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A characterization of the relation between two ℓ-modular correspondences

Une caractérisation de la relation entre deux correspondances ℓ-modulaires

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Abstract. Let F be a non archimedean local field of residual characteristic p and ℓ a prime number different from p. Let V denote Vignéras' ℓ-modular local Langlands correspondence [7], between irreducible ℓ-modular representations of $\mathrm{GL}_n(F)$ and n-dimensional ℓ-modular Deligne representations of the Weil group W_F. In [4], enlarging the space of Galois parameters to Deligne representations with non necessarily nilpotent operators allowed us to propose a modification of the correspondence of Vignéras into a correspondence C, compatible with the formation of local constants in the generic case. In this note, following a remark of Alberto Mínguez, we characterize the modification $\mathrm{C}\circ\mathrm{V}^{-1}$ by a short list of natural properties.

Résumé. Soit F un corps local non archimédien de caractéristique résiduelle p et ℓ un nombre premier différent de p. Soit V la correspondance de Langlands ℓ-modulaire définie par Vignéras en [7], entre représentations irréductibles ℓ-modulaires de $\mathrm{GL}_n(F)$ et représentations de Deligne ℓ-modulaires de dimension n du groupe de Weil W_F. Dans [4], l'élargissement de l'espace des paramètres galoisiens aux représentations de Deligne à opérateur non nécessairement nilpotent, nous a permis de proposer une modification de la correspondance de Vignéras en une correpsondance notée C, compatible aux constantes locales des représentations génériques et de leur paramètre. Dans cette note rédigée à la suite d'une remarque d'Alberto Mínguez, nous caractérisons la modification $\mathrm{C}\circ\mathrm{V}^{-1}$ par une courte liste de propriétés naturelles.

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1. Introduction

Let F be a non-archimedean local field with finite residue field of cardinality q, a power of a prime p, and W_F the Weil group of F. Let ℓ be a prime number different from p. The ℓ-modular local Langlands correspondence established by Vignéras in [7] is a bijection from isomorphism classes of smooth irreducible representations of $\mathrm{GL}_n(F)$ and n-dimensional Deligne representations (Section 2.1) of the Weil group W_F with nilpotent monodromy operator. It is uniquely characterized by a non-naive compatibility with the ℓ-adic local Langlands correspondence ([1, 2, 5, 6]) under reduction modulo ℓ, involving twists by Zelevinsky involutions. In [4], at the cost of having a less direct compatibility with reduction modulo ℓ, we proposed a modification of the correspondence V of Vignéras, by in particular enlarging its target to the space of Deligne representations with non necessarily nilpotent monodromy operator (it is a particularity of the ℓ-modular setting that such operators can live outside the nilpotent world). The modified correspondence C is built to be compatible with local constants on both sides of the corrspondence ([3, 4]) and we proved that it is indeed the case for generic representations in [4]. Here, we show in Section 3 that if we expect a correspondence to have such a property, and some other natural properties, then it will be uniquely determined by V. Namely we characterize the map $\mathrm{C}\circ\mathrm{V}^{-1}$ by a list of five properties in Theorem 8. The map $\mathrm{C}\circ\mathrm{V}^{-1}$ endows the image of C with a semiring structure because the image of V is naturally equipped with semiring laws. We end this note by studying this structure from a different point of view in Section 4.

2. Preliminaries

Let $\nu : \mathrm{W}_F \to \overline{\mathbb{F}_\ell}^\times$ be the unique character trivial on the inertia subgroup of W_F and sending a geometric Frobenius element to q^{-1}, it corresponds to the normalized absolute value $\nu : F^\times \to \overline{\mathbb{F}_\ell}^\times$ via local class field theory.

We consider only smooth representations of locally compact groups, which unless otherwise stated will be considered on $\overline{\mathbb{F}_\ell}$-vector spaces. For $\mathscr{G}$ a locally compact topological group, we let $\mathrm{Irr}(\mathscr{G})$ denote the set of isomorphism classes of irreducible representations of $\mathscr{G}$.

2.1. *Deligne representations*

We follow [4, Section 4], but slightly simplify some notation. A *Deligne-representation* of W_F is a pair (Φ, U) where Φ is a finite dimensional semisimple representation of W_F, and $U \in \mathrm{Hom}_{\mathrm{W}_F}(\nu\Phi,\Phi)$; we call (Φ, U) *nilpotent* if U is a nilpotent endomorphism over $\overline{\mathbb{F}_\ell}$.

The set of morphisms between Deligne representations $(\Phi, U), (\Phi', U')$ (of W_F) is given by $\mathrm{Hom}_{\mathrm{D}}(\Phi,\Phi') = \{f \in \mathrm{Hom}_{\mathrm{W}_F}(\Phi,\Phi') : f\circ U = U'\circ f\}$. This leads to notions of irreducible and indecomposable Deligne representations. We refer to [4, Section 4], for the (standard) definitions of dual and direct sums of Deligne representations.

We let $\mathrm{Rep}_{\mathrm{D,ss}}(\mathrm{W}_F)$ denote the set of isomorphism classes of Deligne-representations; and $\mathrm{Indec}_{\mathrm{D,ss}}(\mathrm{W}_F)$ (resp. $\mathrm{Irr}_{\mathrm{D,ss}}(\mathrm{W}_F)$, $\mathrm{Nilp}_{\mathrm{D,ss}}(\mathrm{W}_F)$) denote the set of isomorphism classes of indecomposable (resp. irreducible, nilpotent) Deligne representations. Thus

$$\mathrm{Irr}_{\mathrm{D,ss}}(\mathrm{W}_F) \subset \mathrm{Indec}_{\mathrm{D,ss}}(\mathrm{W}_F) \subset \mathrm{Rep}_{\mathrm{D,ss}}(\mathrm{W}_F), \qquad \mathrm{Nilp}_{\mathrm{D,ss}}(\mathrm{W}_F) \subset \mathrm{Rep}_{\mathrm{D,ss}}(\mathrm{W}_F).$$

Let $\mathrm{Rep}_{\mathrm{ss}}(\mathrm{W}_F)$ denote the set of isomorphism classes of semisimple representations of W_F, we have a canonical map $\mathrm{Supp}_{\mathrm{W}_F} : \mathrm{Rep}_{\mathrm{D,ss}}(\mathrm{W}_F) \to \mathrm{Rep}_{\mathrm{ss}}(\mathrm{W}_F)$, $(\Phi, U) \mapsto \Phi$; we call Φ the W_F*-support* of (Φ, U).

For $\Psi \in \mathrm{Irr}(\mathrm{W}_F)$ we denote by $o(\Psi)$ the cardinality of the *irreducible line* $\mathbb{Z}_\Psi = \{\nu^k\Psi, k \in \mathbb{Z}\}$; it divides the order of q in $\mathbb{F}_\ell^\times$ hence is prime to ℓ. We let $\mathfrak{l}(\mathrm{W}_F) = \{\mathbb{Z}_\Psi : \Psi \in \mathrm{Irr}(\mathrm{W}_F)\}$.

The fundamental examples of non-nilpotent Deligne representation are the *cycle representations*: let I be an isomorphism from $\nu^{o(\Psi)}\Psi$ to Ψ and define $\mathscr{C}(\Psi, I) = (\Phi(\Psi), C_I) \in \mathrm{Rep}_{\mathrm{ss}}(\mathrm{D}, \overline{\mathbb{F}_\ell})$ by

$$\Phi(\Psi) = \bigoplus_{k=0}^{o(\Psi)-1} \nu^k\Psi, \quad C_I(x_0, \dots, x_{o(\Psi)-1}) = (I(x_{o(\Psi)-1}), x_0, \dots, x_{o(\Psi)-2}),\ x_k \in \nu^k\Psi.$$

Then $\mathscr{C}(\Psi, I) \in \mathrm{Irr}_{\mathrm{ss}}(\mathrm{D}, \overline{\mathbb{F}_\ell})$ and its isomorphism class only depends on $(\mathbb{Z}_\Psi, I)$, by [4, Proposition 4.18].

To remove dependence on I, in [4, Definition 4.6 and Remark 4.9] we define an equivalence relation $\sim$ on $\mathrm{Rep}_{\mathrm{D,ss}}(\mathrm{W}_F)$. We say that

$$(\Phi_1, U_1) \sim (\Phi_2, U_2)$$

for $(\Phi_i, U_i) \in \mathrm{Rep}_{\mathrm{D,ss}}(\mathrm{W}_F)$ if they both admit a decomposition (it is unique up to re-ordering)

$$(\Phi_i, U_i) = \oplus_{k=1}^{r_i} (\Phi_{i,k}, U_{i,k})$$

as a direct sum of elements in $\mathrm{Indec}_{\mathrm{D,ss}}(\mathrm{W}_F)$, such that $r_1 = r_2 =: r$ and for $k = 1, \dots, r$ there exists $\lambda_k \in \overline{\mathbb{F}_\ell}^\times$ such that

$$(\Phi_{2,k}, U_{2,k}) \simeq (\Phi_{1,k}, \lambda_k U_{1,k}).$$

The equivalence class of $\mathscr{C}(\Psi, I)$ is independent of I, and we set

$$\mathscr{C}(\mathbb{Z}_\Psi) := [\mathscr{C}(\Psi, I)] \in [\mathrm{Irr}_{\mathrm{D,ss}}(\mathrm{W}_F)].$$

The sets $\mathrm{Rep}_{\mathrm{D,ss}}(\mathrm{W}_F)$, $\mathrm{Irr}_{\mathrm{D,ss}}(\mathrm{W}_F)$, $\mathrm{Indec}_{\mathrm{D,ss}}(\mathrm{W}_F)$, and $\mathrm{Nilp}_{\mathrm{D,ss}}(\mathrm{W}_F)$ are unions of $\sim$-classes, and if X denotes any of them we set $[X] := X/\sim$. Similarly, for $(\Phi, U) \in \mathrm{Rep}_{\mathrm{D,ss}}(\mathrm{W}_F)$ we write $[\Phi, U]$ for its equivalence class in $[\mathrm{Rep}_{\mathrm{D,ss}}(\mathrm{W}_F)]$. On $\mathrm{Nilp}_{\mathrm{D,ss}}(\mathrm{W}_F)$ the equivalence relation $\sim$ coincides with equality.

The operations $\oplus$ and $(\Phi, U) \mapsto (\Phi, U)^\vee$ on $\mathrm{Rep}_{\mathrm{D,ss}}(\mathrm{W}_F)$ descend to $[\mathrm{Rep}_{\mathrm{D,ss}}(\mathrm{W}_F)]$. Tensor products are more subtle; for example, tensor products of semisimple representations of W_F are not necessarily semisimple. We define a semisimple tensor product operation $\otimes_{\mathrm{ss}}$ on $[\mathrm{Rep}_{\mathrm{D,ss}}(\mathrm{W}_F)]$ in [4, Section 4.4], turning $([\mathrm{Rep}_{\mathrm{D,ss}}(\mathrm{W}_F)], \oplus, \otimes_{\mathrm{ss}})$ into an abelian semiring.

The basic non-irreducible examples of elements of $\mathrm{Nilp}_{\mathrm{D,ss}}(\mathrm{W}_F)$ are called *segments*: For $r \geqslant 1$, set $[0, r-1] := (\Phi(r), N(r))$, where

$$\Phi(r) = \bigoplus_{k=0}^{r-1} \nu^k, \quad N(r)(x_0, \dots, x_{r-1}) = (0, x_0, \dots, x_{r-2}),\ x_k \in \nu^k.$$

We now recall the classification of equivalence classes of Deligne representations of W_F of [4].

Theorem 1 ([4, Section 4]).

1. *Let* $\Phi \in \mathrm{Irr}_{\mathrm{D,ss}}(\mathrm{W}_F)$, *then there is either a unique* $\Psi \in \mathrm{Irr}(\mathrm{W}_F)$ *such that* $\Phi = \Psi$, *or a unique irreducible line* $\mathbb{Z}_\Psi$ *such that* $[\Phi] = \mathscr{C}(\mathbb{Z}_\Psi)$.
2. *Let* $[\Phi, U] \in [\mathrm{Indec}_{\mathrm{D,ss}}(\mathrm{W}_F)]$, *then there exist a unique* $r \geqslant 1$ *and a unique* $\Theta \in [\mathrm{Irr}_{\mathrm{D,ss}}(\mathrm{W}_F)]$ *such that* $[\Phi, U] = [0, r-1] \otimes_{\mathrm{ss}} \Theta$.
3. *Let* $[\Phi, U] \in [\mathrm{Rep}_{\mathrm{D,ss}}(\mathrm{W}_F)]$, *there exist* $[\Phi_i, U_i] \in [\mathrm{Indec}_{\mathrm{D,ss}}(\mathrm{W}_F)]$ *for* $1 \leqslant i \leqslant r$ *such that* $[\Phi, U] = \bigoplus_{i=1}^r [\Phi_i, U_i]$.

We recall the following classical result about tensor products of segments.

Lemma 2. *For* $n \geqslant m \geqslant 1$, *one has*

$$[0, n-1] \otimes_{\mathrm{ss}} [0, m-1] = [0, n+m-2] \oplus [1, n+m-3] \oplus \cdots \oplus [m-1, n-1].$$

Proof. Denote by $\nu_{\overline{\mathbb{Q}_\ell}}$ the normalized absolute value of W_F with values in $\overline{\mathbb{Q}_\ell}^\times$, and by $[0,i-1]_{\overline{\mathbb{Q}_\ell}}$ the ℓ-adic Deligne representation with W_F-support $\oplus_{k=0}^{i-1}\nu^i_{\overline{\mathbb{Q}_\ell}}$ and nilpotent operator $N(i)_{\overline{\mathbb{Q}_\ell}}$ sending $(x_0,\dots,x_{i-1})$ to $(0,x_0,\dots,x_{i-2})$. The relation

$$[0,n-1]_{\overline{\mathbb{Q}_\ell}}\otimes[0,m-1]_{\overline{\mathbb{Q}_\ell}}=[0,n+m-2]_{\overline{\mathbb{Q}_\ell}}\oplus[1,n+m-3]_{\overline{\mathbb{Q}_\ell}}\oplus\cdots\oplus[m-1,n-1]_{\overline{\mathbb{Q}_\ell}} \quad (1)$$

can be translated into a statement on tensor product of irreducible representations of $SL_2(\mathbb{C})$, which is well-known and easily checked by the highest weight theory. Because all powers of $\nu_{\overline{\mathbb{Q}_\ell}}$ take values in $\overline{\mathbb{Z}_\ell}^\times$, the canonical $\overline{\mathbb{Z}_\ell}$-lattice in $\oplus_{k=0}^{i-1}\nu^i_{\overline{\mathbb{Q}_\ell}}$ is stable under both the actions of W_F and $N(i)_{\overline{\mathbb{Q}_\ell}}$, and this defines a $\overline{\mathbb{Z}_\ell}$-Deligne representation $[0,i-1]_{\overline{\mathbb{Z}_\ell}}$. Taking the canonical lattices on both sides of Equation (1) we get

$$[0,n-1]_{\overline{\mathbb{Z}_\ell}}\otimes[0,m-1]_{\overline{\mathbb{Z}_\ell}}=[0,n+m-2]_{\overline{\mathbb{Z}_\ell}}\oplus[1,n+m-3]_{\overline{\mathbb{Z}_\ell}}\oplus\cdots\oplus[m-1,n-1]_{\overline{\mathbb{Z}_\ell}}. \quad (2)$$

Now tensoring Equation (2) by $\overline{\mathbb{F}_\ell}$ we obtain the relation

$$[0,n-1]\otimes[0,m-1]=[0,n+m-2]\oplus[1,n+m-3]\oplus\cdots\oplus[m-1,n-1].$$

Finally by definition (see [4, Defintion 4.37]) one has $[0,n-1]\otimes_{ss}[0,m-1]=[0,n-1]\otimes[0,m-1]$, hence the sought equality. □

2.2. L-*factors*

We set $\mathrm{Irr}_{\mathrm{cusp}}(\mathrm{GL}(F)) := \coprod_{n\geq 0}\mathrm{Irr}_{\mathrm{cusp}}(\mathrm{GL}_n(F))$ where $\mathrm{Irr}_{\mathrm{cusp}}(\mathrm{GL}_n(F))$ is the set of isomorphism classes of irreducible cuspidal representations of $\mathrm{GL}_n(F)$.

Let π and π' be a pair of cuspidal representations of $\mathrm{GL}_n(F)$ and $\mathrm{GL}_m(F)$ respectively. We denote by $L(X,\pi,\pi')$ the Euler factor attached to this pair in [3] via the Rankin–Selberg method, it is a rational function of the form $\frac{1}{Q(X)}$ where $Q\in\overline{\mathbb{F}_\ell}[X]$ satisfies $Q(0)=1$. We recall that a cuspidal representation of $\mathrm{GL}_n(F)$ is called *banal* if $\nu\otimes\pi\not\simeq\pi$. The following is a part of [3, Theorem 4.9].

Proposition 3. *Let* $\pi,\pi'\in\mathrm{Irr}_{\mathrm{cusp}}(\mathrm{GL}(F))$. *If* π *or* π' *is non-banal, then* $L(X,\pi,\pi')=1$.

Let $[\Phi,U]\in[\mathrm{Rep}_{D,ss}(W_F)]$, for brevity from now on we often denote such a class just by Φ, we denote by $L(X,\Phi)$ the L-factor attached to it in [4, Section 5], their most basic property is that

$$L(X,\Phi\oplus\Phi')=L(X,\Phi)L(X,\Phi')$$

for Φ and Φ' in $[\mathrm{Rep}_{D,ss}(W_F)]$. We need the following property of such factors.

Lemma 4. *Let* $\Psi\in\mathrm{Irr}(W_F)$ *and* $a\leqslant b$ *be integers, put* $\Phi=[a,b]\otimes_{ss}\Psi$ *and* $\Phi'=[-b,-a]\otimes_{ss}\Psi^\vee$, *then* $L(X,\Phi\otimes_{ss}\Phi')$ *has a pole at* $X=0$.

Proof. According to [4, Lemma 5.7], it is sufficient to prove that $L(X,\Psi\otimes_{ss}\Psi^\vee)$ has a pole at $X=0$ for $\Psi\in\mathrm{Irr}(W_F)$, but this property follows from the definition of the L-factor in question, and the fact that $\Psi\otimes_{ss}\Psi^\vee$ contains a nonzero vector fixed by W_F. □

2.3. *The map* CV

For $\Psi\in\mathrm{Irr}(W_F)$ we set $\mathrm{St}_0(\mathbb{Z}_\Psi)=\bigoplus_{k=0}^{o(\Psi)-1}\nu^k\Psi$. By Theorem 1, an element $\Phi\in\mathrm{Nilp}_{D,ss}(W_F)$ has a unique decomposition

$$\Phi=\Phi_{\mathrm{acyc}}\oplus\bigoplus_{k\geqslant 1,\mathbb{Z}_\Psi\in\mathfrak{I}(W_F)}[0,k-1]\otimes_{ss} n_{\mathbb{Z}_\Psi,k}\,\mathrm{St}_0(\mathbb{Z}_\Psi),$$

where for all $k\geqslant 1$ and $\mathbb{Z}_\Psi\in\mathfrak{I}(W_F)$, Φ_{acyc} has no summand isomorphic to $[0,k-1]\otimes_{ss}\mathrm{St}_0(\mathbb{Z}_\Psi)$; i.e. we have separated Φ into an acyclic and a cyclic part. Then following [4, Section 6.3], we set:

$$\mathrm{CV}(\Phi)=\Phi_{\mathrm{acyc}}\oplus\bigoplus_{k\geqslant 1,\mathbb{Z}_\Psi\in\mathfrak{I}(W_F)}[0,k-1]\otimes_{ss} n_{\mathbb{Z}_\Psi,k}\mathscr{C}(\mathbb{Z}_\Psi).$$

We denote by $\mathrm{C}_{\mathrm{D,ss}}(\mathrm{W}_F)$ the image of $\mathrm{CV}: \mathrm{Nilp}_{\mathrm{D,ss}}(\mathrm{W}_F) \to [\mathrm{Rep}_{\mathrm{D,ss}}(\mathrm{W}_F)]$, and call $\mathrm{C}_{\mathrm{D,ss}}(\mathrm{W}_F)$ the set of C-*parameters*.

2.4. *ℓ-modular local Langlands*

We let $\mathrm{Irr}(\mathrm{GL}(F)) = \coprod_{n\geqslant 0} \mathrm{Irr}(\mathrm{GL}_n(F))$ where $\mathrm{Irr}(\mathrm{GL}_n(F))$ denotes the set of isomorphism classes of irreducible representations of $\mathrm{GL}_n(F)$.

In [7], Vignéras introduces the ℓ-modular local Langlands correspondence: a bijection

$$\mathrm{V}: \mathrm{Irr}(\mathrm{GL}(F)) \to \mathrm{Nilp}_{\mathrm{D,ss}},$$

characterized in a non-naive way by reduction modulo ℓ. For this note, we recall $\mathrm{Supp}_{\mathrm{W}_F} \circ \mathrm{V}$, the *semisimple ℓ-modular local Langlands correspondence* of Vignéras, induces a bijection between supercuspidal supports elements of $\mathrm{Irr}(\mathrm{GL}_n(F))$ and $\mathrm{Rep}_{\mathrm{ss}}(\mathrm{W}_F)$ compatible with reduction modulo ℓ.

In [4], we introduced the bijection

$$\mathrm{C} = \mathrm{CV} \circ \mathrm{V}: \mathrm{Irr}(\mathrm{GL}(F)) \to \mathrm{C}_{\mathrm{D,ss}}(\mathrm{W}_F);$$

which satisfies $\mathrm{Supp}_{\mathrm{W}_F} \circ \mathrm{V} = \mathrm{Supp}_{\mathrm{W}_F} \circ \mathrm{C}$. Moreover, the correspondence C is compatible with the formation of L-factors for generic representations, a property V does not share; in the cuspidal case:

Proposition 5 ([4, Proposition 6.13]). *For π and π' in $\mathrm{Irr}_{\mathrm{cusp}}(\mathrm{GL}(F))$ one has* $\mathrm{L}(X, \pi, \pi') = \mathrm{L}(X, \mathrm{C}(\pi), \mathrm{C}(\pi'))$.

We note another characterization of non-banal cuspidal representations:

Proposition 6 ([4, Sections 3.2 and 6.2]). *A representation $\pi \in \mathrm{Irr}_{\mathrm{cusp}}(\mathrm{GL}(F))$ is non-banal if and only if $\mathrm{V}(\pi) = \ell^k \mathrm{St}_0(\mathbb{Z}_\Psi)$, or equivalently $\mathrm{C}(\pi) = \ell^k \mathscr{C}(\mathbb{Z}_\Psi)$, for some $k \geqslant 0$ and $\Psi \in \mathrm{Irr}(\mathrm{W}_F)$.*

Amongst non-banal cuspidal representations, those for which $k = 0$ in the above statement, shall play a special role in our characterization. We denote by $\mathrm{Irr}^{\star}_{\mathrm{cusp}}(\mathrm{GL}(F))$ the subset of $\mathrm{Irr}_{\mathrm{cusp}}(\mathrm{GL}(F))$ consisting of those $\pi \in \mathrm{Irr}_{\mathrm{cusp}}(\mathrm{GL}(F))$ such that $\mathrm{C}(\pi) = \mathscr{C}(\mathbb{Z}_\Psi)$, for some $\Psi \in \mathrm{Irr}(\mathrm{W}_F)$

3. The characterization

In this section, we provide a list of natural properties which characterize $\mathrm{CV}: \mathrm{Nilp}_{\mathrm{D,ss}}(\mathrm{W}_F) \to [\mathrm{Rep}_{\mathrm{D,ss}}(\mathrm{W}_F)]$.

Proposition 7. *Let $\mathrm{CV}': \mathrm{Nilp}_{\mathrm{D,ss}}(\mathrm{W}_F) \to [\mathrm{Rep}_{\mathrm{D,ss}}(\mathrm{W}_F)]$ be any map, and $\mathrm{C}' := \mathrm{CV}' \circ \mathrm{V}$. Suppose*

(i) $\mathrm{Supp}_{\mathrm{W}_F} \circ \mathrm{C}'$ *is the semisimple ℓ-modular local Langlands correspondence of Vignéras; in other words, CV' preserves the W_F-support;*
(ii) C' *(or equivalently CV') commutes with taking duals;*
(iii) $\mathrm{L}(X, \pi, \pi^\vee) = \mathrm{L}(X, \mathrm{C}'(\pi), \mathrm{C}'(\pi)^\vee)$ *for all non-banal representations $\pi \in \mathrm{Irr}^{\star}_{\mathrm{cusp}}(\mathrm{GL}(F))$.*

Then for all $\Psi \in \mathrm{Irr}(\mathrm{W}_F)$, one has $\mathrm{CV}'(\mathrm{St}_0(\mathbb{Z}_\Psi)) = \mathscr{C}(\mathbb{Z}_\Psi)$.

Proof. Thanks to (i), $\mathrm{CV}'(\mathrm{St}_0(\mathbb{Z}_\Psi))$ has W_F-support $\bigoplus_{k=0}^{o(\Psi)-1} \nu^k \Psi$. Hence, by Theorem 1, its image under CV' is either $\mathscr{C}(\mathbb{Z}_\Psi)$ or a sum of Deligne representations of the form $[a, b] \otimes_{\mathrm{ss}} \Psi$ for $0 \leqslant a \leqslant b \leqslant o(\Psi) - 1$. If we are in the second situation, writing $\mathrm{CV}'(\mathrm{St}_0(\mathbb{Z}_\Psi)) = ([a, b] \otimes_{\mathrm{ss}} \Psi) \oplus W$, we have $\mathrm{CV}'(\mathrm{St}_0(\mathbb{Z}_\Psi))^\vee = ([-b, -a] \otimes_{\mathrm{ss}} \Psi^\vee) \oplus W^\vee$, thanks to (ii). However, writing τ for the non-banal cuspidal representation $\mathrm{V}^{-1}(\mathrm{St}_0(\mathbb{Z}_\Psi))$, we have $\mathrm{L}(X, \tau, \tau^\vee) = 1$ according to Theorem 3 and Proposition 6, whereas

$$\begin{aligned} \mathrm{L}(X, \mathrm{C}(\tau), \mathrm{C}(\tau^\vee)) &= \mathrm{L}(X, (([a, b] \otimes_{\mathrm{ss}} \Psi) \oplus W) \otimes_{\mathrm{ss}} (([-b, -a] \otimes_{\mathrm{ss}} \Psi^\vee) \oplus W^\vee)) \\ &= \mathrm{L}(X, ([a, b] \otimes_{\mathrm{ss}} \Psi) \otimes_{\mathrm{ss}} ([-b, -a] \otimes_{\mathrm{ss}} \Psi^\vee))\mathrm{L}'(X) \end{aligned}$$

for $\mathrm{L}'(X)$ an Euler factor. Now, observe that $\mathrm{L}(X,([a,b]\otimes_{\mathrm{ss}}\Psi)\otimes_{\mathrm{ss}}([-b,-a]\otimes_{\mathrm{ss}}\Psi^\vee))$ has a pole at $X=0$ according to Lemma 4, hence cannot be equal to 1. The conclusion of this discussion, according to (iii) is $\mathrm{CV}'(\mathrm{St}_0(\mathbb{Z}_\Psi))=\mathscr{C}(\mathbb{Z}_\Psi)$. □

It follows that (i)–(iii) characterize $\mathrm{C}|_{\mathrm{Irr}^\star_{\mathrm{cusp}}(\mathrm{GL}(F))}$ without reference to Vignéras' correspondence V.

On the other hand any map CV' satisfying (i)–(iii) must send each $\nu^k\Psi$ to itself if $o(\Psi)>1$ by (i). So there is no chance that CV' will preserve direct sums because $\bigoplus_{k=0}^{o(\Psi)-1}\mathrm{CV}'(\nu^k\Psi)\neq\mathscr{C}(\mathbb{Z}_\Psi)$. In particular any compatibility property of CV' with direct sums will have to be non-naive. Here is our characterization of the map CV:

Theorem 8. *Suppose* $\mathrm{CV}':\mathrm{Nilp}_{\mathrm{D,ss}}(\mathrm{W}_F)\to[\mathrm{Rep}_{\mathrm{D,ss}}(\mathrm{W}_F)]$ *satisfies* (i)–(iii) *of Proposition 7, and suppose moreover*

(a) *If* $\Phi'\in\mathrm{Im}(\mathrm{CV}')$ *and* $\Phi'=\Phi'_1\oplus\Phi'_2$ *in* $[\mathrm{Rep}_{\mathrm{D,ss}}(\mathrm{W}_F)]$ *then* $\Phi'_1,\Phi'_2\in\mathrm{Im}(\mathrm{CV}')$. *Moreover, if* $\Phi'=\mathrm{CV}'(\Phi)$, $\Phi'_i=\mathrm{CV}'(\Phi_i)$ *for* $\Phi,\Phi_i\in\mathrm{Nilp}_{\mathrm{D,ss}}(\mathrm{W}_F)$, *and* $\Phi'=\Phi'_1\oplus\Phi'_2$, *then* $\Phi=\Phi_1\oplus\Phi_2$.

(b) $\mathrm{CV}'([0,j-1]\otimes_{\mathrm{ss}}\Phi)=[0,j-1]\otimes_{\mathrm{ss}}\mathrm{CV}'(\Phi)$ *for* $j\in\mathbb{N}_{\geqslant1}$ *and* $\Phi\in\mathrm{Nilp}_{\mathrm{D,ss}}(\mathrm{W}_F)$.

Then $\mathrm{CV}'=\mathrm{CV}$.

Proof. For $\Psi\in\mathrm{Irr}(\mathrm{W}_F)$, it follows at once from Proposition 7 and (b) that

$$\mathrm{CV}'([0,j-1]\otimes_{\mathrm{ss}}\Psi)=[0,j-1]\otimes_{\mathrm{ss}}\Psi,\quad\text{if } o(\Psi)>1 \text{ and}$$
$$\mathrm{CV}'([0,j-1]\otimes_{\mathrm{ss}}\mathrm{St}_0(\mathbb{Z}_\Psi))=[0,j-1]\otimes_{\mathrm{ss}}\mathscr{C}(\mathbb{Z}_\Psi).$$

Next we prove that $\mathrm{Im}(\mathrm{CV}')\subset\mathrm{C}_{\mathrm{D,ss}}(\mathrm{W}_F)$. By (a), an element of $\mathrm{Im}(\mathrm{CV}')$ can be decomposed as a direct sum of elements in $\mathrm{Im}(\mathrm{CV}')\cap[\mathrm{Indec}_{\mathrm{D,ss}}]$, and (a) reduces the proof of the inclusion $\mathrm{Im}(\mathrm{CV}')\subset\mathrm{C}_{\mathrm{D,ss}}(\mathrm{W}_F)$ to showing that $[0,j-1]\otimes_{\mathrm{ss}}\mathrm{St}_0(\Psi)\notin\mathrm{Im}(\mathrm{CV}')$ for $\Psi\in\mathrm{Irr}(\mathrm{W}_F)$, $j\geqslant1$.

We first assume that $o(\Psi)=1$, so $\mathrm{St}_0(\Psi)=\Psi$. The only possible pre-image of Ψ by CV' is Ψ by (i), however $\mathrm{CV}'(\Psi)=\mathscr{C}(\mathbb{Z}_\Psi)$ by Proposition 7 so $\mathrm{St}_0(\Psi)\notin\mathrm{Im}(\mathrm{CV}')$. Now suppose $[0,j-1]\otimes_{\mathrm{ss}}\Psi\in\mathrm{Im}(\mathrm{CV}')$ for $j\geqslant2$, then by (b) this would imply that $[0,j-1]\otimes_{\mathrm{ss}}[0,j-1]\otimes_{\mathrm{ss}}\Psi\in\mathrm{Im}(\mathrm{CV}')$, hence that

$$[0,j-1]\otimes_{\mathrm{ss}}[0,j-1]\otimes_{\mathrm{ss}}\Psi=[0,2j-2]\otimes_{\mathrm{ss}}\Psi\oplus\cdots\oplus[j-1,j-1]\otimes_{\mathrm{ss}}\Psi$$

also belongs to $\mathrm{Im}(\mathrm{CV}')$ thanks to Lemma 2. However as $o(\Psi)=1$, the Deligne representation $[j-1,j-1]\otimes_{\mathrm{ss}}\Psi$ is nothing else than Ψ, which does not belong to $\mathrm{Im}(\mathrm{CV}')$, contradicting (a).

If $o(\Psi)>1$, then $\mathrm{CV}'(\nu^k\Psi)=\nu^k\Psi$. If $\mathrm{St}_0(\Psi)$ belonged to $\mathrm{Im}(\mathrm{CV}')$ then (a) would imply that $\mathrm{St}_0(\Psi)=\mathrm{CV}'(\bigoplus_{k=0}^{o(\Psi)-1}\nu^k\Psi)$, which is not the case thanks to Proposition 7. To see that $[0,j-1]\otimes_{\mathrm{ss}}\mathrm{St}_0(\Psi)\notin\mathrm{Im}(\mathrm{CV}')$ for all $j\geqslant2$ we use the same trick as in the $o(\Psi)=1$ case.

Now take $\Phi\in\mathrm{Nilp}_{\mathrm{D,ss}}$, as we just noticed $\mathrm{CV}'(\Phi)$ is a C-parameter and we write it

$$\mathrm{CV}'(\Phi)=\mathrm{CV}'(\Phi)_{\mathrm{acyc}}\oplus\bigoplus_{k\geqslant1,\mathbb{Z}_\Psi\in\mathfrak{l}(\mathrm{W}_F)}[0,k-1]\otimes_{\mathrm{ss}}n_{\mathbb{Z}_\Psi,k}\mathscr{C}(\mathbb{Z}_\Psi)$$

as in Section 2.3, where for each irreducible line $\mathbb{Z}_\Psi$ we have fixed an irreducible $\Psi\in\mathbb{Z}_\Psi$. Then (a) and the beginning of the proof imply that

$$\Phi=\mathrm{CV}'(\Phi)_{\mathrm{acyc}}\bigoplus_{k\geqslant1,\mathbb{Z}_\Psi\in\mathfrak{l}(\mathrm{W}_F)}[0,k-1]\otimes_{\mathrm{ss}}n_{\mathbb{Z}_\Psi,k}\mathrm{St}_0(\mathbb{Z}_\Psi),$$

hence that $\mathrm{CV}'(\Phi)=\mathrm{CV}(\Phi)$. □

4. The semiring structure on the space of C-parameters

As $(\mathrm{Nilp}_{\mathrm{D,ss}}(\mathrm{W}_F),\oplus,\otimes_{\mathrm{ss}})$ is a semiring, the map CV endows $\mathrm{C}_{\mathrm{D,ss}}(\mathrm{W}_F)$ with a semiring structure by transport of structure. We show that this semiring structure on $\mathrm{C}_{\mathrm{D,ss}}(\mathrm{W}_F)$ can be obtained without referring to CV directly, thus shedding a slightly different light on the map CV.

We denote by $\mathscr{G}(\mathrm{Rep}_{\mathrm{D,ss}}(\mathrm{W}_F))$ the Grothendieck group of the monoid $([\mathrm{Rep}_{\mathrm{D,ss}}(\mathrm{W}_F)], \oplus)$. We set

$$\mathscr{G}_0(\mathrm{Rep}_{\mathrm{D,ss}}(\mathrm{W}_F)) = \langle [0,k-1]\otimes_{\mathrm{ss}} \mathrm{St}_0(\mathbb{Z}_\Psi) - [0,k-1]\otimes_{\mathrm{ss}} \mathscr{C}(\mathbb{Z}_\Psi)\rangle_{\mathbb{Z}_\Psi\in \mathfrak{l}(\mathrm{W}_F),\ k\in\mathbb{N}_{\geqslant 1}},$$

the additive subgroup of $\mathscr{G}(\mathrm{Rep}_{\mathrm{D,ss}}(\mathrm{W}_F))$ generated by the differences $[0,k-1]\otimes_{\mathrm{ss}}\mathrm{St}_0(\mathbb{Z}_\Psi) - [0,k-1]\otimes_{\mathrm{ss}}\mathscr{C}(\mathbb{Z}_\Psi)$ for $\mathbb{Z}_\Psi\in\mathfrak{l}(\mathrm{W}_F)$ and $k\in\mathbb{N}_{\geqslant 1}$.

Proposition 9. *The canonical map* $h_{\mathrm{C}}:\mathrm{C}_{\mathrm{D,ss}}(\mathrm{W}_F)\to\mathscr{G}(\mathrm{Rep}_{\mathrm{D,ss}}(\mathrm{W}_F))/\mathscr{G}_0(\mathrm{Rep}_{\mathrm{D,ss}}(\mathrm{W}_F))$*, obtained by composing the canonical projection* $h:\mathscr{G}(\mathrm{Rep}_{\mathrm{D,ss}}(\mathrm{W}_F))\to\mathscr{G}(\mathrm{Rep}_{\mathrm{D,ss}}(\mathrm{W}_F))/\mathscr{G}_0(\mathrm{Rep}_{\mathrm{D,ss}}(\mathrm{W}_F))$ *with the natural injection of* $\mathrm{C}_{\mathrm{D,ss}}(\mathrm{W}_F)\hookrightarrow\mathscr{G}(\mathrm{Rep}_{\mathrm{D,ss}}(\mathrm{W}_F))$*, is injective. Moreover, its image is stable under the operation* $\oplus$*. In particular, this endows the set* $\mathrm{C}_{\mathrm{D,ss}}(\mathrm{W}_F)$ *with a natural monoid structure.*

Proof. Note that h_{C} is the restriction of the canonical surjection h to $\mathrm{C}_{\mathrm{D,ss}}(\mathrm{W}_F)$. Let Φ,Φ' be C-parameters, as in Section 2.3 and the last proof, we write

$$\Phi = \bigoplus_{k\geqslant 1,\mathbb{Z}_\Psi\in\mathfrak{l}(\mathrm{W}_F)} [0,k-1]\otimes_{\mathrm{ss}}\left(\left(\bigoplus_{i=0}^{o(\mathbb{Z}_\Psi)-1} m_{\mathbb{Z}_\Psi,k,i}\nu^i\Psi\right)\oplus n_{\mathbb{Z}_\Psi,k}\mathscr{C}(\mathbb{Z}_\Psi)\right)$$
$$\Phi' = \bigoplus_{k\geqslant 1,\mathbb{Z}_\Psi\in\mathfrak{l}(\mathrm{W}_F)} [0,k-1]\otimes_{\mathrm{ss}}\left(\left(\bigoplus_{i=0}^{o(\mathbb{Z}_\Psi)-1} m'_{\mathbb{Z}_\Psi,k,i}\nu^i\Psi\right)\oplus n'_{\mathbb{Z}_\Psi,k}\mathscr{C}(\mathbb{Z}_\Psi)\right)$$

where for each $(\mathbb{Z}_\Psi,k)$, there are i,i' such that $m_{\mathbb{Z}_\Psi,k,i}=0$ and $m'_{\mathbb{Z}_\Psi,k,i'}=0$. Suppose that both Φ and Φ' have same the image under h_{C}, then $\Phi'-\Phi\in\mathrm{Ker}(h)=\mathscr{G}_0(\mathrm{Rep}_{\mathrm{D,ss}}(\mathrm{W}_F))$. We thus get an equality of the form

$$\Phi-\Phi' = \bigoplus_{k\geqslant 1,\mathbb{Z}_\Psi\in\mathfrak{l}(\mathrm{W}_F)} a_{\mathbb{Z}_\Psi,k}([0,k-1]\otimes_{\mathrm{ss}}\mathrm{St}_0(\mathbb{Z}_\Psi)-[0,k-1]\otimes_{\mathrm{ss}}\mathscr{C}(\mathbb{Z}_\Psi)),$$

where all sums are finite. Set J^+ to be the set of pairs $(\mathbb{Z}_\Psi,k)$ such that $a_{\mathbb{Z}_\Psi,k}\geqslant 0$ and J^- to be the set of pairs $(\mathbb{Z}_\Psi,k)$ such that $b_{\mathbb{Z}_\Psi,k}:=-a_{\mathbb{Z}_\Psi,k}>0$. We obtain

$$\Phi\oplus\bigoplus_{(\mathbb{Z}_\Psi,k)\in J^-} b_{\mathbb{Z}_\Psi,k}[0,k-1]\otimes_{\mathrm{ss}}\mathrm{St}_0(\mathbb{Z}_\Psi)\oplus\bigoplus_{(\mathbb{Z}_\Psi,k)\in J^+} a_{\mathbb{Z}_\Psi,k}[0,k-1]\otimes_{\mathrm{ss}}\mathscr{C}(\mathbb{Z}_\Psi)$$
$$=\Phi'\oplus\bigoplus_{(\mathbb{Z}_\Psi,k)\in J^-} b_{\mathbb{Z}_\Psi,k}[0,k-1]\otimes_{\mathrm{ss}}\mathscr{C}(\mathbb{Z}_\Psi)\oplus\bigoplus_{(\mathbb{Z}_\Psi,k)\in J^+} a_{\mathbb{Z}_\Psi,k}[0,k-1]\otimes_{\mathrm{ss}}\mathrm{St}_0(\mathbb{Z}_\Psi)$$

in $[\mathrm{Rep}_{\mathrm{D,ss}}(\mathrm{W}_F)]$. Now take $(\mathbb{Z}_\Psi,k)\in J^+$, there is i such that $m_{\mathbb{Z}_\Psi,k,i}=0$. Comparing the occurence of $[0,k-1]\otimes_{\mathrm{ss}}\nu^i\Psi$ on the left and right hand sides of the equality we obtain

$$0=m'_{\mathbb{Z}_\Psi,k,i}+a_{\mathbb{Z}_\Psi,k}\Rightarrow a_{\mathbb{Z}_\Psi,k}=0.$$

Hence we just proved thet $a_{\mathbb{Z}_\Psi,k}=0$ for all $(\mathbb{Z}_\Psi,k)\in J^+$. The symmetric argument shows that for $(\mathbb{Z}_\Psi,k)\in J^-$, there is i' such that

$$m_{\mathbb{Z}_\psi,k,i'}+b_{\mathbb{Z}_\Psi,k}=0\Rightarrow b_{\mathbb{Z}_\Psi,k}=0,$$

which is impossible by assumption. Hence $J=J^+$ and $a_{\mathbb{Z}_\Psi,k}=0$ for all $\mathbb{Z}_\Psi\in J$, which implies $\Phi=\Phi'$, so h_{C} is indeed injective.

For the next assertion, suppose that $h_{\mathrm{C}}(\oplus_{\Phi\in[\mathrm{Indec}_{\mathrm{D,ss}}(\mathrm{W}_F)]}n_\Phi\Phi)\in\mathrm{Im}(h_{\mathrm{C}})$. Take $\Phi_0\in[\mathrm{Indec}_{\mathrm{D,ss}}(\mathrm{W}_F)]$ and consider $h_{\mathrm{C}}(\oplus_{\Phi\in[\mathrm{Indec}_{\mathrm{D,ss}}(\mathrm{W}_F)]}n_\Phi\Phi)\oplus h_{\mathrm{C}}(\Phi_0)$. If Φ_0 "completes a cycle" of $\oplus_{\Phi\in\mathrm{Indec}_{\mathrm{D,ss}}(\mathrm{W}_F)}n_\Phi\Phi$, i.e. if $\Phi_0=[0,k]\otimes_{\mathrm{ss}}\Psi$ with Ψ an irreducible representation Ψ of W_F, and if all other elements of $[0,k]\otimes_{\mathrm{ss}}\mathbb{Z}_\Psi$ appear in $\oplus_{\Phi\in\mathrm{Indec}_{\mathrm{D,ss}}(\mathrm{W}_F)}n_\Phi\Phi$ as representations $[0,k]\otimes_{\mathrm{ss}}\nu^j\Psi$ with corresponding multiplicities $n_{[0,k]\otimes_{\mathrm{ss}}\nu^j\Psi}\geqslant 1$, then setting $I=\{[0,k]\otimes_{\mathrm{ss}}\nu^j\Psi, j=1,\dots,o(\Psi)-1\}$, one gets

$$h_{\mathrm{C}}(\oplus_{\Phi\in[\mathrm{Indec}_{\mathrm{D,ss}}(\mathrm{W}_F)]}n_\Phi\Phi)\oplus h_{\mathrm{C}}(\Phi_0)=h_{\mathrm{C}}(\oplus_{\Phi\notin I}n_\Phi\Phi\oplus\oplus_{\Phi\in I}(n_\Phi-1)\Phi\oplus\mathscr{C}(\mathbb{Z}_\Psi)).$$

If Φ_0 does not complete a cycle, one has

$$h_{\mathrm{C}}(\oplus_{\Phi\in[\mathrm{Indec}_{\mathrm{D,ss}}(\mathrm{W}_F)]} n_\Phi \Phi) \oplus h_{\mathrm{C}}(\Phi_0) = h_{\mathrm{C}}(\oplus_{\Phi\in\mathrm{Indec}_{\mathrm{D,ss}}(\mathrm{W}_F)} n_\Phi \Phi \oplus \Phi_0).$$

The assertion follows by induction. □

In fact the tensor product operation descends on $\mathrm{Im}(h_{\mathrm{C}})$.

Proposition 10. *The additive subgroup* $\mathscr{G}_0(\mathrm{Rep}_{\mathrm{D,ss}}(\mathrm{W}_F))$ *of the ring* $\mathscr{G}(\mathrm{Rep}_{\mathrm{D,ss}}(\mathrm{W}_F))$ *is in fact an ideal. Moreover* $\mathrm{Im}(h_{\mathrm{C}})$ *is stable under* $\otimes_{\mathrm{ss}}$. *In particular this endows* $\mathrm{C}_{\mathrm{D,ss}}(\mathrm{W}_F)$ *with a natural semiring structure, and* h_{C} *becomes a semiring isomorphism from* $\mathrm{C}_{\mathrm{D,ss}}(\mathrm{W}_F)$ *to* $\mathrm{Im}(h_{\mathrm{C}})$.

Proof. For the first part, taking $\Psi_0 \in \mathrm{Irr}(\mathrm{W}_F)$, it is enough to prove that for any $\Phi_1 \in \mathrm{Irr}_{\mathrm{D,ss}}(\mathrm{W}_F)$ and $k,\ l \geqslant 0$, the tensor product $[0,k] \otimes_{\mathrm{ss}} (\mathrm{St}_0(\mathbb{Z}_{\Psi_0}) - \mathscr{C}(\mathbb{Z}_{\Psi_0})) \otimes_{\mathrm{ss}} [0,l] \otimes_{\mathrm{ss}} \Phi_1$ belongs to $\mathscr{G}_0(\mathrm{Rep}_{\mathrm{D,ss}}(\mathrm{W}_F))$. By associativity and commutativity of tensor product, and because $[0,i]\otimes_{\mathrm{ss}}[0,j]$ is always a sum of segments by Lemma 2, it is enough to check that $(\mathrm{St}_0(\mathbb{Z}_{\Psi_0}) - \mathscr{C}(\mathbb{Z}_{\Psi_0})) \otimes_{\mathrm{ss}} \Phi_1$ belongs to $\mathscr{G}_0(\mathrm{Rep}_{\mathrm{D,ss}}(\mathrm{W}_F))$. Suppose first that Φ_1 is nilpotent, i.e. $\Phi_1 = \Psi_1 \in \mathrm{Irr}(\mathrm{W}_F)$. Because $\mathrm{St}_0(\mathbb{Z}_{\Psi_0}) \otimes_{\mathrm{ss}} \Psi_1$ is fixed by ν under twisting and because its Deligne operator is zero, we get that

$$\mathrm{St}_0(\mathbb{Z}_{\Psi_0}) \otimes_{\mathrm{ss}} \Psi_1 = \bigoplus_{\mathbb{Z}_\Psi \in \mathfrak{l}(\mathrm{W}_F)} a_{\mathbb{Z}_\Psi} \mathrm{St}_0(\mathbb{Z}_\Psi).$$

On the other hand because $\mathscr{C}(\mathbb{Z}_{\Psi_0}) \otimes_{\mathrm{ss}} \Psi_1$ is fixed by ν and because its Deligne operator is bijective we obtain

$$\mathscr{C}(\mathbb{Z}_{\Psi_0}) \otimes_{\mathrm{ss}} \Psi_1 = \bigoplus_{\mathbb{Z}_\Psi \in \mathfrak{l}(\mathrm{W}_F)} b_{\mathbb{Z}_\Psi} \mathscr{C}(\mathbb{Z}_\Psi).$$

Now observing that both $\mathrm{St}_0(\mathbb{Z}_{\Psi_0}) \otimes_{\mathrm{ss}} \Psi_1$ and $\mathscr{C}(\mathbb{Z}_{\Psi_0}) \otimes_{\mathrm{ss}} \Psi_1$ have the same W_F-support, it implies that $a_{\mathbb{Z}_\Psi} = b_{\mathbb{Z}_\Psi}$ for all lines $\mathbb{Z}_\Psi$, form which we deduce that $(\mathrm{St}_0(\mathbb{Z}_{\Psi_0}) - \mathscr{C}(\mathbb{Z}_{\Psi_0})) \otimes_{\mathrm{ss}} \Phi_1 \in \mathscr{G}_0(\mathrm{Rep}_{\mathrm{D,ss}}(\mathrm{W}_F))$. With the same arguments we obtain that $(\mathrm{St}_0(\mathbb{Z}_{\Psi_0}) - \mathscr{C}(\mathbb{Z}_{\Psi_0})) \otimes_{\mathrm{ss}} \Phi_1 = 0 \in \mathscr{G}_0(\mathrm{Rep}_{\mathrm{D,ss}}(\mathrm{W}_F))$ when Φ_1 is of the form $\mathscr{C}(\mathbb{Z}_{\Psi_1})$ (because in this case both $\mathrm{St}_0(\mathbb{Z}_{\Psi_0}) \otimes_{\mathrm{ss}} \Phi_1$ and $\mathscr{C}(\mathbb{Z}_{\Psi_0}) \otimes_{\mathrm{ss}} \Phi_1$ have bijective Deligne operators). □

The following proposition is proved in a similar, but simpler manner than the propositions above.

Proposition 11. *Let* h_{Nilp} *be the restriction of*

$$h : \mathscr{G}(\mathrm{Rep}_{\mathrm{D,ss}}(\mathrm{W}_F)) \to \mathscr{G}(\mathrm{Rep}_{\mathrm{D,ss}}(\mathrm{W}_F))/\mathscr{G}_0(\mathrm{Rep}_{\mathrm{D,ss}}(\mathrm{W}_F))$$

to $\mathrm{Nilp}_{\mathrm{D,ss}}(\mathrm{W}_F)$, *then* h_{Nilp} *is a semiring isomorphism and* $\mathrm{Im}(h_{\mathrm{Nilp}}) = \mathrm{Im}(h_{\mathrm{C}})$.

The above propositions have the following immediate corollary.

Corollary 12. *One has* $\mathrm{CV} = h_{\mathrm{C}}^{-1} \circ h_{\mathrm{Nilp}}$, *in particular it is a semiring isomorphism from* $\mathrm{Nilp}_{\mathrm{D,ss}}(\mathrm{W}_F)$ *to* $\mathrm{C}_{\mathrm{D,ss}}(\mathrm{W}_F)$.

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References

[1] M. Harris, R. Taylor, *The geometry and cohomology of some simple Shimura varieties*, Annals of Mathematics Studies, vol. 151, Princeton University Press, 2001, with an appendix by Vladimir G. Berkovich, viii+276 pages.

[2] G. Henniart, "Une preuve simple des conjectures de Langlands pour GL(n) sur un corps p-adique", *Invent. Math.* **139** (2000), no. 2, p. 439-455.

[3] R. Kurinczuk, N. Matringe, "Rankin-Selberg local factors modulo ℓ", *Sel. Math., New Ser.* **23** (2017), no. 1, p. 767-811.

[4] ———, "The ℓ-modular local Langlands correspondence and local constants", to appear in *J. Inst. Math. Jussieu*, 2018.

[5] G. Laumon, M. Rapoport, U. Stuhler, “$\mathscr{D}$-elliptic sheaves and the Langlands correspondence”, *Invent. Math.* **113** (1993), no. 2, p. 217-338.
[6] P. Scholze, “The local Langlands correspondence for GL_n over p-adic fields”, *Invent. Math.* **192** (2013), no. 3, p. 663-715.
[7] M.-F. Vignéras, “Correspondance de Langlands semi-simple pour $GL(n,F)$ modulo $l \neq p$”, *Invent. Math.* **144** (2001), no. 1, p. 177-223.