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Number Theory / *Théorie des nombres*

# The number of unimodular roots of some reciprocal polynomials

## *Sur le nombre de racines de module un de certains polynômes réciproques*

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**Abstract.** We introduce a sequence  $P_{2n}$  of monic reciprocal polynomials with integer coefficients having the central coefficients fixed. We prove that the ratio between number of nonunimodular roots of  $P_{2n}$  and its degree  $d$  has a limit when  $d$  tends to infinity. We present an algorithm for calculation the limit and a numerical method for its approximation. If  $P_{2n}$  is the sum of a fixed number of monomials we determine the central coefficients such that the ratio has the minimal limit. We generalise the limit of the ratio for multivariate polynomials. Some examples suggest a theorem for polynomials in two variables which is analogous to Boyd's limit formula for Mahler measure.

**Résumé.** Nous introduisons une suite  $P_{2n}$  de polynômes unitaires réciproques avec des coefficients entiers ayant les coefficients centraux fixes. Nous prouvons que le rapport entre le nombre de racines non unimodulaires de  $P_{2n}$  et son degré  $d$  a une limite lorsque  $d$  tend vers l'infini. Nous présentons un algorithme de calcul de la limite et une méthode numérique pour son approximation. Si  $P_{2n}$  est la somme d'un nombre fixe de monômes, nous déterminons les coefficients centraux de sorte que le rapport ait la limite minimale. Nous généralisons la limite du rapport pour les polynômes de plusieurs variables. Certains exemples suggèrent une conjecture pour les polynômes à deux variables qui est analogue à la formule limite de Boyd pour la mesure de Mahler.

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## 1. Introduction

If  $P(x) = a_d x^d + a_{d-1} x^{d-1} + \dots + a_1 x + a_0$  ( $a_d \neq 0$ ) has zeros  $\alpha_1, \alpha_2, \dots, \alpha_d$  then the Mahler measure of  $P(x)$  is

$$M(P(x)) = |a_d| \prod_{j=1}^d \max(1, |\alpha_j|).$$

Let  $I(P)$  denote the number of complex zeros of  $P(x)$  which are  $< 1$  in modulus, counted with multiplicities. Let  $U(P)$  denote the number of zeros of  $P(x)$  which are  $= 1$  in modulus, (again, counting with multiplicities). Such zeros are called unimodular. Let  $E(P)$  denote the number of complex zeros of  $P(x)$  which are  $> 1$  in modulus, counted with multiplicities. Then it is obviously that  $I(P) + U(P) + E(P) = d$ . Pisot number can be defined as a real algebraic integer greater than 1 having the minimal polynomial  $P(x)$  of degree  $d$  such that  $I(P) = d - 1$ . Salem number is a real algebraic integer  $> 1$  having the minimal polynomial  $P(x)$  of degree  $d$  such that  $U(P) = d - 2$ ,  $I(P) = 1$ . It is well known that the minimal polynomial of a Salem number is reciprocal.

We say that a polynomial of degree  $d$  is reciprocal if  $P(x) = x^d P(1/x)$ . If moduli of coefficients are small then a reciprocal polynomial has many unimodular roots. A Littlewood polynomial is a polynomial all of whose coefficients are 1 or  $-1$ . Mukunda [7] showed that every self-reciprocal Littlewood polynomial of odd degree at least 3 has at least 3 zeros on the unit circle. Drungilas [5] proved that every self-reciprocal Littlewood polynomial of odd degree  $n \geq 7$  has at least 5 zeros on the unit circle and every self-reciprocal Littlewood polynomial of even degree  $n \geq 14$  has at least 4 unimodular zeros. In [1] two types of very special Littlewood polynomials are considered: Littlewood polynomials with one sign change in the sequence of coefficients and Littlewood polynomials with one negative coefficient. The numbers  $U(P)$  and  $I(P)$  of such Littlewood polynomials  $P$  are investigated. In [2] Borwein, Erdélyi, Ferguson and Lockhart showed that there exists a cosine polynomial  $\sum_{m=1}^N \cos(n_m \theta)$  with the  $n_m$  integral and all different so that the number of its real zeros in  $[0, 2\pi)$  is  $O(N^{9/10} (\log N)^{1/5})$  (here the frequencies  $n_m = n_m(N)$  may vary with  $N$ ). However, there are reasons to believe that a cosine polynomial  $\sum_{m=1}^N \cos(n_m \theta)$  always has many zeros in the period.

Clearly, if  $\alpha_j$  is a root of a reciprocal  $P(x)$  then  $1/\alpha_j$  is also a root of  $P(x)$  so that  $I(P) = E(P)$ . Let  $C(P) = \frac{I(P)+E(P)}{2n}$  be the ratio between the number of nonunimodular zeros of  $P$  and its degree. Actually, it is the probability that a randomly chosen zero is not unimodular, and  $C(P) = \frac{E(P)}{n}$ .

We are looking for the sequence  $P_{2n}$  of monic reciprocal polynomials with integer coefficients, such that  $C(P_{2n})$  has a limit when  $n$  tends to  $\infty$  and  $0 < \lim_{n \rightarrow \infty} C(P_{2n}) < 1$ . If  $P_{2n}$  is a sequence of Salem polynomials then this limit is trivially 0. Such sequences are well known: Salem (see [8, Theorem IV, p. 30]) found a simple way to construct infinite sequences of Salem numbers (and Salem polynomials) from Pisot numbers.

Here we will investigate a special sequence of polynomials. Let  $n, k, a_0, a_1, \dots, a_k$ , be integers such that  $n > k \geq 0$ , and let  $P_{2n}(x)$  be a monic, reciprocal polynomial with integer coefficients

$$P_{2n} = x^n \left( x^n + a_0 + \frac{1}{x^n} + \sum_{j=1}^k a_j \left( x^j + \frac{1}{x^j} \right) \right).$$

## 2. The main theorem

**Theorem 1.** *If  $k > 0$  is an integer then for all fixed integers  $a_j$ ,  $j = 1, \dots, k$  there is a limit  $C(P_{2n})$  when  $n$  tends to infinity.*

**Proof.** The theorem will be proved if we show that  $1 - C(P_{2n})$  has a limit when  $n$  tends to  $\infty$ . Since  $1 - C(P_{2n}) = \frac{U(P_{2n})}{2n}$  we have to count the unimodular roots of  $P_{2n}(x)$ . If we use the substitution  $x = e^{it}$  in the equation  $P_{2n}(x) = 0$  we get

$$e^{int} \left( 2 \cos nt + a_0 + \sum_{j=1}^k 2a_j \cos jt \right) = 0$$

Since  $e^{int} \neq 0$  it follows that the equation is equivalent to

$$\cos nt = -\frac{a_0}{2} - \sum_{j=1}^k a_j \cos jt. \quad (1)$$

From the substitution  $x = e^{it}$  it follows that  $x$  is unimodular if and only if  $t$  is real so that we have to count the real roots of (1) ( $t \in [0, 2\pi)$ ). If  $\Gamma_1$  is the graph of  $f_1(t) = \cos nt$  and  $\Gamma_2$  is the graph of  $f_2(t) = -a_0/2 - \sum_{j=1}^k a_j \cos jt$ , the function on the right side of equation (1), then  $U(P)$  is equal to the number of intersection points of these two graphs. These intersection points are obviously settled between lines  $y = -1$  and  $y = 1$ . Graph  $\Gamma_2$  of the continuous function  $f_2$  is fixed i.e. does not depend on  $n$ , therefore we can introduce a partition of  $[0, 2\pi]$  using points  $0 = t_0 < t_1 < \dots < t_p = 2\pi$  such that  $|f_2(t_j)| = 1, 0 < j < p$ . Let us consider subintervals  $I_j = [t_{j-1}, t_j]$  such that if  $t \in I_j$  then  $|f_2(t)| < 1, j \in J = \{j_1, j_2, \dots, j_r\} \subseteq \{1, 2, \dots, p\}$ .

**Definition 2.** *A part of the graph of  $f_1(t) = \cos nt$  such that  $(k - 1)\pi/n \leq t \leq k\pi/n, k \in \mathbb{Z}$  is  $k$ -th branch of  $\cos nt$ . The interval  $[(k - 1)\pi/n, k\pi/n]$  is the domain of the  $k$ -th branch.*

Each branch of  $\cos nt$  obviously has exactly one intersection point with the  $t$ -axis. We are going to prove that if  $n$  is large enough then each branch of  $\cos nt$  also has exactly one intersection point with  $\Gamma_2$ . We need the next lemma which will be proved in the next subsection.

**Lemma 3.** *For all  $B_1, B_2 > 0$  and  $\varepsilon$  such that  $1 > \varepsilon > 0$ , there is  $n_0 \in \mathbb{N}$  such that if  $n \geq n_0$  then*

- (1) *if  $|\cos(nt)| < 1 - \varepsilon$  then  $n|\sin(nt)| > B_1$ ,*
- (2) *if  $|\cos(nt)| > 1 - \varepsilon$  then  $n^2|\cos(nt)| > B_2$ .*

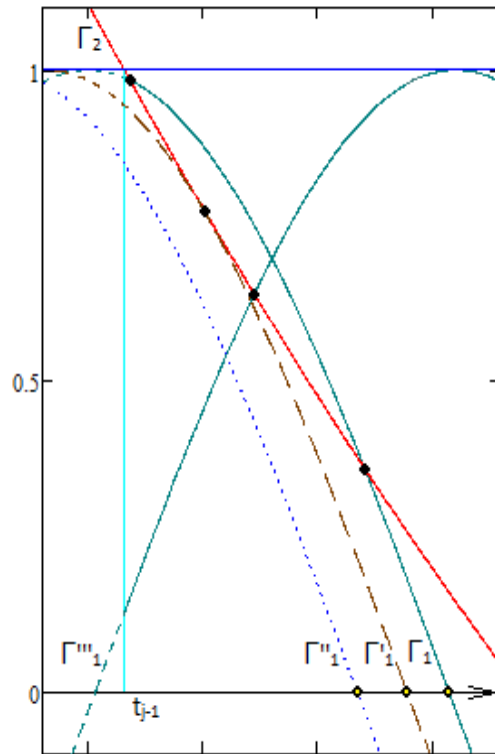
We will also need the following claims.

- (i) There is a bound  $B_1$  of the modulus of the first derivative of  $f_2(t)$ . Indeed  $|f_2'(t)| = |\sum_{j=1}^k j a_j \sin jt| \leq \sum_{j=1}^k j |a_j| =: B_1$ .
- (ii) There is a bound  $B_2$  of the modulus of the second derivative of  $f_2(t)$ . Indeed  $|f_2''(t)| = |\sum_{j=1}^k j^2 a_j \cos jt| \leq \sum_{j=1}^k j^2 |a_j| =: B_2$ .
- (iii) The first derivative of  $f_2(t)$  has a finite number of roots on  $[0, 2\pi]$  so that there is  $\varepsilon_j > 0$  such that  $1 - \varepsilon_j$  is greater than the value at each local maximum and  $-1 + \varepsilon_j$  is less than the value at each local minimum of  $f_2(t)$  on  $(t_{j-1}, t_j)$ .
- (iv) If the domain of a branch of  $\cos nt$  is the subset of the interior of  $I_j$  then  $\cos nt - f_2(t)$  has values of the opposite sign at the end points of the domain so that the branch has at least one intersection point with  $\Gamma_2$ .

Since  $f_2(t)$  is continuous at  $t_{j-1}$  and  $t_j$  it follows that there are  $\delta_{1j} > 0, \delta_{2j} > 0$  such that if  $t \in (t_{j-1}, t_{j-1} + \delta_{1j})$  or  $t \in (t_j - \delta_{2j}, t_j)$  then  $1 - |f_2(t)| < \varepsilon_j$ . If we bring to mind (iii) it follows that  $f_2(t)$  is monotonic on  $(t_{j-1}, t_{j-1} + \delta_{1j})$  and on  $(t_j - \delta_{2j}, t_j)$ . Therefore we can choose  $\delta_{1j} > 0, \delta_{2j} > 0$  such that  $|f_2(t_{j-1} + \delta_{1j})| = 1 - \varepsilon_j, |f_2(t_j - \delta_{2j})| = 1 - \varepsilon_j$

Using Lemma 3(1) there is  $n_j$  such that if  $n \geq n_j$  and  $|f_1(t)| < 1 - \varepsilon_j$  then  $|f_1'(t)| > B_1$ . It follows that on  $E_j := [t_{j-1} + \delta_{1j}, t_j - \delta_{2j}]$  a branch of  $\cos nt$  and  $\Gamma_2$  can not have more than one intersection point: if they have two intersection points  $M_1, M_2$  then using the mean value theorem for the continuous function  $f_1$  the slope  $S$  of the line  $M_1M_2$  is greater than  $B_1$  in modulus. Using the mean value theorem again for the continuous function  $f_2$  it follows that there is a point  $t$  such that  $f_2'(t) = S$  so that  $|f_2'(t)| = |S| > B_1$  which is the contradiction with (i).

It remains to be proved that if the domain of a branch is the subset of  $D_{1j} = (t_{j-1}, t_{j-1} + \delta_{1j}]$  or of  $D_{2j} = [t_j - \delta_{2j}, t_j)$  then the branch of  $\cos nt$  and  $\Gamma_2$  can not have more than one intersection point. Let  $1 > f_2(t) > 1 - \varepsilon_j$  and let the branch has an adjacent branch such that the union of its domains is  $[(k - 1)\pi/n, (k + 1)\pi/n] \subset D_{1j}$  and  $k$  is even. Then using Lemma 3(2) it follows that if  $\cos(nt) > 1 - \varepsilon$  then  $f_1''(t) - f_2''(t) = -n^2 \cos nt - f_2''(t) < -B_2 - f_2''(t)$  is negative. Therefore  $f_1(t) - f_2(t)$  is a concave function so that its graph can have at most two intersection points with the line  $y = 0$ . If such an adjacent branch does not exist which means that  $t_{j-1} \in [k\pi/n, (k+1)\pi/n]$ ,  $k$  is even, then we can prove the concavity of  $f_1(t) - f_2(t)$  in the same manner. We conclude that if  $t_{j-1}$ , the start point of  $I_j$ , is in the domain of a branch of  $\cos nt$  then the branch can have 0, 1, or 2 intersection points with  $\Gamma_2$  (see Figure 1).



**Figure 1.** Graph  $\Gamma_1$ , a branch of  $\cos nt$ , can have 0,  $(\Gamma_1'')$  1,  $(\Gamma_1', \Gamma_1''')$  or 2 ( $\Gamma_1$ ) intersection points with  $\Gamma_2$ , the graph of  $f_2$ .

If  $-1 < f_2(t) < -1 + \varepsilon_j$  after showing the convexity of  $f_1(t) - f_2(t)$  on  $D_1$  the claim follows in the similar manner. Analogously we prove the claim if the domain of a branch is the subset of  $D_2$  as well as the claim for the end point of  $I_j$ : if  $t_j$  is in the domain of a branch of  $\cos nt$  then the branch can have 0, 1, or 2 intersection points with  $\Gamma_2$ .

We conclude that if  $n$  is large enough then each branch of  $\cos nt$ , such that the start and the end point of  $I_j$  are not elements of its domain, has exactly one intersection point with  $\Gamma_2$ . Thus the number  $U_j$  of intersection points of  $\Gamma_1$  and  $\Gamma_2$  differs to the number  $V_j$  of intersection points of  $\Gamma_1$  and the  $t$ -axis,  $t \in I_j$ , by 0,1 or 2, because in the beginning and at the end of  $I_j$  branches are not complete (see Figure 1). If we take the sum  $U_j$  and  $V_j$  over all  $r$  subintervals then it is clear that  $U(P_{2n})$  differs to the number  $V(P_{2n}) = \sum_{j \in J} V_j$  by a number  $\leq 2r$ . Since  $2r$  does not depend on  $n$  it follows that

$$\left( \lim_{n \rightarrow \infty} (1 - C(P_{2n})) \right) = \lim_{n \rightarrow \infty} \frac{U(P_{2n})}{2n} = \lim_{n \rightarrow \infty} \frac{V(P_{2n})}{2n} \left( = \lim_{n \rightarrow \infty} \frac{\sum_{j \in J} V_j}{2n} \right).$$

Since the intersection points of the graphs of  $y = \cos nt$  and the  $t$ -axis are obviously uniformly distributed on  $I_j$  we conclude

$$\lim_{n \rightarrow \infty} \frac{\sum_{j \in J} V_j}{2n} = \frac{\sum_{j \in J} |I_j|}{2\pi}. \quad \square$$

### 2.1. Proof of Lemma 3

Using the symmetry and the periodicity of  $\cos nt$  it is enough to prove the claim for the first branch of  $\cos nt$ ,  $t \in [0, \pi/n]$ . For an arbitrarily chosen  $\varepsilon > 0$  and  $n \in \mathbb{N}$  we determine  $\tau$  such that  $|\cos(n\tau)| = 1 - \varepsilon$ . It follows that  $\tau = \arccos(1 - \varepsilon)/n$  or  $\tau = \arccos(-1 + \varepsilon)/n$  so that

- (1) if  $t \in (\tau, \pi/n - \tau)$  then  $n \sin nt > n \sin n\tau = n \sin(\arccos(1 - \varepsilon)) \rightarrow \infty$  when  $n \rightarrow \infty$ . Therefore the claim follows immediately if we chose

$$n_1 = \left\lceil \frac{B_1}{\sin(\arccos(1 - \varepsilon))} \right\rceil.$$

- (2) if  $t \in (0, \tau) \cup (\pi/n - \tau, \pi/n)$  then  $n^2 |\cos nt| > n^2 |\cos n\tau| = n^2 \cos(\arccos(1 - \varepsilon)) = n^2(1 - \varepsilon) \rightarrow \infty$  when  $n \rightarrow \infty$ . Therefore the claim follows immediately if we chose

$$n_2 = \left\lceil \sqrt{\frac{B_2}{1 - \varepsilon}} \right\rceil.$$

It remains to take  $n_0 = \max(n_1, n_2)$ . □

### 2.2. Algorithm for determination $\lim_{n \rightarrow \infty} C(P_{2n})$

In the proof of Theorem 1 we actually declared steps of an algorithm for determination  $\lim_{n \rightarrow \infty} C(P_{2n})$ :

- (1) determine all real roots  $t_j$  of the equations  $f_2(t) = 1$  and  $f_2(t) = -1$ ,
- (2) arrange them as an increasing sequence  $0 = t_0 < t_1 < \dots < t_p = 2\pi$ ,
- (3) determine  $I_j = [t_{j-1}, t_j]$  such that if  $t_{j-1} < t < t_j$  then  $|f_2(t)| < 1$ ,  $j \in J = \{j_1, j_2, \dots, j_r\} \subseteq \{1, 2, \dots, p\}$ ,
- (4) calculate  $\lim_{n \rightarrow \infty} C(P_{2n}) = 1 - \sum_{j \in J} (t_j - t_{j-1}) / (2\pi)$ .

If  $\overline{f_2(t)}$  is defined:

$$\overline{f_2(t)} = \begin{cases} 1, & |f_2(t)| \geq 1 \\ 0, & \text{otherwise} \end{cases}$$

then

$$\lim_{n \rightarrow \infty} C(P_{2n}) = \frac{1}{2\pi} \int_0^{2\pi} \overline{f_2(t)} dt. \tag{2}$$

### 3. Approximating $\lim_{n \rightarrow \infty} C(P_{2n})$

The equation  $f_2(t) = \pm 1$  i.e.  $-a_0/2 - \sum_{j=1}^k a_j \cos jt = \pm 1$  is algebraic in  $\cos t$  so that  $t_j$  can be expressed by arccosine of an algebraic real number  $\alpha \in [-1, 1]$  thus only solutions of this kind should be taken into account.

We can approximate numerically the integral in (2) i.e.  $\lim_{n \rightarrow \infty} C(P_{2n})$ . Suppose the interval  $[0, 2\pi]$  is divided into  $p$  equal subintervals of length  $\Delta t = 2\pi/p$  so that we introduce a partition of  $[0, 2\pi]$   $0 = t_0 < t_1 < \dots < t_p = 2\pi$  such that  $t_j - t_{j-1} = \Delta t$ . Then we chose numbers  $\xi_j \in [t_j, t_{j-1}]$  and count all  $\xi_j$  such that  $|f_2(\xi_j)| > 1$ ,  $j = 1, 2, \dots, p$ . If there are  $s$  such  $\xi_j$  then  $\lim_{n \rightarrow \infty} C(P_{2n})$  is approximately equal to  $\frac{s}{p}$ .

$$\lim_{n \rightarrow \infty} C(P_{2n}) \approx \frac{1}{p} \sum_{j=1}^p \overline{f_2\left(j \frac{2\pi}{p}\right)}$$

where we chosed  $\xi_j = 2j\pi/p$ .

### 3.1. Small limit points of $C(P_{2n})$

In the case of trinomials i.e. if  $k = 0$ ,  $|a_0| \leq 2$  then all roots of  $P_{2n}(x) = x^{2n} + a_0x^n + 1$  obviously are unimodular. If  $|a_0| > 2$  then  $P_{2n}$  does not have any unimodular root so that  $C(P)$  tends either to zero or to one as  $n$  approaches infinity.

In the case of quadrinomials i.e. if  $k = 1$ ,  $a_0 = 0$ ,  $a_1 = \pm 1$  then  $P_{2n}(x) = x^{2n} \pm x^{n+k} \pm x^{n-k} + 1 = (x^{n-k} \pm 1)(x^{n+k} \pm 1)$  so that obviously all roots are unimodular. If  $|a_1| > 1$  then

$$C(x^{2n} + a_1x^{n+k} + a_1x^{n-k} + 1) = 2 \arccos(1/a_1)/\pi$$

so that it has the minimum value  $2/3$  when  $a_1 = 2$  and  $C(P)$  tends to one as  $a_1$  approaches infinity.

If we exclude trinomials and quadrinomials then it is clear that the limit points of  $C(P_{2n})$  are always greater than zero. A natural question that arises here is what is the smallest value, greater than 0, of the limit points of  $C(P_{2n})$ ?

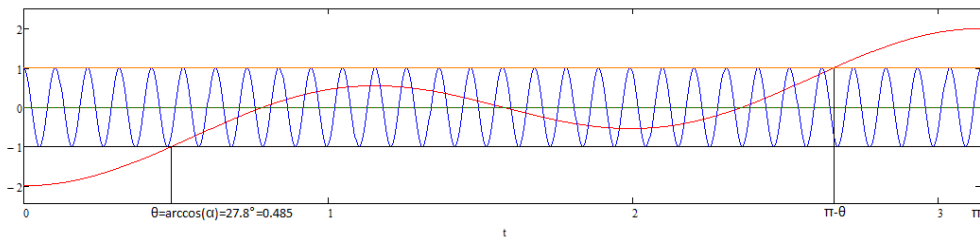
Among all monic reciprocal polynomials  $P_{2n}$ , composed of  $l = 2k + 3 = 5, 7, 9, 11$  monomials, with integer coefficients  $x^{2n} + a_{j_0}x^n + \sum_{i=1}^k a_{j_i}(x^{n+j_i} + x^{n-j_i}) + 1$  an exhaustive search such that  $j_1 = 1, 2, \dots, 10$ ,  $j_i = j_{i-1} + 1, j_{i-1} + 2, \dots, j_{i-1} + 10$ ,  $i = 2, 3, \dots, k$ ;  $a_j = \pm 1, \pm 2, \dots, \pm 10$ ,  $j = j_0, j_1, \dots, j_k$  suggests that  $C(S_l(x))$  has the minimal limit point where we denoted  $S_l(x) = S_{2k+3}(x) = x^{2n} + x^n + \sum_{i=1}^k (x^{n+i} + x^{n-i}) + 1$ . Using the algorithm we solve the equation  $1/2 + \cos t + \cos 2t + \dots + \cos(k)t = \pm 1$ . If we develop  $\cos 4t$ ,  $\cos 3t$  and  $\cos 2t$  and substitute  $\cos t = x$  we get two algebraic equations. By inverse cosine function of its real solutions which are  $\leq 1$  in modulus we get the exact value of  $\lim_{n \rightarrow \infty} C(S_l)$ .

Among all monic reciprocal polynomials  $P_{2n}$ , composed of  $l = 2k + 2 = 6, 8, 10$  monomials, with integer coefficients  $x^{2n} + \sum_{i=1}^k a_{j_i}(x^{n+j_i} + x^{n-j_i}) + 1$  an exhaustive search such that  $j_1 = 1, 2, \dots, 10$ ,  $j_i = j_{i-1} + 1, j_{i-1} + 2, \dots, j_{i-1} + 10$ ,  $i = 2, 3, \dots, k$ ;  $a_j = \pm 1, \pm 2, \dots, \pm 10$ ,  $j = j_1, j_2, \dots, j_k$  suggests that  $C(S_l(x))$  has the minimal limit point where we denoted  $S_l(x) = S_{2k+2}(x) = x^{2n} + \sum_{i=1}^k (x^{n+2i-1} + x^{n-(2i-1)}) + 1$ . Using the algorithm we solve the equation  $\cos t + \cos 3t + \dots + \cos(2k-1)t = \pm 1$ . If we develop  $\cos 7t$ ,  $\cos 5t$  and  $\cos 3t$  and substitute  $\cos t = x$  we get two algebraic equations. Using inverse cosine function of its real solutions which are  $\leq 1$  in modulus we determine the exact value of  $\lim_{n \rightarrow \infty} C(S_l)$ .

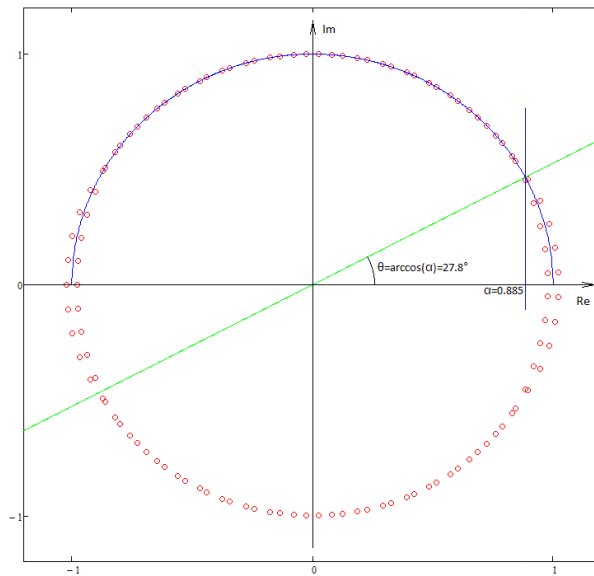
**Table 1.** Among all monic reciprocal polynomials  $P_{2n} \in \mathbb{Z}(x)$ , composed of  $l$  monomials,  $S_l(x)$  has the smallest  $\lim_{n \rightarrow \infty} C(P_{2n})$ .

$l$	Polynomial $S_l(x)$	$\lim_{n \rightarrow \infty} C(S_l)$	Exact value of $\lim_{n \rightarrow \infty} C(S_l)$
5	$x^{2n} + x^{n+1} + x^n + x^{n-1} + 1$	1/3	1/3
6	$x^{2n} + x^{n+3} + x^{n+1} + x^{n-1} + x^{n-3} + 1$	0.308799876	$\frac{2}{\pi} \arccos(\alpha)$ , $\alpha = \frac{\sqrt[3]{\frac{\sqrt{57} + 23}{288} + \frac{1}{6}}}{\sqrt[3]{\frac{\sqrt{57}}{72} + \frac{1}{8}}}$
7	$S_5(x) + x^{n+2} + x^{n-2}$	0.274187115	$(1/\pi) \arccos((\sqrt{13} - 1)/4)$
8	$S_6(x) + x^{n+5} + x^{n-5}$	0.243784699	$\frac{2}{\pi} \arccos(\alpha_8)$
9	$S_7(x) + x^{n+3} + x^{n-3}$	0.218549881	$\frac{1}{\pi} \arccos\left(\frac{36\sqrt[3]{\frac{61\sqrt{29} + 3035}{10368} + \frac{3035}{93312}} - 6\sqrt[3]{\frac{\sqrt{29}}{48} + \frac{61}{432} + 7}}{36\sqrt[3]{\frac{\sqrt{29}}{48} + \frac{61}{432}}}\right)$
10	$S_8(x) + x^{n+7} + x^{n-7}$	0.197681551	$\frac{2}{\pi} \arccos(\alpha_{10})$
11	$S_9(x) + x^{n+4} + x^{n-4}$	0.208201295	$\frac{1}{\pi} (\arccos(\alpha_{11}) + \frac{\pi}{3} - \arccos(\beta_{11}))$

In Table 1 we presented  $S_l(x)$  as results of our experiments and value of  $\lim_{n \rightarrow \infty} C(S_l)$  where  $\alpha_8 = 0.927571571$  is the real solution of  $16x^5 - 16x^3 + 3x = 1$ ,  $\alpha_{10} = 0.952175588$  is the real solution of  $64x^7 - 96x^5 + 40x^3 - 4x = 1$ ,  $\alpha_{11} = 0.843858756$  is the real solution of  $8x^4 - 4x^3 - 6x^2 - 2x + 1/2 = 1$ ,  $\beta_{11} = 0.573949518$  is the real solution of  $8x^4 - 4x^3 - 6x^2 - 2x + 1/2 = -1$ .



**Figure 2.** Each intersection point of graph of  $f_1 = \cos 60t$  (blue curve) and  $f_2 = -\cos t - \cos 2t$  (red curve) corresponds to an unimodular root of  $x^{120} + x^{63} + x^{61} + x^{59} + x^{57} + 1$ .



**Figure 3.** Unit circle sectors where unimodular roots of  $x^{120} + x^{63} + x^{61} + x^{59} + x^{57} + 1$  are located.

Each intersection point of graph of  $f_1 = \cos 60t$  and  $f_2 = -\cos t - \cos 2t$  (see Figure 2) corresponds to an unimodular root of the reciprocal polynomial  $x^{120} + x^{63} + x^{61} + x^{59} + x^{57} + 1$ . Nonunimodular roots have arguments in  $[-\theta, \theta]$  or in  $[\pi - \theta, \pi + \theta]$  where  $\theta = \arccos(\alpha) \approx 0.485 \approx 27.8^\circ$  (see Figure 3) and  $\alpha$  is the real root of  $4x^3 - 2x = 1$  having the exact value presented in Table 1 in the row  $l = 6$ . Since there are 41 intersection point on  $[0, \pi]$  and  $f_1, f_2$  are both even it follows that there are  $120 - 2 \cdot 41 = 38$  nonunimodular roots so that  $C(P_{120}) = 38/120 \approx 0.317$  is close to the limit of  $C(P_{2n}) \rightarrow \frac{2}{\pi} \arccos(\alpha) = \frac{2}{\pi} \theta \approx 0.308799876$ .

### 3.2. Polynomials with smallest limit points of $C(P_{2n})$

Our calculations suggest that the next conjecture seems to be true:

**Conjecture 4.** *If  $P_{2n}$  is a sum of  $l = 2k + 3$  monomials, i.e.  $a_0 \neq 0$ , then the sequence  $C(x^{2n} + x^{n+k} + \dots + x^{n+2} + x^{n+1} + x^n + x^{n-1} + x^{n-2} + \dots + x^{n-k} + 1)$  tends to the smallest limit, greater than zero, of  $C(P_{2n})$ ,  $n \rightarrow \infty$ .*



Let  $k \geq 1$  be an integer. If  $P_{2n}$  is a sum of  $l = 2k + 2$  monomials, i.e.  $a_0 = 0$ , then the sequence  $C(x^{2n} + x^{n+2k-1} + \dots + x^{n+5} + x^{n+3} + x^{n+1} + x^{n-1} + x^{n-3} + x^{n-5} + \dots + x^{n-(2k-1)} + 1)$  tends to the smallest limit, greater than zero, of  $C(P_{2n})$ ,  $n \rightarrow \infty$ .

But in the case of dodecanomials we found that  $C(x^{2n} + x^{n+9} + x^{n+7} + 2x^{n+5} + 2x^{n+3} + 2x^{n+1} + 2x^{n-1} + 2x^{n-3} + 2x^{n-5} + x^{n-7} + x^{n-9} + 1)$  tends to  $2 \arccos(0.943468)/\pi = 0.215085$  which is smaller than  $0.226163 = 2(\arccos(0.966357) + \arccos(0.877575) - \arccos(0.919147))/\pi$  the limit of  $C(x^{2n} + x^{n+9} + x^{n+7} + x^{n+5} + x^{n+3} + x^{n+1} + x^{n-1} + x^{n-3} + x^{n-5} + x^{n-7} + x^{n-9} + 1)$ . Nevertheless the conjecture seems to be true for many  $k$ .

It is natural to ask: do the following limits exist

$$\lim_{k \rightarrow \infty} \lim_{n \rightarrow \infty} C \left( x^n \left( x^n + \frac{1}{x^n} + \sum_{j=1}^k \left( x^{2j-1} + \frac{1}{x^{2j-1}} \right) \right) \right),$$

$$\lim_{k \rightarrow \infty} \lim_{n \rightarrow \infty} C \left( x^n \left( x^n + 1 + \frac{1}{x^n} + \sum_{j=1}^k \left( x^{2j} + \frac{1}{x^{2j}} \right) \right) \right)?$$

Our experiments with  $k \approx$  half of million,  $n \approx$  one hundred million suggest that these limits exist and that they are both equal to 0.20885.

#### 4. Extension of Mahler measure

The definition of the Mahler measure could be extended to polynomials in several variables. We recall Jensen’s formula which states that  $\frac{1}{2\pi} \int_0^{2\pi} \log |P(e^{i\theta})| d\theta = \log |a_0| + \sum_{j=1}^d \log \max(|\alpha_j|, 1)$  Thus

$$M(P) = \exp \left\{ \frac{1}{2\pi} \int_0^{2\pi} \log |P(e^{i\theta})| d\theta \right\},$$

so  $M(P)$  is just the geometric mean of  $|P(z)|$  on the torus  $T$ .

Hence a natural candidate for  $M(F)$  is

$$M(F) = \exp \left\{ \frac{1}{(2\pi)^r} \int_0^{2\pi} \dots \int_0^{2\pi} \log |F(e^{i\theta_1}, \dots, e^{i\theta_r})| d\theta_r \right\}.$$

The smallest known Mahler measures in two variables are (see [3])

$$M((x + 1)y^2 + (x^2 + x + 1)y + x(x + 1)) = 1.25542\dots$$

and

$$M(y^2 + (x^2 + x + 1)y + x^2) = 1.28573\dots$$

Boyd proved [4] the next

**Theorem 5.** As  $m \rightarrow \infty$ ,  $M(P(x, x^m)) \rightarrow M(P(x, y))$ .

Let  $Q(x_1, x_2) = \sum_{j=0}^k \alpha_j x_1^{e_{j1}} x_2^{e_{j2}}$ ,  $\alpha_j \in \mathbb{R}$ ,  $e_{ji} \in \mathbb{Z}$ ,  $j = 0, 1, \dots, k$ ,  $i = 1, 2$  and let

$$W(x_1, x_2) = x_1^n x_2^n (x_1^n x_2^n + x_1^{-n} x_2^{-n} + Q(x_1, x_2) + Q(x_1^{-1}, x_2^{-1}))$$

where  $n \in \mathbb{N}$  is greater than  $\max(|e_{ji}|)$  so that  $W$  is a bivariate polynomial. Let

$$g_2(x_1, x_2) := -1/2(Q(x_1, x_2) + Q(x_1^{-1}, x_2^{-1}))$$

and

$$\overline{g_2(x_1, x_2)} := \begin{cases} 1, & |g_2(x_1, x_2)| \geq 1 \\ 0, & \text{otherwise,} \end{cases} \tag{3}$$

then we can define

$$LC(W) := \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} \overline{g_2(\exp(it_1), \exp(it_2))} dt_1 dt_2. \tag{4}$$

If  $x_2 = 1$  and if we chose the coefficients  $\alpha_j$  of  $Q$  in such a way that  $Q(x_1, 1) = a_0/2 + \sum_{j=1}^k a_j x_1^j$  then  $W(x_1, 1) = P_{2n}(x_1)$ ,  $g_2(\exp(it), 1) = f_2(t)$  and  $\overline{g_2(\exp(it), 1)} = \overline{f_2(t)}$  so that, recalling (2), we conclude that  $LC(P_{2n}) = \lim_{n \rightarrow \infty} C(P_{2n})$ .

If  $Q(x, y) = x + y + 1$  we can prove that the Boyd's property for  $LC$  is valid:  $LC(W(x, x^m)) \rightarrow LC(W(x, y))$  as  $m \rightarrow \infty$ . Indeed  $LC(x^{2n} + x^n(x + x^m + 2 + x^{-1} + x^{-m})) + 1) = \lim_{n \rightarrow \infty} C(x^{2n} + x^n(x + x^m + 2 + x^{-1} + x^{-m})) + 1) = \frac{1}{2\pi} \int_0^{2\pi} \overline{f_{2m}(t)}$ , where  $f_{2m}(t) = -1 - \cos(mt) - \cos(t)$ . Since  $\cos(t) = -\cos(\pi - t)$  it follows that  $\cos(2(m_1 + 1)t) = -\cos(\pi - 2(m_1 + 1)t) = -\cos((2m_1 + 1)(\pi - t))$ . Therefore if  $m$  is odd then for each interval  $I = [a, b] \subseteq [0, \pi]$  such that  $|f_{2m}(t)| > 1$ ,  $a < t < b$ , there is the interval  $I' = [\pi - b, \pi - a]$  of the equal length such that  $|f_{2m}(t)| \leq 1$ ,  $t \in I'$ . We conclude that  $\frac{1}{\pi} \int_0^\pi \overline{f_{2m}(t)} = 0.5$  for  $m$  odd so that  $LC(W(x, x^m)) \rightarrow 0.5$  as  $m \rightarrow \infty$ .

On the other hand

$$LC(W) = \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} \overline{-\frac{1}{2}(\exp(it_1) + \exp(it_2) + 2 + \exp(-it_1) + \exp(-it_2))} dt_1 dt_2$$

$$= \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} \overline{(-1 - \cos t_1 - \cos t_2)} dt_1 dt_2.$$

Since if  $t_1, t_2 \in [0, \pi]$  then  $|1 + \cos t_1 + \cos t_2| \geq 1$  is equivalent with  $t_2 \leq \pi - t_1$  and using the symmetry of the set  $\{(t_1, t_2) \in [0, 2\pi] \times [0, 2\pi] : -1 - \cos t_1 - \cos t_2 \geq 1\}$  it follows that

$$LC(W) = \frac{4}{(2\pi)^2} \int_0^\pi \left( \int_0^{\pi-t_1} dt_2 \right) dt_1 = \frac{1}{\pi^2} \int_0^\pi (\pi - t_1) dt_1 = \frac{1}{2}.$$

This example as well as numerical approximations of  $LC$  of many other polynomials in two variables using the formula (2) and the definition (4) suggest us that the Boyd's limit formula in Theorem 5 is also valid for  $LC$  i.e. we propose the following

**Theorem 6.** *As  $m \rightarrow \infty$ ,  $LC(W(x, x^m)) \rightarrow LC(W(x, y))$ .*

To prove Theorem 6 we use two lemmas which Everest and Ward proved in [6]. Denote as usual the (multiplicative) circle group by  $K = \mathbb{S}^1$ , and the torus by  $K^2 = \mathbb{S}^1 \times \mathbb{S}^1$ . For an integrable function  $f : K \rightarrow \mathbb{C}$ , write

$$\int_0^1 f(e^{2\pi i\theta}) d\theta = \int f(x) d\mu_K = \int f d\mu_K \tag{5}$$

and for an integrable function  $g : K^2 \rightarrow \mathbb{C}$ , write

$$\int_0^1 \int_0^1 g(e^{2\pi i\theta_1}, e^{2\pi i\theta_2}) d\theta_1 d\theta_2 = \int g(x_1, x_2) d\mu_{K^2} = \int g d\mu_{K^2}. \tag{6}$$

We will use the Lebesgue measure  $\mu_K$  on the circle to evaluate the measure of disjoint unions of intervals (whose measure is simply the sum of the lengths).

**Lemma 7.** *Let  $\phi : K^2 \rightarrow \mathbb{R}$  be any continuous function. Then*

$$\lim_{N \rightarrow \infty} \int \phi(x, x^N) d\mu_K = \int \phi d\mu_{K^2}$$

**Lemma 8.** *Let  $\phi : K^2 \rightarrow \mathbb{R}$  be any Riemann-integrable function and  $\delta > 0$  be given. There are finite trigonometric series  $P(x_1, x_2) = \sum_{\|\mathbf{n}\| < M} a_n x_1^{n_1} x_2^{n_2}$  and  $Q(x_1, x_2) = \sum_{\|\mathbf{n}\| < M} b_n x_1^{n_1} x_2^{n_2}$  with the property that*

$$P(x_1, x_2) \leq \phi(x_1, x_2) \leq Q(x_1, x_2)$$

for all  $(x_1, x_2) \in K^2$  and

$$\int (Q - P) d\mu_{K^2} < \delta,$$

where  $\|\mathbf{n}\| = \max\{|n_1|, |n_2|\}$ ,  $M > 0$ .

The finite sums  $P, Q$  used to bound  $\phi$  are called trigonometric polynomials since in the additive group notation the monomial  $x_1^{n_1} x_2^{n_2}$  corresponds to  $e^{2\pi i(n_1\theta_1+n_2\theta_2)}$  under the correspondence  $x_1 = e^{2\pi i\theta_1}, x_2 = e^{2\pi i\theta_2}$ .

**Proof of Theorem 6.** If  $r = 2$  in (3) then function  $\overline{g_2(x_1, x_2)}$  is not continuous but is Riemann-integrable (since it is bounded and the set of discontinuities of  $g_2$  has measure 0). By Lemma 8 there are finite trigonometric series  $P_2(x_1, x_2) = \sum_{\|n\| < M} a_n x_1^{n_1} x_2^{n_2}$  and  $Q_2(x_1, x_2) = \sum_{\|n\| < M} b_n x_1^{n_1} x_2^{n_2}$  with the property that

$$P_2(x_1, x_2) \leq \overline{g_2(x_1, x_2)} \leq Q_2(x_1, x_2)$$

for all  $(x_1, x_2) \in K^2$  and

$$\int (Q_2 - P_2) d\mu_{K^2} < \delta.$$

It follows that

$$\int P_2(x, x^m) d\mu_K \leq \int \overline{g_2(x, x^m)} d\mu_K \leq \int Q_2(x, x^m) d\mu_K. \tag{7}$$

Function  $P_2, Q_2$  are continuous so by Lemma 7

$$\int P_2(x, x^m) d\mu_K \rightarrow \int P_2(x_1, x_2) d\mu_{K^2}, \tag{8}$$

$$\int Q_2(x, x^m) d\mu_K \rightarrow \int Q_2(x_1, x_2) d\mu_{K^2}. \tag{9}$$

Since  $\int Q_2 d\mu_{K^2} - \int P_2 d\mu_{K^2} < \delta$  and  $\delta > 0$  was arbitrary, (7), (8) and (9) then show that

$$\int \overline{g_2(x, x^m)} d\mu_K \rightarrow \int \overline{g_2(x_1, x_2)} d\mu_{K^2}.$$

Recalling (5), (6) and using the substitutions  $t = 2\pi\theta, t_1 = 2\pi\theta_1, t_2 = 2\pi\theta_2$  it follows that

$$\int g_2(x, x^m) d\mu_K = \int_0^1 g_2(e^{2\pi i\theta}, e^{2m\pi i\theta}) d\theta = \frac{1}{2\pi} \int_0^{2\pi} g_2(e^{it}, e^{mit}) dt = LC(W(x, x^m))$$

and

$$\begin{aligned} \int \overline{g_2(x_1, x_2)} d\mu_{K^2} &= \int_0^1 \int_0^1 g_2(e^{2\pi i\theta_1}, e^{2\pi i\theta_2}) d\theta_1 d\theta_2 = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} g_2(e^{it_1}, e^{it_2}) dt_1 dt_2 \\ &= LC(W(x_1, x_2)), \end{aligned}$$

hence the claim follows. □

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