2}$, it follows from the $\frac{3}{2}$-smoothing of the viscous wave operator that the stochastic convolution Ψ defined by

$$\Psi(t) = \int_0^t S(t-t')D^\alpha \mathrm{d}W(t') \tag{36}$$

is a function and hence there is no need for renormalization to study (35). See a recent preprint [27].

[3]For general $0 < \mu < 1$, Theorem 4 also holds for the following equation: $\partial_t^2 u + (1-\Delta)u + 2\mu D\partial_t u + {:}u^k{:} = 2\sqrt{\mu}D^{\frac{1}{2}}\xi$, where the coefficient on the noise is $2\sqrt{\mu}$.

When $\alpha = \frac{3-d}{2}$, the stochastic convolution Ψ in (36) has regularity slightly below 0 and we need to apply the Wick renormalization to study the equation. When $d = 1$, the local well-posedness for general $k \geq 2$ and the pathwise global well-posedness in the cubic case ($k = 3$) as in Theorems 2 and 3, respectively, hold true with the same proofs. When $d = 3$ (corresponding to the space-time white noise forcing), in view of the embedding $H^1(\mathbb{T}^3) \subset L^6(\mathbb{T}^3)$, a slight modification of the proofs of Theorems 2 and 3 shows that (a) for $k = 2,3$, (the renormalized version of) SvNLW (35) (with $\alpha = 0$) on $\mathbb{T}^3$ is locally well-posed in $\mathcal{H}^s(\mathbb{T}^3)$, $s \geq 1$, and (b) (the renormalized version of) the cubic SvNLW (35) (with $k = 3$ and $\alpha = 0$) on $\mathbb{T}^3$ is globally well-posed in $\mathcal{H}^s(\mathbb{T}^3)$, $s \geq 1$. Due to the more restrictive range of Sobolev's inequality on $\mathbb{T}^3$, however, the proof of Theorem 2 does not apply to SvNLW (35) (with $\alpha = 0$) on $\mathbb{T}^3$ for $k \geq 4$. In this case, one needs to make use of the (wave) Strichartz estimates to prove local well-posedness. In higher dimensions, one also needs to use the Strichartz estimates (except for $d = 4$ and $k = 2$, which can be handled by Sobolev's embedding). We, however, do not pursue this issue in this paper.

When $\alpha > \frac{3-d}{2}$, the stochastic convolution Ψ in (36) has even lower regularity, possibly requiring a further renormalization; see Part (ii) below. For values of α close to $\frac{3-d}{2}$, the proof of local well-posedness (Theorem 2) is applicable but the value of α depends on the degree k of the nonlinearity. For higher values of α, one needs to use a more sophisticated approach such as the paracontrolled approach [8, 17, 33, 34] together with the Strichartz estimates. As for the global well-posedness, the proof of Theorem 3 crucially exploits the logarithmic divergence of the stochastic convolution and thus it is not applicable to the case $\alpha > \frac{3-d}{2}$.

(ii) Next, we consider the following equation on $\mathbb{T}^d$:

$$\partial_t^2 u + (1-\Delta)u + D\partial_t u + u^k = \sqrt{2}D^{\frac{1}{2}}\xi \tag{37}$$

with $k \in 2\mathbb{N}+1$, where one may prove almost sure global well-posedness via Bourgain's invariant measure argument. When $d = 1$, the stochastic convolution Φ defined in (26) has spatial regularity $\frac{1}{2} - \varepsilon$ and thus there is no need to introduce renormalization. In this case, local well-posedness easily follows from Sobolev's inequality (without the first order expansion), and Bourgain's invariant measure argument [4] yields the one-dimensional analogue of Theorem 4.

When $d = 3$, we first recall that the Gibbs measure $\vec{\rho}$ in (30) and the Gaussian measure $\vec{\mu}_1 = \mu_1 \otimes \mu_0$ (= the distribution of the random initial data (u_0^ω, u_1^ω) in (24)) are mutually singular; see [2]. Hence, we need to study the Gibbsian initial data in this case. Noting that the Gibbs measure $\vec{\rho}$ corresponds to the Φ_3^4-measure (on u) $\otimes$ the spatial white noise (on $\partial_t u$), we see that the Wick renormalization is not sufficient and that we need to introduce another renormalization to remove the logarithmic divergence; see [20, 30] in the case of the parabolic Φ_3^4-model. The well-posedness theory in this case certainly requires a more sophisticated approach such as the paracontrolled approach [17], which is beyond the scope of this paper.

2. Preliminary lemmas

2.1. *Tools from stochastic analysis*

For readers' convenience, we first recall the Hermite polynomials $H_k(x;\sigma)$, defined through the following generating function:

$$F(t,x;\sigma) = e^{tx - \frac{1}{2}\sigma t^2} = \sum_{k=0}^{\infty} \frac{t^k}{k!} H_k(x;\sigma),$$

which is used in constructing the renormalized powers (13), (28), and (33).

The following lemma establishes the regularity of the stochastic convolution Z and its Wick powers $:Z^k:$, where $Z = \Psi$ in (7) or $Z = \Phi$ in (26).

Lemma 8. *Let $Z = \Psi$ or Φ. Given $k \in \mathbb{N}$ and $N \in \mathbb{N}$, let $:Z_N^k: = :(\mathbf{P}_N Z)^k:$ denote the truncated Wick power defined in* (13) *or* (28)*, respectively. Then, given any $T, \varepsilon > 0$ and finite $p \geq 1$, $\{:Z_N^k:\}_{N\in\mathbb{N}}$ is a Cauchy sequence in $L^p(\Omega; C([0,T]; W^{-\varepsilon,\infty}(\mathbb{T}^2)))$, converging to some limit $:Z^k:$ in $L^p(\Omega; C([0,T]; W^{-\varepsilon,\infty}(\mathbb{T}^2)))$. Moreover, $:Z_N^k:$ converges almost surely to the same limit in $C([0,T]; W^{-\varepsilon,\infty}(\mathbb{T}^2))$. Furthermore, we have the following tails estimates.*

(i) *Given any finite $q \geq 1$, we have*

$$P\Big(\|:Z^k:\|_{L^q_T W_x^{-\varepsilon,\infty}} > \lambda\Big) \leq C \exp\bigg(-c \frac{\lambda^{\frac{2}{k}}}{T^{\frac{2}{qk}}}\bigg) \tag{38}$$

for any $T \geq 1$ and $\lambda > 0$.

(ii) *When $q = \infty$, we have*

$$P\Big(\|:Z^k:\|_{L^\infty([j,j+1]; W_x^{-\varepsilon,\infty})} > \lambda\Big) \leq C \exp\bigg(-c \frac{\lambda^{\frac{2}{k}}}{j+1}\bigg) \tag{39}$$

for any $j \in \mathbb{Z}_{\geq 0}$ and $\lambda > 0$.

(iii) *When $q = \infty$ and $k = 1$, we have*

$$P\Big(\|Z\|_{L^\infty([0,T]; W_x^{-\varepsilon,\infty})} > \lambda\Big) \leq CT \exp\bigg(-c \frac{\lambda^2 \varepsilon}{T}\bigg) \tag{40}$$

for any $T \geq 1$ and $\lambda > 0$.

Part (iii) of this lemma in particular plays an important role in the proof of Theorem 3. See Section 4.

Proof. In view of (9) and (27), the proof of this lemma is essentially identical to that of Lemma 2.3 in [19] for the stochastic (damped) wave equation. Hence, we will be very brief here. As for the convergence part of the statement, see [18, Proposition 2.1] and [17, Lemma 3.1].

(i). As for the exponential tail estimate (38), by repeating the argument in the proof of [18, Proposition 2.1], we have

$$\mathbb{E}\big[|\langle\nabla\rangle^{-\varepsilon} :Z^k(t,x):|^2\big] \lesssim \sum_{n_1,\dots,n_k \in \mathbb{Z}^2} \frac{1}{\langle n_1\rangle^2 \cdots \langle n_k\rangle^2 \langle n_1 + \cdots + n_k\rangle^{2\varepsilon}} \leq C_\varepsilon \tag{41}$$

for any $\varepsilon > 0$, uniformly in $x \in \mathbb{T}^2$ and $t \geq 0$. Then, given finite $q \geq 1$, Sobolev's inequality (with some $r > 4\varepsilon^{-1}$), Minkowski's integral inequality, and the Wiener chaos estimate ([18, Lemmas 2.3 and 2.4]) yield

$$\Big\| \|:Z^k:\|_{L^q_T W_x^{-\varepsilon,\infty}} \Big\|_{L^p(\Omega)} \lesssim \Big\| \|:Z^k:\|_{L^q_T W_x^{-\frac{\varepsilon}{2},r}} \Big\|_{L^p(\Omega)} \lesssim p^{\frac{k}{2}} T^{\frac{1}{q}} \tag{42}$$

for any $p \geq \max(q, r)$. Then, the bound (38) follows from (42) and Chebyshev's inequality (as in [3, Lemma 3]).[4]

(ii). As for the second bound (39), we first write

$$\begin{aligned} P\Big(\|:Z^k:\|_{L^\infty([j,j+1]; W_x^{-\varepsilon,\infty})} > \lambda\Big) \\ \leq P\Big(\|:Z^k(j):\|_{W_x^{-\varepsilon,\infty}} > \tfrac{\lambda}{2}\Big) + P\bigg(\sup_{t\in[j,j+1]} \|:Z^k(t): - :Z^k(j):\|_{W_x^{-\varepsilon,\infty}} > \tfrac{\lambda}{2}\bigg) \end{aligned} \tag{43}$$

[4] Lemma 2.2 in the arXiv version.

for given $j \in \mathbb{Z}_{\geq 0}$ and $\lambda > 0$. Using (41), we can repeat the argument in Part (i) to bound the first term on the right-hand side of (43). As for the second term on the right-hand side of (43), we first recall from the proof of [18, Proposition 2.1] that

$$\Big\| |h|^{-\rho} \|\delta_h(:Z^k(t):)\|_{W_x^{-\varepsilon,\infty}} \Big\|_{L^p(\Omega)} \lesssim p^{\frac{k}{2}}(j+1)^{\frac{k}{2}}$$

for any sufficiently large $p \gg 1$, $t \in [j, j+1]$, and $|h| \leq 1$, where $\delta_h f(t) = f(t+h) - f(t)$ and $0 < \rho < \varepsilon$. Then, the desired bound (39) follows from Chebyshev's inequality and the Garsia–Rodemich–Rumsey inequality ([14, Theorem A.1] and [19, Lemma 2.2]), which provides an exponential tail bound for a Hölder norm (in time). See the proof of Lemma 2.3 in [19] for further details.

(iii). The third bound (40) follows in a similar manner once we make ε-dependence more explicit. By Sobolev's inequality, Minkowski's integral inequality, and the Wiener chaos estimate ([18, Lemmas 2.3 and 2.4]), we have

$$\begin{aligned} \Big\| \|Z(j)\|_{W_x^{-\varepsilon,\infty}} \Big\|_{L^p(\Omega)} &\lesssim \Big\| \|Z(j)\|_{W_x^{-\frac{\varepsilon}{2},r}} \Big\|_{L^p(\Omega)} \lesssim p^{\frac{1}{2}} \Big\| \big\| \langle \nabla \rangle^{-\frac{\varepsilon}{2}} Z(j) \big\|_{L^2(\Omega)} \Big\|_{L_x^r} \\ &\lesssim p^{\frac{1}{2}} \bigg\| \Big(\sum_{n \in \mathbb{Z}^2} \frac{1}{\langle n \rangle^{2+\varepsilon}} \Big)^{\frac{1}{2}} \bigg\|_{L_x^r} \\ &\sim p^{\frac{1}{2}} \varepsilon^{-\frac{1}{2}} \end{aligned}$$

for any $p \geq r > 4\varepsilon^{-1}$. Similarly, by the mean value theorem (with (7) or (26)), we have

$$\Big\| |h|^{-\rho} \|\delta_h Z(t)\|_{W_x^{-\varepsilon,\infty}} \Big\|_{L^p(\Omega)} \lesssim p^{\frac{1}{2}} |h|^{\frac{\varepsilon}{4}-\rho} \bigg(\sum_{n \in \mathbb{Z}^2} \frac{1}{\langle n \rangle^{2+\varepsilon}} \bigg)^{\frac{1}{2}} \lesssim p^{\frac{1}{2}} \varepsilon^{-\frac{1}{2}}$$

for any sufficiently large $p \gg 1$, $t, t+h \in [0, T]$, and $|h| \leq 1$, provided that $0 < \rho < \frac{\varepsilon}{4}$. Then, the rest follows from proceeding as in Part (ii) and summing over the interval $[j, j+1]$. $\square$

2.2. *Tools from deterministic analysis*

We first recall the product estimates.

Lemma 9. *Let* $0 \leq s \leq 1$.

(i) *Suppose that* $1 < p_j, q_j, r < \infty$, $\frac{1}{p_j} + \frac{1}{q_j} = \frac{1}{r}$, $j = 1, 2$. *Then, we have*

$$\|\langle \nabla \rangle^s (fg)\|_{L^r(\mathbb{T}^d)} \lesssim \Big(\|f\|_{L^{p_1}(\mathbb{T}^d)} \|\langle \nabla \rangle^s g\|_{L^{q_1}(\mathbb{T}^d)} + \|\langle \nabla \rangle^s f\|_{L^{p_2}(\mathbb{T}^d)} \|g\|_{L^{q_2}(\mathbb{T}^d)} \Big).$$

(ii) *Suppose that* $1 < p, q, r < \infty$ *satisfy* $\frac{1}{p} + \frac{1}{q} \leq \frac{1}{r} + \frac{s}{d}$. *Then, we have*

$$\big\| \langle \nabla \rangle^{-s} (fg) \big\|_{L^r(\mathbb{T}^d)} \lesssim \big\| \langle \nabla \rangle^{-s} f \big\|_{L^p(\mathbb{T}^d)} \big\| \langle \nabla \rangle^{s} g \big\|_{L^q(\mathbb{T}^d)}.$$

See [18] for the proof. Note that while Lemma 9 (ii) was shown only for $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + \frac{s}{d}$ in [18], the general case $\frac{1}{p} + \frac{1}{q} \leq \frac{1}{r} + \frac{s}{d}$ follows from the inclusion $L^{r_1}(\mathbb{T}^d) \subset L^{r_2}(\mathbb{T}^d)$ for $r_1 \geq r_2$.

Next, we state a Schauder-type estimate for the Poisson kernel

$$P(t) = e^{-\frac{D}{2}t}.$$

Lemma 10. *Let* $1 \leq p \leq q \leq \infty$ *and* $\alpha \geq 0$. *Then, we have*

$$\|D^\alpha P(t) f\|_{L^q(\mathbb{T}^d)} \lesssim t^{-\alpha - d(\frac{1}{p} - \frac{1}{q})} \|f\|_{L^p(\mathbb{T}^d)} \tag{44}$$

for any $0 < t \leq 1$.

Proof. We first prove (44) on $\mathbb{R}^d$ for any $t>0$. Let $K_t(x)$ denote the kernel for $P(t)$ given by $\mathscr{F}_{\mathbb{R}^d}(K_t)(\xi)=e^{-\frac{|\xi|}{2}t}$, where $\mathscr{F}_{\mathbb{R}^d}$ denotes the Fourier transform on $\mathbb{R}^d$. Recall that

$$K_t(x)=c_d\frac{t}{(t^2+|x|^2)^{\frac{d+1}{2}}}. \tag{45}$$

Noting $D^\alpha P(t)f=(D^\alpha K_t)*f$, we need to study the scaling property of $D^\alpha K_t$. On the Fourier side, we have

$$\mathscr{F}_{\mathbb{R}^d}(D^\alpha K_t)(\xi)=|\xi|^\alpha e^{-\frac{|\xi|}{2}t}=t^{-\alpha}(t|\xi|)^\alpha e^{-\frac{|\xi|t}{2}}=t^{-\alpha}\mathscr{F}_{\mathbb{R}^d}(D^\alpha K_1)(t\xi).$$

Namely, we have

$$D^\alpha K_t(x)=t^{-d-\alpha}(D^\alpha K_1)(t^{-1}x). \tag{46}$$

For $1\le r\le\infty$ with $\frac{1}{r}=\frac{1}{q}-\frac{1}{p}+1$, from (46) and (45), we have[5]

$$\|D^\alpha K_t\|_{L^r(\mathbb{R}^d)}=t^{-\alpha-d(1-\frac{1}{r})}\|D^\alpha K_1\|_{L^r(\mathbb{R}^d)}=C_{r,\alpha}t^{-\alpha-d(\frac{1}{p}-\frac{1}{q})}. \tag{47}$$

Then, (44) follows from Young's inequality and (47).

Next, we prove (44) on $\mathbb{T}^d$. Let $R_t(x)$ denote the kernel for $P(t)$ on $\mathbb{T}^d$ given by $\widehat{R}_t(n)=e^{-\frac{|n|}{2}t}=\mathscr{F}_{\mathbb{R}^d}(K_t)(n)$. Then, given any $1\le r\le\infty$, from the Poisson summation formula (with (45)) and Hölder's inequality, we have

$$\begin{aligned}
\|D^\alpha R_t\|_{L^r(\mathbb{T}^d)}&=\Big\|\sum_{n\in\mathbb{Z}^d}\mathscr{F}_{\mathbb{R}^d}(D^\alpha K_t)(n)e^{in\cdot x}\Big\|_{L^r(\mathbb{T}^d)}=\Big\|\sum_{n\in\mathbb{Z}^d}D^\alpha K_t(x+n)\Big\|_{L^r(\mathbb{T}^d)}\\
&\le\bigg\|\Big(\sum_{n\in\mathbb{Z}^d}\langle n\rangle^{-\beta r'}\Big)^{\frac{1}{r'}}\|\langle n\rangle^\beta D^\alpha K_t(x+n)\|_{\ell^r_n}\bigg\|_{L^r_x(\mathbb{T}^d)}\\
&\lesssim\big\|\langle x\rangle^\beta D^\alpha K_t(x)\big\|_{L^r(\mathbb{R}^d)}\\
&\lesssim\big\|D^\alpha K_t(x)\big\|_{L^r(\mathbb{R}^d)}+\big\||\tfrac{x}{t}|^\beta D^\alpha K_t(x)\big\|_{L^r(\mathbb{R}^d)},
\end{aligned} \tag{48}$$

provided that $\beta>d(1-\frac{1}{r})$ and $0<t\le1$. From (46), we have

$$\big\||\tfrac{x}{t}|^\beta D^\alpha K_t(x)\big\|_{L^r(\mathbb{R}^d)}=t^{-\alpha-d(1-\frac{1}{r})}\big\||x|^\beta D^\alpha K_1(x)\big\|_{L^r(\mathbb{R}^d)}=C'_{r,\alpha}t^{-\alpha-d(1-\frac{1}{r})} \tag{49}$$

for some finite $C'_{r,\alpha}>0$, provided that $\beta<d(1-\frac{1}{r})+1$. Hence, the desired bound (44) follows from Young's inequality and (48) with (47) and (49). $\square$

3. Local well-posedness of the stochastic viscous NLW

In this section, we present the proof of Theorem 2. For this purpose, we consider the following deterministic vNLW:

$$\begin{cases}\partial_t^2 v+(1-\Delta)v+D\partial_t v+\sum_{\ell=0}^k\binom{k}{\ell}\Xi_\ell\, v^{k-\ell}=0\\ (v,\partial_t v)|_{t=0}=(v_0,v_1)\end{cases} \tag{50}$$

for given initial data $(v_0,v_1)\in\mathscr{H}^s(\mathbb{T}^2)$, $s\ge1$, and deterministic source terms $(\Xi_0,\dots,\Xi_k)$ with the understanding that $\Xi_0\equiv1$. Define $\mathscr{X}^s(\mathbb{T}^2)$ by

$$\mathscr{X}^s(\mathbb{T}^2)=\mathscr{H}^s(\mathbb{T}^2)\times\prod_{\ell=0}^{k-1}C([0,1];W^{-\frac{\ell}{k},\infty}(\mathbb{T}^2))\times C([0,1];W^{-1+\varepsilon,\infty}(\mathbb{T}^2)) \tag{51}$$

for some small $\varepsilon>0$ and set

$$\|\boldsymbol{\Xi}\|_{\mathscr{X}^s}=\|(v_0,v_1)\|_{\mathscr{H}^s}+\sum_{\ell=1}^{k-1}\|\Xi_\ell\|_{C([0,1];W^{-\frac{\ell}{k},\infty})}+\|\Xi_k\|_{C([0,1];W^{-1+\varepsilon,\infty})}$$

[5]One may first use (45) to show (47) for even $\alpha\ge0$ and then interpolate the result to deduce (47) for general $\alpha\ge0$.

for $\Xi = (v_0, v_1, \Xi_1, \Xi_2, \dots, \Xi_k) \in \mathscr{X}^s(\mathbb{T}^2)$. Then, we have the following local well-posedness result for (50).

Proposition 11. *Let $k \geq 2$ be an integer and $s \geq 1$. Then,* (50) *is locally well-posed in $\mathscr{X}^s(\mathbb{T}^2)$. More precisely, given an enhanced data set:*

$$\Xi = (v_0, v_1, \Xi_1, \Xi_2, \dots, \Xi_k) \in \mathscr{X}^s(\mathbb{T}^2),$$

there exist $0 < T = T(\|\Xi\|_{\mathscr{X}^s}) \leq 1$ and a unique solution $\vec{v} = (v, \partial_t v) \in C([0,T]; \mathscr{H}^1(\mathbb{T}^2))$ to (50), *depending continuously on the enhanced data set Ξ.*

Note that Proposition 11 is completely deterministic. Theorem 2 immediately follows from Proposition 11 and Lemma 8 which states that the (random) enhanced data set $\Xi = (v_0, v_1, \Psi, :\!\Psi^2\!:, \dots, :\!\Psi^k\!:)$ almost surely belongs to $\mathscr{X}^s(\mathbb{T}^2)$ and that the truncated (random) enhanced data set $\Xi_N = (v_0, v_1, \Psi_N, :\!\Psi_N^2\!:, \dots, :\!\Psi_N^k\!:)$ converges almost surely to Ξ in $\mathscr{X}^s(\mathbb{T}^2)$.

Proof. By writing (50) in the Duhamel formulation, we have

$$v(t) = \Gamma(v) \overset{\text{def}}{=} V(t)(v_0, v_1) - \sum_{\ell=0}^{k} \binom{k}{\ell} \int_0^t S(t-t')\big(\Xi_\ell\, v^{k-\ell}\big)(t')\,\mathrm{d}t', \tag{52}$$

where $V(t)$ and $S(t)$ are as in (20) and (8) respectively. Let $\vec{\Gamma}(v) = (\Gamma(v), \partial_t \Gamma(v))$.

Fix $0 < T \leq 1$. We first consider the case $\ell = 0$. From (8) and Sobolev's inequality, we have

$$\begin{aligned} \bigg\| \int_0^t S(t-t') v^k(t')\,\mathrm{d}t' \bigg\|_{C_T H^1_x} &+ \bigg\| \partial_t \int_0^t S(t-t') v^k(t')\,\mathrm{d}t' \bigg\|_{C_T L^2_x} \\ &\lesssim \bigg\| \int_0^t \big\| \sin((t-t')[\![D]\!]) v^k(t') \big\|_{L^2_x} \mathrm{d}t' \bigg\|_{C_T} \\ &\lesssim T \big\| v^k \big\|_{C_T L^2_x} \lesssim T \|v\|^k_{C_T L^{2k}_x} \\ &\lesssim T \|v\|^k_{C_T H^1_x}. \end{aligned} \tag{53}$$

Next, let $1 \leq \ell \leq k-1$. From Lemma 10, Lemma 9 (ii) and then (i) followed by Sobolev's inequality, we have

$$\begin{aligned} \bigg\| \int_0^t S(t-t')\big(\Xi_\ell\, v^{k-\ell}\big)(t')\,\mathrm{d}t' \bigg\|_{C_T H^1_x} &+ \bigg\| \partial_t \int_0^t S(t-t')\big(\Xi_\ell\, v^{k-\ell}\big)(t')\,\mathrm{d}t' \bigg\|_{C_T L^2_x} \\ \lesssim T^{\frac{k-\ell}{k}} \big\| \Xi_\ell\, v^{k-\ell} \big\|_{C_T H_x^{-\frac{\ell}{k}}} &\lesssim T^{\frac{k-\ell}{k}} \big\| \langle \nabla \rangle^{-\frac{\ell}{k}} \Xi_\ell \big\|_{C_T L^{2k}_x} \big\| \langle \nabla \rangle^{\frac{\ell}{k}} v^{k-\ell} \big\|_{C_T L_x^{\frac{2k}{k+\ell-1}}} \\ \lesssim T^{\frac{k-\ell}{k}} \|\Xi\|_{\mathscr{X}^s} \big\| \langle \nabla \rangle^{\frac{\ell}{k}} v \big\|_{C_T L_x^{\frac{2k}{\ell}}} \|v\|^{k-\ell-1}_{C_T L_x^{\frac{2k(k-\ell-1)}{k-1}}} \\ \lesssim T^{\frac{k-\ell}{k}} \|\Xi\|_{\mathscr{X}^s} \|v\|^{k-\ell}_{C_T H^1_x}. \end{aligned} \tag{54}$$

Lastly, when $\ell = k$, it follows from (8) and Lemma 10 that

$$\bigg\| \int_0^t S(t-t')\Xi_k(t')\,\mathrm{d}t' \bigg\|_{C_T H^1_x} + \bigg\| \partial_t \int_0^t S(t-t')\Xi_k(t')\,\mathrm{d}t' \bigg\|_{C_T L^2_x} \lesssim T^\varepsilon \|\Xi_k\|_{C_T H_x^{-1+\varepsilon}} \leq T^\varepsilon \|\Xi\|_{\mathscr{X}^s}. \tag{55}$$

Putting (52), (53), (54), and (55) together, we obtain

$$\|\vec{\Gamma}(v)\|_{C_T \mathscr{H}^1_x} \leq C_1 \|(v_0, v_1)\|_{\mathscr{H}^1} + C_2 T^\theta \big(1 + \|\Xi\|_{\mathscr{X}^s}\big)\big(1 + \|\vec{v}\|_{C_T \mathscr{H}^1_x}\big)^k$$

for some $\theta > 0$. A similar computation yields a difference estimate on $\vec{\Gamma}(v_1) - \vec{\Gamma}(v_2)$. Therefore, by choosing $T = T(\|\Xi\|_{\mathscr{X}^s}) > 0$ sufficiently small, we conclude that Γ is a contraction in the ball $B_R \subset C([0,T]; \mathscr{H}^1(\mathbb{T}^2))$ of radius $R \sim \|(v_0, v_1)\|_{\mathscr{H}^1}$. □

Remark 12.

(i) In view of (51) and an analogue of Lemma 8 (for a rougher noise), we see that the proof of Proposition 11 yields local well-posedness of (18) with a rougher noise $\sqrt{2}D^\alpha\xi$ for $\alpha < \frac{1}{2} + \frac{1}{k}$ such that the corresponding enhanced data set $\Xi = (v_0, v_1, \Psi, :\Psi^2:, \dots, :\Psi^k:)$ belongs to $\mathscr{X}^s(\mathbb{T}^2)$ almost surely. It may be possible to improve the local well-posedness argument above by using the wave Strichartz estimates.

(ii) Suppose that (v_0, v_1) lies in $\mathscr{H}^s(\mathbb{T}^2)$ for some $s < 1$. From (20) and Lemma 10, we have

$$\|V(t)(v_0, v_1)\|_{H^1_x} \lesssim t^{s-1}\|(v_0, v_1)\|_{\mathscr{H}^s}$$

for any $0 < t \le 1$. Then, given $0 < T \le 1$, we define the $Y^{1,s}(T)$-norm by

$$\|(v, \partial_t v)\|_{Y^{1,s}(T)} = \sup_{0\le t\le T} t^{1-s}\|(v, \partial_t v)(t)\|_{\mathscr{H}^1_x},$$

where a function is allowed to blow up at time $t = 0$. By slight modifications of (53), (54), and (55), we obtain

$$\|\vec{\Gamma}(v)\|_{Y^{1,s}(T)} \lesssim \|(v_0, v_1)\|_{\mathscr{H}^s} + T^\theta\big(1 + \|\Xi\|_{\mathscr{X}^s}\big)\big(1 + \|\vec{v}\|_{Y^{1,s}(T)}\big)^k$$

for some $\theta > 0$, provided that $s > \frac{k-1}{k}$. A similar computation also yields a difference estimate. This proves existence of the unique solution $(v, \partial_t v) \in C((0, T]; \mathscr{H}^1(\mathbb{T}^2)) \cap C([0, T]; \mathscr{H}^s(\mathbb{T}^2))$, where the latter regularity may be shown a posteriori.

4. Pathwise global well-posedness in the cubic case

In this section, we prove pathwise global well-posedness of (21) with $(v, \partial_t v)|_{t=0} = (u_0, u_1) \in \mathscr{H}^1(\mathbb{T}^2)$ and $:\Psi^\ell: \in C([0, T]; W^{-\varepsilon,\infty}(\mathbb{T}^2))$, $\varepsilon > 0$, almost surely for $\ell = 1, 2, 3$. As mentioned in Section 1, our main goal is to control the growth of the energy $E(\vec{v})$ in (22) (with $k = 3$).

Given $T > 0$, we fix $0 \le t \le T$ and suppress t-dependence in the following. By (22) and (21), we have

$$\begin{aligned}
\partial_t E(\vec{v}) &= \int_{\mathbb{T}^2} (\partial_t v)\big\{\partial_t^2 v + (1-\Delta)v + v^3\big\}\mathrm{d}x \\
&= -3\int_{\mathbb{T}^2} (\partial_t v) v^2 \Psi \mathrm{d}x - 3\int_{\mathbb{T}^2} (\partial_t v) v{:}\Psi^2{:}\mathrm{d}x - \int_{\mathbb{T}^2} (\partial_t v){:}\Psi^3{:}\mathrm{d}x \\
&\quad - \int_{\mathbb{T}^2} \big(D^{\frac{1}{2}}\partial_t v\big)^2 \mathrm{d}x \\
&=: A_1 + A_2 + A_3 - \|D^{\frac{1}{2}}\partial_t v\|_{L^2}^2.
\end{aligned} \tag{56}$$

Given $T \gg 1$, we set $B(T) = B(T; \Psi)$ by

$$B(T) = 1 + \|{:}\Psi^2{:}\|^2_{L^\infty_T W_x^{-\frac{1}{2},\infty}} + \|{:}\Psi^3{:}\|^2_{L^\infty_T W_x^{-\frac{1}{2},\infty}} + \varepsilon\|\Psi\|^2_{L^\infty_T W_x^{-\varepsilon,\infty}} \tag{57}$$

for some small $\varepsilon > 0$. By Cauchy–Schwarz's inequality, Cauchy's inequality, and Lemma 9 (ii) with (57), we have

$$\begin{aligned}
|A_2| &\le 3\|\langle\nabla\rangle^{\frac{1}{2}}\partial_t v\|_{L^2}\|\langle\nabla\rangle^{-\frac{1}{2}} v{:}\Psi^2{:}\|_{L^2} \\
&\le C\|\partial_t v\|_{L^2}^2 + \delta\|D^{\frac{1}{2}}\partial_t v\|_{L^2}^2 + C_\delta\|\langle\nabla\rangle^{\frac{1}{2}} v\|_{L^2}^2\|\langle\nabla\rangle^{-\frac{1}{2}}{:}\Psi^2{:}\|_{L^\infty}^2 \\
&\le C_\delta B(T)E(\vec{v}) + \delta\|D^{\frac{1}{2}}\partial_t v\|_{L^2}^2
\end{aligned} \tag{58}$$

for some small $\delta > 0$. Similarly, we have

$$\begin{aligned}
|A_3| &\le \|\langle\nabla\rangle^{\frac{1}{2}}\partial_t v\|_{L^2}\|\langle\nabla\rangle^{-\frac{1}{2}}{:}\Psi^3{:}\|_{L^2} \\
&\le C\|\partial_t v\|_{L^2}^2 + \delta\|D^{\frac{1}{2}}\partial_t v\|_{L^2}^2 + C_\delta\|\langle\nabla\rangle^{-\frac{1}{2}}{:}\Psi^3{:}\|_{L^\infty}^2 \\
&\le CE(\vec{v}) + C_\delta B(T) + \delta\|D^{\frac{1}{2}}\partial_t v\|_{L^2}^2.
\end{aligned} \tag{59}$$

It remains to estimate A_1. First, note that from the embedding $L^{\frac{4}{1+\varepsilon}}(\mathbb{T}^2) \subset L^{2+\varepsilon}(\mathbb{T}^2)$ and interpolation, we have

$$\|v\|_{W^{\varepsilon,2+\varepsilon}} \lesssim \|v\|_{W^{\varepsilon,\frac{4}{1+\varepsilon}}} \lesssim \|v\|_{H^1}^{\varepsilon}\|v\|_{L^4}^{1-\varepsilon} \lesssim E^{\frac{\varepsilon}{2}}(\vec{v})E^{\frac{1-\varepsilon}{4}}(\vec{v}) = E^{\frac{1+\varepsilon}{4}}(\vec{v}) \tag{60}$$

for sufficiently small $0 < \varepsilon \ll 1$. For simplicity of notation, we set $E(t) = E(\vec{v}(t))$. In the following, we assume that $E(t) > 1$.

For $0 < t_1 \le t_2 \le T$, let

$$\mathscr{A}_1(t_1, t_2) = \int_{t_1}^{t_2}\int_{\mathbb{T}^2} \partial_t v \cdot v^2 \cdot \Psi \,\mathrm{d}x\mathrm{d}t.$$

Then, from Hölder's inequality, Lemma 9 (i), Cauchy's inequality, Sobolev's inequality, and (60) with (57), we have

$$\begin{aligned}
|\mathscr{A}(t_1,t_2)| &= \Big|\int_{t_1}^{t_2}\int_{\mathbb{T}^2}\langle\nabla\rangle^{\varepsilon}\big(\partial_t v\cdot v^2\big)\cdot\langle\nabla\rangle^{-\varepsilon}\Psi\,\mathrm{d}x\mathrm{d}t\Big|\\
&\le C\int_{t_1}^{t_2}\|\partial_t v(t')\|_{W^{\varepsilon,2+\varepsilon}}\|v(t')\|_{L^4}^2\mathrm{d}t'\cdot\|\Psi\|_{L^\infty_T W_x^{-\varepsilon,\infty}}\\
&\quad + C\int_{t_1}^{t_2}\|\partial_t v(t')\|_{L^4}\|v(t')\|_{W^{\varepsilon,2+\varepsilon}}\|v(t')\|_{L^4}\mathrm{d}t'\cdot\|\Psi\|_{L^\infty_T W_x^{-\varepsilon,\infty}}\\
&\le \int_{t_1}^{t_2}\Big(\delta\|D^{\frac12}\partial_t v(t')\|_{L^2}^2 + \varepsilon^{-1}C_\delta B(T)E(t')\Big)\mathrm{d}t'\\
&\quad + \varepsilon^{-1}C_\delta B(T)\int_{t_1}^{t_2}\|v(t')\|_{W^{\varepsilon,2+\varepsilon}}^2 E(t')^{\frac12}\mathrm{d}t'\\
&\le \delta\int_{t_1}^{t_2}\|D^{\frac12}\partial_t v(t')\|_{L^2}^2 dt' + \varepsilon^{-1}C_\delta B(T)\int_{t_1}^{t_2}E^{1+\frac{\varepsilon}{2}}(t')\mathrm{d}t'
\end{aligned} \tag{61}$$

for any sufficiently small $\varepsilon > 0$. Let us now consider the second term on the right-hand side of (61) with $p = 2\varepsilon^{-1}$:

$$A(p) = p\int_{t_1}^{t_2} E^{1+\frac{1}{p}}(t')\mathrm{d}t'.$$

By optimizing $A(p)$ in p at $p = \log E(t')$ (for each fixed t') and noting that $x^{\frac{c}{\log x}} \lesssim 1$ for any $x > 1$, it follows from (61) that

$$|\mathscr{A}(t_1,t_2)| \le \delta\int_{t_1}^{t_2}\|D^{\frac12}\partial_t v(t')\|_{L^2}^2\mathrm{d}t' + C_\delta B(T)\int_{t_1}^{t_2}E(t')\log E(t')\mathrm{d}t'. \tag{62}$$

Combining (56), (58), (59), and (62), we then obtain the following Gronwall bound:

$$E(t_2) - E(t_1) \le C\cdot B(T)\int_{t_1}^{t_2}E(t')\log E(t')\mathrm{d}t' \tag{63}$$

for any $0 < t_1 \le t_2 \le T$. By solving (63), we obtain

$$E(t) \lesssim e^{e^{C\cdot B(T)t}} \tag{64}$$

for any $0 < t \le T$.

Lastly, note that from Lemma 8, we see that $B(T) < \infty$, almost surely. Furthermore, the choice of $T \gg 1$ was arbitrary. Therefore, we conclude that the $\mathscr{H}^1$-norm of the solution $(v, \partial_t v)(t)$ to (21) remains finite on any finite time interval $[0, T]$, almost surely, thus allowing us to iteratively apply Proposition 11. This concludes the proof of Theorem 3.

Remark 13.

(i) In order to justify the formal computation in this subsection, we need to proceed with the smooth solution $(v_N, \partial_t v_N)$ to the truncated equation (14) with the frequency truncated initial data $(\mathbf{P}_N u_0, \mathbf{P}_N u_1)$ (for example, to guarantee finiteness of the last term on the right-hand side of (56)) and then take $N \to \infty$ in (64) by noting that the implicit constant is independent of $N \in \mathbb{N}$ and by using Proposition 11 (namely, the continuous dependence of a solution on an enhanced data set). This argument, however, is standard and thus we omit details. See, for example, [35].

(ii) By refining the argument, it is possible to obtain a double exponential bound

$$\|(v, \partial_t v)(t)\|_{\mathscr{H}^1} \le C e^{e^{C(\omega) t^\theta}}$$

for some $\theta > 0$ and an almost surely finite random constant $C(\omega) > 0$. We, however, do not pursue this issue here. See, for example, Remark 3.7 in [19].

5. Almost sure global well-posedness

In this section, we first sketch the proof of almost sure global well-posedness with the Gaussian random initial data (u_0^ω, u_1^ω) in (24) (Theorem 4), and then briefly discuss the proof of almost sure global well-posedness with the shifted initial data $(v_0, v_1) + (u_0^\omega, u_1^\omega)$ (Corollary 5).

5.1. *Invariant measure argument*

As mentioned in Section 1, the main strategy for proving Theorem 4 is to (i) use the mutual absolute continuity of $\vec{\mu}_1$ (the law of the random initial data (u_0^ω, u_1^ω) in (24)) and the Gibbs measure $\vec{\rho}$ in (30) and (ii) then apply Bourgain's invariant measure argument. In the following, given a random variable X, let $\mathscr{L}(X)$ denote the law of X.

For this purpose, we first review the construction of the Gibbs measure $\vec{\rho}$. Given $N \in \mathbb{N}$, consider the truncated Gibbs measure:

$$d\vec{\rho}_N(u, \partial_t u) = Z_N^{-1} R_N(u) d\vec{\mu}_1(u, \partial_t u) \tag{65}$$

with the truncated renormalized density:

$$R_N(u) = \exp\left(-\frac{1}{k+1}\int_{\mathbb{T}^2} :(\mathbf{P}_N u)^{k+1}(x): dx\right), \tag{66}$$

where the Wick power $:(\mathbf{P}_N u)^{k+1}:$ is defined by

$$:(\mathbf{P}_N u)^{k+1}(x): \stackrel{\text{def}}{=} H_{k+1}(\mathbf{P}_N u(x); \alpha_N)$$

with α_N as in (27). Then, it is known that $\{R_N\}_{N\in\mathbb{N}}$ converges to some $R(u)$ in $L^p(\vec{\mu}_1)$ for any finite $p \ge 1$ and thus the truncated Gibbs measure $\vec{\rho}_N$ in (65) converges, say in total variation, to the renormalized Gibbs measure $\vec{\rho}$ given by

$$\begin{aligned} d\vec{\rho}(u, \partial_t u) &= Z^{-1} R(u) d\vec{\mu}_1(u, \partial_t u) \\ &= Z^{-1} \exp\left(-\frac{1}{k+1}\int_{\mathbb{T}^2} :u^{k+1}(x): dx\right) d\vec{\mu}_1(u, \partial_t u). \end{aligned} \tag{67}$$

Furthermore, the resulting Gibbs measure $\vec{\rho}$ and $\vec{\mu}_1$ are mutually absolutely continuous. See [13, 15, 40, 42] for details.

Next, we turn our attention to the well-posedness part. Let us consider the following truncated SvNLW:

$$\partial_t^2 u_N + (1-\Delta) u_N + D\partial_t u_N + \mathbf{P}_N\big(:(\mathbf{P}_N u_N)^k:\big) = \sqrt{2} D^{\frac{1}{2}} \xi \tag{68}$$

and its formal limit:

$$\partial_t^2 u + (1-\Delta)u + D\partial_t u + :u^k: = \sqrt{2}D^{\frac{1}{2}}\xi. \tag{69}$$

As we see below, the truncated Gibbs measure $\vec{\rho}_N$ in (65) is invariant under the truncated dynamics (68). Then, Bourgain's invariant measure argument [4, 5] yields the following almost sure global well-posedness.

Theorem 14. *Let $k \in 2\mathbb{N}+1$. Then, the renormalized SvNLW* (69) *is almost surely globally well-posed with respect to the renormalized Gibbs measure $\vec{\rho}$ in* (67). *Furthermore, the renormalized Gibbs measure $\vec{\rho}$ is invariant under the dynamics.*

More precisely, there exists a non-trivial stochastic process $(u, \partial_t u) \in C(\mathbb{R}_+; \mathcal{H}^{-\varepsilon}(\mathbb{T}^2))$ for any $\varepsilon > 0$ such that, given any $T > 0$, the solution $(u_N, \partial_t u_N)$ to the truncated SvNLW (68) *with $\mathcal{L}((u_N(0), \partial_t u_N(0)) = \vec{\rho}_N$ converges in probability to some stochastic process $(u, \partial_t u)$ in $C([0, T]; \mathcal{H}^{-\varepsilon}(\mathbb{T}^2))$. Moreover, we have $\mathcal{L}(u(t), \partial_t u(t)) = \vec{\rho}$ for any $t \geq 0$.*

The proof of Theorem 14 follows exactly the same steps as in the proof of the almost sure global well-posedness to the stochastic damped NLW studied in [19]:

$$\partial_t^2 u + \partial_t u + (1-\Delta)u + :u^k: = \sqrt{2}\xi$$

and thus we only sketch the key steps in the following.

Let us first describe the precise meaning of the renormalizations in (68) and (69). Write the solution u_N to (68) with $\mathcal{L}\big((u_N, \partial_t u_N)|_{t=0}\big) = \vec{\mu}_1$ as

$$u_N = v_N + \Phi = (v_N + \Phi_N) + \mathbf{P}_N^{\perp}\Phi,$$

where $\mathbf{P}_N^{\perp} = \mathrm{Id} - \mathbf{P}_N$ and Φ denotes the stochastic convolution defined in (26) (recall that[6] $\mathcal{L}((\Phi(t), \partial_t\Phi(t))) = \vec{\mu}_1$ for any $t \geq 0$). Then, we see that (68) decouples into the linear dynamics for the high frequency part $\mathbf{P}_N^{\perp}u_N = \mathbf{P}_N^{\perp}\Phi$:

$$\partial_t^2\mathbf{P}_N^{\perp}\Phi + (1-\Delta)\mathbf{P}_N^{\perp}\Phi + D\partial_t\mathbf{P}_N^{\perp}\Phi = \sqrt{2}\mathbf{P}_N^{\perp}D^{\frac{1}{2}}\xi \tag{70}$$

and the nonlinear dynamics for the low frequency part $\mathbf{P}_N u_N$:

$$\partial_t^2\mathbf{P}_N u_N + (1-\Delta)\mathbf{P}_N u_N + D\partial_t\mathbf{P}_N u_N + \mathbf{P}_N\big(:(\mathbf{P}_N u_N)^k:\big) = \sqrt{2}\mathbf{P}_N D^{\frac{1}{2}}\xi. \tag{71}$$

In terms of $v_N = \mathbf{P}_N u_N - \Phi_N$, we can write (71) as

$$\begin{cases} \partial_t^2 v_N + (1-\Delta)v_N + D\partial_t v_N + \sum_{\ell=0}^{k}\binom{k}{\ell}\mathbf{P}_N\big(:\Phi_N^{\ell}:v_N^{k-\ell}\big) = 0 \\ (v_N, \partial_t v_N)|_{t=0} = (0,0), \end{cases} \tag{72}$$

where the Wick power is defined by

$$:\Phi_N^{\ell}(t,x): \stackrel{\text{def}}{=} H_\ell(\Phi_N(t,x); \alpha_N),$$

where α_N is as in (27). By taking $N \to \infty$, we obtain the limiting equation:

$$\begin{cases} \partial_t^2 v + (1-\Delta)v + D\partial_t v + \sum_{\ell=0}^{k}\binom{k}{\ell}:\Phi^{\ell}:v^{k-\ell} = 0 \\ (v, \partial_t v)|_{t=0} = (0,0). \end{cases} \tag{73}$$

In view of Lemma 8, the local well-posedness result (Proposition 11) applies to (72) and (73), uniformly in $N \in \mathbb{N}$. Furthermore, the solution v_N converges to v on the (random) time interval of local existence.

Once we check invariance of the truncated Gibbs measure $\vec{\rho}_N$ in (65) under the truncated SvNLW dynamics (68), the rest of the proof of Theorem 14 follows from a standard application of

[6] A tedious but direct computation, as in (27), shows that $\mathbb{E}\big[|\widehat{\partial_t\Phi}(t,n)|^2\big] = 1$ for any $n \in \mathbb{Z}^2$.

Bourgain's invariant measure argument. See, for example, [8, 33, 34, 38] for details in the context of stochastic nonlinear wave equations. See also [41].

Invariance of the truncated Gibbs measure $\vec{\rho}_N$ under the truncated SvNLW dynamics (68) follows from exactly the same argument presented in Section 4 of [19]. For readers' convenience, however, we sketch the argument. Given $N \in \mathbb{N}$, define the marginal probability measures $\vec{\mu}_{1,N}$ and $\vec{\mu}_{1,N}^{\perp}$ on $\mathbf{P}_N\mathscr{H}^{-\varepsilon}(\mathbb{T}^2)$ and $\mathbf{P}_N^{\perp}\mathscr{H}^{-\varepsilon}(\mathbb{T}^2)$, respectively, as the induced probability measures under the map T_N for $\vec{\mu}_{1,N}$ and $T_N^{\perp}$ for $\vec{\mu}_{1,N}^{\perp}$, where

$$T_N : \omega \in \Omega \longmapsto (\mathbf{P}_N u_0^{\omega}, \mathbf{P}_N u_1^{\omega}) \qquad \text{and} \qquad T_N^{\perp} : \omega \in \Omega \longmapsto (\mathbf{P}_N^{\perp} u_0^{\omega}, \mathbf{P}_N^{\perp} u_2^{\omega}),$$

where $(u_0^{\omega}, u_1^{\omega})$ is as in (24). Then, with $\vec{\mu}_1 = \vec{\mu}_{1,N} \otimes \vec{\mu}_{1,N}^{\perp}$, it follows from (65) that

$$\vec{\rho}_N = \vec{\nu}_N \otimes \vec{\mu}_{1,N}^{\perp}, \tag{74}$$

where $\vec{\nu}_N$ is given by $\mathrm{d}\vec{\nu}_N = Z_N^{-1} R_N(u)\mathrm{d}\vec{\mu}_{1,N}$ with the density R_N as in (66). The high frequency dynamics (70) is linear and thus we can readily verify that the Gaussian measure $\vec{\mu}_{1,N}^{\perp}$ is invariant under the dynamics of (70). Hence, it remains to check invariance of $\vec{\nu}_N$ under the low frequency dynamics (71).

With $(u_N^1, u_N^2) = (\mathbf{P}_N u_N, \partial_t \mathbf{P}_N u_N)$, we can write (71) in the following Ito formulation:

$$\mathrm{d}\begin{pmatrix} u_N^1 \\ u_N^2 \end{pmatrix} = -\left\{ \begin{pmatrix} 0 & -1 \\ 1-\Delta & 0 \end{pmatrix} \begin{pmatrix} u_N^1 \\ u_N^2 \end{pmatrix} + \begin{pmatrix} 0 \\ \mathbf{P}_N\big(:(u_N^1)^k:\big) \end{pmatrix} \right\} \mathrm{d}t + \begin{pmatrix} 0 \\ -Du_N^2 \mathrm{d}t + \sqrt{2}\mathbf{P}_N D^{\frac{1}{2}}\, dW \end{pmatrix}. \tag{75}$$

This shows that the generator $\mathscr{L}^N$ of the Markov semigroup for (75) can be written as $\mathscr{L}^N = \mathscr{L}_1^N + \mathscr{L}_2^N$, where $\mathscr{L}_1^N$ denotes the generator for the deterministic NLW with the truncated nonlinearity:

$$\mathrm{d}\begin{pmatrix} u_N^1 \\ u_N^2 \end{pmatrix} = -\left\{ \begin{pmatrix} 0 & -1 \\ 1-\Delta & 0 \end{pmatrix} \begin{pmatrix} u_N^1 \\ u_N^2 \end{pmatrix} + \begin{pmatrix} 0 \\ \mathbf{P}_N\big(:(u_N^1)^k:\big) \end{pmatrix} \right\} \mathrm{d}t \tag{76}$$

and $\mathscr{L}_2^N$ denotes the generator for the Ornstein–Uhlenbeck process:

$$\mathrm{d}u_N^2 = -Du_N^2 \mathrm{d}t + \sqrt{2}\mathbf{P}_N D^{\frac{1}{2}}\mathrm{d}W. \tag{77}$$

Invariance of $\vec{\nu}_N$ under the dynamics of (76) follows easily from the Hamiltonian structure of (76) with the Hamiltonian:

$$E_N(u_N^1, u_N^2) = \frac{1}{2}\int_{\mathbb{T}^2} \big((u_N^1)^2 + |\nabla u_N^1|^2\big)\mathrm{d}x + \frac{1}{2}\int_{\mathbb{T}^2} (u_N^2)^2 \mathrm{d}x - \log\big(R_N(u_N^1)\big),$$

where R_N is as in (66), in particular, the conservation of the Hamiltonian $E_N(u_N^1, u_N^2)$ and Liouville's theorem (on a finite-dimensional phase space $\mathbf{P}_N\mathscr{H}^{-\varepsilon}(\mathbb{T}^2)$). Hence, we have $(\mathscr{L}_1^N)^*\vec{\nu}_N = 0$.

As for (77), recalling that the Ornstein–Uhlenbeck process preserves the standard Gaussian measure (at each frequency on the Fourier side), we see that $\vec{\nu}_N$ is also invariant under the dynamics of (77), since, on the second component u_N^2, the measure $\vec{\nu}_N$ is nothing but the white noise μ_0 (projected onto the low frequencies $\{|n| \le N\}$). Hence, we have $(\mathscr{L}_2^N)^*\vec{\nu}_N = 0$. Therefore, we conclude that

$$(\mathscr{L}^N)^*\vec{\nu}_N = (\mathscr{L}_1^N)^*\vec{\nu}_N + (\mathscr{L}_2^N)^*\vec{\nu}_N = 0.$$

This shows invariance of $\vec{\nu}_N$ under (75) and hence under (71). Finally, invariance of the truncated Gibbs measure $\vec{\rho}_N$ in (65) under the truncated SvNLW dynamics (68) follows from (74) and the invariance of $\vec{\mu}_{1,N}^{\perp}$ and $\vec{\nu}_N$ under (70) and (75), respectively.

Remark 15. As a consequence of Bourgain's invariant measure argument, the solution $(u, \partial_t u)$ to (18) constructed in Theorem 4 satisfies the following logarithmic growth bound:

$$\|(u(t), \partial_t u(t))\|_{\mathscr{H}^{-\varepsilon}} \le C(\omega)\big(\log(1+t)\big)^{\frac{k}{2}}$$

for any $t \ge 0$. See [38] for details.

5.2. *Almost sure global well-posedness with the shifted initial data*

We conclude this paper by briefly discussing the proof of Corollary 5, using the Cameron–Martin theorem [11]. For this purpose, we first go over the definition of abstract Wiener spaces introduced by Gross [16]. See also Kuo [25]. Let H be a real separable Hilbert space. It is known that the Gauss measure μ with the density $\mathrm{d}\mu = Z^{-1}e^{-\frac{1}{2}\|x\|_H^2}\mathrm{d}x$ is only finitely additive if $\dim H = \infty$.

Let $\mathscr{P}$ denotes the collection of all finite dimensional orthogonal projections of H. A seminorm $|||\cdot|||$ on H is said to be measurable if, for any $\varepsilon > 0$, there exists $\mathbf{P}_\varepsilon \in \mathscr{P}$ such that $\mu(|||\mathbf{P}x||| > \varepsilon) < \varepsilon$ for all $\mathbf{P} \in \mathscr{P}$ with $\mathbf{P} \perp \mathbf{P}_\varepsilon$. Let B be the completion of H with respect to this seminorm $|||\cdot|||$. Then, Gross [16] showed that we can construct μ as a countably additive Gaussian measure on B. In this case, we say that the triplet (B, H, μ) is an *abstract Wiener space*. The original Hilbert space H is referred to as a *Cameron–Martin space* or a reproducing kernel Hilbert space.

Let (B, H, μ) be an abstract Wiener space. Then, the Cameron–Martin Theorem [11] states the following.

Lemma 16. *Given $h \in B$, define a shifted measure μ_h by $\mu_h(\cdot) := \mu(\cdot - h)$. Then, the shifted measure μ_h and the original Gaussian measure are equivalent (namely, mutually absolutely continuous) if and only if $h \in H$.*

Let (u_0^ω, u_1^ω) be as in (24). Then, its distribution is given by $\vec{\mu}_1 = \mu_1 \otimes \mu_0$ defined in (31), with the formal density:

$$d\vec{\mu}_1 = Z^{-1}e^{-\frac{1}{2}\|(u,v)\|^2_{\mathscr{H}^1}}d(u,v).$$

In this context, the Cameron–Martin theorem states that the Gaussian measure $\vec{\mu}_1$ and the shifted measure

$$\vec{\mu}_{1,\vec{v}_0}(\cdot) := \vec{\mu}_1(\cdot - \vec{v}_0) \tag{78}$$

are equivalent if and only if $\vec{v}_0 = (v_0, v_1)$ belongs to the Cameron–Martin space $\mathscr{H}^1(\mathbb{T}^2)$.

Now, fix $\vec{v}_0 = (v_0, v_1) \in \mathscr{H}^1(\mathbb{T}^2)$. Let $\Phi[\vec{v}_0 + \vec{u}_0^\omega]$ be the stochastic convolution defined in (32). With $\Phi_N[\vec{v}_0 + \vec{u}_0^\omega] = \mathbf{P}_N\Phi[\vec{v}_0 + \vec{u}_0^\omega]$, we define the Wick power $:\!(\Phi_N[\vec{v}_0 + \vec{u}_0^\omega])^\ell\!:$ as in (33). Then, from Sobolev's embedding (with large $r \gg 1$ such that $\varepsilon r > 4$), (32), (33), and Lemma 9 (ii) followed by Lemma 9 (i) and Sobolev's inequality, we have

$$\begin{aligned}
\|:\!(\Phi_N[\vec{v}_0 + \vec{u}_0^\omega])^\ell\!:\|_{C_TW_x^{-\varepsilon,\infty}} &\lesssim \|:\!(V(t)\vec{v}_0 + \Phi_N[\vec{u}_0^\omega])^\ell\!:\|_{C_TW_x^{-\frac{\varepsilon}{2},r}} \\
&\lesssim \sum_{j=0}^{\ell} \|(V(t)\vec{v}_0)^j\|_{C_TW_x^{\frac{\varepsilon}{2},p}} \|:\!(\Phi_N[u_0^\omega])^{\ell-j}\!:\|_{C_TW_x^{-\frac{\varepsilon}{2},\infty}} \\
&\lesssim \sum_{j=0}^{\ell} \|(V(t)\vec{v}_0)\|_{C_TW_x^{\frac{\varepsilon}{2},\frac{4}{\varepsilon}}} \|(V(t)\vec{v}_0)\|^{j-1}_{C_TL_x^{\frac{4p(j-1)}{4-\varepsilon p}}} \|:\!(\Phi_N[u_0^\omega])^{\ell-j}\!:\|_{C_TW_x^{-\frac{\varepsilon}{2},\infty}} \\
&\lesssim \sum_{j=0}^{\ell} \|(V(t)\vec{v}_0)\|^j_{C_TH_x^1} \|:\!(\Phi_N[u_0^\omega])^{\ell-j}\!:\|_{C_TW_x^{-\frac{\varepsilon}{2},\infty}} \\
&< \infty,
\end{aligned} \tag{79}$$

almost surely, thanks to the fact that $\vec{v}_0 \in \mathscr{H}^1(\mathbb{T}^2)$ and Lemma 8. Here, we chose $p \geq 1$ such that $\frac{\varepsilon}{4} < \frac{1}{p} < \frac{1}{r} + \frac{\varepsilon}{4}$. We also point out that the implicit constants in (79) are independent of the cutoff size $N \in \mathbb{N}$. A slight modification of (79), combined with Lemma 8, allows us to construct the Wick power $:\!(\Phi[\vec{v}_0 + \vec{u}_0^\omega])^\ell\!:$ by a limiting procedure.

Given $\vec{w}_0 = (w_0, w_1) \in \mathcal{H}^{-\varepsilon}(\mathbb{T}^2)$, consider the following Cauchy problem:

$$\begin{cases} \partial_t^2 v + (1-\Delta)v + D\partial_t v + \sum_{\ell=0}^k \binom{k}{\ell} :(\Phi[\vec{w}_0])^\ell: v^{k-\ell} = 0 \\ (v, \partial_t v)|_{t=0} = (0,0). \end{cases} \tag{80}$$

In the following, we write the solution $\vec{v} = (v, \partial_t v)$ to (80) as $\vec{v}[\vec{w}_0]$ to signify the dependence on $\vec{w}_0$. Here, the Wick power $:(\Phi[\vec{w}_0])^\ell:$, if it exists, is defined by a limiting procedure as above with the divergent constant α_N in (27). For $\vec{w}_0 = \vec{u}_0^\omega$ distributed by $\vec{\mu}_1$, the Wick power exists almost surely. Furthermore, as discussed in Subsection 5.1, by Bourgain's invariant measure argument, the solution $v[\vec{u}_0^\omega]$ to (80) exists globally in time, almost surely. In particular, by a close inspection of the argument (see, for example, [38]), we can also show that

$$\sup_{t\ge 0} \frac{\|\vec{v}[\vec{u}_0^\omega](t)\|_{\mathcal{H}^{1-\varepsilon}}}{C(t)} < \infty, \tag{81}$$

almost surely, for some deterministic increasing function $C(t)$.[7]

We now define a set $\mathscr{A} \subset \mathcal{H}^{-\varepsilon}(\mathbb{T}^2)$ by

$$\mathscr{A} = \left\{ \vec{w}_0 \in \mathcal{H}^{-\varepsilon}(\mathbb{T}^2) : \sup_{t\ge 0} \frac{\|\vec{v}[\vec{w}_0](t)\|_{\mathcal{H}^{1-\varepsilon}}}{C(t)} < \infty \right\}. \tag{82}$$

From the construction, we see that the map $\vec{w}_0 \mapsto \{:(\Phi[\vec{w}_0])^\ell:\}_{\ell=1}^k$ is measurable (but is not continuous). Furthermore, the solution map $(\Phi[\vec{w}_0], :(\Phi[\vec{w}_0])^2:, \dots, :(\Phi[\vec{w}_0])^k:) \mapsto \vec{v}[\vec{w}_0]$ is continuous; see, for example, Proposition 11. Hence, the map $w_0 \mapsto \vec{v}[\vec{w}_0]$ is measurable and thus the set $\mathscr{A}$ in (82) is a measurable set. Therefore, from Lemma 16 with (81) (namely $\vec{\mu}_1(\mathscr{A}) = 1$), we conclude that $\vec{\mu}_{1,\vec{v}_0}(\mathscr{A}) = 1$, where $\vec{\mu}_{1,\vec{v}_0}$ is as in (78). Noting that $\vec{\mu}_{1,\vec{v}_0}$ is the distribution of the shifted initial data $\vec{v}_0 + \vec{u}_0^\omega$, we conclude that the $\mathcal{H}^{1-\varepsilon}$-norm of the solution $\vec{v}$ to (34) remains finite for finite times, almost surely. This proves Corollary 5.

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