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Functional Analysis / *Analyse fonctionnelle*

Convex maps on $\mathbb{R}^n$ and positive definite matrices

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Abstract. We obtain several convexity statements involving positive definite matrices. In particular, if A, B, X, Y are invertible matrices and A, B are positive, we show that the map

$$(s,t) \mapsto \operatorname{Tr}\log\left(X^* A^s X + Y^* B^t Y\right)$$

is jointly convex on $\mathbb{R}^2$. This is related to some exotic matrix Hölder inequalities such as

$$\left\|\sinh\left(\sum_{i=1}^m A_i B_i\right)\right\| \le \left\|\sinh\left(\sum_{i=1}^m A_i^p\right)\right\|^{1/p} \left\|\sinh\left(\sum_{i=1}^m B_i^q\right)\right\|^{1/q}$$

for all positive matrices A_i, B_i, such that $A_i B_i = B_i A_i$, conjugate exponents p, q and unitarily invariant norms $\|\cdot\|$. Our approach to obtain these results consists in studying the behaviour of some functionals along the geodesics of the Riemanian manifold of positive definite matrices.

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1. Convex and log-convex maps

This short note aims to point out some convex maps involving positive definite matrices. We denote by $\mathbb{M}_n$ the space of n-by-n matrices with complex entries, and by $\mathbb{P}_n$ its positive definite cone. A non-negative, continuous function $f(t)$ defined on $[0,\infty)$ is geometrically convex if $f(\sqrt{ab}) \le \sqrt{f(a)f(b)}$ for all $a, b > 0$, equivalently if $\log f(e^t)$ is convex on $\mathbb{R}$. Note that a function $\varphi(t)$ on $(0,\infty)$ satifies the geometric-arithmetic convexity inequality

$$\varphi(\sqrt{ab}) \le \frac{\varphi(a)+\varphi(b)}{2}, \qquad a, b > 0,$$

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if and only if $e^{\varphi(t)}$ is geometrically convex, equivalently $\varphi(e^t)$ is convex on $\mathbb{R}$. This convexity property can be extended to the matrix setting as follows.

Theorem 1. *Let $\varphi(t)$ be a non-decreasing function defined on $(0,\infty)$ such that $\varphi(e^t)$ is convex. Let $A_i \in \mathbb{P}_n$ and $X_i \in \mathbb{M}_n$ be invertible, $i = 1,\dots,m$. Then, the map*

$$(t_1,\dots,t_m) \mapsto \operatorname{Tr}\varphi\left(\sum_{i=1}^m X_i^* A_i^{t_i} X_i\right)$$

is jointly convex on $\mathbb{R}^m$.

Letting $\varphi(t) = \log t$, we get the statement of the Abstract. Theorem 1 can be derived from the following more general log-convexity theorem. Recall that a symmetric norm on $\mathbb{M}_n$ satisfies $\|UAV\| = \|A\|$ for all $A \in \mathbb{M}_n$ and all unitary matrices $U, V \in \mathbb{M}_n$. We denote by $\mathbb{M}_n^+$ the positive semi-definite cone of $\mathbb{M}_n$. A positive linear map $\Phi : \mathbb{M}_n \mapsto \mathbb{M}_d$ satifies $\Phi(\mathbb{M}_n^+) \subset \mathbb{M}_d^+$. A classical example is the Schur multipler $A \mapsto Z \circ A$ with $Z \in \mathbb{M}_n^+$. If $A \in \mathbb{M}_n^+$ is not invertible, we naturally define for $t \ge 0$, the generalized inverse $A^{-t} := (A+F)^{-t}E$ where F is the projection onto the nullspace of A and E is the range projection of A.

Theorem 2. *Let $A_i \in \mathbb{M}_n^+$ and $X_i \in \mathbb{M}_n$, $i = 1,\dots,m$, and let $\Phi : \mathbb{M}_n \to \mathbb{M}_d$ be a positive linear map. Then, for all symmetric norms and all non-decreasing geometrically convex function $g(t)$, the map*

$$(t_1,\dots,t_m) \mapsto \left\| g\left(\Phi\left(\sum_{i=1}^m X_i^* A_i^{t_i} X_i\right)\right)\right\|$$

is jointly log-convex on $\mathbb{R}^m$.

We will prove in the next section these two theorems. Here are some special cases of Theorem 2.

Corollary 3. *Let $A, Z \in \mathbb{M}_n^+$. Then, for all symmetric norms and all non-decreasing geometrically convex function $g(t)$,*

$$\left\|g(Z\circ I)\right\|^2 \le \left\|g(Z\circ A)\right\| \cdot \left\|g(Z\circ A^{-1})\right\|.$$

Corollary 4. *Let $A_i \in \mathbb{M}_n^+$ and $X_i \in \mathbb{M}_n$, $i = 1,\dots,m$. Then, for all symmetric norms and all non-decreasing geometrically convex function $g(t)$,*

$$\left\| g\left(\sum_{i=1}^m X_i^* X_i\right)\right\|^2 \le \left\| g\left(\sum_{i=1}^m X_i^* A_i X_i\right)\right\| \cdot \left\| g\left(\sum_{i=1}^m X_i^* A_i^{-1} X_i\right)\right\|.$$

Corollary 5. *Let $A_i \in \mathbb{M}_n^+$ and $\lambda_i > 0$, $i = 1,\dots,m$, such that $\sum_{i=1}^m \lambda_i = 1$. Let $p > 1$ and $p^{-1}+q^{-1} = 1$. Then, for all symmetric norms and all non-decreasing geometrically convex function $g(t)$,*

$$\left\| g\left(\sum_{i=1}^m \lambda_i A_i\right)\right\| \le \left\|g(I)\right\|^{1/q} \cdot \left\| g\left(\sum_{i=1}^m \lambda_i A_i^p\right)\right\|^{1/p}.$$

If $f(t)$ and $g(t)$ are geometrically convex then so are $f(t)+g(t)$, $\max\{f(t),g(t)\}$, $f(t)g(t)$, $e^{f(t)}$ and $f^{\alpha}(t)$ for all $\alpha > 0$. Hence the above results may be applied to a large class of functions, for instance

$$g(t) = \sum_{k=1}^p c_k t^{\alpha_k}, \qquad c_k > 0,\ \alpha_k \ge 0$$

or

$$g(t) = \max\{c, \beta t^{\alpha}\}, \qquad c,\alpha,\beta \ge 0.$$

Some interesting examples of geometrically convex (also called multiplicatively convex) functions defined on a sub-interval of the positive half-line are given in [5]. These functions can be used to obtain exotic matrix inequalities. A recent study [4] of a two variables log-convex functional have provided many classical and new matrix inequalities.

Remark 6. By using the generalized inverse and a limit argument, Theorem 1 also holds for not necessarily invertible matrices A_i, X_i, $i = 1,\dots,m$, provided that $\varphi(t)$ can be extended as a continuous function on $[0,\infty)$, or the matrix

$$\sum_{i=1}^{m} X_i^* E_i X$$

is positive definite, where E_i stands for the range projection of A_i.

2. Geodesics and log-majorization

The space $\mathbb{P}_n$ of n-by-n positive definite matrices is a symmetric Riemannian manifold. There exists a unique geodesic joining two distinct points $A, B \in \mathbb{P}_n$, that can be parametrized as

$$t \mapsto A\#_t B = A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2}, \qquad t \in (-\infty,\infty). \tag{1}$$

In particular, the middle point between A and B is $A\#_{1/2}B$, the geometric mean, often merely denoted as $A\#B$. For a general t, especially when $t \in (0,1)$, $A\#_t B$ is a weigthed geometric mean. We refer to [3] for a background on the geometric mean and $\mathbb{P}_n$.

Given $S, T \in \mathbb{M}_n^+$, the weak log-majorization relation $S \prec_{w\log} T$ means that

$$\prod_{j=1}^{k} \lambda_j(S) \le \prod_{j=1}^{k} \lambda_j(T)$$

for all $k = 1,\dots,n$, where $\lambda_1(\cdot) \ge \cdots \ge \lambda_n(\cdot)$ stand for the eigenvalues arranged in nonincreasing order. We denote by $S^{\downarrow}$ the diagonal matrix with the eigenvalues $\lambda_1(S),\dots,\lambda_n(S)$ down to the diagonal.

Theorem 7. *Let $A_i, B_i \in \mathbb{P}_n$, $i = 1,\dots,m$ and let $\Phi : \mathbb{M}_n \to \mathbb{M}_d$ be a positive linear map. Then, for all symmetric norms and all non-decreasing geometrically convex function $g(t)$, the map*

$$(t_1,\dots,t_m) \mapsto \left\| g\left(\Phi\left(\sum_{i=1}^{m} A_i \#_{t_i} B_i\right)\right)\right\|$$

is jointly log-convex on $\mathbb{R}^m$.

Proof. Let $A, B \in \mathbb{P}_n$ and let $\Psi : \mathbb{M}_n \to \mathbb{M}_d$ be a positive linear map. We first prove the single variable case of the theorem by showing that the function

$$t \mapsto \|g(\Psi(A\#_t B))\| \tag{2}$$

is log convex on $(-\infty,\infty)$. From Ando's operator inequality [1],

$$\Psi(A\#B) \le \Psi(A)\#\Psi(B),$$

and the relation $\Psi(A)\#\Psi(B) = \Psi(A)^{1/2} V \Psi(B)^{1/2}$ for some unitary $V \in \mathbb{M}_d$, we infer by Horn's inequality (see [2, p. 94]), the weak log-majorization

$$\Psi(A\#B) \prec_{w\log} \Psi(A)^{1/2\downarrow}\Psi(B)^{1/2\downarrow}$$

Since $g(t)$ is geometrically convex, we have $g(e^{(a+b)/2}) \le \sqrt{g(e^a)g(e^b)} \le (g(e^a)+g(e^b))/2$. Hence $t \mapsto g(e^t)$ is a non-decreasing convex function on $(-\infty,\infty)$. The above weak log-majorization then ensures that

$$g(\Psi(A\#B)) \prec_w g(\Psi(A)^{1/2\downarrow}\Psi(B)^{1/2\downarrow})$$

and using that $g(t)$ is geometrically convex, we infer

$$g(\Psi(A\#B)) \prec_w g\left(\Psi(A)\right)^{1/2\downarrow} g\left(\Psi(B)\right)^{1/2\downarrow}.$$

This weak majorization says that

$$\|g(\Psi(A\#B))\| \le \left\| g\Big(\Psi(A)\Big)^{1/2\downarrow} g\left(\Psi(B)\right)^{1/2\downarrow} \right\|$$

for all symmetric norms. The Cauchy–Schwarz inequality for symmetric norms [2, p. 95] yields

$$\|g(\Psi(A\#B))\| \le \|g(\Psi(A))\|^{1/2}\|g(\Psi(B)\|^{1/2}.$$

Since $A\#_{(s+t)/2}B = (A\#_s B)\#(A\#_t B)$, we get

$$\|g(\Psi(A\#_{(s+t)/2}B))\| \le \|g(\Psi(A\#_s B))\|^{1/2}\|g(\Psi(A\#_t B))\|^{1/2}, \tag{3}$$

for all $s, t \in (-\infty,\infty)$, thus (2) is a log-convex function.

We turn to the severable variables case. Let $\Phi : \mathbb{M}_n \to \mathbb{M}_d$ be a positive linear map, and let $A_i, B_i \in \mathbb{P}_n$, $i = 1,\dots,m$. Consider the two block diagonal matices in $\mathbb{M}_m(\mathbb{M}_n)$,

$$A = A_1\#_{s_1}B_1 \oplus\cdots\oplus A_m\#_{s_m}B_m, \quad B = A_1\#_{t_1}B_1 \oplus\cdots\oplus A_m\#_{t_m}B_m,$$

so that

$$A\#_{1/2}B = A_1\#_{\frac{s_1+t_1}{2}}B_1 \oplus\cdots\oplus A_m\#_{\frac{s_m+t_m}{2}}B_m.$$

Define the positive linear map $\Psi : \mathbb{M}_m(\mathbb{M}_n) \to \mathbb{M}_n$,

$$\Psi([A_{i,j}]) := \Phi\left(\sum_{i=1}^m A_{i,i}\right).$$

From (3) with $s = 0$, and $t = 1$, we get

$$\left\|g\left(\Phi\left(\sum_{i=1}^m A_i\#_{\frac{s_i+t_i}{2}}B_i\right)\right)\right\| \le \left\|g\left(\Phi\left(\sum_{i=1}^m A_i\#_{s_i}B_i\right)\right)\right\|^{1/2}\left\|g\left(\Phi\left(\sum_{i=1}^m A_i\#_{t_i}B_i\right)\right)\right\|^{1/2}$$

which completes the proof. □

Corollary 8. *Let $\varphi(t)$ be a non-decreasing function defined on* $(0,\infty)$. *Suppose that* $\exp\varphi(t)$ *is geometrically convex and let $A_i, B_i \in \mathbb{P}_n$, $i = 1,\dots,m$. Then, the map*

$$(t_1,\dots,t_m) \mapsto \operatorname{Tr}\varphi\left(\sum_{i=1}^m A_i\#_{t_i}B_i\right)$$

is jointly convex on $\mathbb{R}^m$.

Proof. Let $\varphi(t) = \log g(t)$, where $g(t)$ is geometrically convex. Since $g^\alpha(t)$ is also geometrically convex for all $\alpha > 0$, Theorem 7 with the normalized trace norm shows that the map

$$(t_1,\dots,t_m) \mapsto \frac{1}{n}\operatorname{Tr} g^\alpha\left(\sum_{i=1}^m A_i\#_{t_i}B_i\right)$$

is jointly log-convex, and so is

$$(t_1,\dots,t_m) \mapsto \left\{\frac{1}{n}\operatorname{Tr} g^\alpha\left(\sum_{i=1}^m A_i\#_{t_i}B_i\right)\right\}^{1/\alpha}.$$

Letting $\alpha \searrow 0$, we infer that the map

$$(t_1,\dots,t_m) \mapsto \det^{1/n} g\left(\sum_{i=1}^m A_i\#_{t_i}B_i\right)$$

is jointly log-convex. Thus the map

$$(t_1,\dots,t_m) \mapsto \log\det g\left(\sum_{i=1}^m A_i\#_{t_i}B_i\right) = \operatorname{Tr}\varphi\left(\sum_{i=1}^m A_i\#_{t_i}B_i\right)$$

is jointly convex. □

Theorem 7 can be regarded as a generalized Hölder inequality. This is more transparent for a single variable and pairs of commuting operators. Note that for two commuting positive definite matrices, $A\#_t B = A^{1-t}B^t$. Letting $t = q^{-1}$ $(= 0p^{-1} + 1q^{-1})$ and using Theorem 7 yields our next and last corollary.

Corollary 9. *Let $A_i, B_i \in \mathbb{M}_n^+$ such that $A_iB_i = B_iA_i$, $i = 1,\dots,m$. Let $p > 1$ and $p^{-1}+q^{-1} = 1$. Then, for all symmetric norms and all non-decreasing geometrically convex function $g(t)$,*

$$\left\| g\left(\sum_{i=1}^m A_iB_i\right)\right\| \le \left\| g\left(\sum_{i=1}^m A_i^p\right)\right\|^{1/p} \cdot \left\| g\left(\sum_{i=1}^m B_i^q\right)\right\|^{1/q}.$$

Choosing $g(t) = \sinh t$, we recapture the Hölder inequality of the Abstract.

We close the paper by showing that Theorem 7 is equivalent to Theorem 2 (and similarly for Corollary 8 and Theorem 1). To this end, first note that by a limit argument we may assume that, in Theorem 2, X_i and A_i are invertible, $i = 1,\dots,m$. Then, using the polar decomposition $X_i = U|X_i|$, observe that

$$X_i^* A^{t_i} X_i = |X_i|(U^* AU)^{t_i}|X_i| = C\#_{t_i} D$$

with $C = |X_i|^2$ and $D = |X_i|U^* AU|X_i| = X_i^* AX_i$.

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