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On homogeneous spaces with finite anti-solvable stabilizers

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Abstract. We say that a group is anti-solvable if all of its composition factors are non-abelian. We consider a particular family of anti-solvable finite groups containing the simple alternating groups for $n \neq 6$ and all 26 sporadic simple groups. We prove that, if K is a perfect field and X is a homogeneous space of a smooth algebraic K-group G with finite geometric stabilizers lying in this family, then X is dominated by a G-torsor. In particular, if $G = \mathrm{SL}_n$, all such homogeneous spaces have rational points.

Keywords. homogeneous spaces, rational points, non-abelian cohomology, finite simple groups.

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We say that a finite group F is *aut-split* if the exact sequence

$$1 \to \mathrm{Inn}(F) \to \mathrm{Aut}(F) \to \mathrm{Out}(F) \to 1,$$

splits, where $\mathrm{Inn}(F)$ denotes the group of inner automorphisms of F and $\mathrm{Out}(F)$ is simply defined as the quotient $\mathrm{Aut}(F)/\mathrm{Inn}(F)$. A full characterization of aut-split finite simple groups was given in [7]. These include the alternating group A_n for $n = 5$ or $n \geq 7$ and all 26 sporadic groups. As for simple groups of Lie type, some notation is needed. If F is a finite simple group of Lie type over $\mathbb{F}_q$ with $q = p^m$ and p prime, denote by d the order of the group of diagonal automorphisms (i.e. those induced by conjugation by diagonal matrices) modulo inner automorphisms. Then F is aut-split if and only if one of the following holds:

- F is a Chevalley group[1], not of type $D_\ell(q)$, and $\left(\frac{q-1}{d}, d, m\right) = 1$;
- $F = D_\ell(q)$ and $\left(\frac{q^\ell-1}{d}, d, m\right) = 1$;
- F is a twisted group, not of type ${}^2D_\ell(q)$, and $\left(\frac{q+1}{d}, d, m\right) = 1$;
- $F = {}^2D_\ell(q)$ and either ℓ is odd or $p = 2$.

[1] In the sense of finite simple groups, not of algebraic groups.

Recall that, by the Jordan–Hölder Theorem (cf. [4, 3.4, Thm. 22]), every finite group admits a composition series and its composition factors (which are simple groups) are unique up to permutation. We say that F is *anti-solvable* if all of its composition factors are non-abelian. In this note we are interested in anti-solvable groups whose composition factors are aut-split (i.e. they belong to the list above). This includes of course the groups in the list themselves. The main result of this note is the following.

Theorem 1. *Let K be a perfect field and let G be a smooth algebraic K-group. Let X be a homogeneous space of G with geometric stabilizer $\bar{F}$. Assume that $\bar{F}$ is finite, anti-solvable, and that all of its composition factors are aut-split. Then X is dominated by a G-torsor. In particular, if $G = \mathrm{SL}_n$, then X has a rational point.*

Of course, the particular case of $G = \mathrm{SL}_n$ is a natural consequence of the triviality of torsors under SL_n, which is valid over any field K. Now, recall that the *Springer class*, which is class in a non-abelian 2-cohomology set associated to every homogeneous space X, corresponds to the obstruction to being dominated by a G-torsor (cf. [3, §2] or [5, §1], or [9] for the original article by Springer). Using this tool we immediately see that Theorem 1 is implied by the following result on non-abelian Galois cohomology.

Theorem 2. *Let K be a perfect field and let L be a finite K-lien (or K-kernel, or K-band) whose underlying group is anti-solvable and all of its composition factors are aut-split. Then $H^2(K,L)$ has exactly one class and it is neutral.*

As an anonymous referee pointed out, the proof of this statement uses nothing in particular about Galois groups and works actually for profinite groups in general. In this context, it is better to consider the point of view of extensions for nonabelian 2-cohomology (cf. for instance [10], or [5, 1.18], or [9, 1.13–14], see also [3, §2.2] for a full comparison between classes of extensions, of gerbes and of nonabelian 2-cocyles). Theorem 2 follows immediately then from the following statement.

Theorem 3. *Let Γ be a profinite group and let L be a finite Γ-lien (or Γ-kernel, or Γ-band) whose underlying group F is anti-solvable and all of its composition factors are aut-split. Then $H^2(\Gamma,L)$ has exactly one class and it is neutral. In particular, every extension of profinite groups*

$$1 \to F \to E \to \Gamma \to 1,$$

(i.e. F is a closed, discrete normal subgroup of E and the quotient arrow is open) is split.

The rest of this note is devoted to proving Theorem 3. We start with the particular case where the group is itself aut-split (for instance, if there is only one composition factor).

Proposition 4. *Let Γ be a profinite group and let L be a finite Γ-lien whose underlying group is aut-split. Then $H^2(\Gamma,L)$ has a neutral class (in particular, the Γ-lien L is representable). If moreover L has a trivial center, then this is the only class in $H^2(\Gamma,L)$.*

The idea of this proof comes from [2, Prop. 3.1], which actually follows ideas from Douai's PhD thesis.

Proof. Let F be the underlying finite group of L. By the definition of a Γ-lien we have a continuous homomorphism $\kappa : \Gamma \to \mathrm{Out}(F)$. Since F is aut-split, we can compose this with a section $\mathrm{Out}(F) \to \mathrm{Aut}(F)$ and get a continuous group action of Γ on F. This defines a an extension of Γ by F by means of the corresponding semi-direct product and, since this action lifts κ, it corresponds to a neutral class $\eta \in H^2(\Gamma,L)$. The second assertion follows immediately from [9, Prop. 1.17] (see also [8, IV, Thm. 8.8]). □

Proof of Theorem 3. Let L be a Γ-lien and let F be its underlying group as in Theorem 3. Since F is anti-solvable, it has trivial center and hence the Γ-lien L is representable (cf. [8, IV, Thm. 8.7]). We fix then a class $\eta \in H^2(\Gamma, L)$, which is unique again by the triviality of the center and [9, Prop. 1.17]. This class corresponds to an extension of Γ by F. We will prove that this extension is split, i.e. that the class is neutral.

Let E be an extension of Γ by F. If F has no proper nontrivial characteristic subgroups (i.e. it is *characteristically simple*), then $F = G^n$ with G a finite simple group (cf. [6, Ch. 2, Thm. 1.4]), which has to be aut-split then by hypothesis. Since every automorphism of such a product permutes its direct factors, we see that $\mathrm{Aut}(F)$ is isomorphic to $\mathrm{Aut}(G)^n \rtimes S_n$ (with the obvious action of S_n) and hence $\mathrm{Out}(F)$ is isomorphic to $\mathrm{Out}(G)^n \rtimes S_n$. It is evident then that the section $s : \mathrm{Out}(G) \to \mathrm{Aut}(G)$ and the obvious section $t : S_n \to \mathrm{Aut}(F)$ define a section $(s^n \rtimes t) : \mathrm{Out}(F) \to \mathrm{Aut}(F)$, proving that F is aut-split as well. We are done then by Proposition 4.

Assume now that F has a proper nontrivial characteristic subgroup N. It is clear that both N and F/N also satisfy the hypotheses of Theorem 3, so we argue by induction. Since N is characteristic, the arrow $\mathrm{Aut}(F) \to \mathrm{Aut}(F/N)$ is well-defined and thus the homomorphism $\kappa : \Gamma \to \mathrm{Out}(F)$ gives a homomorphism $\Gamma \to \mathrm{Out}(F/N)$, which gives us a Γ-lien L' whose underlying group is F/N. By induction hypothesis, we know that $H^2(\Gamma, L')$ has exactly one class η' and that it is neutral. The class η' corresponds to a split extension which, since the arrow $H^2(K, L) \to H^2(K, L')$ clearly sends η to η', fits into a commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & F & \longrightarrow & E & \longrightarrow & \Gamma_K & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow{\scriptstyle \pi} & & \| & & \\
1 & \longrightarrow & F/N & \longrightarrow & E' & \underset{\overset{s}{\longleftarrow}}{\longrightarrow} & \Gamma_K & \longrightarrow & 1.
\end{array}$$

Consider the preimage $\pi^{-1}(s(\Gamma_K)) \subset E$, which fits into the following commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & N & \longrightarrow & \pi^{-1}(s(\Gamma_K)) & \longrightarrow & \Gamma_K & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & F & \longrightarrow & E & \longrightarrow & \Gamma_K & \longrightarrow & 1.
\end{array}$$

The top row corresponds to a class in $H^2(K, L'')$ for some K-lien L'' with underlying group N. By induction hypothesis again we know that this class is neutral and hence the extension is split, giving a section $\Gamma_K \to \pi^{-1}(s(\Gamma_K))$ and hence a section $\Gamma_K \to E$ by composition, proving that the class η is neutral. □

Remark 5. In [1], Bercov characterizes anti-solvable finite groups whose composition factors are aut-split as iterated twisted wreath products of simple groups. Note however that we did not need this result in the proof above.

Remark 6. Theorems 1 and 2 are false if we take away the aut-split hypothesis on the composition factors, already for simple groups. Indeed, consider the group A_6 and the order 2 element $\phi \in \mathrm{Out}(A_6)$ coming from the nontrivial class of outer automorphisms of S_6. Taking $K = \mathbb{R}$, we can define a K-lien L via the homomorphism $\kappa : \Gamma_{\mathbb{R}} \to \mathrm{Out}(A_6)$ that sends the nontrivial element in $\Gamma_{\mathbb{R}}$ to ϕ. By the same arguments used above we see that $H^2(K, L)$ has only one class η. But this class is not neutral, since this would amount to the existence of a lift of ϕ to an automorphism $\varphi \in \mathrm{Aut}(A_6)$ of order 2, which does not exist. Using [3, Cor. 3.3] we may construct a homogeneous space X of SL_n over $\mathbb{R}$ whose Springer class is η. This implies that X is not dominated by a torsor.

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