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On the minimum size of subset and subsequence sums in integers

Jagannath Bhanja[*, a] **and Ram Krishna Pandey**[b]

[a] Harish-Chandra Research Institute, A CI of Homi Bhabha National Institute, Chhatnag Road, Jhunsi, Prayagraj-211019, India

[b] Department of Mathematics, Indian Institute of Technology Roorkee, Roorkee-247667, India

E-mails: jagannathbhanja@hri.res.in, ram.pandey@ma.iitr.ac.in

Abstract. Let $\mathscr{A}$ be a sequence of rk terms which is made up of k distinct integers each appearing exactly r times in $\mathscr{A}$. The sum of all terms of a subsequence of $\mathscr{A}$ is called a subsequence sum of $\mathscr{A}$. For a nonnegative integer $\alpha \le rk$, let $\Sigma_\alpha(\mathscr{A})$ be the set of all subsequence sums of $\mathscr{A}$ that correspond to the subsequences of length α or more. When $r = 1$, we call the subsequence sums as subset sums and we write $\Sigma_\alpha(A)$ for $\Sigma_\alpha(\mathscr{A})$. In this article, using some simple combinatorial arguments, we establish optimal lower bounds for the size of $\Sigma_\alpha(A)$ and $\Sigma_\alpha(\mathscr{A})$. As special cases, we also obtain some already known results in this study.

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1. Introduction

Let A be a set of k integers. The sum of all elements of a subset of A is called a *subset sum* of A. So, the subset sum of the empty set is 0. For a nonnegative integer $\alpha \le k$, let

$$\Sigma_\alpha(A) := \left\{ \sum_{a \in A'} a : A' \subset A,\ \left|A'\right| \ge \alpha \right\}$$

and

$$\Sigma^\alpha(A) := \left\{ \sum_{a \in A'} a : A' \subset A,\ \left|A'\right| \le k - \alpha \right\}.$$

That is, $\Sigma_\alpha(A)$ is the set of subset sums corresponding to the subsets of A that are of the size at least α and $\Sigma^\alpha(A)$ is the set of subset sums corresponding to the subsets of A that are of the size at most $k-\alpha$. So, $\Sigma_\alpha(A) = \sum_{a \in A} a - \Sigma^\alpha(A)$. Therefore $|\Sigma_\alpha(A)| = |\Sigma^\alpha(A)|$.

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[*] Corresponding author.

Now, we extend the above definitions for sequences of integers. Before we go for extension, we mention some notation that are used throughout the paper.

Let

$$\mathscr{A} = \left(\underbrace{a_1, \ldots, a_1}_{r \text{ copies}}, \underbrace{a_2, \ldots, a_2}_{r \text{ copies}}, \ldots, \underbrace{a_k, \ldots, a_k}_{r \text{ copies}} \right)$$

be a sequence of rk terms, where $a_1, a_2, \ldots, a_k$ are distinct integers each appearing exactly r times in $\mathscr{A}$. We denote this sequence by $\mathscr{A} = (a_1, a_2, \ldots, a_k)_r$. If $\mathscr{A}'$ is a subsequence of $\mathscr{A}$, then we write $\mathscr{A}' \subset \mathscr{A}$. By $x \in \mathscr{A}$, we mean x is a term in $\mathscr{A}$. For the number of terms in a sequence $\mathscr{A}$, we use the notation $|\mathscr{A}|$. For an integer x, we let $x * \mathscr{A}$ be the sequence which is obtained from by multiplying each term of $\mathscr{A}$ by x. For two nonempty sequences $\mathscr{A}$, $\mathscr{B}$, by $\mathscr{A} \cap \mathscr{B}$, we mean the sequence of all those terms that are in both $\mathscr{A}$ and $\mathscr{B}$. Furthermore, for integers a, b with $b \geq a$, by $[a, b]_r$, we mean the sequence $(a, a+1, \ldots, b)_r$.

Let $\mathscr{A} = (a_1, a_2, \ldots, a_k)_r$ be a sequence of integers with rk terms. The sum of all terms of a subsequence of $\mathscr{A}$ is called a *subsequence sum* of $\mathscr{A}$. For a nonnegative integer $\alpha \leq rk$, let

$$\Sigma_\alpha(\mathscr{A}) := \left\{ \sum_{a \in \mathscr{A}'} a : \mathscr{A}' \subset \mathscr{A},\ \left|\mathscr{A}'\right| \geq \alpha \right\}$$

and

$$\Sigma^\alpha(\mathscr{A}) := \left\{ \sum_{a \in \mathscr{A}'} a : \mathscr{A}' \subset \mathscr{A},\ \left|\mathscr{A}'\right| \leq rk - \alpha \right\}.$$

That is, $\Sigma_\alpha(\mathscr{A})$ is the set of subsequence sums corresponding to the subsequences of $\mathscr{A}$ that are of the size at least α and $\Sigma^\alpha(\mathscr{A})$ is the set of subsequence sums corresponding to the subsequences of $\mathscr{A}$ that are of the size at most $rk - \alpha$. Then in the same line with the subset sums, we have $|\Sigma_\alpha(\mathscr{A})| = |\Sigma^\alpha(\mathscr{A})|$ for all $0 \leq \alpha \leq rk$.

The set of subset sums $\Sigma_\alpha(A)$ and $\Sigma^\alpha(A)$ and the set of subsequence sums $\Sigma_\alpha(\mathscr{A})$ and $\Sigma^\alpha(\mathscr{A})$ may also be written as unions of sumsets:

For a finite set A of k integers and for positive integers h, r, the *h-fold sumset hA* is the collection of all sums of h not-necessarily-distinct elements of A, the *h-fold restricted sumset $h\hat{}A$* is the collection of all sums of h distinct elements of A, and the *generalized sumset $h^{(r)}A$* is the collection of all sums of h elements of A with at most r repetitions for each element (see [15]). Then $\Sigma_\alpha(A) = \bigcup_{h=\alpha}^{k} h\hat{}A$, $\Sigma^\alpha(A) = \bigcup_{h=0}^{k-\alpha} h\hat{}A$, $\Sigma_\alpha(\mathscr{A}) = \bigcup_{h=\alpha}^{rk} h^{(r)}A$, and $\Sigma^\alpha(\mathscr{A}) = \bigcup_{h=0}^{rk-\alpha} h^{(r)}A$, where $\mathscr{A} = (A)_r$ and $0\hat{}A = 0^{(r)}A = \{0\}$.

An important problem in additive number theory is to find the optimal lower bound for $|\Sigma_\alpha(A)|$ and $|\Sigma_\alpha(\mathscr{A})|$. Such problems are very useful in some other combinatorial problems such as the zero-sum problems (see [6, 8, 11, 21]). Nathanson [18] established the optimal lower bound for $|\Sigma_1(A)|$ for sets of integers A. Mistri and Pandey [16, 17] and Jiang and Li [14] extended Nathanson's results to $\Sigma_1(\mathscr{A})$ for sequences of integers $\mathscr{A}$. Note that these subset and subsequence sums may also be studied in any abelian group (for earlier works, in case $\alpha = 0$ and $\alpha = 1$, see [1, 3, 7–10, 12, 13, 20]). Recently, Balandraud [2] proved the optimal lower bound for $|\Sigma_\alpha(A)|$ in the finite prime field $\mathbb{F}_p$, where p is a prime number. Inspired by Balandraud's work [2], in this paper we establish optimal lower bounds for $|\Sigma_\alpha(A)|$ and $|\Sigma_\alpha(\mathscr{A})|$ in the group of integers. Note that, in [5], we have already settled this problem when the set A (or sequence $\mathscr{A}$) contains nonnegative or nonpositive integers. So, in this paper we consider the sets (or sequences) which may contain both positive and negative integers.

In Section 2, we prove optimal lower bounds for $|\Sigma_\alpha(A)|$ for finite sets of integers A. In Section 3, we extend the results of Section 2 to sequences of integers.

The following results are used to prove the results in this paper.

Theorem 1 ([19, Theorem 1.4]). *Let A, B be nonempty finite sets of integers. Set $A+B=\{a+b : a \in A, b \in B\}$. Then*
$$|A+B| \geq |A|+|B|-1.$$
This lower bound is optimal.

Theorem 2 ([19, Theorem 1.3]). *Let A be a nonempty finite set of integers and h be a positive integer. Then*
$$|hA| \geq h|A|-h+1.$$
This lower bound is optimal.

Theorem 3 ([2, Theorem 4]). *Let A be a nonempty subset of $\mathbb{F}_p$ such that $A \cap (-A)=\emptyset$. Then for any integer $\alpha \in [0,|A|]$, we have*
$$|\Sigma_\alpha(A)| \geq \min\left\{p, \frac{|A|(|A|+1)}{2}-\frac{\alpha(\alpha+1)}{2}+1\right\}.$$
This lower bound is optimal.

2. Subset sum

In Theorem 4 and Corollary 5, we prove optimal lower bound for $|\Sigma_\alpha(A)|$ under the assumptions $A \cap (-A)=\emptyset$ and $A \cap (-A)=\{0\}$, respectively. In Theorem 7 and Corollary 8, we prove optimal lower bound for $|\Sigma_\alpha(A)|$ for arbitrary finite sets of integers A. The bounds in Theorem 7 and Corollary 8 depends on the number of positive elements and the number of negative elements in set A. In Corollary 9, we prove lower bounds for $|\Sigma_\alpha(A)|$, which holds for arbitrary finite sets of integers A and only depend on the total number of elements of A not the number of positive and negative elements of A.

Theorem 4. *Let A be a set of k integers such that $A \cap (-A)=\emptyset$. For any integer $\alpha \in [0,k]$, we have*
$$|\Sigma_\alpha(A)| \geq \frac{k(k+1)}{2}-\frac{\alpha(\alpha+1)}{2}+1. \tag{1}$$
This lower bound is optimal.

Proof. Let p be a prime number such that
$$p > \max\left\{2\max^*(A), \tfrac{k(k+1)}{2}-\tfrac{\alpha(\alpha+1)}{2}+1\right\},$$
where $\max^*(A)=\max\{|a| : a \in A\}$. Now, the elements of A can be thought of residue classes modulo p. Since $p > 2\max^*(A)$, any two elements of A are different modulo p. Furthermore $A \cap (-A)=\emptyset$. Hence, by Theorem 3, we get
$$|\Sigma_\alpha(A)| \geq \frac{k(k+1)}{2}-\frac{\alpha(\alpha+1)}{2}+1.$$
Next, to verify that the lower bound in (1) is optimal, let $A=[1,k]$. Then $A \cap (-A)=\emptyset$ and
$$\Sigma_\alpha(A) \subset [1+2+\cdots+\alpha, 1+2+\cdots+k]=\left[\frac{\alpha(\alpha+1)}{2}, \frac{k(k+1)}{2}\right].$$
Therefore
$$|\Sigma_\alpha(A)| \leq \frac{k(k+1)}{2}-\frac{\alpha(\alpha+1)}{2}+1.$$
This together with (1) gives
$$|\Sigma_\alpha(A)| = \frac{k(k+1)}{2}-\frac{\alpha(\alpha+1)}{2}+1.$$
Thus, the lower bound in (1) is optimal. This completes the proof of the theorem. □

Corollary 5. *Let A be a set of k integers such that $A \cap (-A) = \{0\}$. For any integer $\alpha \in [0, k]$, we have*

$$|\Sigma_\alpha(A)| \geq \frac{k(k-1)}{2} - \frac{\alpha(\alpha-1)}{2} + 1. \tag{2}$$

This lower bound is optimal.

Proof. If $A = \{0\}$, then $\Sigma_\alpha(A) = \{0\}$. Therefore $|\Sigma_\alpha(A)| = 1$, and (2) holds. So, let $A \neq \{0\}$ and set $A' = A \setminus \{0\}$. Then it is easy to see that $\Sigma_0(A) = \Sigma_0(A')$ and $\Sigma_\alpha(A) = \Sigma_{\alpha-1}(A')$ for $\alpha \geq 1$. Since $A' \cap (-A') = \emptyset$, by Theorem 4, we get

$$|\Sigma_0(A)| = \left|\Sigma_0\left(A'\right)\right| \geq \frac{k(k-1)}{2} + 1$$

and

$$|\Sigma_\alpha(A)| = \left|\Sigma_{\alpha-1}\left(A'\right)\right| \geq \frac{k(k-1)}{2} - \frac{\alpha(\alpha-1)}{2} + 1$$

for $\alpha \geq 1$. Hence (2) is established.

Now, let $A = [0, k-1]$. Then

$$A \cap (-A) = \{0\} \quad \text{and} \quad \Sigma_\alpha(A) \subset \left[\frac{\alpha(\alpha-1)}{2}, \frac{k(k-1)}{2}\right].$$

Therefore

$$|\Sigma_\alpha(A)| \leq \frac{k(k-1)}{2} - \frac{\alpha(\alpha-1)}{2} + 1.$$

This together with (2) gives that the lower bound in (2) is optimal. □

Remark 6. Nathanson's theorem [18, Theorem 3] is a particular case of Theorem 4 and Corollary 5, for $\alpha = 1$.

Theorem 7. *Let n and p be positive integers and A be a set of n negative and p positive integers. Let $\alpha \in [0, n+p]$ be an integer.*

(i) *If $\alpha \leq n$ and $\alpha \leq p$, then $|\Sigma_\alpha(A)| \geq \frac{n(n+1)}{2} + \frac{p(p+1)}{2} + 1$.*
(ii) *If $\alpha \leq n$ and $\alpha > p$, then $|\Sigma_\alpha(A)| \geq \frac{n(n+1)}{2} + \frac{p(p+1)}{2} - \frac{(\alpha-p)(\alpha-p+1)}{2} + 1$.*
(iii) *If $\alpha > n$ and $\alpha \leq p$, then $|\Sigma_\alpha(A)| \geq \frac{n(n+1)}{2} + \frac{p(p+1)}{2} - \frac{(\alpha-n)(\alpha-n+1)}{2} + 1$.*
(iv) *If $\alpha > n$ and $\alpha > p$, then $|\Sigma_\alpha(A)| \geq \frac{n(n+1)}{2} + \frac{p(p+1)}{2} - \frac{(\alpha-n)(\alpha-n+1)}{2} - \frac{(\alpha-p)(\alpha-p+1)}{2} + 1$.*

These lower bounds are optimal.

Proof. Let $A = A_n \cup A_p$, where $A_n = \{b_1, \ldots, b_n\}$ and $A_p = \{c_1, \ldots, c_p\}$ such that $b_n < b_{n-1} < \cdots < b_1 < 0 < c_1 < c_2 < \cdots < c_p$.

(i) If $\alpha \leq n$ and $\alpha \leq p$, then

$$\left(\Sigma_\alpha\left(A_n\right) + \Sigma_0\left(A_p\right)\right) \cup \left(\Sigma^1\left(\{b_1, \ldots, b_\alpha\}\right) + \sum_{j=1}^{p} c_j\right) \subset \Sigma_\alpha(A)$$

with

$$\left(\Sigma_\alpha\left(A_n\right) + \Sigma_0\left(A_p\right)\right) \cap \left(\Sigma^1\left(\{b_1, \ldots, b_\alpha\}\right) + \sum_{j=1}^{p} c_j\right) = \emptyset.$$

Hence, by Theorem 1 and Theorem 4, we have

$$\begin{aligned}
|\Sigma_\alpha(A)| &\geq \left|\Sigma_\alpha\left(A_n\right) + \Sigma_0\left(A_p\right)\right| + \left|\Sigma^1\left(\{b_1, \ldots, b_\alpha\}\right)\right| \\
&\geq \left|\Sigma_\alpha\left(A_n\right)\right| + \left|\Sigma_0\left(A_p\right)\right| + \left|\Sigma^1\left(\{b_1, \ldots, b_\alpha\}\right)\right| - 1 \\
&\geq \left(\frac{n(n+1)}{2} - \frac{\alpha(\alpha+1)}{2} + 1\right) + \left(\frac{p(p+1)}{2} + 1\right) + \frac{\alpha(\alpha+1)}{2} - 1 \\
&= \frac{n(n+1)}{2} + \frac{p(p+1)}{2} + 1.
\end{aligned}$$

(ii) If $\alpha \le n$ and $\alpha > p$, then

$$\left(\Sigma_\alpha\left(A_n\right)+\Sigma_0\left(A_p\right)\right) \cup \left(\Sigma_{\alpha-p}\left(\{b_1,\ldots,b_\alpha\}\right)+\sum_{j=1}^{p} c_j\right) \subset \Sigma_\alpha(A)$$

with

$$\left(\Sigma_\alpha\left(A_n\right)+\Sigma_0\left(A_p\right)\right) \cap \left(\Sigma_{\alpha-p}\left(\{b_1,\ldots,b_\alpha\}\right)+\sum_{j=1}^{p} c_j\right) = \left\{\sum_{j=1}^{\alpha} b_j+\sum_{j=1}^{p} c_j\right\}.$$

Hence, by Theorem 1 and Theorem 4, we have

$$\begin{aligned}
|\Sigma_\alpha(A)| &\ge \left|\Sigma_\alpha\left(A_n\right)+\Sigma_0\left(A_p\right)\right|+\left|\Sigma_{\alpha-p}\left(\{b_1,\ldots,b_\alpha\}\right)\right|-1\\
&\ge \left|\Sigma_\alpha\left(A_n\right)\right|+\left|\Sigma_0\left(A_p\right)\right|+\left|\Sigma_{\alpha-p}\left(\{b_1,\ldots,b_\alpha\}\right)\right|-2\\
&\ge \left(\frac{n(n+1)}{2}-\frac{\alpha(\alpha+1)}{2}+1\right)+\left(\frac{p\left(p+1\right)}{2}+1\right)+\left(\frac{\alpha(\alpha+1)}{2}-\frac{\left(\alpha-p\right)\left(\alpha-p+1\right)}{2}+1\right)-2\\
&= \frac{n(n+1)}{2}+\frac{p\left(p+1\right)}{2}-\frac{\left(\alpha-p\right)\left(\alpha-p+1\right)}{2}+1.
\end{aligned}$$

(iii) If $\alpha > n$ and $\alpha \le p$, then by applying the result of (ii) for $(-A)$, we obtain

$$|\Sigma_\alpha(A)| = |\Sigma_\alpha(-A)| \ge \frac{n(n+1)}{2}+\frac{p\left(p+1\right)}{2}-\frac{(\alpha-n)(\alpha-n+1)}{2}+1.$$

(iv) If $\alpha > n$ and $\alpha > p$, then

$$\left(\sum_{j=1}^{n} b_j+\Sigma_{\alpha-n}(A_p)\right) \cup \left(\Sigma_{\alpha-p}(A_n)+\sum_{j=1}^{p} c_j\right) \subset \Sigma_\alpha(A)$$

with

$$\left(\sum_{j=1}^{n} b_j+\Sigma_{\alpha-n}(A_p)\right) \cap \left(\Sigma_{\alpha-p}(A_n)+\sum_{j=1}^{p} c_j\right) = \left\{\sum_{j=1}^{n} b_j+\sum_{j=1}^{p} c_j\right\}.$$

Hence, by Theorem 4, we get

$$\begin{aligned}
|\Sigma_\alpha(A)| &\ge \left|\Sigma_{\alpha-n}\left(A_p\right)\right|+\left|\Sigma_{\alpha-p}\left(A_n\right)\right|-1\\
&\ge \left(\frac{p\left(p+1\right)}{2}-\frac{(\alpha-n)(\alpha-n+1)}{2}+1\right)+\left(\frac{n(n+1)}{2}-\frac{\left(\alpha-p\right)\left(\alpha-p+1\right)}{2}+1\right)-1\\
&= \frac{n(n+1)}{2}+\frac{p\left(p+1\right)}{2}-\frac{(\alpha-n)(\alpha-n+1)}{2}-\frac{\left(\alpha-p\right)\left(\alpha-p+1\right)}{2}+1.
\end{aligned}$$

It can be easily verified that all the lower bounds mentioned in the theorem are optimal for $A = [-n, p] \setminus \{0\}$. □

Corollary 8. *Let n and p be positive integers and A be a set of n negative integers, p positive integers and zero. Let $\alpha \in [0, n+p+1]$ be an integer.*

(i) *If $\alpha \le n$ and $\alpha \le p$, then $|\Sigma_\alpha(A)| \ge \frac{n(n+1)}{2}+\frac{p(p+1)}{2}+1$.*
(ii) *If $\alpha \le n$ and $\alpha > p$, then $|\Sigma_\alpha(A)| \ge \frac{n(n+1)}{2}+\frac{p(p+1)}{2}-\frac{(\alpha-p)(\alpha-p-1)}{2}+1$.*
(iii) *If $\alpha > n$ and $\alpha \le p$, then $|\Sigma_\alpha(A)| \ge \frac{n(n+1)}{2}+\frac{p(p+1)}{2}-\frac{(\alpha-n)(\alpha-n-1)}{2}+1$.*
(iv) *If $\alpha > n$ and $\alpha > p$, then $|\Sigma_\alpha(A)| \ge \frac{n(n+1)}{2}+\frac{p(p+1)}{2}-\frac{(\alpha-n)(\alpha-n-1)}{2}-\frac{(\alpha-p)(\alpha-p-1)}{2}+1$.*

These lower bounds are optimal.

Proof. The lower bounds for $|\Sigma_\alpha(A)|$ easily follows from Theorem 7 and the fact that $\Sigma_0(A) = \Sigma_0(A')$ and $\Sigma_\alpha(A) = \Sigma_{\alpha-1}(A')$ for $\alpha \ge 1$, where $A' = A \setminus \{0\}$. Furthermore, the optimality of these bounds can be verified by taking $A = [-n, p]$. □

Corollary 9. *Let $k \geq 2$ and A be a set of k integers. Let $\alpha \in [0, k]$ be an integer. If $0 \notin A$, then*

$$|\Sigma_\alpha(A)| \geq \left\lfloor \frac{(k+1)^2}{4} \right\rfloor - \frac{\alpha(\alpha+1)}{2} + 1. \tag{3}$$

If $0 \in A$, then

$$|\Sigma_\alpha(A)| \geq \left\lfloor \frac{k^2}{4} \right\rfloor - \frac{\alpha(\alpha-1)}{2} + 1. \tag{4}$$

Proof.
Case 1. $0 \notin A$. If $k = 2$, then $A = \{a_1, a_2\}$ for some integers a_1, a_2 with $a_1 < a_2$. Therefore $\Sigma_1(A) = \{a_1, a_2, a_1 + a_2\} \subset \Sigma_0(A)$ and $\Sigma_2(A) = \{a_1 + a_2\}$. Hence (3) holds for $k = 2$. So, assume that $k \geq 3$. As $k(k+1)/2 > (k+1)^2/4$ for $k \geq 3$, if $|\Sigma_\alpha(A)| \geq k(k+1)/2 - \alpha(\alpha+1)/2 + 1$, then we are done. So, let $|\Sigma_\alpha(A)| < k(k+1)/2 - \alpha(\alpha+1)/2 + 1$. Then, Theorem 4 implies that A contains both positive and negative integers. Let A_n and A_p be subsets of A that contain respectively, all negative and all positive integers of A. Let also $|A_n| = n$ and $|A_p| = p$. Then $n \geq 1$ and $p \geq 1$. By Theorem 7, we have

$$|\Sigma_\alpha(A)| \geq \frac{n(n+1)}{2} + \frac{p\left(p+1\right)}{2} - \frac{\alpha(\alpha+1)}{2} + 1$$

for all $\alpha \in [0, k]$. Since $k = n + p$, without loss of generality we may assume that $n \geq k/2$. Therefore

$$\begin{aligned} |\Sigma_\alpha(A)| &\geq \frac{n(n+1)}{2} + \frac{(k-n)(k-n+1)}{2} - \frac{\alpha(\alpha+1)}{2} + 1 \\ &= \left(n - \frac{k}{2}\right)^2 + \frac{(k+1)^2 - 1}{4} - \frac{\alpha(\alpha+1)}{2} + 1 \\ &\geq \frac{(k+1)^2 - 1}{4} - \frac{\alpha(\alpha+1)}{2} + 1. \end{aligned}$$

Hence

$$|\Sigma_\alpha(A)| \geq \left\lfloor \frac{(k+1)^2}{4} \right\rfloor - \frac{\alpha(\alpha+1)}{2} + 1.$$

Case 2. $0 \in A$. Set $A' = A \setminus \{0\}$. Then $\Sigma_0(A) = \Sigma_0(A')$ and $\Sigma_\alpha(A) = \Sigma_{\alpha-1}(A')$ for $\alpha \geq 1$. Hence, by Case 1, we get

$$|\Sigma_0(A)| = \left|\Sigma_0\left(A'\right)\right| \geq \left\lfloor \frac{k^2}{4} \right\rfloor + 1$$

and

$$|\Sigma_\alpha(A)| = \left|\Sigma_{\alpha-1}\left(A'\right)\right| \geq \left\lfloor \frac{k^2}{4} \right\rfloor - \frac{\alpha(\alpha-1)}{2} + 1$$

for $\alpha \geq 1$. Hence

$$|\Sigma_\alpha(A)| \geq \left\lfloor \frac{k^2}{4} \right\rfloor - \frac{\alpha(\alpha-1)}{2} + 1$$

for all $\alpha \in [0, k]$. This completes the proof of the corollary. □

Remark 10. Nathanson [18] have already proved this corollary for $\alpha = 1$. The purpose of this corollary is to prove a similar result for every $\alpha \in [0, k]$. Note that the lower bounds in Corollary 9 are not optimal for all $\alpha \in [0, k]$, except for $\alpha = 0$ and $\alpha = 1$.

Remark 11. The lower bounds in Corollary 9 can also be written in the following form: If $0 \notin A$, then

$$|\Sigma_\alpha(A)| \geq \begin{cases} \dfrac{(k+1)^2}{4} - \dfrac{\alpha(\alpha+1)}{2} + 1 & \text{if } k \equiv 1 \pmod 2 \\ \dfrac{(k+1)^2 - 1}{4} - \dfrac{\alpha(\alpha+1)}{2} + 1 & \text{if } k \equiv 0 \pmod 2. \end{cases}$$

If $0 \in A$, then

$$|\Sigma_\alpha(A)| \geq \begin{cases} \dfrac{k^2-1}{4} - \dfrac{\alpha(\alpha-1)}{2} + 1 & \text{if } k \equiv 1 \pmod 2 \\ \dfrac{k^2}{4} - \dfrac{\alpha(\alpha-1)}{2} + 1 & \text{if } k \equiv 0 \pmod 2. \end{cases}$$

3. Subsequence sum

In this section, we extend the results of the previous section from sets of integers to sequences of integers. In Theorem 12, we establish optimal lower bound for $|\Sigma_\alpha(\mathscr{A})|$ under the assumptions $\mathscr{A} \cap (-\mathscr{A}) = \emptyset$ and $\mathscr{A} \cap (-\mathscr{A}) = (0)_r$. In Theorem 14 and Corollary 15, we prove optimal lower bound for $|\Sigma_\alpha(\mathscr{A})|$ for arbitrary finite sequences of integers $\mathscr{A}$. The bounds in Theorem 14 and Corollary 15 depends on the number of negative terms and the number of positive terms in sequence $\mathscr{A}$. In Corollary 16, we prove lower bounds for $|\Sigma_\alpha(\mathscr{A})|$, which holds for arbitrary finite sequences of integers $\mathscr{A}$ and only depend on the total number of terms of $\mathscr{A}$ not the number of positive and negative terms of $\mathscr{A}$.

If $\alpha = rk$ and $\mathscr{A} = (a_1, \ldots, a_k)_r$, then $\Sigma_\alpha(\mathscr{A}) = \{ra_1 + ra_2 + \cdots + ra_k\}$. Therefore $|\Sigma_\alpha(\mathscr{A})| = 1$. So, in the rest of this section, we assume that $\alpha < rk$.

Theorem 12. *Let $k \geq 2$, $r \geq 1$, and $\alpha \in [0, rk-1]$ be integers. Let $m \in [1,k]$ be an integer such that $(m-1)r \leq \alpha < mr$. Let $\mathscr{A}$ be a sequence of rk terms which is made up of k distinct integers each repeated exactly r times. If $\mathscr{A} \cap (-\mathscr{A}) = \emptyset$, then*

$$|\Sigma_\alpha(\mathscr{A})| \geq r\left(\frac{k(k+1)}{2} - \frac{m(m+1)}{2}\right) + m(mr-\alpha) + 1. \tag{5}$$

If $\mathscr{A} \cap (-\mathscr{A}) = (0)_r$, then

$$|\Sigma_\alpha(\mathscr{A})| \geq r\left(\frac{k(k-1)}{2} - \frac{m(m-1)}{2}\right) + (m-1)(mr-\alpha) + 1. \tag{6}$$

These lower bounds are optimal.

Proof. Let A be the set of all distinct terms of sequence $\mathscr{A}$. Since $(m-1)r \leq \alpha < mr$, we can write α as $\alpha = (m-1)r + u$ for some integer $0 \leq u < r$. Then

$$(r-u)\Sigma_{m-1}(A) + u\Sigma_m(A) \subset \Sigma_\alpha(\mathscr{A}),$$

where $(r-u)\Sigma_{m-1}(A)$ is the $(r-u)$-fold sumset of $\Sigma_{m-1}(A)$ and $u\Sigma_m(A)$ is the u-fold sumset of $\Sigma_m(A)$. So, by Theorem 1 and Theorem 2, we have

$$|\Sigma_\alpha(\mathscr{A})| \geq |(r-u)\Sigma_{m-1}(A)| + |u\Sigma_m(A)| - 1 \geq (r-u)\,|\Sigma_{m-1}(A)| + u\,|\Sigma_m(A)| - r + 1.$$

If $\mathscr{A} \cap (-\mathscr{A}) = \emptyset$, then $A \cap (-A) = \emptyset$. Thus, by Theorem 4, we have

$$\begin{aligned} |\Sigma_\alpha(\mathscr{A})| &\geq (r-u)\left(\frac{k(k+1)}{2} - \frac{m(m-1)}{2} + 1\right) + u\left(\frac{k(k+1)}{2} - \frac{m(m+1)}{2} + 1\right) - r + 1 \\ &= r\left(\frac{k(k+1)}{2} - \frac{m(m+1)}{2}\right) + m(r-u) + 1 \\ &= r\left(\frac{k(k+1)}{2} - \frac{m(m+1)}{2}\right) + m(mr-\alpha) + 1. \end{aligned}$$

Similarly, if $\mathscr{A} \cap (-\mathscr{A}) = (0)_r$, then $A \cap (-A) = \{0\}$. Thus, by Corollary 5, we have

$$\begin{aligned}
|\Sigma_\alpha(\mathscr{A})| &\geq (r-u)\left(\frac{k(k-1)}{2} - \frac{(m-1)(m-2)}{2} + 1\right) + u\left(\frac{k(k-1)}{2} - \frac{m(m-1)}{2} + 1\right) - r + 1 \\
&= r\left(\frac{k(k-1)}{2} - \frac{m(m-1)}{2}\right) + (m-1)(r-u) + 1 \\
&= r\left(\frac{k(k-1)}{2} - \frac{m(m-1)}{2}\right) + (m-1)(mr-\alpha) + 1.
\end{aligned}$$

Hence (5) and (6) are established.

Next, to verify that the lower bounds in (5) and (6) are optimal, let $\mathscr{A} = [1,k]_r$ and $\mathscr{B} = [0,k-1]_r$. Then $\mathscr{A} \cap (-\mathscr{A}) = \emptyset$ and $\mathscr{B} \cap (-\mathscr{B}) = (0)_r$ with

$$\Sigma_\alpha(\mathscr{A}) \subset \left[r \cdot 1 + \cdots + r \cdot (m-1) + (\alpha - (m-1)r) \cdot m, r \cdot 1 + \cdots + r \cdot k\right]$$

and

$$\Sigma_\alpha(\mathscr{B}) \subset \left[r \cdot 1 + \cdots + r \cdot (m-2) + (\alpha - (m-1)r) \cdot (m-1), r \cdot 1 + \cdots + r \cdot (k-1)\right].$$

Therefore

$$|\Sigma_\alpha(\mathscr{A})| \leq \frac{rk(k+1)}{2} - \frac{rm(m+1)}{2} + m(mr-\alpha) + 1$$

and

$$|\Sigma_\alpha(\mathscr{B})| \leq \frac{rk(k-1)}{2} - \frac{rm(m-1)}{2} + (m-1)(mr-\alpha) + 1.$$

These two inequalities together with (5) and (6) implies that the lower bounds in (5) and (6) are optimal. This completes the proof of Theorem 12. □

Remark 13. Mistri and Pandey's result [16, Theorem 1] is a particular case of Theorem 12, for $\alpha = 1$.

Theorem 14. *Let $k \geq 2$, $r \geq 1$, and $\alpha \in [0, rk-1]$ be integers. Let $m \in [1,k]$ be an integer such that $(m-1)r \leq \alpha < mr$. Let $\mathscr{A}$ be a sequence of rk terms which is made up of n negative integers and p positive integers each repeated exactly r times.*

(i) *If $m \leq n$ and $m \leq p$, then*

$$|\Sigma_\alpha(\mathscr{A})| \geq r\left(\frac{n(n+1)}{2} + \frac{p(p+1)}{2}\right) + 1.$$

(ii) *If $m \leq n$ and $m > p$, then*

$$|\Sigma_\alpha(\mathscr{A})| \geq r\left(\frac{n(n+1)}{2} + \frac{p\left(p+1\right)}{2} - \frac{(m-p)\left(m-p+1\right)}{2}\right) + \left(m-p\right)(mr-\alpha) + 1.$$

(iii) *If $m > n$ and $m \leq p$, then*

$$|\Sigma_\alpha(\mathscr{A})| \geq r\left(\frac{n(n+1)}{2} + \frac{p\left(p+1\right)}{2} - \frac{(m-n)(m-n+1)}{2}\right) + (m-n)(mr-\alpha) + 1.$$

(iv) *If $m > n$ and $m > p$, then*

$$\begin{aligned}
&|\Sigma_\alpha(\mathscr{A})| \\
&\geq r\left(\frac{n(n+1)}{2} + \frac{p\left(p+1\right)}{2} - \frac{(m-n)(m-n+1)}{2} - \frac{\left(m-p\right)\left(m-p+1\right)}{2}\right) + \left(2m-n-p\right)(mr-\alpha) + 1.
\end{aligned}$$

These lower bounds are optimal.

Proof. Let A_n and A_p be sets that contain respectively, all distinct negative terms and all distinct positive terms of $\mathscr{A}$. Then $|A_n| = n$ and $|A_p| = p$. Let also $A_n = \{b_1, b_2, \ldots, b_n\}$ and $A_p = \{c_1, c_2, \ldots, c_p\}$, where $b_n < b_{n-1} < \cdots < b_1 < 0 < c_1 < c_2 < \cdots < c_p$.

(i) If $m \le n$ and $m \le p$, then

$$r\left(\Sigma_m(A_n) + \Sigma_0(A_p)\right) \cup \left(\Sigma^1\left((b_1, \ldots, b_m)_r\right) + \sum_{j=1}^{p} rc_j\right) \subset \Sigma_\alpha(\mathscr{A})$$

with

$$r\left(\Sigma_m(A_n) + \Sigma_0(A_p)\right) \cap \left(\Sigma^1\left((b_1, \ldots, b_m)_r\right) + \sum_{j=1}^{p} rc_j\right) = \emptyset.$$

Hence, by Theorem 1, Theorem 2, Theorem 4, and Theorem 12, we have

$$\begin{aligned}
|\Sigma_\alpha(\mathscr{A})| &\ge \left|r\left(\Sigma_m(A_n) + \Sigma_0(A_p)\right)\right| + \left|\Sigma^1\left((b_1, \ldots, b_m)_r\right)\right| \\
&\ge r\left|\Sigma_m(A_n)\right| + r\left|\Sigma_0(A_p)\right| + \left|\Sigma^1\left((b_1, \ldots, b_m)_r\right)\right| - 2r + 1 \\
&\ge r\left(\frac{n(n+1)}{2} - \frac{m(m+1)}{2} + 1\right) + r\left(\frac{p(p+1)}{2} + 1\right) + \frac{rm(m+1)}{2} - 2r + 1 \\
&= r\left(\frac{n(n+1)}{2} + \frac{p(p+1)}{2}\right) + 1.
\end{aligned}$$

(ii) If $m \le n$ and $m > p$, then

$$r\left(\Sigma_m(A_n) + \Sigma_0(A_p)\right) \cup \left(\Sigma_{\alpha-pr}\left((b_1, \ldots, b_m)_r\right) + \sum_{j=1}^{p} rc_j\right) \subset \Sigma_\alpha(\mathscr{A})$$

with

$$r\left(\Sigma_m(A_n) + \Sigma_0(A_p)\right) \cap \left(\Sigma_{\alpha-pr}\left((b_1, \ldots, b_m)_r\right) + \sum_{j=1}^{p} rc_j\right) = \left\{\sum_{j=1}^{m} rb_j + \sum_{j=1}^{p} rc_j\right\}.$$

Hence, by Theorem 1, Theorem 2, Theorem 4, and Theorem 12, we have

$$\begin{aligned}
|\Sigma_\alpha(\mathscr{A})| &\ge \left|r\left(\Sigma_m(A_n) + \Sigma_0(A_p)\right)\right| + \left|\Sigma_{\alpha-pr}\left((b_1, \ldots, b_m)_r\right)\right| - 1 \\
&\ge r\left|\Sigma_m(A_n)\right| + r\left|\Sigma_0(A_p)\right| + \left|\Sigma_{\alpha-pr}\left((b_1, \ldots, b_m)_r\right)\right| - 2r + 1 \\
&\ge r\left(\frac{n(n+1)}{2} - \frac{m(m+1)}{2} + 1\right) + r\left(\frac{p(p+1)}{2} + 1\right) + r\left(\frac{m(m+1)}{2} - \frac{(m-p)(m-p+1)}{2}\right) \\
&\quad + (m-p)(mr-\alpha) - 2r + 1 \\
&= r\left(\frac{n(n+1)}{2} + \frac{p(p+1)}{2} - \frac{(m-p)(m-p+1)}{2}\right) + (m-p)(mr-\alpha) + 1.
\end{aligned}$$

(iii) If $m > n$ and $m \le p$, then by applying the result of (ii) for $(-\mathscr{A})$, we obtain

$$|\Sigma_\alpha(\mathscr{A})| = |\Sigma_\alpha(-\mathscr{A})| \ge r\left(\frac{n(n+1)}{2} + \frac{p(p+1)}{2} - \frac{(m-n)(m-n+1)}{2}\right) + (m-n)(mr-\alpha) + 1.$$

(iv) If $m > n$ and $m > p$, then

$$\left(\sum_{j=1}^{n} rb_j + \Sigma_{\alpha-nr}\left((A_p)_r\right)\right) \cup \left(\Sigma_{\alpha-pr}\left((A_n)_r\right) + \sum_{j=1}^{p} rc_j\right) \subset \Sigma_\alpha(\mathscr{A})$$

with

$$\left(\sum_{j=1}^{n} rb_j + \Sigma_{\alpha-nr}\left((A_p)_r\right)\right) \cap \left(\Sigma_{\alpha-pr}\left((A_n)_r\right) + \sum_{j=1}^{p} rc_j\right) = \left\{\sum_{j=1}^{n} rb_j + \sum_{j=1}^{p} rc_j\right\}.$$

Hence, by Theorem 12, we have

$$\begin{aligned}
|\Sigma_\alpha(\mathscr{A})| &\geq \left|\Sigma_{\alpha-nr}\left(\left(A_p\right)_r\right)\right| + \left|\Sigma_{\alpha-pr}\left((A_n)_r\right)\right| - 1 \\
&\geq r\left(\frac{p\left(p+1\right)}{2} - \frac{(m-n)(m-n+1)}{2}\right) + (m-n)(mr-\alpha) \\
&\quad + r\left(\frac{n(n+1)}{2} - \frac{\left(m-p\right)\left(m-p+1\right)}{2}\right) + \left(m-p\right)(mr-\alpha) + 1 \\
&= r\left(\frac{n(n+1)}{2} + \frac{p\left(p+1\right)}{2} - \frac{(m-n)(m-n+1)}{2} - \frac{\left(m-p\right)\left(m-p+1\right)}{2}\right) \\
&\quad + \left(2m-n-p\right)(mr-\alpha) + 1.
\end{aligned}$$

Furthermore, the optimality of the lower bounds in (i)–(iv) can be verified by taking $\mathscr{A} = [-n,p]_r \setminus (0)_r$. □

Corollary 15. *Let $k \geq 2$, $r \geq 1$, and $\alpha \in [0, rk-1]$ be integers. Let $m \in [1,k]$ be an integer such that $(m-1)r \leq \alpha < mr$. Let $\mathscr{A}$ be a sequence of rk terms which is made up of n negative integers, p positive integers and zero, each repeated exactly r times.*

(i) *If $m \leq n$ and $m \leq p$, then*

$$|\Sigma_\alpha(\mathscr{A})| \geq r\left(\frac{n(n+1)}{2} + \frac{p\left(p+1\right)}{2}\right) + 1.$$

(ii) *If $m \leq n$ and $m > p$, then*

$$|\Sigma_\alpha(\mathscr{A})| \geq r\left(\frac{n(n+1)}{2} + \frac{p\left(p+1\right)}{2} - \frac{\left(m-p\right)\left(m-p-1\right)}{2}\right) + \left(m-p-1\right)(mr-\alpha) + 1.$$

(iii) *If $m > n$ and $m \leq p$, then*

$$|\Sigma_\alpha(\mathscr{A})| \geq r\left(\frac{n(n+1)}{2} + \frac{p\left(p+1\right)}{2} - \frac{(m-n)(m-n-1)}{2}\right) + (m-n-1)(mr-\alpha) + 1.$$

(iv) *If $m > n$ and $m > p$, then*

$$\begin{aligned}
|\Sigma_\alpha(\mathscr{A})| \geq r\left(\frac{n(n+1)}{2} + \frac{p\left(p+1\right)}{2} - \frac{(m-n)(m-n-1)}{2} - \frac{\left(m-p\right)\left(m-p-1\right)}{2}\right) \\
+ \left(2m-n-p-2\right)(mr-\alpha) + 1.
\end{aligned}$$

These lower bounds are optimal.

Proof. The lower bounds for $|\Sigma_\alpha(\mathscr{A})|$ easily follows from Theorem 14 and the fact that $\Sigma_\alpha(\mathscr{A}) = \Sigma_0(\mathscr{A}')$ for $0 \leq \alpha < r$ and $\Sigma_\alpha(\mathscr{A}) = \Sigma_{\alpha-r}(\mathscr{A}')$ for $r \leq \alpha < rk$, where $\mathscr{A}' = \mathscr{A} \setminus (0)_r$. Furthermore, the optimality of these bounds can be verified by taking $\mathscr{A} = [-n,p]_r$. □

Corollary 16. *Let $k \geq 3$, $r \geq 1$, and $\alpha \in [0, rk-1]$ be integers. Let $m \in [1,k]$ be an integer such that $(m-1)r \leq \alpha < mr$. Let $\mathscr{A}$ be a sequence of rk terms which is made up of k distinct integers each repeated exactly r times. If $0 \notin \mathscr{A}$, then*

$$|\Sigma_\alpha(\mathscr{A})| \geq \begin{cases} r\left(\dfrac{(k+1)^2}{4} - \dfrac{m(m+1)}{2}\right) + 1 & \text{if } k \equiv 1 \pmod 2 \\ r\left(\dfrac{(k+1)^2-1}{4} - \dfrac{m(m+1)}{2}\right) + 1 & \text{if } k \equiv 0 \pmod 2. \end{cases}$$

If $0 \in \mathscr{A}$, then

$$|\Sigma_\alpha(\mathscr{A})| \geq \begin{cases} r\left(\dfrac{k^2-1}{4} - \dfrac{m(m-1)}{2}\right) + 1 & \text{if } k \equiv 1 \pmod 2 \\ r\left(\dfrac{k^2}{4} - \dfrac{m(m-1)}{2}\right) + 1 & \text{if } k \equiv 0 \pmod 2. \end{cases}$$

Proof. Note that

$$\frac{rk(k+1)}{2} \geq \begin{cases} \dfrac{r(k+1)^2}{4}+1 & \text{if } k \equiv 1 \pmod 2 \\ \dfrac{r((k+1)^2-1)}{4}+1 & \text{if } k \equiv 0 \pmod 2 \end{cases}$$

and

$$\frac{rk(k-1)}{2}+1 \geq \begin{cases} \dfrac{r(k^2-1)}{4}+1 & \text{if } k \equiv 1 \pmod 2 \\ \dfrac{rk^2}{4}+1 & \text{if } k \equiv 0 \pmod 2 \end{cases}$$

for $k \geq 3$. If $0 \notin \mathscr{A}$ and

$$|\Sigma_\alpha(\mathscr{A})| \geq r\left(\frac{k(k+1)}{2} - \frac{m(m+1)}{2}\right) + m(mr-\alpha)+1,$$

then we are done. So, let

$$|\Sigma_\alpha(\mathscr{A})| < r\left(\frac{k(k+1)}{2} - \frac{m(m+1)}{2}\right) + m(mr-\alpha)+1$$

when $0 \notin \mathscr{A}$. Then, Theorem 12 implies that $\mathscr{A}$ contains both positive and negative integers. By similar arguments, when $0 \in \mathscr{A}$ also, we can assume that $\mathscr{A}$ contains both positive and negative integers. So, in both the cases $0 \in \mathscr{A}$ and $0 \notin \mathscr{A}$, we can assume that $\mathscr{A}$ contains both positive and negative integers. Let A_n and A_p be sets that contain respectively, all distinct negative terms and all distinct positive terms of sequence $\mathscr{A}$. Let also $|A_n| = n$ and $|A_p| = p$. Then $n \geq 1$ and $p \geq 1$.

Case 1. $0 \notin \mathscr{A}$**.** By Theorem 14, we have

$$|\Sigma_\alpha(\mathscr{A})| \geq r\left(\frac{n(n+1)}{2} + \frac{p(p+1)}{2} - \frac{m(m+1)}{2}\right)+1$$

for all $\alpha \in [0, rk-1]$. Therefore

$$\begin{aligned} |\Sigma_\alpha(\mathscr{A})| &\geq r\left(\frac{n(n+1)}{2} + \frac{(k-n)(k-n+1)}{2} - \frac{m(m+1)}{2}\right)+1 \\ &= r\left(\left(n-\frac{k}{2}\right)^2 + \frac{k^2+2k}{4} - \frac{m(m+1)}{2}\right)+1. \end{aligned}$$

Since $k = n+p$, without loss of generality we may assume that $n \geq \lceil k/2 \rceil$. If $k \equiv 1 \pmod 2$, then $k = 2t+1$ for some positive integer t. Hence

$$\begin{aligned} |\Sigma_\alpha(\mathscr{A})| &\geq r\left(\left(n-t-\frac{1}{2}\right)^2 + \frac{k^2+2k}{4} - \frac{m(m+1)}{2}\right)+1 \\ &= r\left((n-t)(n-t-1) + \frac{(k+1)^2}{4} - \frac{m(m+1)}{2}\right)+1 \\ &\geq r\left(\frac{(k+1)^2}{4} - \frac{m(m+1)}{2}\right)+1. \end{aligned}$$

If $k \equiv 0 \pmod 2$, then $k = 2t$ for some positive integer t. Without loss of generality we may assume that $n \geq t$. Hence

$$\begin{aligned} |\Sigma_\alpha(\mathscr{A})| &\geq r\left((n-t)^2 + \frac{k^2+2k}{4} - \frac{m(m+1)}{2}\right)+1 \\ &\geq r\left(\frac{(k+1)^2-1}{4} - \frac{m(m+1)}{2}\right)+1. \end{aligned}$$

Case 2. $0 \in \mathscr{A}$**.** By Corollary 15, we have

$$|\Sigma_\alpha(\mathscr{A})| \geq r\left(\frac{n(n+1)}{2} + \frac{p(p+1)}{2} - \frac{m(m-1)}{2}\right) + 1$$

for all $\alpha \in [0, rk-1]$. Therefore

$$|\Sigma_\alpha(\mathscr{A})| \geq r\left(\left(n - \frac{k-1}{2}\right)^2 + \frac{k^2-1}{4} - \frac{m(m-1)}{2}\right) + 1.$$

Since $k = n+p+1$, without loss of generality we may assume that $n \geq \lceil (k-1)/2 \rceil$. If $k \equiv 1 \pmod 2$, then $k = 2t+1$ for some positive integer t. Hence

$$\begin{aligned}|\Sigma_\alpha(\mathscr{A})| &\geq r\left((n-t)^2 + \frac{k^2-1}{4} - \frac{m(m-1)}{2}\right) + 1\\ &\geq r\left(\frac{k^2-1}{4} - \frac{m(m-1)}{2}\right) + 1.\end{aligned}$$

If $k \equiv 0 \pmod 2$, then $k = 2t$ for some positive integer t. Hence

$$\begin{aligned}|\Sigma_\alpha(\mathscr{A})| &\geq r\left(\left(n - t + \frac{1}{2}\right)^2 + \frac{k^2-1}{4} - \frac{m(m-1)}{2}\right) + 1\\ &= r\left((n-t)(n-t+1) + \frac{k^2}{4} - \frac{m(m-1)}{2}\right) + 1\\ &\geq r\left(\frac{k^2}{4} - \frac{m(m-1)}{2}\right) + 1.\end{aligned}$$

This completes the proof of Corollary 16. □

Remark 17. Mistri and Pandey [16] have already proved this corollary for $\alpha = 1$. The purpose of this corollary is to prove a similar result for every $\alpha \in [0, rk-1]$. Note that the lower bounds in Corollary 16 are not optimal for all $\alpha \in [0, rk-1]$, except for $\alpha = 0$ and $\alpha = 1$.

4. Open problems

(1) Along this line, it is important to find the optimal lower bound for $|\Sigma_\alpha(\mathscr{A})|$, for arbitrary finite sequence of integers

$$\mathscr{A} = \left(\underbrace{a_1, \ldots, a_1}_{r_1 \text{ copies}}, \underbrace{a_2, \ldots, a_2}_{r_2 \text{ copies}}, \ldots, \underbrace{a_k, \ldots, a_k}_{r_k \text{ copies}}\right).$$

When the sequence $\mathscr{A}$ contains nonnegative or nonpositive integers, we already have the optimal lower bound for $|\Sigma_\alpha(\mathscr{A})|$ (see [5]). So, the only case that remains to study is when the sequence $\mathscr{A}$ contains both positive and negative integers. Note that, in this paper we settled this problem in the special case $r_i = r$ for all $i = 1, 2, \ldots, k$.

(2) It is also an important problem to study the structure of the sequence $\mathscr{A}$ for which the lower bound for $|\Sigma_\alpha(\mathscr{A})|$ is optimal. When $\mathscr{A}$ contains nonnegative or nonpositive integers this problem has already been established (see [5]). So, it remains to solve this problem when the sequence $\mathscr{A}$ contains both positive and negative integers.

(3) For a finite set H of nonnegative integers and a finite set A of k integers, define the sumsets

$$HA := \bigcup_{h \in H} hA,\ H\hat{}A := \bigcup_{h \in H} h\hat{}A \quad \text{and} \quad H^{(r)}A := \bigcup_{h \in H} h^{(r)}A.$$

Then $H\hat{}A = \Sigma_\alpha(A)$ for $H = [\alpha, k]$, $H\hat{}A = \Sigma^\alpha(A)$ for $H = [0, k-\alpha]$, $H^{(r)}A = \Sigma_\alpha(\mathscr{A})$ for $H = [\alpha, rk]$, and $H^{(r)}A = \Sigma^\alpha(\mathscr{A})$ for $H = [0, rk-\alpha]$, where $\mathscr{A} = (A)_r$. Along the same line with the sumsets hA, $h\hat{}A$, and $\Sigma_\alpha(A)$, the first named author of this article established optimal lower bounds for $|HA|$ and $|H\hat{}A|$, when A contains nonnegative or nonpositive integers (see [4]). The author also characterized the sets H and A for which the lower bounds are achieved [4]. It will be interesting to generalize such results to the sumset $H^{(r)}A$.

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