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Continuity of the dual Haar measure

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Continuité de la mesure de Haar duale

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Abstract. Given a continuous field of locally compact groups, we show that the field of the Plancherel weights of their C*-algebras is lower semi-continuous. As a corollary, we obtain that the dual Haar system of a continuous Haar system of a locally compact abelian group bundle is also continuous.

Résumé. Étant donné un champ continu de groupes localement compacts, on montre que le champ des poids de Plancherel de leurs C*-algèbres est semi-continu inférieurement. On en déduit que, lorsque les groupes sont abéliens, le système de Haar dual d'un système de Haar continu est aussi continu.

Version française abrégée

Soient G un groupe localement compact abélien et $\widehat{G}$ son groupe dual. La mesure de Haar duale d'une mesure de Haar de G est la mesure de Haar de $\widehat{G}$ qui rend la transformation de Fourier isométrique. Supposons maintenant que $G \to X$ soit un fibré localement compact de groupes abéliens muni d'un système de mesures de Haar continu pour la topologie vague des mesures. Alors le fibré des groupes duaux a une topologie localement compact et est muni du système des mesures de Haar duales. Nous montrons que ce système dual est lui aussi continu.

La démonstration est basée sur l'observation que, dans le cas d'un groupe localement compact abélien G, la mesure de Haar duale peut être vue comme le poids de Plancherel du groupe G, c'est-à-dire le poids canonique de l'algèbre hilbertienne de la représentation régulière gauche de G. La théorie de Tomita–Takesaki fournit la formule (1), valide pour tout groupe localement compact G.

Si $G \to X$ est un fibré localement compact de groupes non nécessairement abéliens, muni d'un système de Haar continu, on déduit de cette formule et de l'existence d'unité approchée que, pour tout élément positif a de la C*-algèbre $C^*(G)$, la fonction $x \mapsto \varphi_x(a_x)$, où φ_x est le poids de Plancherel du groupe G_x au dessus de x et a_x est l'image de a dans $C^*(G_x)$, est semi-continue inférieurement sur X. On définit ainsi un $C_0(X)$-poids sur la $C_0(X)$-algèbre $C^*(G)$. Comme dans la théorie usuelle, donnée dans [1], des poids scalaires sur les C*-algèbres, son domaine de

définition est un idéal à gauche dense. Dans le cas commutatif, il contient l'idéal de Pedersen. Cette propriété est exactement la continuité du système de Haar dual. Outre le $C_0(X)$-poids de Plancherel d'un fibré de groupes, il existe des exemples naturels de $C_0(X)$-poids; en particulier, les $\mathcal{Z}$-traces que Dixmier a introduites dans [3] dans le cadre des algèbres de von Neumann ont leur analogue pour une C*-algèbre A: c'est une $C_b(X)$-trace où X est l'espace des idéaux primitifs de A.

1. Introduction

Let G be a locally compact abelian group and let $\widehat{G}$ be its dual group. A Haar measure λ of G determines a dual Haar measure $\widehat{\lambda}$ of $\widehat{G}$: namely, this the Haar measure of $\widehat{G}$ which makes the Fourier transform $\mathcal{F}$ an isometry from $L^2(G, \lambda)$ to $L^2(\widehat{G}, \widehat{\lambda})$. Suppose now that $p: G \to X$ is a locally compact abelian group bundle in the sense of [7]. A Haar system for G is a family of measures $(\lambda^x)_{x \in X}$, where for all $x \in X$, λ^x is a Haar measure of $G_x = p^{-1}(x)$; it is said to be continuous if for all $f \in C_c(G)$, the function $x \mapsto \int f d\lambda^x$ is continuous. The dual group bundle $\widehat{p}: \widehat{G} \to X$ is then endowed with the dual Haar system. We show that if the initial Haar system is continuous, then so is the dual Haar system. This was stated as [7, Proposition 3.6]. However it was recently pointed to the authors by Henrik Kreidler that their proof is defective. This note corrects this and gives a more general result, based on the fact that, given a locally compact abelian group G, the Haar weight of $\widehat{G}$ corresponds via the Gelfand transform to the Plancherel weight of G. Our main result says that given a a locally compact group bundle $p: G \to X$ with continuous Haar system, the Plancherel weight of G_x varies continuously in a suitable sense. This lead us to the definition of a lower semi-continuous $C_0(X)$-weight on a $C_0(X)$-C*-algebra which we illustrate by three examples.

2. The Plancherel weight of a locally compact group

We recall first some elements of the Tomita–Takesaki theory, using the standard notation of [11]. Given a left Hilbert algebra $\mathcal{A}$ where the product, the involution and the scalar product are respectively denoted by ab, $a^\sharp$ and $(a \,|\, b)$, we denote by $\mathcal{H}$ the Hilbert space completion of $\mathcal{A}$ and by $\pi: \mathcal{A} \to \mathcal{L}(\mathcal{H})$ the left representation. We denote by $\mathcal{M} = \pi(\mathcal{A})''$ the left von Neumann algebra of $\mathcal{A}$. We denote by S the closure of the involution $a \mapsto a^\sharp$ and by F its adjoint. The domain of S [resp. F] is denoted by $\mathcal{D}^\sharp$ [resp. $\mathcal{D}^\flat$] and one writes $\xi^\sharp = S\xi$ [resp. $\eta^\flat = F\eta$] for $\xi \in \mathcal{D}^\sharp$ [resp. for $\eta \in \mathcal{D}^\flat$]. An element $\eta \in \mathcal{H}$ is called right bounded if there exists a bounded operator $\pi'(\eta)$ on $\mathcal{H}$ such that $\pi'(\eta)a = \pi(a)\eta$ for all $a \in \mathcal{A}$. One then writes $\xi\eta = \pi'(\eta)\xi$ for all $\xi \in \mathcal{H}$. The set of right bounded elements is denoted by $\mathcal{B}'$. One shows that $\mathcal{A}' = \mathcal{B}' \cap \mathcal{D}^\flat$ with involution $\xi^\flat$ is a right Hilbert algebra in the same Hilbert space $\mathcal{H}$. An element $\xi \in \mathcal{H}$ is called left bounded if there exists a bounded operator $\pi(\xi)$ on $\mathcal{H}$ such that $\pi(\xi)\eta = \pi'(\eta)\xi$ for all $\eta \in \mathcal{A}'$. One then writes $\xi\eta = \pi(\xi)\eta$ for all $\eta \in \mathcal{H}$. If ξ is left bounded and η is right bounded, the notation is consistent: $\pi(\xi)\eta = \xi\eta = \pi'(\eta)\xi$. The set of left bounded elements is denoted by $\mathcal{B}$. According to [2, Section 2], it is a linear subspace of $\mathcal{H}$ containing $\mathcal{A}$, it is stable under $\mathcal{M}$ and $\pi(\mathcal{B})$ is contained in $\mathcal{M}$. We recall that the canonical weight τ of the left Hilbert algebra $\mathcal{A}$ is the map $\tau: \mathcal{M}_+ \to [0,\infty]$ defined by $\tau(T) = \|\xi\|^2$ if T is of the form $\pi(\xi)^*\pi(\xi)$ for some $\xi \in \mathcal{B}$ and $\tau(T) = \infty$ otherwise. It is a faithful, semi-finite, σ-weakly lower semi-continuous weight.

Using the polar decomposition of an operator in a von Neumann algebra and the techniques of [11, Lemma 3.4], one obtains:

Lemma 1. *For all $\xi \in \mathcal{B}$, one has $\tau(\pi(\xi)\pi(\xi)^*) = \|\xi^\sharp\|^2$ if $\xi \in \mathcal{D}^\sharp$ and is equal to ∞ otherwise.*

We consider now the left Hilbert algebra associated with the left regular representation of a locally compact group G endowed with a left Haar measure λ as defined in [9]: the left Hilbert algebra is $\mathcal{A} = C_c(G)$, where the product is the usual convolution product, the involution, denoted by f^* instead of $f^\sharp$, is $f^*(\gamma) = \overline{f(\gamma^{-1})}$ and the scalar product is $(f \mid g) = \int f\overline{g} d\lambda^{-1}$. The left representation is denoted by L instead of π. It acts on the Hilbert space $\mathcal{H} = L^2(G, \lambda^{-1})$ by $L(f)g = f * g$ for $f, g \in C_c(G)$. The left von Neumann algebra $\mathcal{M}$ is the group von Neumann algebra $VN(G)$. The canonical weight τ of this left Hilbert algebra is called the Plancherel weight of the group G ([10, 12]). One has for $f \in C_c(G)$:

$$\tau\left(L\left(f^* * f\right)\right) = \|f\|^2 = \int |f|^2 d\lambda^{-1} = \left(f^* * f\right)(e).$$

Let us express Lemma 1 in the case of the left regular representation of a locally compact group G with a left Haar measure λ.

Corollary 2. *For every left bounded element ξ of $L^2(G, \lambda^{-1})$, one has*

$$\tau\left(L(\xi)L(\xi)^*\right) = \int |\xi|^2 d\lambda. \tag{1}$$

3. The Plancherel weight of a group bundle

We consider now the case of a locally compact group bundle $p : G \to X$. We assume that the groups G_x have a left Haar measure λ^x such that $x \mapsto \lambda^x$ is continuous. As a locally compact groupoid with Haar system, we can construct its C*-algebra $C^*(G)$ as in [9]. According to [5, Section 5], this is a $C_0(X)$-C*-algebra. Its elements are continuous fields $x \mapsto a_x$ where $a_x \in C^*(G_x)$. For $x \in X$, we let L_x be the left regular representation of G_x and τ_x be the Plancherel weight of G_x.

Proposition 3. *Let $G \to X$ be a locally compact group bundle with a continuous Haar system $(\lambda^x)_{x \in X}$. Given $a \in C^*(G)_+$, the function $\mathcal{T}(a) : x \mapsto \tau_x(L_x(a_x))$ is lower semi-continuous.*

One first show that the function

$$x \mapsto \tau_x\left(L_x\left(a_x^{1/2} * \left(f * f^*\right)_x * a_x^{1/2}\right)\right)$$

is lower semi-continuous for $f \in C_c(G)$ by using the equation (1). One deduces that

$$x \mapsto \tau_x\left(L_x\left(a_x^{1/2} * (e_i)_x * a_x^{1/2}\right)\right)$$

is lower semi-continuous where (e_i) is an approximate unit as in [6, Proposition 2.10], which we can assume bounded by 1. Then, our function can be written as a lower upper bound of a family of lower semi-continuous functions.

4. Examples of $C_0(X)$-weights on $C_0(X)$-C*-algebras

Definition 4. *Let X be a locally compact Hausdorff space. A $C_0(X)$-weight on a $C_0(X)$-C*-algebra A is a map $\Phi : A_+ \to LSC(X)_+$, where $LSC(X)_+$ denotes the convex cone of lower semi-continuous functions $f : X \to [0, \infty]$, which is additive and such that $\Phi(ha) = h\Phi(a)$ for all $h \in C_0(X)_+$ and $a \in A_+$. It is called lower semi-continuous if $a_n \to a$ implies $\Phi(a) \leq \liminf \Phi(a_n)$ and densely defined if its domain $P = \{a \in A_+ : \Phi(a) \text{ is finite and continuous}\}$ is dense in A_+.*

We note as in [4, Lemme 4.4.2.i] that the domain is hereditary. Here are examples.

(1) The map $\mathcal{T} : C^*(G)_+ \to LSC(X)_+$ of the previous section is a lower semi-continuous and densely defined $C_0(X)$-weight. It can be called the Plancherel $C_0(X)$-weight of the group bundle $G \to X$.

(2) Let $\pi : Y \to X$ be a continuous, open and surjective map, where Y and X are locally compact Hausdorff spaces. For $x \in X$, let $Y_x = \pi^{-1}(x)$ be the fibre over x. Endowed with the fundamental family of continuous sections $C_c(Y)$, $x \mapsto C_0(Y_x)$ is a continuous field of C*-algebras. Its C*-algebra of continuous sections is identified with $C_0(Y)$. Thus $C_0(Y)$ is a $C_0(X)$-C*-algebra. Let $(\alpha_x)_{x \in X}$ be a family of Radon measures on Y, with α_x supported on $\pi^{-1}(x)$, and such that for all $f \in C_c(Y)$, the function $x \mapsto \int f \, d\alpha_x$ is continuous. Then $\Phi : C_0(Y) \to LSC(X)_+$ such that $\Phi(f)(x) = \int f \, d\alpha_x$ for $f \in C_0(Y)$ is a densely defined and lower semi-continuous $C_0(X)$-weights of $C_0(Y)$. The converse is also true and uses the fact that $C_c(Y)$ is the Pedersen ideal, i.e. the minimal dense ideal, of $C_0(Y)$ ([8, 5.6.3.]). In particular, given a a locally compact group bundle $G \to X$, a continuous Haar system $(\lambda^x)_{x \in X}$ defines a densely defined and lower semi-continuous $C_0(X)$-weight of $C_0(G)$ called the Haar $C_0(X)$-weight of the group bundle $G \to X$.

We now have all the elements to prove the continuity of the dual Haar system.

Corollary 5. *Let $p : G \to X$ a locally compact bundle of abelian groups, equipped with a continuous Haar system $(\lambda^x)_{x \in X}$. Then the family of dual Haar measures $(\widehat{\lambda}^x)_{x \in X}$ is a continuous Haar system for $\widehat{p} : \widehat{G} \to X$.*

Indeed, the Gelfand transform identifies the Plancherel $C_0(X)$-weight of $C^*(G)$ and the Haar $C_0(X)$-weight of $C^*(\widehat{G})$.

(3) In order to give our last example, we replace $C_0(X)$ by its multiplier algebra $C_b(X)$ whose elements are the complex-valued bounded continuous functions on X. We say that a C*-algebra A is a $C_b(X)$-algebra if it is endowed with a nondegenerate morphism of $C_b(X)$ into the centre of the multiplier algebra of A. Then we define a $C_b(X)$-weight by replacing $C_0(X)$ by $C_b(X)$ in the above definition. By the Dauns–Hofmann theorem, a C*-algebra A is a $C_b(X)$-algebra, where X is the primitive ideal space $\mathrm{Prim}(A)$ endowed with the Jacobson topology. Given $a \in A_+$, according to [8, Proposition 4.4.9], one can define $\Phi(a)(P) = \mathrm{Trace}(\pi(a))$ where P is a primitive ideal and π is any irreducible representation admitting P as kernel, and $\Phi(a)$ is lower semi-continuous on X. Then $\Phi : A_+ \to LSC(X)_+$ is a lower semi-continuous $C_b(X)$-weight. It is densely defined if and only if A is a continuous-trace C*-algebra. This is the C*-algebraic version of the $\mathcal{Z}$-trace introduced by Dixmier in [3] for von Neumann algebras.

Acknowledgements

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