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Académie des sciences

# Comptes Rendus
# Mathématique

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Volume 360 (2022), p. 583-588

<https://doi.org/10.5802/crmath.301>

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Algebra / *Algèbre*

# On the number of prime divisors of character degrees and conjugacy classes of a finite group

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**Abstract.** A result of Gluck is that any finite group $G$ has an abelian subgroup $A$ such that $|G:A|$ is bounded by a polynomial function of the largest irreducible character degree of $G$. Moretó presented a variation of this result that looks at the number of prime factors of the irreducible character degrees and obtained an almost quadratic bound. The author improved the result of Moretó to almost linear. In this note, we further improve the bound, and also study the related problem on conjugacy class sizes.

**Mathematical subject classification (2010).** 20C15, 20C20.

**Funding.** This work was partially supported by the Simons Foundation (No 499532).

*Manuscript received 28th September 2021, accepted 25th November 2021.*

## 1. Introduction

Let $G$ be a finite group, $\mathbf{F}(G)$ be the Fitting subgroup of $G$, and denote by $b(G) = \max\{\psi(1) \mid \psi \in \mathrm{Irr}(G)\}$ the largest degree of an irreducible character of $G$. Gluck proved in [6] that any finite group $G$ has an abelian subgroup $A$ such that $|G:A|$ is bounded by a polynomial function of $b(G)$. For solvable groups, Gluck showed that $|G:\mathbf{F}(G)| \le b(G)^{13/2}$ and conjectured that $|G:\mathbf{F}(G)| \le b(G)^2$. This is a well-known conjecture for solvable groups.

In more recent work, Moretó [8] explored a different notion of "large". The obvious meaning for the word "large" refers to the absolute value of an integer $n$. The meaning that Moretó considered involves the prime factorization of $n$; $n$ is "large" if it has many prime divisors (counting multiplicities). Given an integer $n = p_1^{a_1} \cdots p_t^{a_t}$ as a product of powers of different primes, we define $\omega(n) = a_1 + \cdots + a_n$ and $\omega_{p_i}(n) = a_i$. Let $G$ be a finite group, we use $\mathrm{Irr}(G)$ to denote the set of complex irreducible characters of $G$ and $\mathrm{cl}(G)$ to denote the set of conjugacy classes of $G$. We set $\omega(G) = \max\{\omega(\chi(1)) \mid \chi \in \mathrm{Irr}(G)\}$ and $\omega_{cl}(G) = \max\{\omega(|C|) \mid C \in \mathrm{cl}(G)\}$.

Moretó [8] proved the following variation of the theorem of Gluck. In this paper, the base of the log will always be 2.

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**Theorem 1.** *There exist (universal) constants $D_1$ and $D_2$ such that if $G$ is a non-abelian finite group then $G$ has an abelian subgroup $A$ satisfying*

$$\omega(|G:A|) \le D_1\omega(G)^2 \log\omega(G) + D_2.$$

Moretó speculated that a linear bound might exist. In [12], the author improved the bound in Theorem 1 to almost linear.

**Theorem 2.** *There exists a (universal) constant $D$ such that $G$ has an abelian subgroup $A$ satisfying $\omega(|G:A|) \le D\omega(G)\log\omega(G)$ when $\omega(G) > 1$.*

In this paper, we first strengthen Theorem 2.

**Theorem 3.** *There exist (universal) constants $D_3$ and $D_4$ such that $G$ has an abelian subgroup $A$ satisfying $\omega(|G:A|) \le D_3\omega(G)(\log\log(\omega(G)) + D_4)$ when $\omega(G) > 1$.*

After this, we also study the related question about conjugacy classes.

**Theorem 4.** *There exist (universal) constants $D_5$ and $D_6$ such that $\omega(|G:\mathbf{F}(G)|) \le D_5\omega_{cl}(G)(\log\log(\omega_{cl}(G)) + D_6)$ when $\omega_{cl}(G) > 1$.*

## 2. Preliminary Results

**Lemma 5.** *Let $N$ be a normal subgroup of $G$.*

(1) *If $x \in N$, then $|x^N|$ divides $|x^G|$.*
(2) *If $x \in G$, then $|(xN)^{G/N}|$ divides $|x^G|$.*

**Lemma 6.** *Let $G$ be a finite simple group. Then $\omega(|\mathrm{Aut}(G)|) \le 2\omega(|G|)$.*

**Proof.** This is [8, Lemma 2.7]. □

**Lemma 7.** *Let $n \ge 2$ be a positive integer. Then $\omega(n!) \le 2n(\log(\log n) + C)$ for some fixed constant $C$.*

**Proof.** This is a standard result in number theory. We have

$$\omega_p(n!) = \sum_{k=1}^{\lfloor \log_p n\rfloor} \left\lfloor \frac{n}{p^k}\right\rfloor \le \sum_{k=1}^{\infty} \left\lfloor \frac{n}{p^k}\right\rfloor \le \frac{n}{p-1} \le \frac{2n}{p}.$$

Hence by [5], we have

$$\omega(n!) \le 2n\sum_{p\le n}\frac{1}{p} \le 2n\cdot\big(\log\log n + C\big).$$

□

**Lemma 8.** *There exist (universal) constants $C_1$ and $C_2$ such that if $G$ is a nonabelian simple group, then $\omega(|G|) \le C_1\omega(G)(\log\log\omega(G) + C_2)$.*

**Proof.** Let $G$ be a finite simple group which is not an alternating group, then $\omega(|G|) \le 5\omega(G)$ by [8, Lemmas 2.2, 2.3, 2.4].

Let $G$ be an alternating group of degree $n \ge 5$. Note that $\omega(|G|) = \omega(n!/2)$. Since $\omega_2(n!) \ge \frac{n}{2}$, we have $2\omega(G) \ge \omega_2(n!) \ge \frac{n}{2}$ by [4, Lemma 1.2] (for the exceptional cases in the table one may verify this by a direct calculation). By Lemma 7, we have

$$\omega(|G|) \le 8\omega(G)\cdot\big(\log\big(\log 4\omega(G)\big) + C\big).$$

We note that $\omega(G) \ge 2$ since $G$ is a nonabelian simple group. Thus,

$$\omega(|G|) \le 8\omega(G)\cdot\big(\log\big(\log\omega(G)\big) + C + 2\big).$$

We may choose $C_1 = 8$, $C_2 = C + 2$ and the result follows. □

**Lemma 9.** *The alternating groups* $\mathrm{Alt}_n$ *of degree* $n \geq 5$ *has conjugacy classes* $cl_1$ *and* $cl_2$ *such that* $|\mathrm{Alt}_n| \mid |cl_1||cl_2|$.

**Proof.** For $n = 5, 6$, this can be checked by a direct calculation.

For the case when $n \geq 7$, set $\alpha = (1, 2, \ldots, n)$ and $\beta = (1, 2, \ldots, n-2)$ if $n$ is odd. Then,

$$\frac{1}{2}\left|\alpha^{\mathrm{Sym}_n}\right| = \frac{1}{2}(n)!/(n) \,\Big|\, \left|\alpha^{\mathrm{Alt}_n}\right|$$

and

$$\frac{1}{2}\left|\beta^{\mathrm{Sym}_n}\right| = \frac{1}{2}(n)!/(2(n-2)) \,\Big|\, \left|\beta^{\mathrm{Alt}_n}\right|.$$

Since $2(n-2)n \mid \frac{1}{2}(n)!$, we have $\frac{1}{2}(n)! \mid |\alpha^{\mathrm{Alt}_n}| \cdot |\beta^{\mathrm{Alt}_n}|$, and the result follows.

Set $\alpha = (1, 2, \ldots, n-1)$ and $\beta = (1, 2, \ldots, n-3)$ if $n$ is even. Then,

$$\frac{1}{2}\left|\alpha^{\mathrm{Sym}_n}\right| = \frac{1}{2}(n)!/(n-1) \,\Big|\, \left|\alpha^{\mathrm{Alt}_n}\right|$$

and

$$\frac{1}{2}\left|\beta^{\mathrm{Sym}_n}\right| = \frac{1}{2}(n)!/(3(n-3)) \,\Big|\, \left|\beta^{\mathrm{Alt}_n}\right|.$$

Since $3(n-3)(n-1) \mid \frac{1}{2}(n)!$, we have $\frac{1}{2}(n)! \mid |\alpha^{\mathrm{Alt}_n}| \cdot |\beta^{\mathrm{Alt}_n}|$, and the result follows. □

**Theorem 10.** *Let* $S \neq {}^2F_4(2)'$ *be a simple group of Lie type. There exist semisimple elements* $s_1, s_2 \in S$ *such that* $|s_1^S||s_2^S|$ *is divisible by* $|S|$.

**Proof.** The case $S = \mathrm{PSL}_2(5)$ can be checked directly using [2] for instance. For $S = \mathrm{PSL}_2(q)$ with $q > 5$, it is well known that $S$ has a class of size $q(q+1)$ and another of size $q(q-1)$ (see [3]), and we are done. We therefore may assume that $S \neq \mathrm{PSL}_2(q)$ from now on.

Let $\mathbf{G}$ be a simple algebraic group of simply connected type and $F : \mathbf{G} \to \mathbf{G}$ be a Steinberg endomorphism such that $S = \bar{G} = G/Z$ with $G := \mathbf{G}^F$ and $Z := \mathbf{Z}(G)$. For $g \in G$, let $\overline{g}$ denote by the image of $g$ in $S$ under the natural projection $\tau : G \to S$.

Let $g \in G$ and denote $\mathscr{C}_g$ the preimage of $\mathbf{C}_{\overline{G}}(\overline{g})$ under $\tau$. Following [1, § 4.2], we consider the map $T_g : \mathscr{C}_g \to Z$ by $h \mapsto [h, g] := h^{-1}g^{-1}hg$. It is easy to show that $T_g$ is in fact a group homomorphism, and hence it follows that

$$\left|\mathbf{C}_{\overline{G}}\left(\overline{g}\right)\right||Z| = \left|\mathscr{C}_g\right| = \left|\mathbf{C}_G(g)\right|\left|\mathrm{Im}\left(T_g\right)\right|.$$

In particular, $|\mathbf{C}_{\overline{G}}(\overline{g})|$ divides $|\mathbf{C}_G(g)|$, and therefore, it suffices to prove that there exist semisimple elements $g_1, g_2 \in G$ such that $|S|$ is divisible by $|\mathbf{C}_G(g_1)||\mathbf{C}_G(g_2)|$.

First, we consider the case when $G$ is of exceptional type. Centralizers of semisimple elements in such group are well known, see [7] for instance. One then can easily find the required semisimple elements $g_1$ and $g_2$ such that $|\mathbf{C}_G(g_1)||\mathbf{C}_G(g_2)|$ divides $|S|$. As an example, the group $G = E_8(q)$ has two elements whose centralizers have orders $\Phi_1(q)\Phi_2(q)\Phi_3(q)\Phi_5(q)$ and $\Phi_1(q)\Phi_2(q)\Phi_6(q)\Phi_{10}(q)$, where $\Phi_k(q)$ denotes the $k$th cyclotomic polynomial evaluated at $q$, and thus they satisfy the desired condition. It remains to consider classical groups of rank at least 2. We will produce the required semisimple elements $g_1, g_2 \in G$ using known knowledge on the structure of semisimple elements in finite classical groups.

Let $S = \mathrm{PSL}_n(q)$ with $n \geq 3$. Using [10, Lemma 3.2], we can find $g_1 \in G = \mathrm{SL}_n(q)$ such that $\mathbf{C}_{\mathrm{GL}_n(q)}(g_1) \cong \mathrm{GL}_1(q^n)$ and $|\mathbf{C}_G(g_1)| = (q^n - 1)/(q-1)$. (Here the centralizer is in fact a Coxter torus of $G$.) Also, we can find $g_2$ so that $|\mathbf{C}_G(g_2)| = q^{n-1} - 1$.

Let $S = \mathrm{PSU}_n(q)$ with $n \geq 3$. By [10, Lemma 3.3], we know that the centralizer of a semisimple element in $\mathrm{GU}_n(q)$ is a direct product of groups of the forms $\mathrm{GL}_m(q^{2k})$ and $\mathrm{GU}_m(q^{2k-1})$ for $k \in \mathbb{Z}^{\geq 0}$. Therefore we may choose $g_1$ and $g_2$ in $\mathrm{SU}_n(q)$ such that the orders of their centralizers are $(q^n+1)/(q+1)$ and $(q^{n-2}+1)(q-1)$ when $n$ is odd and $q^{n-1}+1$ and $(q^n-1)/(q+1)$ when $n$ is even.

Let $S = \mathrm{PSp}_{2n}(q)$ or $\Omega_{2n+1}(q)$ with $n \ge 2$. By [9, Lemmas 2.2 and 2.4], there exist semisimple elements $g_1, g_2 \in G$ such that $\mathbf{C}_G(g_1) = \mathrm{GL}_1(q^n)$ and $\mathbf{C}_G(g_2) = \mathrm{GU}_1(q^n)$, so that $|S|$ is divisible by $|\mathbf{C}_G(g_1)||\mathbf{C}_G(g_2)|$.

Lastly, let $S = P\Omega^{\epsilon}_{2n}(q)$ with $n \ge 4$ and $\epsilon = \pm$. Then $G = \mathrm{P}(\mathrm{CO}^{\epsilon}_{2n}(q))^0$. By [9, Lemmas 2.3 and 2.5], there are $g_1, g_2 \in G$ such that $|\mathbf{C}_G(g_1)| = q^n - \epsilon 1$ and $|\mathbf{C}_G(g_2)| = (q^{n-1}+\epsilon 1)(q+1)$. The proof of Theorem 10 is complete. □

**Theorem 11.** *Let $S$ be a nonabelian simple group. There exist two elements $s_1, s_2 \in S$ such that $|s_1^S||s_2^S|$ is divisible by $|S|$.*

**Proof.** The sporadic groups can be checked by using [2]. The other cases follow from Lemma 9 and Theorem 10. □

## 3. Main Results

We first improve the main result of [12].

**Proposition 12.** *Assume that $N \lhd G$ where $N = L_1 \times \ldots \times L_m$ is the product of $m$ finite non-abelian simple groups $L_i$ permuted transitively by $G$. Let $K = \bigcap_i \mathbf{N}_G(L_i)$. Then $\omega(|G|) \le 2m(\log\log m + C) + 2\omega(|N|)$.*

**Proof.** We observe that $G/K$ is a permutation group on $m$ letters and $K/N \le \mathrm{Out}(L_1) \times \ldots \times \mathrm{Out}(L_m)$. Then the result follows from Lemma 7 and Lemma 6. □

**Proposition 13.** *Let $G$ be a group with no non-trivial solvable normal subgroup. Then $\omega(|G|) \le (2+2C_1)\omega(F^*(G))(\log\log\omega(F^*(G)) + C_2)$.*

**Proof.** The generalized Fitting subgroup $F^*(G) = E_1 \times E_2 \times \cdots \times E_n$ is a direct product of $n$ non-abelian simple groups on which $G$ acts faithfully by conjugation.

Let $K_0 = G$, $\{E_{11}, E_{12}, \ldots, E_{1m_1}\}$ be an orbit of $K_0$ on the set of simple direct factors of $F^*(G)$, and set $L_1 = E_{11} \times E_{12} \times \cdots \times E_{1m_1}$. Let $K_1 = \mathbf{C}_{K_0}(L_1)$, $\{E_{21}, E_{22}, \ldots, E_{2m_2}\}$ be an orbit of $K_1$ on the set of simple direct factors of $F^*(G)$, and set $L_2 = E_{21} \times E_{22} \times \cdots \times E_{2m_2}$. Let $K_2 = \mathbf{C}_{K_1}(L_2)$, and inductively, we may define $L_3, K_3 \ldots L_t, K_t$ where $K_t = 1$.

Clearly $F^*(G) = L_1 \times L_2 \times \cdots \times L_t$. We know that $K_{i-1}/K_i$ acts transitively and faithfully on $L_i$ where $i = 1, \ldots, t$ and $L_i \cap K_i = 1$. Thus by Proposition 12, we have that $\omega(|K_{i-1}/K_i|) \le 2m_i(\log\log m_i + C) + 2\omega(|L_i|)$.

Thus

$$\omega(|G|) \le \omega(|K_0/K_1|) + \cdots + \omega(|K_{t-1}/K_t|) \le 2\sum_{i=1}^{t} m_i \left(\log\log m_i + C\right) + 2\sum_{i=1}^{t} \omega(|L_i|).$$

Since $n = m_1 + \cdots + m_t$, we have

$$\omega(|G|) \le 2n\left(\log\log n + C\right) + 2\omega\left(\left|F^*(G)\right|\right).$$

Since $F^*(G)$ is a direct product of $n$ simple groups, $\omega(F^*(G)) \ge n$. By Lemma 8, we have

$$\omega(|G|) \le (2+2C_1)\,\omega\left(F^*(G)\right)\left(\log\log\omega\left(F^*(G)\right) + C_2\right).$$ □

We now prove Theorem 3.

**Proof.** Let $T$ be the maximal solvable normal subgroup of $G$. Then there exists an abelian subgroup $A$ of $T$ such that $\omega(|T:A|) \le 19\omega(T) \le 19\omega(G)$ by [12, Theorem 1.3].

Let $\bar{G} = G/T$, and we have that

$$\omega\left(\left|\bar{G}\right|\right) \le (2+2C_1)\,\omega\left(F^*\left(\bar{G}\right)\right)\left(\log\log\omega\left(F^*\left(\bar{G}\right)\right)+C_2\right) \le (2+2C_1)\,\omega(G)\left(\log\log\omega(G)+C_2\right)$$

by Proposition 13.

Thus $\omega(|G:A|) = \omega(|\bar{G}|)+\omega(|T:A|) \le D_3\omega(G)(\log\log(\omega(G))+D_4)$ for some fixed constants $D_3$ and $D_4$. □

We now study the related problem on class sizes. We would expect a similar result would hold. However, as the extra-special group shows, one cannot have a bound by just moduloing out an abelian subgroup. On the other hand, if we replace an abelian subgroup by the Fitting subgroup, we will get something along the same line.

**Proposition 14.** *Let $G$ be a finite solvable group. Then $\omega(|G:\mathbf{F}(G)|) \le 15\omega_{cl}(G)$.*

**Proof.** Since $G$ is solvable, then there exist 15 conjugacy classes $cl_1,\dots,cl_{15}$ such that $|G:\mathbf{F}(G)|$ divides $|cl_1|\cdots|cl_{15}|$ by [11, Theorem 3.5]. Thus we have $\omega(|G:\mathbf{F}(G)|) \le 15\omega_{cl}(G)$. □

**Proposition 15.** *Let $G$ be a group with no non-trivial solvable normal subgroup. Then $\omega(|G|) \le 2\omega_{cl}(F^*(G))(\log\log\omega_{cl}(F^*(G))+C_2+2)$.*

**Proof.** The generalized Fitting subgroup $F^*(G) = E_1\times E_2\times\cdots\times E_n$ is a direct product of $n$ non-abelian simple groups on which $G$ acts faithfully by conjugation.

Let $K_0 = G$, $\{E_{11},E_{12},\dots,E_{1m_1}\}$ be an orbit of $K_0$ on the set of simple direct factors of $F^*(G)$, and set $L_1 = E_{11}\times E_{12}\times\cdots\times E_{1m_1}$. Let $K_1 = \mathbf{C}_{K_0}(L_1)$, $\{E_{21},E_{22},\dots,E_{2m_2}\}$ be an orbit of $K_1$ on the set of simple direct factors of $F^*(G)$, and set $L_2 = E_{21}\times E_{22}\times\cdots\times E_{2m_2}$. Let $K_2 = \mathbf{C}_{K_1}(L_2)$, and inductively, we may define $L_3,K_3\dots L_t,K_t$ where $K_t = 1$.

Clearly $F^*(G) = L_1\times L_2\cdots\times L_t$. We know that $K_{i-1}/K_i$ acts transitively and faithfully on $L_i$ where $i=1,\dots,t$ and $L_i\cap K_i = 1$. Thus by Proposition 12, we have that

$$\omega\left(|K_{i-1}/K_i|\right) \le 2m_i\left(\log\log m_i + C\right) + 2\omega\left(|L_i|\right).$$

Thus

$$\omega(|G|) \le \omega\left(|K_0/K_1|\right)+\cdots+\omega\left(|K_{t-1}/K_t|\right) \le 2\sum_{i=1}^{t} m_i\left(\log\log m_i + C\right) + 2\sum_{i=1}^{t}\omega\left(|L_i|\right).$$

Since $n = m_1+\cdots+m_t$, we have

$$\omega(|G|) \le 2n\left(\log\log n + C_2\right) + 2\omega\left(\left|F^*(G)\right|\right).$$

Since $F^*(G)$ is a direct product of $n$ simple groups, $\omega_{cl}(F^*(G)) \ge n$. By Theorem 11, we have $\omega(|F^*(G)|) \le 2\omega_{cl}(F^*(G))$. Thus,

$$\omega(|G|) \le 2\omega_{cl}\left(F^*(G)\right)\left(\log\log\omega_{cl}\left(F^*(G)\right)+C_2+2\right).$$

□

We now prove Theorem 4.

**Proof.** Let $T$ be the maximal solvable normal subgroup of $G$. Then $\omega(|T:\mathbf{F}(G)|) \le 15\omega_{cl}(T) \le 15\omega_{cl}(G)$ by Proposition 14.

Let $\bar{G} = G/T$, and we have that

$$\omega(|\bar{G}|) \le 2\omega_{cl}\left(F^*\left(\bar{G}\right)\right)\left(\log\log\omega_{cl}\left(F^*\left(\bar{G}\right)\right)+C_2+2\right) \le 2\omega_{cl}(G)\left(\log\log\omega_{cl}(G)+C_2+2\right)$$

by Proposition 15.

Thus $\omega(|G:\mathbf{F}(G)|) = \omega(|\bar{G}|)+\omega(|T:\mathbf{F}(G)|) \le D_5\omega_{cl}(G)(\log\log(\omega_{cl}(G))+D_6)$ for some fixed constants $D_5$ and $D_6$. □

*Acknowledgement*

The author is grateful to the referee for the valuable suggestions which greatly improved the manuscript.

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