INSTITUT DE FRANCE
Académie des sciences

Comptes Rendus

Mathématique

Arnaud Beauville

A non-hyperelliptic curve with torsion Ceresa class

Volume 359, issue 7 (2021), p. 871-872

Published online: 17 September 2021

https://doi.org/10.5802/crmath.226

Les Comptes Rendus. Mathématique *sont membres du*
Centre Mersenne pour l'édition scientifique ouverte
www.centre-mersenne.org
e-ISSN : 1778-3569

Comptes Rendus
Mathématique
2021, 359, n° 7, p. 871-872
https://doi.org/10.5802/crmath.226

Algebraic geometry / *Geometrie algébrique*

A non-hyperelliptic curve with torsion Ceresa class

Arnaud Beauville[*, a]

[a] Université Côte d'Azur, CNRS – Laboratoire J.-A. Dieudonné, Parc Valrose, F-06108 Nice cedex 2, France.
E-mail: arnaud.beauville@unice.fr

Abstract. We exhibit a non-hyperelliptic curve C of genus 3 such that the class of the Ceresa cycle $[C]-[-C]$ in the intermediate Jacobian of JC is torsion.

Manuscript received 21st May 2021, accepted 24th May 2021.

1. Introduction

Let C be a complex curve of genus $g \geq 3$, and p a point of C. We embed C into its Jacobian J by the Abel–Jacobi map $x \mapsto [x]-[p]$. The *Ceresa cycle* $\mathfrak{z}_p(C)$ is the cycle $[C]-[(-1_J)^*C]$ in the Chow group $CH_1(J)_{\text{hom}}$ of homologically trivial 1-cycles. The *Ceresa class* $\mathfrak{c}_p(C)$ is the image of $\mathfrak{z}_p(C)$ in the intermediate Jacobian $\mathfrak{J}_1(J)$ parameterizing 1-cycles under the Abel–Jacobi map $CH_1(J)_{\text{hom}} \to \mathfrak{J}_1(J)$.

When C is general, $\mathfrak{z}_p(C)$ is not algebraically trivial [2]. On the other hand, if C is hyperelliptic $\mathfrak{z}_p(C)$ is algebraically trivial – in fact it is zero if one chooses for p a Weierstrass point. Not much is known besides these two extreme cases. There are few curves for which $\mathfrak{z}_p(C)$ is known to be not algebraically trivial: Fermat curves of degree ≤ 1000 [4], and the Klein quartic [5]. An essential ingredient of these results is the fact that $\mathfrak{c}_p(C)$ is not a torsion class.

It is an open question whether there are non-hyperelliptic curves with $\mathfrak{z}_p(C)$ algebraically trivial. As observed in [3, Remark 2.4], this condition is equivalent to a number of interesting properties: in particular the existence of a *multiplicative Chow–Künneth decomposition* modulo algebraic equivalence, or the fact that the class $[C] \in CH_1(J) \otimes \mathbb{Q}$ is algebraically equivalent to the minimal class $\frac{\theta^{g-1}}{(g-1)!}$, where $\theta \in CH^1(J)$ is the class of the principal polarization.

In this note we exhibit a curve C of genus 3 with the weaker property that the Ceresa class $\mathfrak{c}_p(C)$ is torsion (under the Bloch–Beilinson conjectures, this actually implies the algebraic triviality of $\mathfrak{z}_p(C)$ up to torsion). The construction is very simple: the curve C has an automorphism σ which

[*] Corresponding author.

fixes a point p, and therefore preserves $\mathfrak{c}_p(C)$; we just have to check that the fixed point set of σ acting on $\mathfrak{J}_1(J)$ is finite.

A similar example, based on a much more sophisticated approach, appears in [1, Remark 3.6].

2. The result

Proposition 1. *Let $C \subset \mathbb{P}^2$ be the genus 3 curve defined by $X^4 + XZ^3 + Y^3 Z = 0$, and let $p = (0,0,1)$. The Ceresa class $\mathfrak{c}_p(C)$ is torsion.*

Proof. Let ω be a primitive 9^{th} root of unity. We consider the automorphism σ of C defined by $\sigma(X,Y,Z) = (X, \omega^2 Y, \omega^3 Z)$. We have $\sigma(p) = p$; therefore σ preserves the Ceresa cycle $\mathfrak{z}_p(C)$, and also its class $\mathfrak{c}_p(C)$ in $\mathfrak{J} := \mathfrak{J}_1(J)$.

Thus it suffices to prove that σ has finitely many fixed points on $\mathfrak{J}$; equivalently, that the eigenvalues of σ acting on the tangent space $T_0(\mathfrak{J})$ are $\neq 1$.

Now $T_0(\mathfrak{J})$ is identified with $H^{0,3}(J) \oplus H^{1,2}(J) = \bigwedge^3 V^* \oplus (\bigwedge^2 V^* \otimes V)$, where $V = H^{1,0}(J) = H^0(C, K_C)$. We first compute the eigenvalues of σ on V. The elements of V are of the form $L \cdot \frac{XdZ - ZdX}{Y^2 Z}$, with $L \in H^0(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}}(1))$; it follows that the eigenvalues of σ on V are $\omega^5, \omega^7, \omega^8$. Therefore the eigenvalue on $\bigwedge^3 V^*$ is ω^7, and the eigenvalues on $\bigwedge^2 V^*$ are $\omega^3, \omega^5, \omega^6$. Thus each product of an eigenvalue on $\bigwedge^2 V^*$ and one on V is $\neq 1$, hence the Proposition. □

References

[1] D. Bisogno, W. Li, D. Litt, P. Srinivasan, “Group-theoretic Johnson classes and non-hyperelliptic curves with torsion Ceresa class”, https://arxiv.org/abs/2004.06146, 2020.

[2] G. Ceresa, “C is not algebraically equivalent to C^- in its Jacobian”, *Ann. Math.* **117** (1983), no. 2, p. 285-291.

[3] L. Fu, R. Laterveer, C. Vial, “Multiplicative Chow–Künneth decompositions and varieties of cohomological K3 type”, *Ann. Mat. Pura Appl. (4)* **200** (2021), no. 5, p. 2085-2126.

[4] N. Otsubo, “On the Abel–Jacobi maps of Fermat Jacobians”, *Math. Z.* **270** (2012), no. 1-2, p. 423-444.

[5] Y. Tadokoro, “A nontrivial algebraic cycle in the Jacobian variety of the Klein quartic”, *Math. Z.* **260** (2008), no. 2, p. 265-275.