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On the exponential generating function of labelled trees

Sur la série génératrice des arbres étiquetés

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Abstract. We show that the generating function of labelled trees is not D^{∞}-finite.

Résumé. Nous montrons que la série génératrice des arbres étiquetés n'est pas D^{∞}-finie.

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Version française abrégée

Nous montrons que la série génératrice exponentielle des arbres étiquetés, $T(x) = \sum_{n\geq 1} \frac{n^{n-1}}{n!} x^n$, n'est pas D^{∞}-finie. En particulier, cela implique que, bien que $T(x)$ vérifie des équations différentielles non-linéaires, ces dernières ne peuvent pas être « trop simples ». En particulier, $T(x)$ n'est pas un quotient de deux fonctions D-finies (vérifiant des équations différentielles à coefficients polynomiaux), et plus généralement, $T(x)$ ne vérifie aucune équation différentielle linéaire à coefficients des fonctions D-finies. La preuve repose ultimement sur un résultat de théorie de Galois différentielle. Plusieurs questions ouvertes sont proposées, dont une sur la nature de la série génératrice ordinaire des arbres étiquetés, $\sum_{n\geq 1} n^{n-1} x^n$.

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1. Context and main result

A formal power series $f(x) = \sum_{n\geq 0} a_n x^n$ in $\mathbb{C}[[x]]$ is called *differentially finite*, or simply *D-finite* [23], if it satisfies a *linear* differential equation with polynomial coefficients in $\mathbb{C}[x]$. Many generating functions in combinatorics and many special functions in mathematical physics are D-finite [2,9].

DD-finite series and more generally D^n-finite series are larger classes of power series, recently introduced in [13]. DD-finite power series satisfy linear differential equations, whose coefficients are themselves D-finite power series. One of the simplest examples is $\tan(x)$, which is DD-finite (because it satisfies $\cos(x) f(x) - \sin(x) = 0$), but is not D-finite (because it has an infinite number of complex singularities, a property which is incompatible with D-finiteness). Another basic example is the exponential generating function of the Bell numbers B_n, which count partitions of $\{1,2,\dots,n\}$, namely:

$$B(x) := \sum_{n\geq 0} \frac{B_n}{n!} x^n. \tag{1}$$

Indeed, it is classical [9, p. 109] that $B(x) = e^{e^x - 1}$, therefore $B(x)$ is DD-finite. On the other hand, $B(x)$ is not D-finite: this can be proved either analytically (using the too fast growth of $B(x)$ as $x \to \infty$), or purely algebraically (using [22], and the fact that the power series e^x is not algebraic).

More generally, given a differential ring R, the set of *differentially definable* functions over R, denoted by $\mathrm{D}(R)$, is the differential ring of formal power series satisfying linear differential equations with coefficients in R. In particular, $\mathrm{D}(\mathbb{C}[x])$ is the ring of D-finite power series, $\mathrm{D}^2(\mathbb{C}[x]) := \mathrm{D}(\mathrm{D}(\mathbb{C}[x]))$ is the ring of DD-finite power series, and $\mathrm{D}^n(\mathbb{C}[x]) := \mathrm{D}(\mathrm{D}^{n-1}(\mathbb{C}[x]))$ is the ring of D^n-finite power series. We say that a power series $f(x) \in \mathbb{C}[[x]]$ is D^∞*-finite* if there exists an n such that $f(x)$ is D^n-finite.

It is known [14] that D^n-finite power series form a strictly increasing sequence of sets and that any D^∞-finite power series is *differentially algebraic*, in short D-*algebraic*, that is, it satisfies a differential equation, possibly *non-linear*, with polynomial coefficients in $\mathbb{C}[x]$. This class, as well as its complement (of *D-transcendental* series), are quite well studied [11,21].

Let now $(t_n)_{n\geq 0} = (0,1,2,9,64,625,7776,\dots)$ be the sequence whose general term t_n counts *labelled rooted trees* with n nodes. It is well known that $t_n = n^{n-1}$, for any $n \geq 1$. This beautiful and non-trivial result is usually attributed to Cayley [6], although an equivalent result had been proved earlier by Borchardt [4], and even earlier by Sylvester, see [3, Ch. 4]. Due to the importance of the combinatorial class of trees, and to the simplicity of the formula, Cayley's result has attracted a lot of interest over the time, and it admits several different proofs, see e.g., [16, §4] and [1, §30]. One of the more conceptual proofs goes along the following lines (see [9, §II. 5.1] for details). Let

$$T(x) := \sum_{n\geq 0} \frac{t_n}{n!} x^n \tag{2}$$

be the exponential generating function of the sequence $(t_n)_n$. The class $\mathscr{T}$ of all rooted labelled trees is definable by a *symbolic equation* $\mathscr{T} = \mathscr{Z} \star \mathrm{SET}(\mathscr{T})$ reflecting their recursive definition, where $\mathscr{Z}$ represents the atomic class consisting of a single labelled node, and $\star$ denotes the labelled product on combinatorial classes. This symbolic equation provides, by syntactic translation, an implicit equation on the level of exponential generating functions:

$$T(x) = x\,\mathrm{e}^{T(x)}, \tag{3}$$

which can be solved using *Lagrange inversion*

$$t_n = n! \cdot [x^n]\, T(x) = n! \cdot \left(\frac{1}{n}[z^{n-1}](e^z)^n\right) = n^{n-1}. \tag{4}$$

From (3), it follows easily that $T(x)$ is D-algebraic and satisfies the non-linear equation

$$x(1-T(x))T'(x) = T(x),$$

and from there, that the sequence $(t_n)_{n\geq 0}$ satisfies the non-linear recurrence relation

$$t_{n+1} = \frac{n+1}{n}\cdot\sum_{i=1}^{n}\binom{n}{i} t_i\, t_{n-i+1}, \quad \text{for all } n\geq 1.$$

This recurrence can also be proved using (4), by taking $y = n$, $x = w = 1$ in Abel's identity [12]

$$(x+y)^n = \sum_{k=0}^{n}\binom{n}{k} x(x+wk)^{k-1}(y-wk)^{n-k},$$

and then by isolating the term $k = n$ in the resulting equality.

On the other hand, it is known that the power series $T(x)$ is not D-finite, see [10, Thm. 7], or [8, Thm. 2]. This raises the natural question whether $T(x)$ is DD-finite, or D^n-finite for some $n \geq 2$. Our main result is that this is not the case:

Theorem 1. *The power series* $T(x) = \sum_{n\geq 1}\frac{n^{n-1}}{n!}x^n$ *in* (2) *is not* D^∞*-finite.*

To our knowledge, this is the first explicit example of a natural combinatorial generating function which is provably D-algebraic but not D^∞-finite. In particular, Theorem 1 implies that $T(x)$ is not equal to the quotient of two D-finite functions, and more generally, that it does not satisfy any linear differential equation with D-finite coefficients.

2. Proof of the main result

Our proof of Theorem 1 builds upon the following recent result by Noordman, van der Put and Top.

Theorem 2 ([18]). *Assume that* $u(x) \in \mathbb{C}[[x]]\setminus\mathbb{C}$ *is a solution of* $u' = u^3 - u^2$. *Then* u *is not* D^∞*-finite.*

The proof of Theorem 2 is based on two ingredients. The first one is a result by Rosenlicht [20] stating that any set of non-constant solutions (in any differential field) of the differential equation $u' = u^3 - u^2$ is algebraically independent over $\mathbb{C}$ (see also [18, Prop. 7.1]); the proof is elementary. The second one [18, Prop. 7.1] is that any non-constant power series solution of an autonomous first-order differential equation with this independence property cannot be D^∞-finite; the proof is based on differential Galois theory.

Proof of Theorem 1. We will use Theorem 2 and a few facts about the (principal branch of the) Lambert W function, satisfying $W(x)\cdot e^{W(x)} = x$ for all $x \in \mathbb{C}$.

Recall [7] that the Taylor series of W around 0 is given by

$$W(x) = \sum_{n=1}^{\infty}\frac{(-n)^{n-1}}{n!}x^n = x - x^2 + \tfrac{3}{2}x^3 - \tfrac{8}{3}x^4 + \tfrac{125}{24}x^5 - \cdots.$$

In other words, our $T(x)$ and $W(x)$ are simply related by $W(x) = -T(-x)$.

The function defined by this series can be extended to a holomorphic function defined on all complex numbers with a branch cut along the interval $(-\infty, -\frac{1}{e}]$; this holomorphic function defines the principal branch of the Lambert W function.

We can substitute $x \mapsto e^{x+1}$ in the functional equation for $W(x)$ obtaining then

$$W(e^{x+1})e^{W(e^{x+1})} = e^{x+1},$$

or, renaming $Y(x) = W(e^{x+1})$, we have a new functional equation: $Y(x)e^{Y(x)-1} = e^x$. From this equality it follows by logarithmic differentiation that $Y'(x)\cdot(1+Y(x)) = Y(x)$.

Take now $U(x) := \frac{1}{1+Y(x)} = \frac{1}{2} - \frac{1}{8}x + \frac{1}{64}x^2 + \frac{1}{768}x^3 + \cdots$. We have that

$$U'(x) = \frac{-Y'(x)}{(1+Y(x))^2} = \frac{-Y(x)}{(1+Y(x))^3} = U(x)^3 - U(x)^2.$$

By Theorem 2, $U(x)$ is not D^∞-finite. By closure properties of D^∞-finite functions (see [14, Thm. 4] and [13, §3]), it follows that $Y(x)$ is not D^∞-finite either.

To conclude, note that by definition, for real x in the neighborhood of 0, we have $W(x) = Y(\log(x) - 1)$, and by Theorem 10 in [14], it follows that $W(x)$ and $T(x)$ are not D^∞-finite either, proving Theorem 1. □

3. Open questions

The class of D-finite power series is closed under Hadamard (term-wise) product. This is false for D^∞-finite power series; for instance, Klazar showed in [15] that the ordinary generating function $\sum_{n\geq 0} B_n x^n$ of the Bell numbers is not differentially algebraic, contrary to its exponential generating function (1), which is DD-finite.

Moreover, it was conjectured by Pak and Yeliussizov [19, Open Problem 2.4] that this is an instance of a more general phenomenon.

Conjecture 3 ([19, Open Problem 2.4]). *If for a sequence $(a_n)_{n\geq 0}$ both ordinary and exponential generating functions $\sum_{n\geq 0} a_n x^n$ and $\sum_{n\geq 0} a_n \frac{x^n}{n!}$ are D-algebraic, then both are D-finite. (Equivalently, $(a_n)_{n\geq 0}$ satisfies a linear recurrence with polynomial coefficients in n.)*

This conjecture has been recently proven for large (infinite) classes of generating functions [5]. However, the very natural example of the generating function for labelled trees escapes the method in [5].

We therefore leave the following as an open question.

Open question 4. *Is the power series $\sum_{n\geq 1} n^{n-1} x^n$ D^∞-finite? Is it at least differentially algebraic?*

According to Conjecture 3, the answer should be "no" for both questions in Open question 4.

Another natural question concerns the generating function for partition numbers:

$$\sum_{n\geq 0} p_n x^n := \prod_{n\geq 1} \frac{1}{1-x^n} = 1 + x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + 11x^6 + \cdots,$$

which is known to be differentially algebraic [17].

Open question 5. *Is it true that $\sum_{n\geq 0} p_n x^n$ is not D^∞-finite?*

One may also ask for the nature of the exponential variant of the generating function for partition numbers.

Open question 6. *Is the power series $\sum_{n\geq 0} \frac{p_n}{n!} x^n$ D^∞-finite, or at least differentially algebraic?*

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