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Algebra / *Algèbre*

On the monoidal invariance of the cohomological dimension of Hopf algebras

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Abstract. We discuss the question of whether the global dimension is a monoidal invariant for Hopf algebras, in the sense that if two Hopf algebras have equivalent monoidal categories of comodules, then their global dimensions should be equal. We provide several positive new answers to this question, under various assumptions of smoothness, cosemisimplicity or finite dimension. We also discuss the comparison between the global dimension and the Gerstenhaber–Schack cohomological dimension in the cosemisimple case, obtaining equality in the case the latter is finite. One of our main tools is the new concept of twisted separable functor.

Résumé. Nous étudions la question de l'invariance monoïdale de la dimension globale des algèbres de Hopf : si deux algèbres de Hopf ont des catégories de comodules monoïdalement équivalentes, ont-elles même dimension globale ? Nous apportons plusieurs nouvelles réponses positives dans les cas d'algèbres de Hopf lisses, cosemisimples ou de dimension finie. Nous comparons également la dimension globale et la dimension cohomologique de Gerstenhaber–Schack dans le cas cosemisimple. Un outil important pour obtenir ces divers résultats est la nouvelle notion de foncteur séparable twisté.

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1. Introduction

A classical invariant of an algebra A is its (right) global dimension

$$\mathrm{r.gldim}(A) = \max\left\{\mathrm{pd}_A(M),\ M \in \mathscr{M}_A\right\} \in \mathbb{N} \cup \{\infty\}$$

where for a (right) A-module M, $\mathrm{pd}_A(M)$ stands for its projective dimension, i.e. the smallest possible length for a resolution of M by projective A-modules.

The global dimension is a key ingredient in the analysis of certain geometric properties of discrete groups [10, 16], and often serves as a good analogue of the dimension of a smooth affine variety. However in some noncommutative situations, it is better to replace it by the Hochschild cohomological dimension, which has similar geometric significance, and is defined by:

$$\begin{aligned}\mathrm{cd}(A) &= \max\left\{n :\ H^n(A,M) \neq 0 \text{ for some } A\text{-bimodule } M\right\} \in \mathbb{N} \cup \{\infty\}\\ &= \min\left\{n :\ H^{n+1}(A,M) = 0 \text{ for some } A\text{-bimodule } M\right\}\\ &= \mathrm{pd}_{{}_A\mathscr{M}_A}(A)\end{aligned}$$

where $H^*(A,\cdot)$ denotes Hochschild cohomology and $\mathrm{pd}_{{}_A\mathcal{M}_A}(A)$ is the projective dimension of A in the category of A-bimodules.

Indeed, for example if $A = A_1(k)$ is the first Weyl algebra (k is, as in all the paper, an algebraically closed field), we have $\mathrm{r.gldim}(A_1(k)) = 1$ (in characteristic zero) and $\mathrm{cd}(A_1(k)) = 2$, while $A_1(k)$ should definitively be considered as a 2-dimensional object.

When A is a Hopf algebra, it is well-known that we have

$$\mathrm{r.gldim}(A) = \mathrm{pd}_A(k_\varepsilon) = \mathrm{cd}(A) = \mathrm{l.gldim}(A) = \mathrm{pd}_A({}_\varepsilon k)$$

where k_ε and ${}_\varepsilon k$ denote the respective right and left trivial A-modules, and $\mathrm{l.gldim}(A)$ is the left global dimension. See [28] for the equalities at the extreme left and right, and, for example, [22] for the other equality. We simply will denote this number by $\mathrm{cd}(A)$, and call it the cohomological dimension of A.

A general classical problem is whether the global dimension or the Hochschild cohomological dimension remain preserved under various kind of “deformations" of A, and the question we are particularly interested in, originally asked in [7] and suggested by examples studied in [5], is the following one.

Question 1. *If A and B are Hopf algebras having equivalent linear tensor categories of comodules, do we have* $\mathrm{cd}(A) = \mathrm{cd}(B)$*?*

Some remarks immediately arise on the significance and interest of Question 1.

(1) The word “tensor” is crucial in the question, since this is what captures information about the algebra structure inside the category of comodules. Dropping it would make the question meaningless, as shown by the example of group algebras: if two group algebras have equivalent categories of comodules, the only conclusion, in lack of additional information, is that the groups have the same cardinality.

(2) Tannaka–Krein duality [23] enables one to reconstruct a Hopf algebra from its tensor category of comodules together with the forgetful functor to vector spaces. However, it is not assumed here that the given monoidal equivalence is compatible with the respective forgetful functors, and so the Hopf algebras are non-isomorphic in general. There are many instances of the situation, see for example [6, 42] for a large review of examples, and [27, 32, 33, 37] for more recent ones.

(3) As just said, the Hopf algebras in Question 1 are non-isomorphic in general, but worst, some of their ring-theoretical properties, such as Gelfand–Kirillov dimension, can be very different, see [15]. The interest in the question is thus both theoretical, in the investigation of which properties of a Hopf algebra are preserved under monoidal equivalence of the category of comodules, and practical, in the determination of the global dimension of new Hopf algebras from known old ones.

There are, to the best of our knowledge, two partial positive answers to Question 1 in the literature.

(1) In [7, 8], it is shown that when A, B are cosemisimple with antipode satisfying $S^4 = \mathrm{id}$, then $\mathrm{cd}(A) = \mathrm{cd}(B)$.

(2) In [47], Wang, Yu and Zhang show that when A is twisted Calabi–Yau and B is homologically smooth, then $\mathrm{cd}(A) = \mathrm{cd}(B)$.

The aim of this paper is to provide several new positive answers to Question 1, together with application to the determination of the cohomological dimension of some Hopf algebras in some new situations (universal cosovereign Hopf algebras and free wreath products). Indeed, we show that Question 1 has a positive answer in the following cases.

(1) The smooth case: we show that if A, B have bijective antipode and are (homologically) smooth, then $\mathrm{cd}(A) = \mathrm{cd}(B)$. This improves on [47, Theorem 2.4.5], which assumed

moreover that A is twisted Calabi–Yau (and then proved that B is twisted Calabi–Yau as well), see Theorem 8. The proof is done by carefully inspecting the arguments in [47].

(2) The cosemisimple case: we show that if A, B are cosemisimple and both have finite cohomological dimension, then $\mathrm{cd}(A) = \mathrm{cd}(B)$. See Theorem 24. Removing the assumption $S^4 = \mathrm{id}$ from [8] (with instead the finiteness assumption on cohomological dimensions) enables us to compute cohomological dimension in a number of new situations, see Section 8.

(3) The finite-dimensional case: we show that under natural characteristic assumption on the base field or the assumption that A^* is unimodular, then $\mathrm{cd}(A) = \mathrm{cd}(B)$, see Theorem 45. Here, since finite-dimensional Hopf algebras are self-injective, we have $\mathrm{cd}(A) \in \{0, \infty\}$, and the interest of Question 1 is more on the theoretical side, but, as Etingof pointed out, understanding the finite-dimensional situation should be an important aspect. The proof of Theorem 45 is a rather direct consequence of previous results [1,18,26], but an interesting aspect is that it connects Question 1 to a weak form of an important historical conjecture of Kaplansky saying that a finite-dimensional cosemisimple Hopf algebra is unimodular (the strong form says that a cosemisimple Hopf algebra satisfies $S^2 = \mathrm{id}$).

Our method in the smooth and cosemisimple cases is based on the fact that if $\mathcal{M}^A \simeq^{\otimes} \mathcal{M}^B$ as above, results by Schauenburg [40] ensures that there exists an A-B Galois object R, and then on proving that $\mathrm{cd}(A) = \mathrm{cd}(R) = \mathrm{cd}(B)$, which is achieved in the smooth case by following arguments of Yu [50]. In general one notices furthermore that $\mathrm{cd}(A) = \mathrm{pd}_{{}_R\mathcal{M}_R^B}(R)$, the projective dimension of R in the category of R-bimodules inside B-comodules, and then the main question is to compare $\mathrm{pd}_{{}_R\mathcal{M}_R^B}(R)$ and $\mathrm{pd}_{{}_R\mathcal{M}_R}(R) = \mathrm{cd}(R)$. The main ingredient in this comparison in the cosemisimple case is a twisted averaging trick, Lemma 20, that we believe to be quite non-straightforward. The averaging lemma leads to the concept of twisted separable functor we define in Section 4, a generalization of the notion of separable functor introduced in [34].

Of course, the above considerations lead to the following question.

Question 2. *Let A be a Hopf algebra. Under which conditions on A do we have* $\mathrm{cd}(A) = \mathrm{cd}(R)$ *for any left or right A-Galois object R?*

Theorem 24 in the cosemisimple case has the drawback, in concrete situations, that we need to know in advance that both Hopf algebras have finite cohomological dimension, an information that is not necessarily available. This leads us back to our initial idea to tackle Question 1 in [7], which was to use an auxiliary cohomological dimension for the Hopf algebra A, the Gerstenhaber–Schack cohomological dimension, defined by

$$\mathrm{cd}_{\mathrm{GS}}(A) = \max\left\{n : \mathrm{Ext}^n_{\mathcal{YD}_A^A}(k, V) \neq 0 \text{ for some } V \in \mathcal{YD}_A^A\right\} \in \mathbb{N} \cup \{\infty\}$$

where $\mathcal{YD}_A^A$ is the category of Yetter–Drinfeld modules over A and k is the trivial Yetter–Drinfeld module. It was shown in [7, Theorem 5.6, Corollary 5.7] that $\mathrm{cd}(A) \le \mathrm{cd}_{\mathrm{GS}}(A)$ and that if A, B are Hopf algebras with $\mathcal{M}^A \simeq^{\otimes} \mathcal{M}^B$, then

$$\max(\mathrm{cd}(A), \mathrm{cd}(B)) \le \mathrm{cd}_{\mathrm{GS}}(A) = \mathrm{cd}_{\mathrm{GS}}(B)$$

Therefore, comparing $\mathrm{cd}(A)$ and $\mathrm{cd}_{\mathrm{GS}}(A)$ can be a key step towards answers to Question 1. In this direction, we show (Theorem 31) that if A is a cosemisimple Hopf algebra with $\mathrm{cd}_{\mathrm{GS}}(A)$ finite, then $\mathrm{cd}(A) = \mathrm{cd}_{\mathrm{GS}}(A)$. Again the method of proof is based on a twisted averaging trick and uses an appropriate twisted separable functor. Theorem 31 has, as a corollary, a weak form of Theorem 24, which is probably sufficient in dealing with numerous examples, see Corollary 32.

We expect that the equality $\mathrm{cd}(A) = \mathrm{cd}_{\mathrm{GS}}(A)$ holds for any cosemisimple Hopf algebra, but as already pointed in [7], it cannot hold for any Hopf algebra over any field, as we see by taking a semisimple non cosemisimple Hopf algebra over a field of positive characteristic, so we asked there whether the equality was true in characteristic zero. Etingof pointed out that it does not hold in characteristic zero even for the very simple example $A = k[x]$ with x primitive. Hence we have now the following question.

Question 3. *What are the Hopf algebras such that* $\mathrm{cd}(A) = \mathrm{cd}_{\mathrm{GS}}(A)$*?*

The paper is organized as follows. Section 2 recalls the connection between Hopf–Galois objects and monoidal equivalences and proves our first result on the monoidal invariance of the cohomological dimension, in the smooth case. Section 3 provides the necessary material on categories of bimodules inside categories of comodules. Section 4 introduces the notion of twisted separable functor. This is used in Section 5 to prove Theorem 24, our second partial positive answer to Question 1, in the cosemisimple case. Section 6 discusses the comparison between cohomological dimension and Gerstenhaber–Schack cohomological dimension, together with the necessary material on Yetter–Drinfeld modules. Section 7 studies the behaviour of Gerstenhaber–Schack cohomological dimension under Hopf subalgebras in the cosemisimple case. Section 8 is devoted to applications to some examples. Section 9 discusses the finite-dimensional situation in Question 1. The reader only interested in this case might go directly to this section. The concluding Section 10 summarizes the known positive answers to Question 1.

Notations and conventions

We work over an algebraically closed field k. We assume that the reader is familiar with the theory of Hopf algebras and their tensor categories of comodules, as e.g. in [19, 24, 31], and with the basics of homological algebra [10,48]. If A is a Hopf algebra, as usual, Δ, ε and S stand respectively for the comultiplication, counit and antipode of A. We use Sweedler's notations in the standard way. The category of right A-comodules is denoted $\mathscr{M}^A$, the category of right A-modules is denoted $\mathscr{M}_A$, etc... The trivial (right) A-module is denoted k_ε. The set of A-module morphisms (resp. A-comodule morphisms) between two A-modules (resp. two A-comodules) V and W is denoted $\mathrm{Hom}_A(V,W)$ (resp. $\mathrm{Hom}^A(V,W)$).

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2. Hopf–Galois objects and monoidal equivalences

2.1. *Hopf–Galois objects*

Let A be a Hopf algebra. Recall that a *left A-Galois object* is a non-zero left A-comodule algebra R such that the canonical map

$$\begin{aligned} R \otimes R &\longrightarrow A \otimes R \\ x \otimes y &\longmapsto x_{(-1)} \otimes x_{(0)} y \end{aligned}$$

is bijective. Similarly a *right A-Galois object* is a non-zero right A-comodule algebra such that the obvious analogue of the previous canonical map is bijective. If B is another Hopf algebra, an *A-B-bi-Galois object* is an A-B-bicomodule algebra which is simultaneously left A-Galois and right B-Galois. See [40, 42].

As said in the introduction, it is important, in view of Question 1, to determine whether a Hopf algebra and its left or right Galois object have the same cohomological dimension, which lead us to Question 2, and for which we list a number of basic remarks.

Remark 4. Let A be a Hopf algebra and let R be a left or right A-Galois object. Then we have $\mathrm{cd}(R) \le \mathrm{cd}(A)$. This follows from Stefan's spectral sequence [44, Theorem 3.3], or can be checked directly at the level of complexes defining Hochschild cohomology [6, Theorem 7.12]. See [50, Lemma 2.2] as well.

Remark 5. There is indeed the need of assumptions on A in Question 2, as the example of the Taft algebra H_n shows, which admits the matrix algebra $M_n(k)$ as a Galois object [29], and for which we have $\mathrm{cd}(M_n(k)) = 0 < \mathrm{cd}(H_n) = \infty$.

Remark 6. In the setting of Question 2, as the Weyl algebra example shows, which is a Galois object over $k[x, y]$, the good dimension to consider is indeed the Hochschild cohomological dimension, and not the global dimension.

Recall that an algebra is said to be (homologically) *smooth* if the trivial bimodule has a finite resolution by finitely generated projective bimodules. For Hopf algebras, this is equivalent to say that the trivial left or right A-module has a finite resolution by finitely generated projective modules.

A partial positive answer to Question 2 was obtained by Yu [50]. Indeed, if A is a Hopf algebra with bijective antipode and R be a left of right A-Galois object, [50, Theorem 2.4.5] states that if A is twisted Calabi–Yau of dimension d, then so is R, and hence in particular $d = \mathrm{cd}(A) = \mathrm{cd}(R)$. Our first observation is that, inspecting carefully the arguments in [50], what is needed to ensure the equality of the cohomological dimensions is smoothness only.

Theorem 7. *Let A be a Hopf algebra with bijective antipode, and let R be a left or right A-Galois object. If A is smooth, then we have* $\mathrm{cd}(A) = \mathrm{cd}(R)$.

Proof. Since A is homologically smooth, we have that $\mathrm{cd}(A)$ is finite, hence $\mathrm{cd}(A) = \max\{n : \mathrm{Ext}^n_{{}_A\mathcal{M}}({}_\varepsilon k, F) \neq 0$ for some free module $F\}$, and moreover the functor $\mathrm{Ext}^*_{{}_A\mathcal{M}}({}_\varepsilon k, \cdot)$ commutes with direct limits, see e.g. [10, VIII, Theorem 4.8]). Hence

$$\mathrm{cd}(A) = \mathrm{pd}_{{}_A\mathcal{M}}({}_\varepsilon k) = \max\left\{n : \mathrm{Ext}^n_{{}_A\mathcal{M}}({}_\varepsilon k, A) \neq 0\right\}$$

The algebra R is homologically smooth since A is, by [50, Lemma 2.4], hence we have similarly

$$\mathrm{cd}(R) = \mathrm{pd}_{{}_R\mathcal{M}_R}(R) = \max\left\{n : \mathrm{Ext}^n_{{}_R\mathcal{M}_R}(R, R \otimes R) \neq 0\right\}$$

We have by [50, Lemma 2.2, Lemma 2.1]

$$\mathrm{Ext}^*_{{}_R\mathcal{M}_R}(R, R \otimes R) \simeq \mathrm{Ext}^*_{{}_A\mathcal{M}}({}_\varepsilon k, {}_A A \otimes R) \simeq \mathrm{Ext}^*_{{}_A\mathcal{M}}({}_\varepsilon k, {}_A A) \otimes R$$

where the first isomorphism is obtained from [50, Lemma 2.2, Lemma 2.1], with the left A-module structure on ${}_A A \otimes R$ being simply by multiplying in A on the left, and the second one follows from the smoothness of A, since then $\mathrm{Ext}^*_{{}_A\mathcal{M}}({}_\varepsilon k, \cdot)$ commutes with direct limits. Hence we obtain $\mathrm{cd}(A) = \mathrm{cd}(R)$.

If we start with a right Hopf–Galois object R over A, it is well-known that R^{op} is a left A-Galois object in a natural way (if the antipode of A is bijective), so that we can use the result for left A-Galois objects to conclude that $\mathrm{cd}(A) = \mathrm{cd}(R)$ as well. □

2.2. *Monoidal equivalences*

Let A, B be Hopf algebras. Schauenburg [40, Corollary 5.7] has shown the equivalence of the following assertions:

(1) There exists an equivalence of monoidal categories $\mathcal{M}^A \simeq^\otimes \mathcal{M}^B$;
(2) There exists an A-B-bi-Galois object.

It therefore follows that finding answers to Question 2 has immediate applications to Question 1. We thus obtain, via Theorem 7, a partial positive answer to Question 1, only assuming that the Hopf algebras are smooth, while [47, Theorem 2.4.5] assumed furthermore that one of the Hopf algebras is twisted Calabi–Yau, and then proved that the other one is twisted Calabi–Yau with the same dimension as well.

Theorem 8. *Let A, B be Hopf algebras that have equivalent linear tensor categories of comodules: $\mathscr{M}^A \simeq^{\otimes} \mathscr{M}^B$. If A and B are smooth and have bijective antipode, we have* $\mathrm{cd}(A) = \mathrm{cd}(B)$.

Proof. Since there exists an A-B-bi-Galois object R, we have $\mathrm{cd}(A) = \mathrm{cd}(R) = \mathrm{cd}(B)$ by Theorem 7. □

Remark 9. It is pointed out in Remark 5 that the matrix algebra $M_n(k)$ is a Galois object over the Taft algebra H_n, and in fact $M_n(k)$ is an H_n-H_n-bi-Galois object [41]. This indicates that the approach via Hopf–Galois objects cannot cover all the possible situations in Question 1.

3. Equivariant bimodule categories and projective dimensions

In this section we explain how one can use bimodule categories in order to obtain informations on Question 2 and hence on Question 1.

Let A be a Hopf algebra, let R be a right A-comodule algebra (recall that this means that R is an algebra in the monoidal category $\mathscr{M}^A$) and let ${}_R\mathscr{M}_R^A$ be the category of R-bimodules in the category $\mathscr{M}^A$: the objects are the A-comodules V with an R-bimodule structure having the Hopf bimodule compatibility conditions

$$(x\cdot v)_{(0)}\otimes(x\cdot v)_{(1)} = x_{(0)}\cdot v_{(0)}\otimes x_{(1)}v_{(1)}, \quad (v\cdot x)_{(0)}\otimes(v\cdot x)_{(1)} = v_{(0)}\cdot x_{(0)}\otimes v_{(1)}x_{(1)}$$

for any $x\in R$ and $v\in V$. The morphisms are the A-colinear and R-bilinear maps. The category ${}_R\mathscr{M}_R^A$ is obviously abelian, and the tensor product of bimodules induces a monoidal structure on it.

The following basic property is certainly well-known, and a straightforward verification.

Proposition 10. *Let A be a Hopf algebra and let R be a right A-comodule algebra.*

1. *The forgetful functor $\Omega^A : {}_R\mathscr{M}_R^A \to \mathscr{M}^A$ has a left adjoint, which associates to a comodule V the object $R\otimes V\otimes R$ whose bimodule structure is given by left and right multiplication of R and whose comodule structure is the tensor product of the underlying comodules.*
2. *The forgetful functor $\Omega_R : {}_R\mathscr{M}_R^A \to {}_R\mathscr{M}_R$ has a right adjoint, which associates to an R-bimodule V the object $V\otimes A$ whose R-bimodule structure is given by*

$$x\cdot(v\otimes a) = x_{(0)}\cdot v\otimes x_{(1)}a, \quad (v\otimes a)\cdot x = v\cdot x_{(0)}\otimes ax_{(1)}$$

and whose A-comodule structure is induced by the comultiplication of A.

Objects in ${}_R\mathscr{M}_R^A$ that are images of the above left adjoint functor are called *free*, they are indeed free as bimodules. Any object in ${}_R\mathscr{M}_R^A$ is a quotient of a free object.

As usual, if $\mathscr{C}$ is an abelian category having enough projective objects, the notation $\mathrm{pd}_{\mathscr{C}}(V)$ refers to the projective dimension of the object V, i.e. the smallest length of a resolution of V by projective objects in $\mathscr{C}$, with, as well

$$\mathrm{pd}_{\mathscr{C}}(V) = \max\left\{n : \mathrm{Ext}^n_{\mathscr{C}}(V,W)\neq 0 \text{ for some object } W \text{ in } \mathscr{C}\right\}$$

The following corollary is a direct consequence of Proposition 10 and of the standard properties of pairs of adjoint functors.

Corollary 11. *Let A be a Hopf algebra and let R be a right A-comodule algebra.*

(1) *The category ${}_R\mathcal{M}_R^A$ has enough injective objects, since ${}_R\mathcal{M}_R$ has, and we have, for any object V in ${}_R\mathcal{M}_R^A$ and any R-bimodule W:*

$$\mathrm{Ext}^*_{{}_R\mathcal{M}_R}(\Omega_R(V), W) \simeq \mathrm{Ext}^*_{{}_R\mathcal{M}_R^A}(V, W \otimes A)$$

(2) *If $\mathcal{M}^A$ has enough projective objects (in which case one says that A is co-Frobenius), so has ${}_R\mathcal{M}_R^A$. In that case, the previous isomorphism ensures that for any object V in ${}_R\mathcal{M}_R^A$, we have*

$$\mathrm{pd}_{{}_R\mathcal{M}_R}(\Omega_R(V)) \leq \mathrm{pd}_{{}_R\mathcal{M}_R^A}(V)$$

(3) *If A is cosemisimple, then ${}_R\mathcal{M}_R^A$ has enough projective objects, and the projective objects are the direct summands of the free ones.*

The connection between our problem and bimodules is now given by the following result.

Proposition 12. *Let A be a Hopf algebra, let R be a left A-Galois object, and let B a Hopf algebra such that R is A-B-bi-Galois. Then the category ${}_R\mathcal{M}_R^B$ has enough projective objects, and we have*

$$\mathrm{cd}(A) = \mathrm{pd}_{{}_R\mathcal{M}_R^B}(R) \geq \mathrm{pd}_{{}_R\mathcal{M}_R}(R) = \mathrm{cd}(R)$$

Proof. First recall that it follows from the right version of [39, Theorem 5.7] (the structure theorem for Hopf modules) that the functor

$$\begin{aligned} {}_A\mathcal{M} &\longrightarrow {}_A\mathcal{M}_A^A \\ V &\longmapsto V \circledcirc A \end{aligned}$$

is a monoidal equivalence of categories, where $V \circledcirc A$ is $V \otimes A$ as a vector space, has the tensor product left A-module structure and the right module and comodule structures are induced by the multiplication and comultiplication of A respectively. This monoidal equivalence transforms the trivial module ${}_\varepsilon k$ into the A-bimodule A.

Now let R be an A-B-bi-Galois object. The corresponding monoidal equivalence $\mathcal{M}^A \simeq^\otimes \mathcal{M}^B$ in [40] is given by the cotensor product $-\square_A R$, and sends A to R, and thus clearly induces an equivalence between the bimodule categories ${}_A\mathcal{M}_A^A$ and ${}_R\mathcal{M}_R^B$. Composing with the equivalence at the beginning of the proof, we get a monoidal equivalence

$${}_A\mathcal{M} \simeq^\otimes {}_R\mathcal{M}_R^B$$

sending ${}_\varepsilon k$ to R. It is then clear that ${}_R\mathcal{M}_R^B$ has enough projective objects, and that $\mathrm{cd}(A) = \mathrm{pd}_A({}_\varepsilon k) = \mathrm{pd}_{{}_R\mathcal{M}_R^B}(R)$. The last inequality has been given in Remark 4. □

It is therefore crucial to compare $\mathrm{pd}_{{}_R\mathcal{M}_R^B}(R)$ and $\mathrm{pd}_{{}_R\mathcal{M}_R}(R)$ for R a right B-Galois object. Notice that the problem makes sense and is interesting for any comodule algebra, as soon as ${}_R\mathcal{M}_R^B$ has enough projective projects. This is the motivation for the tools we develop in the next section.

4. Twisted separable functors

In this section we introduce the notion of twisted separable functor.

If $\mathcal{C}$ is category, we say that a subclass $\mathcal{F}$ of objects of $\mathcal{C}$ is generating if for every object V of $\mathcal{C}$, there exists an object P of $\mathcal{F}$ together with an epimorphism $P \to V$.

Definition 13. *Let $\mathcal{C}$ and $\mathcal{D}$ be some categories. We say that a functor $F : \mathcal{C} \to \mathcal{D}$ is* twisted separable *if there exist*

(1) *an autoequivalence Θ of the category $\mathcal{D}$;*
(2) *a generating subclass $\mathcal{F}$ of objects of $\mathcal{C}$ together with, for any object P of $\mathcal{F}$, an isomorphism $\theta_P : F(P) \to \Theta F(P)$;*

(3) *a natural morphism* $\mathbf{M}_{\cdot,\cdot} : \mathrm{Hom}_{\mathscr{D}}(F(\cdot), \Theta F(\cdot)) \to \mathrm{Hom}_{\mathscr{C}}(\cdot,\cdot)$ *such that for any object* P *of* $\mathscr{F}$*, we have* $\mathbf{M}_{P,P}(\theta_P) = \mathrm{id}_P$.

The naturality condition above means that for any morphisms $\alpha : V' \to V$, $\beta : W \to W'$ in $\mathscr{C}$ and any morphism $f : F(V) \to \Theta F(W)$ in $\mathscr{D}$, we have

$$\beta \circ \mathbf{M}_{V,W}(f) \circ \alpha = \mathbf{M}_{V',W'}(\Theta F(\beta) \circ f \circ F(\alpha))$$

When $\mathscr{F}$ is the whole class of objects of $\mathscr{C}$, the autoequivalence Θ is the identity and the isomorphisms θ_P all are the identity, we get the notion of separable functor from [34], which is known to be provide a convenient setting for various types of generalized Maschke theorems, see [12]. A basic example of a separable functor is, when A is a cosemisimple Hopf algebra, the forgetful functor $\mathscr{M}^A \to \mathrm{Vec}_k$: this is the content of Proposition 18 in the next section.

Our motivation to introduce the present notion of twisted separable functor is the following result.

Proposition 14. *Let* $\mathscr{C}$ *and* $\mathscr{D}$ *be abelian categories having enough projective objects, and let* $F : \mathscr{C} \to \mathscr{D}$ *be a functor. Assume that the following conditions hold:*

(1) *the functor* F *is exact and preserves projective objects;*
(2) *the functor* F *is twisted separable and* $\mathscr{F}$*, the corresponding class of objects of* $\mathscr{C}$*, contains a generating subclass* $\mathscr{F}_0$ *consisting of projective objects.*

Then, for any object V *of* $\mathscr{C}$ *such that* $\mathrm{pd}_{\mathscr{C}}(V)$ *is finite, we have* $\mathrm{pd}_{\mathscr{C}}(V) = \mathrm{pd}_{\mathscr{D}}(F(V))$.

We begin with some preliminaries.

Lemma 15. *Let* $\mathscr{C}$ *be an abelian category and let* $\mathscr{F}_0$ *be a generating subclass of* $\mathscr{C}$ *consisting of projective objects. If* $\mathrm{pd}_{\mathscr{C}}(V)$ *is finite, we have*

$$\mathrm{pd}_{\mathscr{C}}(V) = \max\{n : \mathrm{Ext}^n_{\mathscr{C}}(V,F) \neq 0 \textit{ for some object } F \textit{ in } \mathscr{F}_0\}$$

Proof. Every object X fits into an exact sequence $0 \to W \to F \to X \to 0$ with F an object of $\mathscr{F}_0$, hence projective. The result is thus obtained via a classical argument: if $n = \mathrm{pd}_{\mathscr{C}}(V)$, the long Ext exact sequence gives that the functor $\mathrm{Ext}^n_{\mathscr{C}}(V,-)$ is right exact, and hence $\mathrm{Ext}^n_{\mathscr{C}}(V,F) \neq 0$ for some object F of $\mathscr{F}_0$. □

Lemma 16. *Assume we are in the setting of Proposition 14. For any objects* X, W *of* $\mathscr{C}$*, we have a morphism*

$$\mathrm{Ext}^*_{\mathscr{D}}(F(X), \Theta F(W)) \longrightarrow \mathrm{Ext}^*_{\mathscr{C}}(X,W)$$

which is surjective if W *is an object of* $\mathscr{F}$.

Proof. Start with a projective resolution

$$\cdots \longrightarrow P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} X \to 0$$

of X by objects in $\mathscr{C}$. Since the functor F is exact and preserves projectives, we get a projective resolution

$$\cdots \longrightarrow F(P_n) \xrightarrow{F(d_n)} F(P_{n-1}) \xrightarrow{F(d_{n-1})} \cdots \xrightarrow{F(d_2)} F(P_1) \xrightarrow{F(d_1)} F(P_0) \xrightarrow{F(d_0)} F(X) \to 0$$

of $F(X)$ in $\mathscr{D}$. For all $i \geq 0$, we have, by the naturality assumption, commutative diagrams

$$\begin{tikzcd} \mathrm{Hom}_{\mathscr{D}}(F(P_i), \Theta F(W)) \arrow[r, "-\circ F(d_{i+1})"] \arrow[d, "\mathbf{M}_{P_i,W}"] & \mathrm{Hom}_{\mathscr{D}}(F(P_{i+1}), \Theta F(W)) \arrow[d, "\mathbf{M}_{P_{i+1},W}"] \\ \mathrm{Hom}_{\mathscr{C}}(P_i, W) \arrow[r, "-\circ d_{i+1}"] & \mathrm{Hom}_{\mathscr{C}}(P_{i+1}, W) \end{tikzcd}$$

that induce a morphism of complexes

$$\widetilde{\mathbf{M}} : \mathrm{Hom}_{\mathscr{D}}(F(P_*), \Theta F(W)) \to \mathrm{Hom}_{\mathscr{C}}(P_*, W)$$

and hence a morphism between the corresponding cohomologies:

$$H^*(\widetilde{\mathbf{M}}) : \mathrm{Ext}^*_{\mathscr{D}}(F(X), \Theta F(W)) \longrightarrow \mathrm{Ext}^*_{\mathscr{C}}(X, W)$$

Assume now that W is an object of $\mathscr{F}$, and let $f \in \mathrm{Hom}_{\mathscr{C}}(P_i, W)$. We have

$$\mathbf{M}_{P_i,W}(\theta_W \circ F(f)) = M_{W,W}(\theta_W) \circ f = f$$

and if moreover $f \circ d_{i+1} = 0$, we have also $\theta_W \circ F(f) \circ F(d_{i+1}) = 0$. This shows that $H^*(\widetilde{\mathbf{M}})$ is surjective. □

Remark 17. Assume, as the setting of Proposition 14 allows us to, that in the proof of the previous lemma, we have started with a projective resolution

$$\cdots \longrightarrow P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} X \to 0$$

of X by objects in $\mathscr{F}$. Then, for $f \in \mathrm{Hom}_{\mathscr{C}}(P_i, W)$, we have

$$\mathbf{M}_{P_i,W}(\Theta(F(f)) \circ \theta_{P_i}) = f \circ M_{P_i,P_i}(\theta_{P_i}) = f$$

This shows that the morphism of complexes $\widetilde{\mathbf{M}} : \mathrm{Hom}_{\mathscr{D}}(F(P_*), \Theta F(W)) \to \mathrm{Hom}_{\mathscr{C}}(P_*, W)$ in the above proof is surjective in general. However, since we see no reason that $\Theta F(f) \circ \theta_{P_i} \circ F(d_{i+1}) = 0$, we cannot conclude that the corresponding morphism in cohomology is surjective without our assumption on W.

Proof of Proposition 14. Let V be an object of $\mathscr{C}$, and let

$$\cdots \longrightarrow P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} V \to 0$$

be a projective resolution of V. Since the functor F is exact and preserves projectives, we get a projective resolution

$$\cdots \longrightarrow F(P_n) \xrightarrow{F(d_n)} F(P_{n-1}) \xrightarrow{F(d_{n-1})} \cdots \xrightarrow{F(d_2)} F(P_1) \xrightarrow{F(d_1)} F(P_0) \xrightarrow{F(d_0)} F(V) \to 0$$

of $F(V)$ in $\mathscr{D}$. This shows that $\mathrm{pd}_{\mathscr{D}}(F(V)) \leq \mathrm{pd}_{\mathscr{C}}(V)$. To prove the converse inequality, we can assume that $n = \mathrm{pd}_{\mathscr{D}}(F(V))$ is finite. We then have in particular $\mathrm{Ext}^{n+1}_{\mathscr{D}}(F(V), \Theta F(P)) = \{0\}$ for any object P in $\mathscr{F}$, and by Lemma 16, we have $\mathrm{Ext}^{n+1}_{\mathscr{C}}(V, P) = \{0\}$ as well. Hence, assuming that $\mathrm{pd}_{\mathscr{C}}(V)$ is finite, Lemma 15 shows that $\mathrm{pd}_{\mathscr{C}}(V) \leq n$, concluding the proof. □

In this paper we will not develop any more theory on twisted separable functors, and will focus on applications of Proposition 14.

5. Question 1 in the cosemisimple case

Recall that a Hopf algebra is cosemisimple if and only if it admits a Haar integral, i.e. a linear map $h : A \to k$ such that for any $a \in A$, we have

$$h(a_{(1)})a_{(2)} = h(a) = h(a_{(2)})a_{(1)} \quad \text{and} \quad h(1) = 1$$

The proof of the semisimplicity of the category of comodules from the existence of a Haar integral is a consequence of the following classical averaging construction, which shows that the forgetful functor $\mathscr{M}^A \to \mathrm{Vec}_k$ is separable, and that we record for future use.

Proposition 18. *Let V, W be right A-comodules over a cosemisimple Hopf algebra A, and let $f : V \to W$ be a linear map. The map*

$$\begin{aligned}\mathbf{M}_{V,W}(f) : V &\longrightarrow W\\ v &\longmapsto h\big(f(v_{(0)})_{(1)}S(v_{(1)})\big)\, f(v_{(0)})_{(0)}\end{aligned}$$

is a morphism of comodules, with $M(f) = f$ if and only if f is a morphism of comodules and with, for any morphisms of comodules $\alpha : V' \to V$ and $\beta : W \to W'$, $\beta \circ \mathbf{M}_{V,W}(f) \circ \alpha = \mathbf{M}_{V',W'}(\beta \circ f \circ \alpha)$. The above construction therefore defines a projection

$$\mathbf{M}_{V,W} : \mathrm{Hom}(V,W) \to \mathrm{Hom}^A(V,W)$$

that we call the averaging with respect to V and W.

The Haar integral is not a trace in general, but satisfies a KMS type property, discovered by Woronowicz [49] in the setting of compact quantum groups.

Theorem 19. *Let A be a cosemisimple Hopf algebra with Haar integral h. There exists a convolution invertible linear map $\psi : A \to k$, called a modular functional on A, satisfying the following conditions:*

- $S^2 = \psi * \mathrm{id} * \psi^{-1}$;
- $\sigma := \psi * \mathrm{id} * \psi$ *is an algebra automorphism of A;*
- *we have $h(ab) = h(b\sigma(a))$ for any $a, b \in A$.*

The proof relies on the orthogonality relations, whose first occurrence is due to Larson [25], and were completed by Woronowicz [49], see [24, Proposition 11.15] for the statement. In all the treatments we are aware of [24, 35], the setting is over the field of complex numbers, but inspecting the proof shows that the orthogonality relations are valid for any cosemisimple Hopf algebra over any algebraically closed field, and then proving the modularity property is just as in [35, Corollary 5.10].

We now present our key averaging lemma for bimodules. If R is an A-comodule algebra over a cosemisimple Hopf algebra A, we denote by ρ the algebra automorphism of R defined by $\rho = \mathrm{id} * \psi^{-2}$, i.e. $\rho(x) = \psi^{-2}(x_{(1)})x_{(0)}$, with ψ a modular functional as in Theorem 19.

Lemma 20. *Let A be a cosemisimple Hopf algebra and let R be a right A-comodule algebra. Let V, W be objects of ${}_R\mathcal{M}_R^A$. If $f : V \to W$ is a linear map satisfying*

$$f(v \cdot x) = f(v) \cdot x \quad \text{and} \quad f(x \cdot v) = \rho(x) \cdot f(v)$$

for any $v \in V$ and $x \in R$, then $\mathbf{M}_{V,W}(f) : V \to W$ is a morphism in the category ${}_R\mathcal{M}_R^A$.

Proof. We already know that $\mathbf{M}_{V,W}(f) : V \to W$ is colinear and there remains to prove that $\mathbf{M}_{V,W}(f)$ is left and right R-linear as well. Let $v \in V$ and $x \in R$. We have, using our condition on f and the compatibility between the comodule and right module structure:

$$\begin{aligned}\mathbf{M}_{V,W}(f)(v \cdot x) &= h\big(f((v \cdot x)_{(0)})_{(1)}S((v \cdot x)_{(1)})\big)\, f((v \cdot x)_{(0)})_{(0)}\\ &= h\big(f(v_{(0)} \cdot x_{(0)})_{(1)}S(v_{(1)}x_{(1)})\big)\, f(v_{(0)} \cdot x_{(0)})_{(0)}\\ &= h\big((f(v_{(0)}) \cdot x_{(0)})_{(1)}S(v_{(1)}x_{(1)})\big)\, (f(v_{(0)}) \cdot x_{(0)})_{(0)}\\ &= h\big((f(v_{(0)})_{(1)}x_{(1)}S(v_{(1)}x_{(2)})\big)\, f(v_{(0)})_{(0)} \cdot x_{(0)}\\ &= h\big((f(v_{(0)})_{(1)}x_{(1)}S(x_{(2)})S(v_{(1)})\big)\, f(v_{(0)})_{(0)} \cdot x_{(0)}\\ &= \mathbf{M}_{V,W}(f)(v) \cdot x\end{aligned}$$

Hence f is right R-linear. We also have, using our condition on f and the compatibility between the comodule and left module structure:

$$\begin{aligned}\mathbf{M}_{V,W}(f)(x\cdot v) &= h\big(f((x\cdot v)_{(0)})_{(1)}S((x\cdot v)_{(1)})\big)\, f((x\cdot v)_{(0)})_{(0)}\\ &= h\big((f(x_{(0)}\cdot v_{(0)}))_{(1)}S(x_{(1)}v_{(1)})\big)\, f(x_{(0)}\cdot v_{(0)})_{(0)}\\ &= \psi^{-2}(x_{(1)})h\big((x_{(0)}\cdot f(v_{(0)}))_{(1)}S(x_{(2)}v_{(1)})\big)\,(x_{(0)}\cdot f(v_{(0)})_{(0)}\\ &= \psi^{-2}(x_{(2)})h\big(x_{(1)}f(v_{(0)})_{(1)}S(x_{(3)}v_{(1)}))\big)\, x_{(0)}\cdot f(v_{(0)})_{(0)}\end{aligned}$$

Using the properties of the modular functional, this gives:

$$\begin{aligned}\mathbf{M}_{V,W}(f)(x\cdot v) &= \psi^{-2}(x_{(4)})h\big(f(v_{(0)})_{(1)}S(v_{(1)})S(x_{(5)})\psi(x_{(1)})x_{(2)}\psi(x_{(3)})\big)\, x_{(0)}\cdot f(v_{(0)})_{(0)}\\ &= h\big(f(v_{(0)})_{(1)}S(v_{(1)})S(x_{(4)})\psi(x_{(1)})x_{(2)}\psi^{-1}(x_{(3)})\big)\, x_{(0)}\cdot f(v_{(0)})_{(0)}\\ &= h\big(f(v_{(0)})_{(1)}v_{(1)}S(x_{(2)})S^2(x_{(1)})\big)\, x_{(0)}\cdot f(v_{(0)})_{(0)}\\ &= x\cdot\mathbf{M}_{V,W}(f)(v)\end{aligned}$$

and this shows that $\mathbf{M}_{V,W}(f)$ is left R-linear as well. □

Lemma 21. *Let V be a comodule over a cosemisimple Hopf algebra A, let R be an A-comodule algebra and consider the map $\rho_V=\rho\otimes\mathrm{id}_V\otimes\mathrm{id}_R : R\otimes V\otimes R\to R\otimes V\otimes R$. We have $\mathbf{M}(\rho_V)=\mathrm{id}_{R\otimes V\otimes R}$, where $\mathbf{M}$ stands for averaging with respect to $R\otimes V\otimes R$.*

Proof. It is immediate that $\rho_V=\rho\otimes\mathrm{id}_V\otimes\mathrm{id}_R : R\otimes V\otimes R\to R\otimes V\otimes R$ satisfies the assumption of Lemma 20, hence $\mathbf{M}(\rho_V)$ is left and right R-linear. Since it is clear that $\mathbf{M}(\rho_V)(1\otimes v\otimes 1)=1\otimes v\otimes 1$ for any $v\in V$, we get the result by the R-bilinearity of $\mathbf{M}(\rho_V)$. □

We now have all the ingredients to prove the following result.

Proposition 22. *Let A be a cosemisimple Hopf algebra and let R be a right A-comodule algebra. The forgetful functor $\Omega_R : {}_R\mathcal{M}_R^A\to {}_R\mathcal{M}_R$ is twisted separable, and we have $\mathrm{pd}_{{}_R\mathcal{M}_R^A}(V)=\mathrm{pd}_{{}_R\mathcal{M}_R}(V)$ for any object V in ${}_R\mathcal{M}_R^A$ such that $\mathrm{pd}_{{}_R\mathcal{M}_R^A}(V)$ is finite. In particular, if $\mathrm{pd}_{{}_R\mathcal{M}_R^A}(R)$ is finite, we have $\mathrm{pd}_{{}_R\mathcal{M}_R^A}(R)=\mathrm{pd}_{{}_R\mathcal{M}_R}(R)=\mathrm{cd}(R)$.*

Proof. In order to show that the forgetful functor $\Omega_R : {}_R\mathcal{M}_R^A\to {}_R\mathcal{M}_R$ is twisted separable, consider

1. the class $\mathscr{F}=\mathscr{F}_0$ of free bimodules in ${}_R\mathcal{M}_R^A$;
2. the autoequivalence Θ of the category ${}_R\mathcal{M}_R$ that associates to an R-bimodule W the R-bimodule ${}_\rho W$ having W as underlying vector space and R-bimodule structure given by $x\cdot' w\cdot' y=\rho(x)\cdot w\cdot y$, and is trivial on morphisms;
3. for a free object $R\otimes V\otimes R$, the R-bimodule isomorphism $\rho_V : R\otimes V\otimes R\to {}_\rho(R\otimes V\otimes R)$ in Lemma 21;
4. for objects V,W in ${}_R\mathcal{M}_R^A$, the averaging map
$$\mathbf{M}_{V,W} : \mathrm{Hom}_{{}_R\mathcal{M}_R}(V,{}_\rho W)\to\mathrm{Hom}_{{}_R\mathcal{M}_R^A}(V,W)$$
from Lemma 20.

It follows from Lemma 20, Lemma 21 and Proposition 18 that the functor $\Omega_R : {}_R\mathcal{M}_R^A\to {}_R\mathcal{M}_R$ is indeed twisted separable. Moreover, as already said, the class $\mathscr{F}$ of free objects consists of projective objects, the projective objects in ${}_R\mathcal{M}_R^A$ are direct summands of free objects and hence are preserved by Ω_R, which is clearly exact. Hence we are in the situation of Proposition 14, and we obtain the equality of projective dimensions. □

We obtain the following partial answer to Question 2.

Theorem 23. *Let A be Hopf algebra and let R be a left or right A-Galois object. If A is cosemisimple and* $\mathrm{cd}(A)$ *is finite, we have* $\mathrm{cd}(A) = \mathrm{cd}(R)$

Proof. The result is obtained by combining Proposition 12 and Proposition 22. □

We now obtain our partial answer to Question 1 in the cosemisimple case. The proof is similar to that of Theorem 8.

Theorem 24. *Let A, B be Hopf algebras that have equivalent linear tensor categories of comodules: $\mathcal{M}^A \simeq^{\otimes} \mathcal{M}^B$. If A and B are cosemisimple and* $\mathrm{cd}(A)$, $\mathrm{cd}(B)$ *are finite, we have* $\mathrm{cd}(A) = \mathrm{cd}(B)$.

We finish the section by noticing that Proposition 22 can be strengthened in the case $S^4 = \mathrm{id}$.

Proposition 25. *Let A be a cosemisimple Hopf algebra with* $S^4 = \mathrm{id}$, *and let R be a right A-comodule algebra.*

1. *The forgetful functor $\Omega_R : {}_R\mathcal{M}_R^A \to {}_R\mathcal{M}_R$ is separable. We thus have* $\mathrm{pd}_{{}_R\mathcal{M}_R^A}(V) = \mathrm{pd}_{{}_R\mathcal{M}_R}(V)$ *for any object V in ${}_R\mathcal{M}_R^A$, and* $\mathrm{pd}_{{}_R\mathcal{M}_R^A}(R) = \mathrm{cd}(R)$.
2. *Let $F : \mathcal{M}^A \simeq^{\otimes} \mathcal{M}^B$ be a monoidal equivalence with B satisfying* $S^4 = \mathrm{id}$ *as well. We then have, for the B-comodule algebra $T = F(R)$,* $\mathrm{cd}(R) = \mathrm{cd}(T)$.
3. *Let $F : \mathcal{M}^A \to \mathrm{Vec}_k$ be a fibre functor. If* $\mathrm{cd}(R)$ *is finite, we have, for the algebra $T = F(R)$,* $\mathrm{cd}(R) = \mathrm{cd}(T)$.

Proof. As in the proof of Lemma 20, using the properties of the modular functional,we see that for any $a, x \in A$

$$h(S(a_{(1)})xa_{(2)}) = \psi^{-2}(a_{(2)})h\left(xa_{(3)}S^{-1}(a_{(1)})\right)$$

At $x = 1$ this gives $\varepsilon(a) = \psi^{-2}(a_{(2)})h\left(a_{(3)}S^{-1}(a_{(1)})\right)$. If $S^4 = \mathrm{id}$, then ψ^{-2} convolution commutes with the identity, hence we get $\psi^{-2} = \varepsilon$. Hence the automorphism ρ associated to an A-comodule algebra R is the identity, the autoequivalence Θ in the proof of Proposition 22 is the identity, and the class $\mathscr{F}$ is the class of all objects, and it follows that $\Omega_R : {}_R\mathcal{M}_R^A \to {}_R\mathcal{M}_R$ is separable. The result about projective dimensions is then either well-known or follows from the obvious improvement of Proposition 22 in the separable case, having in mind that the conclusion of Lemma 16 now holds for any object.

A monoidal equivalence $F : \mathcal{M}^A \simeq^{\otimes} \mathcal{M}^B$ induces, as before, an equivalence between the bimodule categories ${}_R\mathcal{M}_R^A$ and ${}_T\mathcal{M}_T^B$ for $T = F(R)$, sending R to T, and then the assumption $S^4 = \mathrm{id}$ on A and B ensures that $\mathrm{cd}(R) = \mathrm{pd}_{{}_R\mathcal{M}_R^A}(R) = \mathrm{pd}_{{}_T\mathcal{M}_T^B}(T) = \mathrm{cd}(T)$.

Start now with a fibre functor $F : \mathcal{M}^A \to \mathrm{Vec}_k$, i.e. a k-linear monoidal exact faithful functor that commutes with colimits. Such a functor induces, by Tannaka–Krein duality (see e.g. [19,23]) or by the results in [40], a monoidal equivalence $\mathcal{M}^A \simeq^{\otimes} \mathcal{M}^B$ for some Hopf algebra B, with as well a monoidal equivalence ${}_R\mathcal{M}_R^A \simeq^{\otimes} {}_T\mathcal{M}_T^B$. The assumption that $S^4 = \mathrm{id}$ for A then gives $\mathrm{pd}_{{}_R\mathcal{M}_R^A}(R) = \mathrm{cd}(R)$. Since $\mathrm{pd}_{{}_R\mathcal{M}_R^A}(R) = \mathrm{pd}_{{}_T\mathcal{M}_T^B}(T)$, Proposition 22 ensures, under the assumption that $\mathrm{cd}(R)$ is finite, that $\mathrm{pd}_{{}_T\mathcal{M}_T^B}(T) = \mathrm{cd}(T)$, and thus this gives the expected result. □

Example 26. Let $\sigma : A \otimes A \to k$ be (Hopf, right) 2-cocycle on a Hopf algebra A (see [31]), i.e. σ is a convolution invertible linear map $\sigma : A \otimes A \to k$ satisfying, for any $a, b, c \in A$

$$\sigma(a,1) = \varepsilon(a)1 = \sigma(1,a),\ \sigma(a_{(2)}, b_{(2)})\sigma(a_{(1)}b_{(1)}, c) = \sigma(a, b_{(1)}c_{(1)})\sigma(b_{(2)}, c_{(2)})$$

If R is a right A-comodule algebra, we obtain a new (associative) algebra R_σ by letting

$$x.y = \sigma(x_{(1)}, y_{(1)})x_{(0)}y_{(0)}$$

We then have, if A is cosemisimple with $S^4 = \mathrm{id}$, $\mathrm{cd}(R) = \mathrm{cd}(R_\sigma)$ if $\mathrm{cd}(R)$ is finite.

Proof. The algebra R_σ is the image of R under the fibre functor $\mathcal{M}^A \to \mathrm{Vec}_k$ which has the forgetful functor as underlying functor and monoidal constraint $V \otimes W \to V \otimes W$, $v \otimes w \mapsto \sigma(v_{(1)}, w_{(1)})v_{(0)} \otimes w_{(0)}$. The result is thus a consequence of Proposition 25. □

Remark 27. If the Hopf algebra A^σ (see [17, 40]) satisfies as well $S^4 = \mathrm{id}$, we can conclude that $\mathrm{cd}(R) = \mathrm{cd}(R_\sigma)$ without the finiteness assumption. This applies in particular, when A is a group algebra, to the 2-cocycle twisting of a group-graded algebra, which therefore has the same Hochschild cohomological dimension as the original algebra. This was probably well-known, but we are not aware of an explicit reference for this fact.

6. Yetter—Drinfeld modules over cosemisimple Hopf algebras

Our aim in this section is to compare the cohomological dimension and the Gerstenhaber–Schack cohomological dimension of a cosemisimple Hopf algebra, providing in this way a version of Theorem 24 that looks slightly weaker, but that is probably more useful in concrete situations (Corollary 32).

Recall that a (right-right) Yetter–Drinfeld module over a Hopf algebra A is a right A-comodule and right A-module V satisfying the condition, $\forall\, v \in V$, $\forall\, a \in A$,

$$(v\cdot a)_{(0)} \otimes (v\cdot a)_{(1)} = v_{(0)}\cdot a_{(2)} \otimes S(a_{(1)})v_{(1)}a_{(3)}$$

The category of Yetter–Drinfeld modules over A is denoted $\mathscr{YD}_A^A$: the morphisms are the A-linear and A-colinear maps. The category $\mathscr{YD}_A^A$ is obviously abelian, and, endowed with the usual tensor product of modules and comodules, is a tensor category, with unit the trivial Yetter–Drinfeld module, denoted k.

The forgetful functor $\Omega^A : \mathscr{YD}_A^A \to \mathscr{M}^A$ has a left adjoint [11], the free Yetter–Drinfeld module functor, which sends a comodule V to the Yetter–Drinfeld module $V \boxtimes A$, which as a vector space is $V\otimes A$, has the right module structure given by multiplication on the right, and right coaction given by

$$(v\otimes a)_{(0)} \otimes (v\otimes a)_{(1)} = v_{(0)} \otimes a_{(2)} \otimes S(a_{(1)})v_{(1)}a_{(3)}$$

A Yetter–Drinfeld module isomorphic to some $V\boxtimes A$ as above is said to be free. Let us record the following facts, that are straightforward consequences of standard properties of pairs of adjoint functors.

(1) Every Yetter–Drinfeld module is a quotient of a free Yetter–Drinfeld module. Indeed, for a Yetter–Drinfeld V, the A-module structure of V induces a surjective morphism $\Omega^A(V)\boxtimes A \to V$.
(2) If the category $\mathscr{M}^A$ has enough projective objects, then so has $\mathscr{YD}_A^A$.
(3) If A is cosemisimple, then $\mathscr{YD}_A^A$ has enough projective objects, and the projective objects are precisely the direct summands of the free Yetter–Drinfeld modules.

Similarly, the forgetful functor $\Omega_A : \mathscr{YD}_A^A \to \mathscr{M}_A$ has a right adjoint [11], the cofree Yetter–Drinfeld module functor, which sends a module V to the Yetter–Drinfeld module $V\#A$, which as a vector space is $V\otimes A$, has the right comodule structure given by the comultiplication of A on the right, and right A-module structure given by

$$(v\otimes a)\cdot b = v\cdot b_{(2)} \otimes S(b_{(1)})ab_{(3)}$$

Again, as a consequence of general properties of adjoint functors, it follows that the category $\mathscr{YD}_A^A$ has enough injective objects, since $\mathscr{M}_A$ has.

Recall that we have defined the Gerstenhaber–Schack cohomological dimension of a Hopf algebra A by

$$\mathrm{cd}_{\mathrm{GS}}(A) = \max\left\{n:\ \mathrm{Ext}^n_{\mathscr{YD}_A^A}(k,V)\neq 0 \text{ for some } V\in\mathscr{YD}_A^A\right\}\in\mathbb{N}\cup\{\infty\}$$

The name comes from the fact, proved in [45], that if V is a Yetter–Drinfeld over A, then $\mathrm{Ext}^*_{\mathscr{YD}_A^A}(k,V)$ is isomorphic with $H^*_{\mathrm{GS}}(A,V)$, the Gerstenhaber–Schack cohomology of A with coefficients in V [21, 43].

Notice that since $\mathscr{YD}_A^A$ has enough injective objects, the above Ext can be computed using injective resolutions of V, and if $\mathscr{YD}_A^A$ has enough projective objects, using projective resolutions of k in $\mathscr{YD}_A^A$. Another consequence of general properties of pairs of adjoint functors is that we have, for any Yetter–Drinfeld module V and any A-module W, natural isomorphisms

$$\mathrm{Ext}_A^*(\Omega_A(V), W) \simeq \mathrm{Ext}^*_{\mathscr{YD}_A^A}(V, W\# A)$$

This is what proves that $\mathrm{cd}(A) \leq \mathrm{cd}_{\mathrm{GS}}(A)$ [7].

We now present an averaging lemma for Yetter–Drinfeld modules over cosemisimple Hopf algebras, in the same spirit as Lemma 20, which will be the key tool towards the proof of Theorem 31. If A is a cosemisimple Hopf algebra with modular functional ψ, we denote by θ the algebra automorphism of A defined by $\theta = \psi^2 * \mathrm{id}$.

Lemma 28. *Let V, W be Yetter–Drinfeld modules over a cosemisimple Hopf algebra A. If $f : V \to W$ is a linear map satisfying $f(v\cdot a) = f(v)\cdot\theta(a)$ for any $v \in V$ and $a \in A$, then $\mathbf{M}_{V,W}(f) : V \to W$ is a morphism of Yetter–Drinfeld modules.*

Proof. We already know that $\mathbf{M}_{V,W}(f) : V \to W$ is colinear and there remains to prove that $\mathbf{M}_{V,W}(f)$ is A-linear as well. Let $v \in V$ and $a \in A$. We have, using our condition on f and the Yetter–Drinfeld property:

$$\begin{aligned}
\mathbf{M}_{V,W}(f)(v\cdot a) &= h\big(f((v\cdot a)_{(0)})_{(1)}S((v\cdot a)_{(1)})\big)\, f((v\cdot a)_{(0)})_{(0)}\\
&= h\big(f(v_{(0)}\cdot a_{(2)})_{(1)}S(S(a_{(1)})v_{(1)}a_{(3)})\big)\, f(v_{(0)}\cdot a_{(2)})_{(0)}\\
&= h\big((f(v_{(0)})\cdot\theta(a_{(2)}))_{(1)}S(S(a_{(1)})v_{(1)}a_{(3)})\big)\,(f(v_{(0)})\cdot\theta(a_{(2)}))_{(0)}\\
&= \psi^2(a_{(2)})h\big((f(v_{(0)})\cdot a_{(3)})_{(1)}S(S(a_{(1)})v_{(1)}a_{(4)})\big)\,(f(v_{(0)})\cdot a_{(3)})_{(0)}\\
&= \psi^2(a_{(2)})h\big(S(a_{(3)})f(v_{(0)})_{(1)}a_{(5)}S(S(a_{(1)})v_{(1)}a_{(6)})\big)\, f(v_{(0)})_{(0)}\cdot a_{(4)}\\
&= \psi^2(a_{(2)})h\big(S(a_{(3)})f(v_{(0)})_{(1)}a_{(5)}S(a_{(6)})S(v_{(1)})S^2(a_{(1)})\big)\, f(v_{(0)})_{(0)}\cdot a_{(4)}\\
&= \psi^2(a_{(2)})h\big(S(a_{(3)})f(v_{(0)})_{(1)}S(v_{(1)})S^2(a_{(1)})\big)\, f(v_{(0)})_{(0)}\cdot a_{(4)}
\end{aligned}$$

Using the properties of the modular functional, and since $\sigma\circ S = \sigma^{-1} = \psi^{-1} * S * \psi^{-1}$ because σ is an algebra map, this gives:

$$\begin{aligned}
\mathbf{M}_{V,W}(f)(v\cdot a) &= \psi^2(a_{(2)})h\big(f(v_{(0)})_{(1)}S(v_{(1)})S^2(a_{(1)})\sigma(S(a_{(3)}))\big)\, f(v_{(0)})_{(0)}\cdot a_{(4)}\\
&= \psi^2(a_{(2)})h\big(f(v_{(0)})_{(1)}S(v_{(1)})S^2(a_{(1)})\psi^{-1}(a_{(3)})S(a_{(4)})\psi^{-1}(a_5)\big)\, f(v_{(0)})_{(0)}\cdot a_{(6)}\\
&= h\big(f(v_{(0)})_{(1)}S(v_{(1)})S^2(a_{(1)})\psi(a_{(2)})S(a_{(3)})\psi^{-1}(a_4))\big)\, f(v_{(0)})_{(0)}\cdot a_{(5)}\\
&= h\big(f(v_{(0)})_{(1)}S(v_{(1)})S^2(a_{(1)})S^3(a_2)\big)\, f(v_{(0)})_{(0)}\cdot a_{(3)}\\
&= h\big(f(v_{(0)})_{(1)}S(v_{(1)})\big)\, f(v_{(0)})_{(0)}\cdot a = \mathbf{M}_{V,W}(f)(v)\cdot a
\end{aligned}$$

and this shows that $\mathbf{M}_{V,W}(f)$ is A-linear. □

Lemma 29. *Let V be a right comodule over the cosemisimple Hopf algebra A, and consider the linear map $\theta_V = \mathrm{id}_V\otimes\theta : V\boxtimes A \to V\boxtimes A$. We have $\mathbf{M}(\theta_V) = \mathrm{id}_{V\boxtimes A}$, where $\mathbf{M}(\theta_V)$ stands for $\mathbf{M}_{V\boxtimes A, V\boxtimes A}(\theta_V)$.*

Proof. It is immediate that $\mathrm{id}_V\otimes\theta : V\boxtimes A \to V\boxtimes A$ satisfies the assumption of Lemma 28, hence $\mathbf{M}(\mathrm{id}_V\otimes\theta)$ is A-linear. Since it is clear that $\mathbf{M}(\mathrm{id}_V\otimes\theta)(v\otimes 1) = v\otimes 1$ for any $v\in V$, we get the result by the A-linearity of $\mathbf{M}(\mathrm{id}_V\otimes\theta)$. □

We now have all the ingredients to prove the following result.

Proposition 30. *Let A be a cosemisimple Hopf algebra. The forgetful functor $\Omega_A : \mathscr{YD}_A^A \to \mathscr{M}_A$ is twisted separable, and we have $\mathrm{pd}_{\mathscr{YD}_A^A}(V) = \mathrm{pd}_A(V)$ for any Yetter–Drinfeld module V such that $\mathrm{pd}_{\mathscr{YD}_A^A}(V)$ is finite.*

Proof. In order to show that the forgetful functor $\Omega_A : \mathscr{YD}_A^A \to \mathscr{M}_A$ is twisted separable, consider

(1) the class $\mathscr{F} = \mathscr{F}_0$ of free Yetter–Drinfeld modules;
(2) the autoequivalence Θ of the category $\mathscr{M}_A$ that associates to a right A-module W the A-module W_θ having W as underlying vector space and A-module structure given by $w \cdot' a = w \cdot \theta(a)$, and is trivial on morphisms;
(3) for a free Yetter–Drinfeld module $V \boxtimes A$, the A-module isomorphism $\theta_V : V \otimes A \to (V \otimes A)_\theta$ in Lemma 29.
(4) for Yetter–Drinfeld modules V, W, the averaging map
$$\mathbf{M}_{V,W} : \mathrm{Hom}_A(V, W_\theta) \to \mathrm{Hom}_{\mathscr{YD}_A^A}(V, W)$$
from Lemma 28.

It follows from Lemma 28, Lemma 29 and Proposition 18 that the functor $\Omega_A : \mathscr{YD}_A^A \to \mathscr{M}_A$ is indeed twisted separable. Moreover, as already said, the class $\mathscr{F}$ of free Yetter–Drinfeld modules consists of projective objects, the projective objects in $\mathscr{YD}_A^A$ are direct summands of free objects and hence are preserved by Ω_A, which is exact. Hence we are in the situation of Proposition 14, and we obtain the equality of projective dimensions. □

We thus obtain the main result in the section.

Theorem 31. *Let A be a cosemisimple Hopf algebra. If $\mathrm{cd}_{\mathrm{GS}}(A)$ is finite, we have $\mathrm{cd}(A) = \mathrm{cd}_{\mathrm{GS}}(A)$.*

Proof. Let A be a cosemisimple Hopf algebra. Since $\mathrm{cd}_{\mathrm{GS}}(A) = \mathrm{pd}_{\mathscr{YD}_A^A}(k)$ and $\mathrm{cd}(A) = \mathrm{pd}_A(k_\varepsilon)$, we have $\mathrm{cd}(A) = \mathrm{cd}_{\mathrm{GS}}(A)$ if $\mathrm{cd}_{\mathrm{GS}}(A)$ is finite, by Proposition 30. □

We get the following weak form of Theorem 24, whose formulation is useful.

Corollary 32. *Let A, B be Hopf algebras such that $\mathscr{M}^A \simeq^{\otimes} \mathscr{M}^B$. If A and B are cosemisimple and $\mathrm{cd}_{\mathrm{GS}}(A)$ is finite, we have $\mathrm{cd}(A) = \mathrm{cd}(B)$.*

Proof. We have $\mathrm{cd}_{\mathrm{GS}}(A) = \mathrm{cd}_{\mathrm{GS}}(B)$, hence $\mathrm{cd}(A) = \mathrm{cd}(B)$ by Theorem 31. □

As in Section 3, Proposition 30 can be strengthened when $S^4 = \mathrm{id}$.

Theorem 33. *Let A be Hopf algebra. The forgetful functor $\Omega_A : \mathscr{YD}_A^A \to \mathscr{M}_A$ is separable if and only if A is cosemisimple and $S^4 = \mathrm{id}$, and in that case we have $\mathrm{pd}_{\mathscr{YD}_A^A}(V) = \mathrm{pd}_A(V)$ for any Yetter–Drinfeld module V.*

Proof. If A is cosemisimple and $S^4 = \mathrm{id}$,we see, as in the proof of Proposition 25, that the automorphism θ of A is the identity, and that $\Omega_A : \mathscr{YD}_A^A \to \mathscr{M}_A$ is indeed separable, and the assertion on projective dimensions, which was already proved in [8, Section 6], follows similarly.

Assume now that $\Omega_A : \mathscr{YD}_A^A \to \mathscr{M}_A$ is separable. Since Ω_A admits the right adjoint $-\#A$, the characterization of separability for functors that admit a right adjoint in [38] gives in particular an A-colinear and A-linear map
$$\eta : k\#A \longrightarrow k \quad \text{with } \eta(1) = 1$$
By the A-collinearity and $\eta(1) = 1$, we have that $\eta = h$ is a Haar integral on A, which is thus cosemisimple. The A-linearity of h gives, for any $a, x \in A$,
$$h(S(a_{(1)})xa_{(2)}) = \varepsilon(a)h(x)$$
We have seen in the proof of Proposition 25 that for any $a, x \in A$,
$$h(S(a_{(1)})xa_{(2)}) = \psi^{-2}(a_{(2)})h\left(xa_{(3)}S^{-1}(a_{(1)})\right)$$
Hence we have for any $a, x \in A$
$$h\left(x(\varepsilon(a) - \psi^{-2}(a_{(2)})a_{(3)}S^{-1}(a_{(1)}))\right) = 0$$

The non-degeneracy of the Haar integral (which follows from the orthogonality relations) then gives, for any $a \in A$

$$\varepsilon(a)1 = \psi^{-2}(a_{(2)})a_{(3)}S^{-1}(a_{(1)})$$

Hence applying ε gives $\varepsilon = \psi^{-2}$, and we thus have $S^4 = \mathrm{id}$. □

We finish the section by noticing that Yetter–Drinfeld modules are also useful outside the cosemisimple case. Recall [7] that a Yetter–Drinfeld module is said to be relative projective if it is a direct summand of a free Yetter–Drinfeld module, and let us say that a Hopf algebra is *Yetter–Drinfeld smooth* if the trivial object k has a finite resolution by relative projective Yetter–Drinfeld modules that are finitely generated as modules.

Theorem 34. *Let A, B be Hopf algebras that have equivalent linear tensor categories of comodules: $\mathcal{M}^A \simeq^{\otimes} \mathcal{M}^B$. If A and B have bijective antipode and A is Yetter–Drinfeld smooth, then we have* $\mathrm{cd}(A) = \mathrm{cd}(B)$.

Proof. Clearly A is smooth since it is Yetter–Drinfeld smooth, and if we start from a resolution of k be finitely generated relative projective Yetter–Drinfeld modules in $\mathcal{YD}_A^A$, [5, Theorem 4.1] ensures that one can transport this resolution to a resolution of k to a finitely generated relative projective Yetter–Drinfeld modules in $\mathcal{YD}_B^B$. Hence B is smooth as well and Theorem 8 concludes the proof. □

7. Hopf subalgebras and cohomological dimension

Let $B \subset A$ be a Hopf subalgebra. Under the assumption of faithful flatness of A as a B-module, which holds in many situations and in particular if A is cosemisimple [13], we have $\mathrm{cd}(B) \leq \mathrm{cd}(A)$ [7, Proposition 3.1]. In this section we prove, in view of an example in the next section, an analogue inequality for Gerstenhaber–Schack cohomological dimension, in the cosemisimple case. Of course, if the conclusion of Theorem 31 was known to hold for any cosemisimple Hopf algebra, this would become trivial.

We begin with some results of independent interest. Recall [7] that a Yetter–Drinfeld module is said to be relative projective if it is a direct summand in a free one.

Proposition 35. *Let A be a Hopf algebra, let V be a Yetter–Drinfeld over A and let W be a right A-comodule. Then we have an isomorphism of Yetter–Drinfeld modules*

$$(\Omega^A(V) \otimes W) \boxtimes A \simeq V \otimes (W \boxtimes A)$$

In particular, if P is a relative projective Yetter–Drinfeld module, so is the Yetter–Drinfeld module $V \otimes P$.

Proof. The map

$$\begin{aligned}(\Omega^A(V) \otimes W) \boxtimes A &\longmapsto V \otimes (W \boxtimes A)\\ v \otimes w \otimes a &\longmapsto v \cdot a_{(1)} \otimes w \otimes a_{(2)}\end{aligned}$$

is easily seen to be a morphism of Yetter–Drinfeld modules, and its inverse is given by $v \otimes w \otimes a \mapsto v \cdot S(a_{(1)}) \otimes w \otimes a_{(2)}$. If P is relative projective, let W be a right A-comodule and Q be a Yetter–Drinfeld module such that $W \boxtimes A \simeq P \oplus Q$. We then have $(V \otimes P) \oplus (V \otimes Q) \simeq V \otimes (W \boxtimes A) \simeq (\Omega^A(V) \otimes W) \boxtimes A$, which proves that $V \otimes P$ is relative projective. □

Corollary 36. *If A is a cosemisimple Hopf algebra, we have*

$$\begin{aligned}\mathrm{cd}_{\mathrm{GS}}(A) &= \mathrm{pd}_{\mathscr{YD}_A^A}(k) = \max\left\{n : \mathrm{Ext}^n_{\mathscr{YD}_A^A}(k,V) \neq 0 \textit{ for some } V \in \mathscr{YD}_A^A\right\} \\ &= \max\left\{\mathrm{pd}_{\mathscr{YD}_A^A}(V),\ V \in \mathscr{YD}_A^A\right\} = \max\left\{n : \mathrm{Ext}^n_{\mathscr{YD}_A^A}(V,W) \neq 0 \textit{ for some } V,W \in \mathscr{YD}_A^A\right\} \\ &= \min\left\{n : \mathrm{Ext}^{n+1}_{\mathscr{YD}_A^A}(V,W) = 0 \textit{ for any } V,W \in \mathscr{YD}_A^A\right\} \\ &= \max\left\{\mathrm{injd}_{\mathscr{YD}_A^A}(V),\ V \in \mathscr{YD}_A^A\right\}\end{aligned}$$

where $\mathrm{injd}_{\mathscr{YD}_A^A}$ *is the injective dimension in the category* $\mathscr{YD}_A^A$.

Proof. The first two equalities have already been discussed. Let $P_* \to k$ be resolution of k by projective objects, of length $n = \mathrm{pd}_{\mathscr{YD}_A^A}(k)$. Since A is cosemisimple, the projective objects are the relative projectives, so if V is a Yetter–Drinfeld module, tensoring the above resolution with V yields, by Proposition 35, a length n resolution of V by projective objects. This gives the third equality, and the other ones then follow by classical arguments. □

Let $B \subset A$ be a Hopf subalgebra. Recall [8] that there is a pair of adjoint functors

$$\begin{aligned} \mathscr{YD}_A^A &\longrightarrow \mathscr{YD}_B^B \qquad & \mathscr{YD}_B^B &\longrightarrow \mathscr{YD}_A^A \\ X &\longmapsto X^{(B)} & V &\longmapsto V \boxtimes_B A \end{aligned}$$

where

1. for a Yetter–Drinfeld module X over A, $X^{(B)} = \{x \in X \mid x_{(0)} \otimes x_{(1)} \in X \otimes B\}$ has the restricted B-module structure;
2. for a Yetter–Drinfeld module V over B, $V \boxtimes_B A$ is the induced module $V \otimes_B A$, with A-comodule structure given by

$$(v \otimes_B a)_{(0)} \otimes (v \otimes_B a)_{(1)} = v_{(0)} \otimes_B a_{(2)} \otimes S(a_{(1)}) v_{(1)} a_{(3)}$$

Lemma 37. *Let $B \subset A$ be a Hopf subalgebra, and assume that A is cosemisimple. Let V be a Yetter–Drinfeld module over B. Then V is isomorphic to a direct summand of $(V \boxtimes_B A)^{(B)}$.*

Proof. It is immediate to check thatwe have a morphism of Yetter–Drinfeld modules

$$i : V \to (V \boxtimes_B A)^{(B)},\ v \mapsto v \otimes_B 1$$

Assume now that A is cosemisimple. Then, by the proof of Theorem 2.1 in [13], there exists a sub-B-bimodule $T \subset A$, which is as well a subcoalgebra, such that $A = B \oplus T$. Let $E : A \to B$ be the corresponding projection: $E(b) = b$ for $b \in B$ and $E(a) = 0$ for $a \in T$. By construction E is a B-bimodule map and a coalgebra map, and it is immediate to check that we have for any $a \in A$

$$S(E(a)_{(1)}) \otimes E(a)_{(2)} \otimes E(a)_{(3)} = S(a_{(1)}) \otimes E(a_{(2)}) \otimes a_{(3)}$$

From this, we see that the map

$$(V \boxtimes_B A)^{(B)} \to V,\ v \otimes_B a \to v \cdot E(a)$$

is a morphism of Yetter–Drinfeld modules. Since this map is clearly a retraction to i, this proves the lemma. □

We now have all the ingredients to prove the expected result.

Proposition 38. *Let $B \subset A$ be a Hopf subalgebra. If A is cosemisimple, we have* $\mathrm{cd}_{\mathrm{GS}}(B) \leq \mathrm{cd}_{\mathrm{GS}}(A)$.

Proof. We can assume that $\mathrm{cd}_{\mathrm{GS}}(A) = n$ is finite. Since A is cosemisimple, [13, Theorem 2.1] ensures that A is flat as a left B-module, and $B \subset A$ is coflat. Hence, by [8, Proposition 3.3] we have

$$\mathrm{Ext}^*_{\mathcal{YD}_A^A}(V \boxtimes_B A, X) \simeq \mathrm{Ext}^*_{\mathcal{YD}_B^B}(V, X^{(B)})$$

for any Yetter–Drinfeld module X over A, and any Yetter–Drinfeld module V over B. Hence, for $V = k$, Corollary 36 yields

$$\mathrm{Ext}^{n+1}_{\mathcal{YD}_B^B}(k, X^{(B)}) \simeq \mathrm{Ext}^{n+1}_{\mathcal{YD}_A^A}(k \boxtimes_B A, X) = \{0\}$$

for any Yetter–Drinfeld module X over A. Lemma 37 ensures that any Yetter–Drinfeld module over B is a direct summand in one of type $X^{(B)}$, so we get $\mathrm{cd}_{\mathrm{GS}}(B) \leq n$, as required. □

8. Examples

We now use the previous results to examine some examples that were not covered by the literature.

8.1. *Universal cosovereign Hopf algebras*

In this subsection we complete some of the results of [8] on the cohomological dimension of the universal cosovereign Hopf algebras. Recall that for $n \geq 2$ and $F \in \mathrm{GL}_n(k)$, the algebra $H(F)$ is the algebra generated by $(u_{ij})_{1\leq i,j\leq n}$ and $(v_{ij})_{1\leq i,j\leq n}$, with relations:

$$uv^t = v^t u = I_n; \quad vFu^tF^{-1} = Fu^tF^{-1}v = I_n,$$

where $u = (u_{ij})$, $v = (v_{ij})$ and I_n is the identity $n \times n$ matrix. The algebra $H(F)$ has a Hopf algebra structure defined by

$$\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}, \quad \Delta(v_{ij}) = \sum_k v_{ik} \otimes v_{kj},$$
$$\varepsilon(u_{ij}) = \varepsilon(v_{ij}) = \delta_{ij}, \quad S(u) = v^t, \quad S(v) = Fu^tF^{-1}.$$

We refer the reader to [4, 8] for more information and background on the universal cosovereign Hopf algebras $H(F)$.

Recall [8] that we say that a matrix $F \in \mathrm{GL}_n(k)$ is
- normalizable if $\mathrm{tr}(F) \neq 0$ and $\mathrm{tr}(F^{-1}) \neq 0$ or $\mathrm{tr}(F) = 0 = \mathrm{tr}(F^{-1})$;
- generic if it is normalizable and the solutions of the equation $q^2 - \sqrt{\mathrm{tr}(F)\,\mathrm{tr}(F^{-1})}\,q + 1 = 0$ are generic, i.e. are not roots of unity of order ≥ 3 (this property does not depend on the choice of the above square root);
- an asymmetry if there exists $E \in \mathrm{GL}_n(k)$ such that $F = E^tE^{-1}$.

Theorem 39. *Let $F \in \mathrm{GL}_n(k)$, $n \geq 2$. If F is an asymmetry or F is generic, we have* $\mathrm{cd}(H(F)) = 3$.

Proof. We know from [8, Theorem 2.1], that $\mathrm{cd}(H(F)) = 3$if F is an asymmetry and that $\mathrm{cd}_{\mathrm{GS}}(H(F) = 3$ if F is generic, in which case $H(F)$ is cosemisimple [4], so Theorem 31 gives the result in that case. □

As an illustration of Theorem 23, consider, for $E \in \mathrm{GL}_n(k)$ and $F \in \mathrm{GL}_m(k)$, $n, m \geq 2$, the algebra $H(E,F)$ presented by generators u_{ij}, v_{ij}, $1 \leq i \leq m, 1 \leq j \leq n$, and relations

$$uv^t = I_m = vFu^tE^{-1} \quad ; \quad v^tu = I_n = Fu^tE^{-1}v.$$

Theorem 40. *If E, F are generic,* $\mathrm{tr}(E) = \mathrm{tr}(F)$ *and* $\mathrm{tr}(E^{-1}) = \mathrm{tr}(F^{-1})$, *then we have* $\mathrm{cd}(H(E,F)) = 3$.

Proof. The assumption $\mathrm{tr}(E) = \mathrm{tr}(F)$ and $\mathrm{tr}(E^{-1}) = \mathrm{tr}(F^{-1})$ ensures that $H(E,F)$ is an $H(E)$-$H(F)$-bi-Galois object [4]. Hence, since the genericity assumption ensures that $H(E)$ cosemisimple and we know from the previous result that $\mathrm{cd}(H(E))$ and $\mathrm{cd}(H(F))$ are finite, the result follows from Theorem 23. □

8.2. *Free wreath products*

In this subsection we assume that the base field is $k = \mathbb{C}$, since the monoidal equivalences on which we rely [20, 27] were obtained in this framework. Before going to the general setting of Theorem 42, we feel it is probably worth to present a particular example. So for $n, p \geq 1$, consider, following the notation of [2], the algebra $A_h^p(n)$ presented by generators u_{ij}, $1 \leq i, j \leq n$, and relations

$$\sum_{j=1}^n u_{ij}^p = 1 = \sum_{j=1}^n u_{ji}^p, \quad u_{ij}u_{ik} = 0 = u_{ji}u_{ki}, \quad \text{for } k \neq j,$$

At $p = 1$, $A_h^1(n) = A_s(n)$, the coordinate algebra of Wang's quantum permutation group [46]. In general $A_h^p(n)$ is a Hopf algebra with [3]

$$\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}, \quad \varepsilon(u_{ij}) = \delta_{ij}, \quad S(u_{ij}) = u_{ji}^{p-1}$$

The following result, for which the $p = 1$ case was obtained in [7] (see [9] as well, where it is shown that $A_s(n)$ is Calabi–Yau of dimension 3), will be a particular instance of the forthcoming Theorem 42.

Theorem 41. *We have, for $p \geq 1$ and $n \geq 4$, $\mathrm{cd}(A_h^p(n)) = 3$.*

Let A be a Hopf algebra, and consider A^{*n}, the free product algebra of n copies of A, which inherits a natural Hopf algebra structure such that the canonical morphisms $\nu_i : A \to A^{*n}$ are Hopf algebras morphisms. The free wreath product $A *_w A_s(n)$ [3] is the quotient of the algebra $A^{*n} * A_s(n)$ by the two-sided ideal generated by the elements:

$$\nu_k(a)u_{ki} - u_{ki}\nu_k(a)\,, \quad 1 \leq i, k \leq n\,, \quad a \in A.$$

The free wreath product $A *_w A_s(n)$ admits a Hopf algebra structure given by

$$\Delta(u_{ij}) = \sum_{k=1}^n u_{ik} \otimes u_{kj}, \quad \Delta(\nu_i(a)) = \sum_{k=1}^n \nu_i(a_{(1)})u_{ik} \otimes \nu_k(a_{(2)}),$$

$$\varepsilon(u_{ij}) = \delta_{ij}, \quad \varepsilon(\nu_i(a)) = \varepsilon(a),\ S(u_{ij}) = u_{ji}, \quad S(\nu_i(a)) = \sum_{k=1}^n \nu_k(S(a))u_{ki}.$$

When A is a compact Hopf algebra (i.e. arises from a compact quantum, we do not need the precise definition here), the free wreath product is as well a compact Hopf algebra. In that case the monoidal categories of comodules have been described for $n \geq 4$ by Lemeux–Tarrago [27] in the case $S^2 = \mathrm{id}$ and Fima–Pittau [20] in general.

Taking A to be the group algebra $\mathbb{C}[\mathbb{Z}/p\mathbb{Z}]$, we have $A_h^p(n) \simeq \mathbb{C}[\mathbb{Z}/p\mathbb{Z}] *_w A_s(n)$ by [3, Example 2.5], hence Theorem 41 is a particular instance of the following result.

Theorem 42. *We have* $\mathrm{cd}(A *_w A_s(n)) = \max(\mathrm{cd}(A), 3)$ *for any compact Hopf algebra A such that* $\mathrm{cd}(A) = \mathrm{cd}_{\mathrm{GS}}(A)$ *and any $n \geq 4$.*

Proof. First notice that there is a Hopf algebra map $\pi : A *_w A_s(n) \to A_s(n)$ such that $\pi(u_{ij}) = u_{ij}$ and $\pi(a) = \varepsilon(a)$, hence $A_s(n)$ stands as Hopf subalgebra of $A *_w A_s(n)$. We thus have, by [7, Proposition 3.1], $3 = \mathrm{cd}(A_s(n)) \leq \mathrm{cd}(A *_w A_s(n))$. Similarly the natural map $A^{*n} \to A *_w A_s(n)$ has a retraction, and hence A^{*n} stands as left coideal $*$-subalgebra of $A *_w A_s(n)$. By the results in [14], $A *_w A_s(n)$ is thus faithfully flat as A^{*n}-module, hence projective [30]. We then have, using [8, Corollary 5.3], $\mathrm{cd}(A) = \mathrm{cd}(A^{*n}) \leq \mathrm{cd}(A *_w A_s(n))$, since restricting a resolution by projective $A *_w A_s(n)$-modules to A^{*n}-modules remains a projective resolution. Hence we have

$$\max(\mathrm{cd}(A), 3) \leq \mathrm{cd}(A *_w A_s(n))$$

The converse inequality obviously holds if $\mathrm{cd}(A)$ is infinite, hence we can assume that $\mathrm{cd}(A)$ is finite, and hence, in view of our assumption, that $\mathrm{cd}_{\mathrm{GS}}(A)$ is finite.

The results in [20, 27] ensure the existence, for q satisfying $q + q^{-1} = \sqrt{n}$, of a monoidal equivalence between the category of comodules over $A *_w A_s(n)$ and the category of comodules over a certain Hopf subalgebra H of the free product $A * \mathcal{O}(SU_q(2))$. We have, combining Proposition 38 and [8, Corollary 5.10]

$$\mathrm{cd}_{\mathrm{GS}}(H) \leq \mathrm{cd}_{\mathrm{GS}}(A * \mathcal{O}(SU_q(2))) = \max(\mathrm{cd}_{\mathrm{GS}}(A), \mathrm{cd}_{\mathrm{GS}}(\mathcal{O}(SU_q(2)))$$

Since $\mathrm{cd}_{\mathrm{GS}}(\mathcal{O}(SU_q(2)) = 3$ by [5, 7], we get $\mathrm{cd}_{\mathrm{GS}}(H) \leq \max(\mathrm{cd}_{\mathrm{GS}}(A), 3)$, and since we assume that $\mathrm{cd}_{\mathrm{GS}}(A)$ is finite, we get that $\mathrm{cd}_{\mathrm{GS}}(H)$ is finite. Hence by Corollary 32 and Theorem 31, we get

$$\mathrm{cd}(A *_w A_s(n)) = \mathrm{cd}(H) = \mathrm{cd}_{\mathrm{GS}}(H) \leq \max(\mathrm{cd}_{\mathrm{GS}}(A), 3) = \max(\mathrm{cd}(A), 3)$$

which concludes the proof. □

Remark 43. At $n = 2$, using the simple description of the free wreath product as a crossed coproduct in [3], it is not difficult to show directly that $\mathrm{cd}(A *_w A_s(2)) = \max(\mathrm{cd}(A), 1)$ if A is non trivial.

Remark 44. Fima–Pittau [20] define more generally a free wreath product $A *_w A_{\mathrm{aut}}(R, \varphi)$, for suitable pairs (R, φ) consisting of a finite-dimensional C^*-algebra and a faithful state, and prove a similar monoidal equivalence result, so that Theorem 42 should generalize to this setting.

9. Question 1 in the finite-dimensional case

In this section we provide a partial answer to Question 1 in the finite-dimensional case. Recall that a Hopf algebra A is said to be unimodular if there is a non-zero two-sided integral in A, i.e. there exists a non-zero $t \in A$ such that $ta = at = \varepsilon(a)t$ for any a. If A is cosemisimple and finite-dimensional, then A^* is unimodular.

Theorem 45. *Let A, B be finite-dimensional Hopf algebras such that $\mathcal{M}^A \simeq^{\otimes} \mathcal{M}^B$. Then we have* $\mathrm{cd}(A) = \mathrm{cd}(B)$ *if one of the following condition holds.*

(1) *The characteristic of k is zero, or satisfies $p > d^{\frac{\varphi(d)}{2}}$, where $d = \dim(A)$.*
(2) *A^* is unimodular.*

Proof. First notice that since a finite-dimensional Hopf algebra is self-injective (projective modules are injective), we have $\mathrm{cd}(A), \mathrm{cd}(B) \in \{0, \infty\}$ and hence there are only few cases to consider. Moreover, for the Drinfeld double $D(A)$, we have $\mathrm{cd}(D(A)) = 0$ if and only if $D(A)$ is semisimple, if and only if A is semisimple and cosemisimple [36, Proposition 7], and $\mathrm{cd}(D(A)) = \infty$ otherwise. Moreover, we have $\mathrm{cd}(D(A)) = \mathrm{cd}(D(B))$ since our monoidal equivalence $\mathcal{M}^A \simeq^{\otimes} \mathcal{M}^B$ induces a monoidal equivalence between the monoidal centers of these categories (notice that $\mathrm{cd}(D(A)) = \mathrm{cd}_{\mathrm{GS}}(A)$).

If k has characteristic zero or satisfies $p > d^{\frac{\varphi(d)}{2}}$, then by [26, Theorem 3.3] and [18, Theorem 4.2] respectively, we have that A is semisimple if and only if A is semisimple and cosemisimple, if and only if $\mathrm{cd}(D(A)) = 0$. Hence under one of these assumptions we have $\mathrm{cd}(A) = \mathrm{cd}(B)$ because $\mathrm{cd}(D(A)) = \mathrm{cd}(D(B))$.

Since $\mathcal{M}^A \simeq^{\otimes} \mathcal{M}^B$ and A, B are finite-dimensional, We have, by [40, Corollary 5.9], $B \simeq A^{\sigma}$ for some Hopf 2-cocycle σ. At the dual level this means that $B^* \simeq (A^*)^J$ for some Drinfeld twist

J. Hence if $\mathrm{cd}(A) = 0$, i.e. A is semisimple, we have that A^* is cosemisimple, and assuming that A^* is unimodular, we have that B^* is cosemisimple as well by [1, Corollary 3.6], and hence B is semisimple, so that $\mathrm{cd}(B) = 0$, as needed. The assumption that A^* is unimodular is stable under Drinfeld twist since the multiplication does not change, thus B^* is unimodular as well, and hence we also have $\mathrm{cd}(B) = 0 \Rightarrow \mathrm{cd}(A) = 0$, concluding the proof. □

As we see in the proof of the previous theorem, a complete answer to Question 1 in the finite-dimensional case reduces to the question whether the class of finite-dimensional cosemisimple Hopf algebras is stable under Drinfeld twists. Remark 3.9 in [1] claimed that this is expected to be true, and would follow from a weak form of an important conjecture of Kaplansky saying that a finite-dimensional cosemisimple Hopf algebra is unimodular (the strong form says that a cosemisimple Hopf algebra satisfies $S^2 = \mathrm{id}$), but we are not aware of a proof since then.

10. Summary of positive answers to Question 1

In this last section, for the convenience of the reader, we summarize what are, to the best of our knowledge, the known positive answers to Question 1, most of which are in this paper. Let A, B be Hopf algebras having equivalent linear tensor categories of comodules. Then we have $\mathrm{cd}(A) = \mathrm{cd}(B)$ in the following situations.

(1) A, B have bijective antipode and are smooth.
(2) A, B are cosemisimple and their antipodes satisfy $S^4 = \mathrm{id}$.
(3) A, B are cosemisimple and $\mathrm{cd}(A)$, $\mathrm{cd}(B)$ are finite.
(4) A, B are finite-dimensional, and the characteristic of k is zero, or satisfies $p > d^{\frac{\varphi(d)}{2}}$, where $d = \dim(A)$.
(5) A, B are finite-dimensional and A^* is unimodular.

References

[1] E. Aljadeff, P. Etingof, S. Gelaki, D. Nikshych, "On twisting of finite-dimensional Hopf algebras", *J. Algebra* **256** (2002), no. 2, p. 484-501.

[2] T. Banica, R. Vergnioux, "Fusion rules for quantum reflection groups", *J. Noncommut. Geom.* **3** (2009), no. 3, p. 327-359.

[3] J. Bichon, "Free wreath product by the quantum permutation group", *Algebr. Represent. Theory* **7** (2004), no. 4, p. 343-362.

[4] ———, "Co-representation theory of universal co-sovereign Hopf algebras", *J. Lond. Math. Soc.* **75** (2007), no. 1, p. 83-98.

[5] ———, "Hochschild homology of Hopf algebras and free Yetter-Drinfeld resolutions of the counit", *Compos. Math.* **149** (2013), no. 4, p. 658-678.

[6] ———, "Hopf-Galois objects and cogroupoids", *Rev. Unión Mat. Argent.* **55** (2014), no. 2, p. 11-69.

[7] ———, "Gerstenhaber-Schack and Hochschild cohomologies of Hopf algebras", *Doc. Math.* **21** (2016), p. 955-986.

[8] ———, "Cohomological dimensions of universal cosovereign Hopf algebras", *Publ. Mat., Barc.* **62** (2018), no. 2, p. 301-330.

[9] J. Bichon, U. Franz, M. Gerhold, "Homological properties of quantum permutation algebras", *New York J. Math.* **23** (2017), p. 1671-1695.

[10] K. S. Brown, *Cohomology of groups*, Graduate Texts in Mathematics, vol. 87, Springer, 1994, Corrected reprint of the 1982 original, x+306 pages.

[11] S. Caenepeel, G. Militaru, S. Zhu, "Crossed modules and Doi–Hopf modules", *Isr. J. Math.* **100** (1997), p. 221-247.

[12] ———, *Frobenius and separable functors for generalized module categories and nonlinear equations*, Lecture Notes in Mathematics, vol. 1787, Springer, 2002, xiv+354 pages.

[13] A. Chirvasitu, "Cosemisimple Hopf algebras are faithfully flat over Hopf subalgebras", *Algebra Number Theory* **8** (2014), no. 5, p. 1179-1199.

[14] ———, "Relative Fourier transforms and expectations on coideal subalgebras", *J. Algebra* **516** (2018), p. 271-297.

[15] A. Chirvasitu, C. Walton, X. Wang, "Gelfand-Kirillov dimension of cosemisimple Hopf algebras", *Proc. Am. Math. Soc.* **147** (2019), no. 11, p. 4665-4672.

[16] W. Dicks, M. J. Dunwoody, *Groups acting on graphs*, Cambridge Studies in Advanced Mathematics, vol. 17, Cambridge University Press, 1989, xvi+283 pages.

[17] Y. Doi, "Braided bialgebras and quadratic bialgebras", *Commun. Algebra* **21** (1993), no. 5, p. 1731-1749.

[18] P. Etingof, S. Gelaki, "On finite-dimensional semisimple and cosemisimple Hopf algebras in positive characteristic", *Int. Math. Res. Not.* (1998), no. 16, p. 851-864.

[19] P. Etingof, S. Gelaki, D. Nikshych, V. Ostrik, *Tensor categories*, Mathematical Surveys and Monographs, vol. 205, American Mathematical Society, 2015, xvi+343 pages.

[20] P. Fima, L. Pittau, "The free wreath product of a compact quantum group by a quantum automorphism group", *J. Funct. Anal.* **271** (2016), no. 7, p. 1996-2043.

[21] M. Gerstenhaber, S. D. Schack, "Bialgebra cohomology, deformations, and quantum groups", *Proc. Natl. Acad. Sci. USA* **87** (1990), no. 1, p. 478-481.

[22] V. Ginzburg, S. Kumar, "Cohomology of quantum groups at roots of unity", *Duke Math. J.* **69** (1993), no. 1, p. 179-198.

[23] A. Joyal, R. Street, "An introduction to Tannaka duality and quantum groups", in *Category theory (Como, 1990)*, Lecture Notes in Mathematics, vol. 1488, Springer, 1991, p. 413-492.

[24] A. Klimyk, K. Schmüdgen, *Quantum groups and their representations*, Texts and Monographs in Physics, Springer, 1997, xx+552 pages.

[25] R. G. Larson, "Characters of Hopf algebras", *J. Algebra* **17** (1971), p. 352-368.

[26] R. G. Larson, D. E. Radford, "Finite-dimensional cosemisimple Hopf algebras in characteristic 0 are semisimple", *J. Algebra* **117** (1988), no. 2, p. 267-289.

[27] F. Lemeux, P. Tarrago, "Free wreath product quantum groups: the monoidal category, approximation properties and free probability", *J. Funct. Anal.* **270** (2016), no. 10, p. 3828-3883.

[28] M. E. Lorenz, M. Lorenz, "On crossed products of Hopf algebras", *Proc. Am. Math. Soc.* **123** (1995), no. 1, p. 33-38.

[29] A. Masuoka, "Cleft extensions for a Hopf algebra generated by a nearly primitive element", *Commun. Algebra* **22** (1994), no. 11, p. 4537-4559.

[30] A. Masuoka, D. Wigner, "Faithful flatness of Hopf algebras", *J. Algebra* **170** (1994), no. 1, p. 156-164.

[31] S. Montgomery, *Hopf algebras and their actions on rings*, CBMS Regional Conference Series in Mathematics, vol. 82, American Mathematical Society, 1993, xiv+238 pages.

[32] C. Mrozinski, "Quantum groups of GL(2) representation type", *J. Noncommut. Geom.* **8** (2014), no. 1, p. 107-140.

[33] ———, "Quantum automorphism groups and $SO(3)$-deformations", *J. Pure Appl. Algebra* **219** (2015), no. 1, p. 1-32.

[34] C. Năstăsescu, M. Van den Bergh, F. Van Oystaeyen, "Separable functors applied to graded rings", *J. Algebra* **123** (1989), no. 2, p. 397-413.

[35] P. Podleś, E. Müller, "Introduction to quantum groups", *Rev. Math. Phys.* **10** (1998), no. 4, p. 511-551.

[36] D. E. Radford, "Minimal quasitriangular Hopf algebras", *J. Algebra* **157** (1993), no. 2, p. 285-315.

[37] T. Raedschelders, M. Van den Bergh, "The Manin Hopf algebra of a Koszul Artin-Schelter regular algebra is quasi-hereditary", *Adv. Math.* **305** (2017), p. 601-660.

[38] M. D. Rafael, "Separable functors revisited", *Commun. Algebra* **18** (1990), no. 5, p. 1445-1459.

[39] P. Schauenburg, "Hopf modules and Yetter-Drinfel′d modules", *J. Algebra* **169** (1994), no. 3, p. 874-890.

[40] ———, "Hopf bi-Galois extensions", *Commun. Algebra* **24** (1996), no. 12, p. 3797-3825.

[41] ———, "Bi-Galois objects over the Taft algebras", *Isr. J. Math.* **115** (2000), p. 101-123.

[42] ———, "Hopf-Galois and bi-Galois extensions", in *Galois theory, Hopf algebras, and semiabelian categories*, Fields Institute Communications, vol. 43, American Mathematical Society, 2004, p. 469-515.

[43] S. Shnider, S. Sternberg, *Quantum groups from coalgebras to Drinfeld algebras. A guided tour*, International Press, 1993, xxii+496 pages.

[44] D. Ştefan, "Hochschild cohomology on Hopf Galois extensions", *J. Pure Appl. Algebra* **103** (1995), no. 2, p. 221-233.

[45] R. Taillefer, "Injective Hopf bimodules, cohomologies of infinite dimensional Hopf algebras and graded-commutativity of the Yoneda product", *J. Algebra* **276** (2004), no. 1, p. 259-279.

[46] S. Wang, "Quantum symmetry groups of finite spaces", *Commun. Math. Phys.* **195** (1998), no. 1, p. 195-211.

[47] X. Wang, X. Yu, Y. Zhang, "Calabi-Yau property under monoidal Morita-Takeuchi equivalence", *Pac. J. Math.* **290** (2017), no. 2, p. 481-510.

[48] C. A. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, 1994, xiv+450 pages.

[49] S. L. Woronowicz, "Compact matrix pseudogroups", *Commun. Math. Phys.* **111** (1987), no. 4, p. 613-665.

[50] X. Yu, "Hopf-Galois objects of Calabi-Yau Hopf algebras", *J. Algebra Appl.* **15** (2016), no. 10, article no. 1650194 (19 pages).