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Motivic classes and the integral Hodge Question

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Abstract. We prove that the obstruction to the integral Hodge Question factors through the completion of the Grothendieck ring of varieties for the dimension filtration. As an application, combining work of Peyre, Colliot-Thélène and Voisin, we give the first example of a finite group G such that the motivic class of its classifying stack BG in Ekedahl's Grothendieck ring of stacks over $\mathbb{C}$ is non-trivial and BG has trivial unramified Brauer group.

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1. Introduction

Let X be a smooth projective complex variety, and let $d := \dim(X)$. For every integer i, we write $CH^i(X)$ for the group of algebraic cycles of codimension i on X modulo rational equivalence, and we set $CH_i(X) := CH^{d-i}(X)$. We have the cycle class maps

$$\mathrm{cl}^i_X : CH^i(X) \to H^{2i}(X(\mathbb{C}), \mathbb{Z}).$$

By convention, we set $CH^i(X) = 0$ and $H^{2i}(X(\mathbb{C}), \mathbb{Z}) = 0$ when $i < 0$ and $i > d$.

A cohomology class $\alpha \in H^{2i}(X(\mathbb{C}), \mathbb{Z})$ is called an integral Hodge class if its image in $H^{2i}(X(\mathbb{C}), \mathbb{C})$ is of type (i, i) with respect to the Hodge decomposition of $H^{2i}(X(\mathbb{C}), \mathbb{C})$. We denote by $\mathrm{Hdg}^{2i}(X, \mathbb{Z})$ the subgroup of integral Hodge classes of $H^{2i}(X(\mathbb{C}), \mathbb{Z})$. We have an inclusion $\mathrm{Im}(\mathrm{cl}^i_X) \subseteq \mathrm{Hdg}^{2i}(X, \mathbb{Z})$. We set

$$Z^{2i}(X) := \mathrm{Hdg}^{2i}(X, \mathbb{Z}) / \mathrm{Im}(\mathrm{cl}^i_X), \qquad Z_{2i}(X) := Z^{2d-2i}(X).$$

For every integer i, the abelian group $Z^{2i}(X)$ is finitely generated. The Hodge Conjecture for cycles of codimension i on X predicts that $Z^{2i}(X)$ is finite. The integral Hodge Question for cycles of codimension i on X asks whether $Z^{2i}(X)$ is zero. By the Lefschetz Theorem on $(1,1)$-classes, the integral Hodge Question has an affirmative answer when $i = 1$. When $i = 2$, the integral Hodge Question has a negative answer in general, as shown by examples of M. Atiyah and F. Hirzebruch [1].

We denote by $K_0(\mathrm{Var}_{\mathbb{C}})$ the Grothendieck ring of complex varieties, introduced by A. Grothendieck in a 1964 letter to J.-P. Serre. We also let $K_0(\mathrm{Ab})$ the abelian group generated by isomorphism classes $[A]$ of finitely generated abelian groups A, modulo the relations $[A\oplus B]=[A]+[B]$. We show that the obstruction to the integral Hodge Question factors through the completion $\widehat{K}_0(\mathrm{Var}_{\mathbb{C}})$ of $K_0(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^{-1}]$ with respect to the dimension filtration topology.

Theorem 1. *Let i be an integer.*

(a) *There exists a unique group homomorphism*

$$Z_{2i}: K_0(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^{-1}] \to K_0(\mathrm{Ab})$$

which sends $\{X\}/\mathbb{L}^m \mapsto [Z_{2i+2m}(X)]$ for every smooth projective complex variety X and every $m\geq 0$.

(b) *The homomorphism Z_{2i} is continuous with respect to the dimension filtration topology on $K_0(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^{-1}]$ and the discrete topology on $K_0(\mathrm{Ab})$. It thus extends uniquely to a group homomorphism*

$$\widehat{Z}_{2i}: \widehat{K}_0(\mathrm{Var}_{\mathbb{C}}) \to K_0(\mathrm{Ab}).$$

We now describe the application that motivated Theorem 1. The Grothendieck ring of complex algebraic stacks $K_0(\mathrm{Stacks}_{\mathbb{C}})$ was defined by T. Ekedahl in [6]. By definition, every algebraic stack $\mathscr{X}$ of finite type over $\mathbb{C}$ and with affine stabilizers has a class $\{\mathscr{X}\}$ in $K_0(\mathrm{Stacks}_{\mathbb{C}})$. The multiplicative identity of $K_0(\mathrm{Stacks}_{\mathbb{C}})$ is $1=\{\mathrm{Spec}\,\mathbb{C}\}$.

Let G be a finite group. It is an interesting problem to compute the class $\{BG\}$ in $K_0(\mathrm{Stacks}_{\mathbb{C}})$, and in particular to understand whether the equality $\{BG\}=1$ holds. Although no formal implication is known, the equality $\{BG\}=1$ appears to be related with the stable rationality of the field of invariants $\mathbb{C}(V)^G$, where V is a faithful complex representation of G. (By the no-name lemma, the stable rationality of $\mathbb{C}(V)^G$ does not depend on the faithful representation V.)

It turns out that $\{BG\}=1$ in $K_0(\mathrm{Stacks}_{\mathbb{C}})$ in many cases. For example, this is true when G is a cyclic group, a symmetric group, or a finite subgroup of $\mathrm{GL}_3(\mathbb{C})$; see [5, Proposition 3.2, Theorem 4.1] and [7, Theorem 2.4]. For all these G, the fields of invariants $\mathbb{C}(V)^G$ are known to be stably rational.

There are also examples of finite groups G for which $\{BG\}\neq 1$ in $K_0(\mathrm{Stacks}_{\mathbb{C}})$. It was shown by Ekedahl that, if the unramified Brauer group $\mathrm{Br}_{\mathrm{nr}}(\mathbb{C}(V)^G/\mathbb{C})$ is not trivial, then $\{BG\}\neq 1$; see [5, Theorem 5.1]. The unramified Brauer group had famously been used in [10] by D. J. Saltman to give the first examples of finite groups G for which $\mathbb{C}(V)^G$ is not stably rational. It follows from Ekedahl's result that Saltman's examples also satisfy $\{BG\}\neq 1$.

In [9], E. Peyre gave the first examples of finite groups G for which $\mathbb{C}(V)^G$ is not stably rational and $\mathrm{Br}_{\mathrm{nr}}(\mathbb{C}(V)^G/\mathbb{C})=0$. It is natural to wonder whether similar counterexamples to $\{BG\}=1$ can be constructed.

Question 2. *Does there exist a finite group G such that* $\mathrm{Br}_{\mathrm{nr}}(\mathbb{C}(V)^G/\mathbb{C})$ *is trivial, but* $\{BG\}\neq 1$ *in* $K_0(\mathrm{Stacks}_{\mathbb{C}})$*?*

To our knowledge, this question was first asked by Ekedahl. It was posed to us by A. Vistoli.

If $K/\mathbb{C}$ is a finitely generated field extension, we denote by $H^i_{\mathrm{nr}}(K/\mathbb{C},\mathbb{Q}/\mathbb{Z})$ the i-th unramified cohomology group of K over $\mathbb{C}$ with $\mathbb{Q}/\mathbb{Z}$ coefficients. We have $H^2_{\mathrm{nr}}(K/\mathbb{C},\mathbb{Q}/\mathbb{Z})=\mathrm{Br}_{\mathrm{nr}}(K/\mathbb{C})$. If $H^i_{\mathrm{nr}}(\mathbb{C}(V)^G/\mathbb{C},\mathbb{Q}/\mathbb{Z})\neq 0$ for some i, then $\mathbb{C}(V)^G$ is not stably rational; see [8, Proposition 3.4]. When $i\geq 3$, it is not known whether $H^i_{\mathrm{nr}}(\mathbb{C}(V)^G/\mathbb{C},\mathbb{Q}/\mathbb{Z})\neq 0$ implies $\{BG\}\neq 1$ in $K_0(\mathrm{Stacks}_{\mathbb{C}})$. We show that this is the case if $i=3$.

Theorem 3. *Let V be a faithful complex representation of G. Assume that $H^3_{\mathrm{nr}}(\mathbb{C}(V)^G/\mathbb{C},\mathbb{Q}/\mathbb{Z})$ is not trivial. Then $\{BG\}\neq 1$ in $K_0(\mathrm{Stacks}_{\mathbb{C}})$.*

Our proof of Theorem 3 combines Theorem 1 with a result of J.-L. Colliot-Thélène and C. Voisin [4]; see Theorem 5 below.

For every odd prime p, Peyre constructed a central extension

$$1 \to (\mathbb{Z}/p\mathbb{Z})^6 \to G \to (\mathbb{Z}/p\mathbb{Z})^6 \to 1$$

such that $H^3_{\mathrm{nr}}(\mathbb{C}(V)^G/\mathbb{C}, \mathbb{Q}/\mathbb{Z}) \neq 0$ for any faithful complex representation V of G; see [9, Theorem 6.1]. By Theorem 3, the examples of Peyre provide an affirmative answer to Question 2.

In [11] B. Totaro asked, among other things, whether the stable rationality of $\mathbb{C}(V)^G$ is equivalent to the condition $\{BG\} = 1$ in $K_0(\mathrm{Stacks}_{\mathbb{C}})$. An affirmative answer to Totaro's question is supported by all known examples, and also by Theorem 3. However, a proof of the equivalence seems to be out of reach of current techniques.

2. The Grothendieck rings of varieties and stacks

By definition, the Grothendieck ring of varieties $K_0(\mathrm{Var}_{\mathbb{C}})$ is the abelian group generated by isomorphism classes $\{X\}$ of schemes X of finite type over $\mathbb{C}$, subject to the relations $\{X\} = \{Y\} + \{X \setminus Y\}$ for every closed immersion $Y \hookrightarrow X$. The multiplication in $K_0(\mathrm{Var}_{\mathbb{C}})$ is defined on generators by $\{X\} \cdot \{Y\} := \{X \times_{\mathbb{C}} Y\}$, and we have $1 = \{\mathrm{Spec}\, \mathbb{C}\}$. We set $\mathbb{L} := \{\mathbb{A}^1_{\mathbb{C}}\}$.

Following Ekedahl [6], we define the Grothendieck ring of stacks $K_0(\mathrm{Stacks}_{\mathbb{C}})$ as the abelian group generated by isomorphism classes $\{\mathscr{X}\}$ of algebraic stacks $\mathscr{X}$ with affine stabilizers and of finite type over $\mathbb{C}$, modulo the relations $\{\mathscr{X}\} = \{\mathscr{Y}\} + \{\mathscr{X} \setminus \mathscr{Y}\}$ for every closed immersion $\mathscr{Y} \hookrightarrow \mathscr{X}$, and the relations $\{\mathscr{E}\} = \{\mathbb{A}^r_{\mathbb{C}} \times_{\mathbb{C}} \mathscr{X}\}$ for every vector bundle $\mathscr{E} \to \mathscr{X}$ of constant rank r. The multiplication is defined on generators by $\{\mathscr{X}\} \cdot \{\mathscr{Y}\} := \{\mathscr{X} \times_{\mathbb{C}} \mathscr{Y}\}$, and we have $1 = \{\mathrm{Spec}\, \mathbb{C}\}$. By [6, Theorem 1.2], the canonical ring homomorphism $K_0(\mathrm{Var}_{\mathbb{C}}) \to K_0(\mathrm{Stacks}_{\mathbb{C}})$ induces an isomorphism

$$K_0(\mathrm{Stacks}_{\mathbb{C}}) \cong K_0(\mathrm{Var}_{\mathbb{C}})[\{\mathbb{L}^{-1}, (\mathbb{L}^n - 1)^{-1} : n \geq 1\}].$$

The dimension filtration $\mathrm{Fil}^\bullet K_0(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^{-1}]$ of $K_0(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^{-1}]$ is defined as follows: for every $n \in \mathbb{Z}$, $\mathrm{Fil}^n K_0(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^{-1}]$ is the subgroup generated by the elements $\{X\}/\mathbb{L}^m$, where X is a complex variety and $\dim(X) - m \leq n$. Using resolution of singularities, we see that $\mathrm{Fil}^n K_0(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^{-1}]$ is generated by elements of the form $\{X\}/\mathbb{L}^m$, where X is a smooth projective complex variety and $\dim(X) - m \leq n$; see [6, Lemma 3.1]. We denote by $\widehat{K}_0(\mathrm{Var}_{\mathbb{C}})$ the completion of $K_0(\mathrm{Var}_{\mathbb{C}})$ with respect to the dimension filtration. For every $n, n' \in \mathbb{Z}$, we have

$$\mathrm{Fil}^n K_0(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^{-1}] \cdot \mathrm{Fil}^{n'} K_0(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^{-1}] \subseteq \mathrm{Fil}^{n+n'} K_0(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^{-1}].$$

It follows that the multiplication on $K_0(\mathrm{Var}_{\mathbb{C}})$ extends to $\widehat{K}_0(\mathrm{Var}_{\mathbb{C}})$, making the latter into a commutative ring with identity.

For every $n \geq 1$, we have $(1 - \mathbb{L}^n) \sum_{i \geq 0} \mathbb{L}^{ni} = 1$ in $\widehat{K}_0(\mathrm{Var}_{\mathbb{C}})$. Therefore, we have canonical ring homomorphisms

$$K_0(\mathrm{Var}_{\mathbb{C}}) \to K_0(\mathrm{Stacks}_{\mathbb{C}}) \to \widehat{K}_0(\mathrm{Var}_{\mathbb{C}}).$$

The following result was observed by Ekedahl in [6, p. 14]. It follows from Bittner's presentation of $K_0(\mathrm{Var}_{\mathbb{C}})$, given in [2, Theorem 3.1].

Lemma 4. *As an abelian group, $K_0(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^{-1}]$ may be presented as the abelian group generated by formal fractions of the form $\{X\}/\mathbb{L}^m$, where X is a smooth projective complex variety and $m \geq 0$, modulo the following relations:*

(i) $\{\varnothing\} = 0$,

(ii) $\{\widetilde{X}\}/\mathbb{L}^m - \{X\}/\mathbb{L}^m = \{E\}/\mathbb{L}^m - \{Y\}/\mathbb{L}^m$*, for every smooth projective complex variety X, every blow-up $\widetilde{X} \to X$ at a smooth closed subscheme $Y \subseteq X$, with exceptional divisor $E \to Y$, and every $m \geq 0$,*

(iii) $\{X \times_{\mathbb{C}} \mathbb{P}^1_{\mathbb{C}}\}/\mathbb{L}^{m+1} - \{X\}/\mathbb{L}^{m+1} = \{X\}/\mathbb{L}^m$, *for every smooth projective complex variety X and every $m \geq 0$.*

3. Proof of Theorem 1

Proof of Theorem 1. (a). To show that $Z_{2i} : K_0(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^{-1}] \to K_0(\mathrm{Ab})$ is well-defined, we verify that the association $\{X\}/\mathbb{L}^m \mapsto [Z_{2i+2m}(X)]$ respects the relations of Lemma 4. It is clear that (i) is satisfied.

Let $m \geq 0$, let $Y \hookrightarrow X$ be a closed immersion of smooth projective complex varieties, let $\widetilde{X} \to X$ be the blow-up of X at Y, and let E be the exceptional divisor of the blow-up. Denote by d the dimension of X, and by r the codimension of Y in X. We want to show that

$$[Z_{2i+2m}(X)] - [Z_{2i+2m}(Y)] = [Z_{2i+2m}(\widetilde{X})] - [Z_{2i+2m}(E)]. \tag{1}$$

in $K_0(\mathrm{Ab})$. Letting $j = d - i - m$, we see that (1) is equivalent to:

$$[Z^{2j}(X)] - [Z^{2j-2r}(Y)] = [Z^{2j}(\widetilde{X})] - [Z^{2j-2}(E)]. \tag{2}$$

By [13, Theorem 9.27], we have a group isomorphism

$$\varphi^j : \bigoplus_{0 \leq h \leq r-2} CH^{j-1-h}(Y) \oplus CH^j(X) \xrightarrow{\sim} CH^j(\widetilde{X}).$$

By [12, Theorem 7.31], we have an isomorphism of Hodge structures

$$\bigoplus_{0 \leq h \leq r-2} H^{2j-2-2h}(Y(\mathbb{C}), \mathbb{Z}) \oplus H^{2j}(X(\mathbb{C}), \mathbb{Z}) \xrightarrow{\sim} H^{2j}(\widetilde{X}(\mathbb{C}), \mathbb{Z}),$$

where the Hodge structure on $H^{2j-2-2h}(Y(\mathbb{C}), \mathbb{Z})$ is shifted by $(h+1, h+1)$, and so has weight $2j$. In particular, we have an isomorphism of groups

$$\psi^j : \bigoplus_{0 \leq h \leq r-2} \mathrm{Hdg}^{2j-2-2h}(Y, \mathbb{Z}) \oplus \mathrm{Hdg}^{2j}(X, \mathbb{Z}) \xrightarrow{\sim} \mathrm{Hdg}^{2j}(\widetilde{X}, \mathbb{Z}).$$

Comparing the explicit descriptions of these isomorphisms, as given in the references, we see that φ^j and ψ^j are compatible with the cycle class maps. In other words, we have a commutative square

$$\begin{tikzcd}
\bigoplus_{0 \leq h \leq r-2} CH^{j-1-h}(Y) \oplus CH^j(X) \arrow[r, "\varphi^j"] \arrow[d, "(\oplus_h \mathrm{cl}^h_Y) \oplus \mathrm{cl}^j_X"] & CH^j(\widetilde{X}) \arrow[d, "\mathrm{cl}^j_{\widetilde{X}}"] \\
\bigoplus_{0 \leq h \leq r-2} \mathrm{Hdg}^{2j-2-2h}(Y, \mathbb{Z}) \oplus \mathrm{Hdg}^{2j}(X, \mathbb{Z}) \arrow[r, "\psi^j"] & \mathrm{Hdg}^{2j}(\widetilde{X}, \mathbb{Z}).
\end{tikzcd}$$

We deduce that

$$Z^{2j}(\widetilde{X}) \cong \bigoplus_{0 \leq h \leq r-2} Z^{2j-2-2h}(Y) \oplus Z^{2j}(X). \tag{3}$$

The morphism $E \to Y$ identifies E with the projectivization of the normal bundle of Y inside X. By [13, Theorem 9.25][1] and [12, Lemma 7.32], the pullback along $E \to Y$ induces a commutative diagram

$$\begin{tikzcd}
\bigoplus_{0 \leq h \leq r-1} CH^{j-1-h}(Y) \arrow[r, "\sim"] \arrow[d, "\oplus_h \mathrm{cl}^h_Y"] & CH^{j-1}(E) \arrow[d, "\mathrm{cl}^{j-1}_E"] \\
\bigoplus_{0 \leq h \leq r-1} \mathrm{Hdg}^{2j-2-2h}(Y, \mathbb{Z}) \arrow[r, "\sim"] & \mathrm{Hdg}^{2j-2}(E, \mathbb{Z}),
\end{tikzcd}$$

where the horizontal arrows are isomorphisms. We deduce that

$$Z^{2j-2}(E) \cong \bigoplus_{0 \leq h \leq r-1} Z^{2j-2-2h}(Y). \tag{4}$$

[1] Note that the formula of [13, Theorem 9.25] contains a typographical error: $CH_{l-r-1+k}(-)$ should be $CH_{l-r+1+k}(-)$.

Now (2) follows from (3) and (4). Therefore, Z_{2i} respects all relations of type (ii).

It remains to show that Z_{2i} is compatible with relations of type (iii). Let X be a smooth projective variety of dimension d, and let $m \geq 0$ be an integer. We must show that

$$[Z_{2i+2m+2}(X \times_{\mathbb{C}} \mathbb{P}^1_{\mathbb{C}})] - [Z_{2i+2m+2}(X)] = [Z_{2i+2m}(X)].$$

Setting $j = d - i - m$, the claim becomes

$$[Z^{2j}(X \times_{\mathbb{C}} \mathbb{P}^1_{\mathbb{C}})] - [Z^{2j-2}(X)] = [Z^{2j}(X)]. \tag{5}$$

Applying [13, Theorem 9.25] and [12, Lemma 7.32] to the trivial projective bundle $X \times_{\mathbb{C}} \mathbb{P}^1_{\mathbb{C}} \to X$, we obtain a commutative square

$$\begin{tikzcd} CH^j(X) \oplus CH^{j-1}(X) \arrow[r, "\sim"] \arrow[d, "\mathrm{cl}^j_X \oplus \mathrm{cl}^{j-1}_X"] & CH^j(X \times_{\mathbb{C}} \mathbb{P}^1_{\mathbb{C}}) \arrow[d, "\mathrm{cl}^j_{X \times_{\mathbb{C}} \mathbb{P}^1_{\mathbb{C}}}"] \\ \mathrm{Hdg}^{2j}(X,\mathbb{Z}) \oplus \mathrm{Hdg}^{2j-2}(X,\mathbb{Z}) \arrow[r, "\sim"] & \mathrm{Hdg}^{2j}(X \times_{\mathbb{C}} \mathbb{P}^1_{\mathbb{C}}, \mathbb{Z}). \end{tikzcd}$$

Thus

$$Z^{2j}(X \times_{\mathbb{C}} \mathbb{P}^1_{\mathbb{C}}) \cong Z^{2j}(X) \oplus Z^{2j-2}(X),$$

which implies (5). It follows that Z_{2i} respects relations of type (iii) as well, hence Z_{2i} is a well-defined group homomorphism.

(b). Let X be a smooth projective variety of dimension d, and let $m \geq d - i$. Then $2i + 2m \geq 2d$, and so

$$Z_{2i}(\{X\}/\mathbb{L}^m) = [Z_{2i+2m}(X)] = 0.$$

This means that Z_{2i} sends $\mathrm{Fil}^i K_0(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^{-1}]$ to zero. Therefore, if we endow $K_0(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^{-1}]$ with the dimension filtration topology and $K_0(\mathrm{Ab})$ with the discrete topology, the homomorphism Z_{2i} is continuous. It follows that Z_{2i} extends uniquely to a homomorphism $\widehat{Z}_{2i} : \widehat{K}_0(\mathrm{Var}_{\mathbb{C}}) \to K_0(\mathrm{Ab})$. □

We also denote by Z_{2i} the composition

$$K_0(\mathrm{Stacks}_{\mathbb{C}}) \to \widehat{K}_0(\mathrm{Var}_{\mathbb{C}}) \xrightarrow{\hat{Z}_{2i}} K_0(\mathrm{Ab}).$$

4. Proof of Theorem 3

Theorem 5 (Colliot-Thélène, Voisin). *Let X be a smooth projective complex variety, of dimension d. Assume that there exists a smooth closed subvariety $S \subseteq X$ of dimension ≤ 2, such that the pushforward map $CH_0(S) \to CH_0(X)$ is surjective. Then we have an isomorphism of finite groups*

$$H^3_{\mathrm{nr}}(\mathbb{C}(X)/\mathbb{C}, \mathbb{Q}/\mathbb{Z}) \cong Z^4(X).$$

Proof. See [4, Théorème 1.1]. □

Remark 6. It is well known that the assumptions of Theorem 5 are satisfied when X is unirational. We have learned the following argument from J.-L. Colliot-Thélène.

If X is a smooth projective unirational complex variety, then there exist a dense open subscheme $U \subseteq X$ and a surjective morphism $\varphi : V \to U$, where V is an open subscheme of some affine space. If $p_1, p_2 \in U(\mathbb{C})$, we may find $q_1, q_2 \in V(\mathbb{C})$ such that $\varphi(q_i) = p_i$ for $i = 1, 2$. There is a line connecting q_1 and q_2, hence, since X is complete, we find a morphism $\mathbb{P}^1_{\mathbb{C}} \to X$ whose image contains p_1 and p_2. It follows that any two zero-cyles of degree 1 in U are rationally equivalent.

Now, if $p \in X(\mathbb{C})$, a moving lemma shows that p is rationally equivalent to a zero-cycle whose support is contained in U; see [3, Complément, p. 599]. We conclude that the degree map $\deg : CH_0(X) \to \mathbb{Z}$ is an isomorphism. Thus, the hypotheses of Theorem 5 are satisfied, with S a closed point of X.

Proposition 7. *Let G be a finite group and let V be a faithful complex representation of G. Then $Z_{2i}(\{BG\}) = 0$ for every $i \geq -1$, and*

$$Z_{-4}(\{BG\}) = [H^3_{\mathrm{nr}}(\mathbb{C}(V)^G/\mathbb{C}, \mathbb{Q}/\mathbb{Z})].$$

Proof. Let $d \geq 1$ be the dimension of V. By [5, Proposition 3.1(ii)], we have

$$\{BG\} = \lim_{m\to\infty} \{V^m/G\}\mathbb{L}^{-md}.$$

Fix an integer i. By Theorem 1(b), the homomorphism $\widehat{Z}_{2i} : \widehat{K}_0(\mathrm{Var}_{\mathbb{C}}) \to K_0(\mathrm{Ab})$ is continuous. Therefore, if m is sufficiently large, we have

$$Z_{2i}(\{BG\}) = \widehat{Z}_{2i}(\{BG\}) = \widehat{Z}_{2i}(\{V^m/G\}\mathbb{L}^{-md}) = Z_{2i}(\{V^m/G\}\mathbb{L}^{-md}).$$

We fix one such m. Using resolution of singularities, we may write

$$\{V^m/G\} = \{X\} + \sum_q n_q\{X_q\} \tag{6}$$

in $K_0(\mathrm{Var}_{\mathbb{C}})$, where X and the X_q are smooth projective varieties over $\mathbb{C}$, X is birationally equivalent to V^m/G, $\dim(X_q) \leq md-1$, and $n_q \in \mathbb{Z}$ for every q. Applying Z_{2i}, we obtain:

$$Z_{2i}(\{BG\}) = [Z_{2i+2md}(X)] + \sum_q n_q[Z_{2i+2md}(X_q)]. \tag{7}$$

By the Lefschetz Theorem on $(1,1)$-classes, we have $Z_{2md-2}(X) = Z^2(X) = 0$. Therefore, if $i \geq -1$ every term on the right hand side of (7) is zero. This shows that $Z_{2i}(\{BG\}) = 0$ for all $i \geq -1$.

If $i = -2$, another application of the Lefschetz Theorem on $(1,1)$-classes shows that the right hand side of (7) reduces to $[Z_{2md-4}(X)] = [Z^4(X)]$. Since X is birationally equivalent to V/G, by Theorem 5 we have:

$$Z^4(X) \cong H^3_{\mathrm{nr}}(\mathbb{C}(X)/\mathbb{C}, \mathbb{Q}/\mathbb{Z}) \cong H^3_{\mathrm{nr}}(\mathbb{C}(V)^G/\mathbb{C}, \mathbb{Q}/\mathbb{Z}).$$

Using (7) with $i = -2$, we conclude that

$$Z_{-4}(\{BG\}) = [Z^4(X)] = [H^3_{\mathrm{nr}}(\mathbb{C}(V)^G/\mathbb{C}, \mathbb{Q}/\mathbb{Z})]. \qquad \square$$

Proof of Theorem 3. As an abelian group, $K_0(\mathrm{Ab})$ is freely generated by $[\mathbb{Z}]$ and $[\mathbb{Z}/p^n\mathbb{Z}]$, where p ranges among all prime numbers and $n \geq 1$; see [6, Proposition 3.3(i)]. Since $H^3_{\mathrm{nr}}(\mathbb{C}(V)^G/\mathbb{C}, \mathbb{Q}/\mathbb{Z})$ is not trivial, $[H^3_{\mathrm{nr}}(\mathbb{C}(V)^G/\mathbb{C}, \mathbb{Q}/\mathbb{Z})] \neq 0$ in $K_0(\mathrm{Ab})$. By Proposition 7, we deduce that $Z_{-4}(\{BG\}) \neq 0$ in $K_0(\mathrm{Ab})$. On the other hand, it is clear that $Z_{-4}(\{\mathrm{Spec}\,\mathbb{C}\}) = 0$. We conclude that $\{BG\} \neq 1$ in $K_0(\mathrm{Stacks}_{\mathbb{C}})$. $\square$

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