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Partial Differential Equations / *Équations aux dérivées partielles*

# Concerning the pathological set in the context of probabilistic well-posedness

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**Abstract.** We prove a complementary result to the probabilistic well-posedness for the nonlinear wave equation. More precisely, we show that there is a dense set $S$ of the Sobolev space of super-critical regularity such that (in sharp contrast with the probabilistic well-posedness results) the family of global smooth solutions, generated by the convolution with some approximate identity of the elements of $S$, does not converge in the space of super-critical Sobolev regularity.

**Résumé.** On démontre un résultat complémentaire à ceux manifestant le caractère bien posé probabiliste de l'équation des ondes avec des données initiales de régularité de Sobolev super critique par rapport au changement d'échelle laissant invariant l'équation.

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## 1. Introduction

In this work, we are interested in the three dimensional nonlinear wave equation

$$\begin{cases} \partial_t^2 u - \Delta u + |u|^{2\sigma} u = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{T}^3, \\ (u, \partial_t u)|_{t=0} = (f,g) \in \mathscr{H}^s\left(\mathbb{T}^3\right), \end{cases} \tag{1}$$

where $u$ is a real-valued function and

$$\mathscr{H}^s\left(\mathbb{T}^3\right) := H^s\left(\mathbb{T}^3\right) \times H^{s-1}\left(\mathbb{T}^3\right).$$

The nonlinear wave equation (1) is a Hamiltonian system with conserved energy

$$H[u] := \frac{1}{2}\int_{\mathbb{T}^3} |\nabla u|^2 dx + \frac{1}{2\sigma+2}\int_{\mathbb{T}^3} |u|^{2\sigma+2} dx.$$

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It was shown (see [8, 15]) that when $\sigma \le 2$, the problem (1) possesses a global strong solution in the energy space $\mathscr{H}^1(\mathbb{T}^3)$. By replacing $\mathbb{T}^3$ to $\mathbb{R}^3$, the scaling

$$u \mapsto u_\lambda(t,x) := \lambda^{\frac{1}{\sigma}} u(\lambda t, \lambda x)$$

keeps the equation (1) invariant. This leads to the critical regularity index $s_c = \frac{3}{2} - \frac{1}{\sigma} \le 1$. Intuitively, for $s < s_c$ if the initial data is concentrated at the frequency scale $\gg 1$ and is of size 1 measured by the $\mathscr{H}^s$ norm, then the nonlinear part in the dynamics of (1) is dominant and it causes instability of the $\mathscr{H}^s$ norm of the solution. This is called a norm inflation and it was extensively studied, see [6, 10, 11] in the context of nonlinear wave equations. For instance, it was shown in [6] that there exists a sequence of smooth initial data whose $\mathscr{H}^s$ norms converge to zero, while the $\mathscr{H}^s$ norms of the obtained sequence of solutions amplifies at very short time. We also refer to [12] where a different concentration phenomenon, related to the Lorentz invariance of the wave equation, is observed.

In [4] and [5], by using probabilistic tools, N. Burq and the second author showed that problem (1) with cubic nonlinearity still possesses global strong solutions for a "large class" of functions of super-critical regularity. The result was further extended to $1 \le \sigma \le 2$ in [14] and [17]. More precisely, the following statement follows from [5, 14, 17].

**Theorem 1.** *Let $1 \le \sigma \le 2$ and $1-\frac{1}{\sigma} < s < s_c = \frac{3}{2}-\frac{1}{\sigma}$. Then there is a dense set $\Sigma \subset \mathscr{H}^s(\mathbb{T}^3)$ satisfying $\Sigma \cap \mathscr{H}^{s'}(\mathbb{T}^3) = \emptyset$ for every $s' > s$ such that the following holds true. For every $(f,g) \in \Sigma$, let $(f_n, g_n)$ be the sequence in $C^\infty(\mathbb{T}^3) \times C^\infty(\mathbb{T}^3)$ defined by the regularization by convolution, i.e.*

$$f_n = \rho_n * f, \quad g_n = \rho_n * g,$$

*where $(\rho_n)_{n\in\mathbb{N}}$ is an approximate identity. Denote by $(u_n(t), \partial_t u_n(t))$ the smooth solutions of (1) with the smooth initial data $(f_n, g_n)$. Then there exists a limit object $u(t)$ such that for any $T > 0$,*

$$\lim_{n\to\infty} \|(u_n(t), \partial_t u_n(t)) - (u(t), \partial_t u(t))\|_{L^\infty([-T,T];\mathscr{H}^s(\mathbb{T}^3))} = 0.$$

*Moreover $u(t)$ solves (1) in the distributional sense.*

When $1 \le \sigma < 2$, the above Theorem 1 can be extended to $s = 1 - \frac{1}{\sigma}$, thanks to [5] (the case $\sigma = 1$) and a recent result [9] (the case $1 < \sigma < 2$).

In Theorem 1 the set $\Sigma$ is a full measure set with respect to a suitable non degenerate probability measure $\mu$ on the Sobolev space $\mathscr{H}^s(\mathbb{T}^3)$ such that $\mu(\mathscr{H}^{s'}(\mathbb{T}^3)) = 0$ for every $s' > s$. One proves more than Theorem 1 in [5, 14, 17] but the statement of Theorem 1 is the suitable one for our purpose here.

Theorem 1 is inspired by the seminal contribution of Bourgain [3]. There are however several new features with respect to [3]. The first one is that more general randomisations compared to [3] are allowed. This led to results similar to Theorem 1 in the context of a non compact spatial domains (see e.g. [2, 13]). Next, the argument allowing to pass from local to global solutions in Theorem 1 is very different from [3]. It is based on a probabilistic energy estimate introduced in [5] (see also [7]) while the argument giving the globalisation of the local solutions in [3] is restricted to a very particular distribution of the initial data. Finally, Theorem 1 deals with functions of positive Sobolev regularity which avoids a renormalization of the equation, making the results more natural from a purely PDE perspective.

Strictly speaking, the result of Theorem 1 is not stated as such in [5, 14, 17]. One may however adapt the argument presented in [18] which proves Theorem 1 for $\sigma = 1$ to the case of $\sigma \in [1, 2]$.

The regularization by convolution used in Theorem 1 is essential. We refer to [18, 19] for results showing that other regularizations of $(f,g) \in \Sigma$ may give divergent sequences of smooth solutions.

The main result of this paper is that even if we naturally regularize the data by convolution, there is a dense set of (pathological) initial data giving not converging smooth solutions. This is in some sense a complementary to Theorem 1 result.

In order to state our result, we fix a bump function $\rho \in C_c^\infty(\mathbb{R}^3)$ such that

$$0 \le \rho(x) \le 1, \quad \rho|_{|x|>\frac{1}{100}} \equiv 0, \quad \int_{\mathbb{R}^3} \rho(x)\,dx = 1.$$

For any $\epsilon > 0$, we define $\rho_\epsilon(x) := \epsilon^{-3}\rho(x/\epsilon)$. With this notation, we have the following statement.

**Theorem 2.** *Let $\frac{1}{2} \le \sigma \le 2$ and $\max\{0, \frac{3}{2} - \frac{2}{2\sigma-1}\} < s < s_c = \frac{3}{2} - \frac{1}{\sigma}$. There exists a dense set $S \subset \mathcal{H}^s(\mathbb{T}^3)$, such that for every $(f,g) \in S$, the family of global smooth solutions $(u^\epsilon)_{t>0}$ of* (1) *with initial data $(\rho_\epsilon * f, \rho_\epsilon * g)$ does not converge. More precisely*

$$\limsup_{\epsilon\to 0} \left\| u^\epsilon(t) \right\|_{L^\infty([0,1];\,H^s(\mathbb{T}^3))} = +\infty.$$

The main ingredient of the proof of Theorem 2 is a refined version of the ill-posedness construction in [4] (see also [16]) which uses an idea of Lebeau [10] exploiting the property of the finite propagation speed of the wave equation. It is an interesting problem to extend the result of Theorem 2 to the case of the nonlinear Schrödinger equation. Such a result would be a significant extension of [1].

The results of Theorem 1 and Theorem 2 show that for data of supercritical regularity two opposite behaviours coexiste. Both behaviours are manifested on dense sets which makes that it would be probably interesting to try to observe these behaviours by numerical simulations.

## 2. Unstable profile

### 2.1. *Explicit estimates for the ODE profile*

Let $V(t)$ be the unique solution of the following ODE:

$$V'' + |V|^{2\sigma} V = 0, \quad V(0) = 1,\ V'(0) = 0. \tag{2}$$

It can be shown that $V(t)$ is periodic (see [16, Lemma 6.2]). We choose the following parameters:

$$\kappa_n = (\log n)^{-\delta_1},\ \epsilon_n = \frac{1}{100n},\ t_n = \left((\log n)^{\delta_2} n^{-\left(\frac{d}{2}-s\right)}\right)^\sigma,\ \lambda_n = \left(\kappa_n n^{\frac{d}{2}-s}\right)^\sigma, \tag{3}$$

where $0 < \delta_1 < \delta_2 < 1$ and their precise values are to be chosen according to different context.

Take $\varphi \in C_c^\infty(|x| \le 1)$, radial, $0 \le \varphi \le 1$, and $\nabla\varphi \ne 0$ on $0 < |x| < 1$. Let

$$v_n(0,x) := \kappa_n n^{\frac{d}{2}-s}\varphi(nx), \quad v_n^\epsilon(0) := \rho_\epsilon * v_n(0). \tag{4}$$

Define

$$v_n^\epsilon(t,x) = v_n^\epsilon(0,x) V\left(t\left(v_n^\epsilon(0,x)\right)^\sigma\right). \tag{5}$$

Then one verifies that $v_n^\epsilon$ solves

$$\partial_t^2 v_n^\epsilon + \left|v_n^\epsilon\right|^{2\sigma} v_n^\epsilon = 0, \quad \left(v_n^\epsilon, \partial_t v_n^\epsilon\right)\big|_{t=0} = \left(v_n^\epsilon(0), 0\right). \tag{6}$$

**Lemma 3.** *Let $0 \le s < s_c$, then for parameters defined in* (3)*, we have*

(1) $\|v_n^{\epsilon_n}(t_n)\|_{H^s(\mathbb{T}^3)} \gtrsim \kappa_n(\lambda_n t_n)^s$.
(2) $\|v_n^{\epsilon_n}(t)\|_{H^k(\mathbb{T}^3)} \lesssim \kappa_n(\lambda_n t_n)^k n^{k-s}$, *for $k = 0,1,2,3,\cdots$ and $t \in [0,t_n]$.*
(3) $\|\partial^\alpha v_n^{\epsilon_n}(t)\|_{L^\infty(\mathbb{T}^3)} \lesssim \lambda_n^{\frac{1}{\sigma}} n^{|\alpha|}(1+\lambda_n t)$, *for $\alpha \in \mathbb{N}^3, |\alpha| = 0,1$ and $t \in [0,t_n]$.*

**Proof.** The proof follows from a direct calculation as in [4], with an additional attention to the convolution. We denote by $T_\lambda$, the scaling operator $T_\lambda(f) := f(\lambda\cdot)$. Without loss of generality, we will do all the computation in $\mathbb{R}^3$ instead of $\mathbb{T}^3$, since all the functions involved are compactly supported near the origin.

By definition, for $\alpha\in\mathbb{N}^3, |\alpha|=k$,

$$v_n^\epsilon(0,x)=\lambda_n^{\frac{1}{\sigma}}\int_{\mathbb{R}^3}\varphi(n(x-y))\frac{1}{\epsilon^3}\rho\left(\frac{y}{\epsilon}\right)dy,\quad \partial^\alpha v_n^\epsilon(0,x)=\lambda_n^{\frac{1}{\sigma}}n^k\int_{\mathbb{R}^3}T_n(\partial^\alpha\varphi)(x-y)\frac{1}{\epsilon^3}\rho\left(\frac{y}{\epsilon}\right)dy.$$

Using Young's convolution inequality, we have from (5) that

$$\left\|\partial^\alpha v_n^{\epsilon_n}(0)\right\|_{L^\infty}\lesssim\lambda_n^{\frac{1}{\sigma}}n^{|\alpha|},\quad \left\|\partial^\alpha v_n^{\epsilon_n}(0)\right\|_{L^2}\lesssim\kappa_n n^{|\alpha|-s},\quad \left\|v_n^{\epsilon_n}(t)\right\|_{L^\infty}\lesssim\lambda_n^{\frac{1}{\sigma}},$$

and

$$\left\|v_n^{\epsilon_n}(t)\right\|_{L^2}\le\|V\|_{L^\infty}\left\|v_n^{\epsilon_n}(0)\right\|_{L^2}\lesssim\kappa_n n^{-s}.$$

This proves Lemmas 3 (2) and (3) for the case $k=0$. From direct calculation using (5),

$$\nabla v_n^{\epsilon_n}(t,x)=\sigma t\left(v_n^{\epsilon_n}(0,x)\right)^\sigma\nabla v_n^\epsilon(0,x)V'\left(t\left(v_n^{\epsilon_n}(0,x)\right)^\sigma\right)+\nabla v_n^{\epsilon_n}(0,x)V\left(t\left(v_n^{\epsilon_n}(0,x)\right)^\sigma\right).\tag{7}$$

Thus $\|\nabla v_n^{\epsilon_n}(t)\|_{L^\infty}\lesssim(\lambda_n t+1)\lambda_n^{\frac{1}{\sigma}}n$. Note that $\lambda_n t_n=(\log n)^{\sigma(\delta_2-\delta_1)}\gg1$, the dominant part in $\partial^\alpha v_n^{\epsilon_n}(t,x)$ comes from

$$\left(\left(v_n^{\epsilon_n}(0)\right)^{\sigma-1}\nabla v_n^{\epsilon_n}(0)\right)^{|\alpha|}t^{|\alpha|}v_n^{\epsilon_n}(0)V^{(|\alpha|)}(\cdot),$$

if we estimate $t$ by $t_n$, hence $\|v_n^{\epsilon_n}(t)\|_{H^k}\lesssim\kappa_n(\lambda_n t_n)^k n^{k-s}$, for all $k=0,1,2,\cdots$. This proves Lemma 3 (2).

The only non-trivial part is Lemma 3 (1). Since $0<s<1$, from the interpolation

$$\left\|v_n^{\epsilon_n}(t)\right\|_{H^1}\lesssim\left\|v_n^{\epsilon_n}(t)\right\|_{H^s}^{\frac{1}{2-s}}\left\|v_n^{\epsilon_n}(t)\right\|_{H^2}^{\frac{1-s}{2-s}}$$

and the upper bound of $\|v_n^{\epsilon_n}(t)\|_{H^2}$ that we have proved, it suffices to show that

$$\left\|v_n^{\epsilon_n}(t_n)\right\|_{H^1}\gtrsim\kappa_n(\lambda_n t_n)n^{1-s}.\tag{8}$$

It is reduced to get a lower bound for the dominant part

$$\begin{aligned}&\left\|\sigma t_n\left(v_n^{\epsilon_n}(0,x)\right)^\sigma\nabla v_n^{\epsilon_n}(0,x)V'\left(t_n\left(v_n^{\epsilon_n}(0,x)\right)^\sigma\right)\right\|_{L^2}\\&\qquad=\sigma t_n n\lambda_n^{1+\frac{1}{\sigma}}\left\|\left[\left(T_n(\nabla\varphi)\right)*\rho_{\epsilon_n}\right]\left[\left(T_n(\varphi)\right)*\rho_{\epsilon_n}\right]^\sigma V'\left(\lambda_n t_n\left(\left(T_n\varphi\right)*\rho_{\epsilon_n}\right)^\sigma\right)\right\|_{L^2}\end{aligned}\tag{9}$$

Note that $(T_nf)*\rho_{\epsilon_n}(x)=\int f(nx-n\epsilon_n y)\rho(y)dy$, hence

$$(\text{RHS. of }(9))\sim t_n n^{1-\frac{d}{2}}\lambda_n^{1+\frac{1}{\sigma}}\left\|\nabla\left(\varphi*\widetilde{\rho}\right)\cdot\left(\varphi*\widetilde{\rho}\right)^\sigma V'\left(\lambda_n t_n(n\epsilon_n)^{\sigma d}\left(\varphi*\widetilde{\rho}\right)^\sigma(x)\right)\right\|_{L^2},$$

where $\widetilde{\rho}=T_{\frac{1}{n\epsilon_n}}\rho=T_{100}\rho$. Note that $t_n n^{-\frac{d}{2}}\lambda_n^{1+\frac{1}{\sigma}}=\lambda_n t_n n^{1-s}$, hence (22) follows from the following Lemma 4:

**Lemma 4.** *Assume that $\psi\in C_c^\infty(\mathbb{R}^d)$ and $\psi(x)>0$ for all $|x|<1$. Assume that there exist two constants $0<a<b<1$, such that $d\psi\neq0$ on $\{x:a\le|x|\le b\}$. Let $W$ be a non-trivial periodic function (i.e. $W\neq0$). Then there exist $c_0>0,\lambda_0>0$, such that for all $\lambda\ge\lambda_0$,*

$$\left\|\nabla\psi(x)|\psi(x)|^\sigma W(\lambda\psi(x))\right\|_{L^2(\mathbb{R}^d)}\ge c_0>0.$$

**Proof.** We follow the geometric argument in [16]. Denote by $\mathscr{C}_{a,b}:=\{x:a\le|x|\le b\}$. By shrinking $a,b$ if necessary, we may assume that $\psi(\mathscr{C}_{a,b})$ is foliated by $\Sigma_s:=\{x:\psi(x)=s\}$. From the hypothesis on $\psi$, there exist $0<c_1<C_1<\infty$, such that $c_1\le|\nabla\psi|\le C_1$ on $\mathscr{C}_{a,b}$. Let $B=\max_{\mathscr{C}_{a,b}}\psi$ and $A=\min_{\mathscr{C}_{a,b}}\psi$, then we have for $F(s)=|s|^{2\sigma}|W(\lambda s)|^2$ that

$$\left\|\nabla\psi(F\circ\psi)^{1/2}\right\|_{L^2}^2\ge c_1^2\int_{\mathscr{C}_{a,b}}F(\psi(x))dx.$$

By the co-area formula,

$$\int_{\mathscr{C}_{a,b}} F(\psi(x))dx = \int_A^B F(s)ds \int_{\Sigma_s} \frac{d\sigma_{\Sigma_s}}{|\nabla\psi|} \geq c' \int_A^B |s|^{2\sigma}|W(\lambda s)|^2 ds,$$

thanks to the fact that the mapping $s \mapsto \mathscr{M}^{d-1}(\Sigma_s)$ is continuous, where $\mathscr{M}^{d-1}$ is the surface measure on $\Sigma_s$. By changing variables, we obtain that

$$\int_A^B |s|^{2\sigma}|W(\lambda s)|^2 ds = \frac{1}{\lambda^{2\sigma+1}} \int_{\lambda A}^{\lambda B} |s|^{2\sigma}|W(s)|^2 ds \geq C_{A,B} \frac{1}{\lambda(B-A)} \int_{\lambda A}^{\lambda B} |W(s)|^2 ds \geq C'_{A,B},$$

where the last constant does not depend on $\lambda$, if $\lambda$ is large enough. This completes the proof of Lemma 4. □

The proof of Lemma 3 is now complete. □

## 2.2. *Perturbative analysis*

Fix $(u_0, u_1) \in C^\infty(\mathbb{T}^3) \times C^\infty(\mathbb{T}^3)$, denote by $u_n^{\epsilon_n}$ the solution of

$$\partial_t^2 u_n^{\epsilon_n} - \Delta u_n^{\epsilon_n} + \left|u_n^{\epsilon_n}\right|^{2\sigma} u_n^{\epsilon_n} = 0$$

with the initial data $(u_n^{\epsilon_n}(0), \partial_t u_n^{\epsilon_n}(0)) = \rho_{\epsilon_n} * \big((u_0, u_1) + (v_n(0), 0)\big)$, where $v_n(0)$ is given by (4). We denote by

$$S(t)(f,g) := \cos(t\sqrt{-\Delta})f + \frac{\sin\sqrt{-\Delta}}{\sqrt{-\Delta}} g$$

the propagator of the linear wave equation.

**Proposition 5.** *Assume that* $\max\{\frac{3}{2} - \frac{2}{2\sigma-1}, 0\} \leq s < s_c = \frac{3}{2} - \frac{1}{\sigma}$, *then for any* $0 < \theta < \frac{\sigma}{2}(\frac{3}{2} - s) - \frac{1}{2}$ *and* $(u_0, u_1) \in C^\infty(\mathbb{T}^3) \times C^\infty(\mathbb{T}^3)$, *there exist* $C > 0$, $\delta_2 > 0$, *such that for any* $\delta_1 \in (0, \delta_2)$, *we have*

$$\sup_{t\in[0,t_n]} \left\| u_n^{\epsilon_n}(t) - S(t)(u_0, u_1) - v_n^{\epsilon_n}(t) \right\|_{H^\nu(\mathbb{T}^3)} \leq C n^{(\nu-s)-\theta}, \forall\, \nu = 0, 1, 2,$$

*where the function* $v_n^{\epsilon_n}(t)$ *is defined in* (5) *with parameters as in* (3), *and the constant* $C$ *only depends on the smooth data* $(u_0, u_1)$ *and* $\theta > 0$. *Consequently, we have*

$$\sup_{t\in[0,t_n]} \left\| u_n^{\epsilon_n}(t) - S(t)(u_0, u_1) - v_n^{\epsilon_n}(t) \right\|_{H^s(\mathbb{T}^3)} \leq C n^{-\theta}.$$

*In particular, for* $\delta_1$ *sufficiently small,*

$$\left\| u_n^{\epsilon_n}(t_n) \right\|_{H^s(\mathbb{T}^3)} \gtrsim (\log n)^{s\sigma(\delta_2-\delta_1)-\delta_1} \to \infty, \text{ as } n \to \infty.$$

**Proof.** Denote by $u_L(t) = S(t)(u_0, u_1)$ the linear solution and $f(v) = |v|^{2\sigma} v$. Consider the difference $w_n = u_n^{\epsilon_n} - u_L - v_n^{\epsilon_n}$, then it satisfies the equation

$$\partial_t^2 w_n - \Delta w_n = \Delta v_n^{\epsilon_n} - \left(f\left(v_n^{\epsilon_n} + u_L + w_n\right) - f(v_n^{\epsilon_n})\right),\ (w_n, \partial_t w_n)|_{t=0} = 0.$$

Define the semi-classical energy for $w_n$ as in [4]

$$\begin{aligned} E_n(t) :=& \frac{1}{n^{2(1-s)}} \left( \|\partial_t w_n(t)\|^2_{L^2(\mathbb{T}^3)} + \|\nabla w_n(t)\|^2_{L^2(\mathbb{T}^3)} \right) \\ &+ \frac{1}{n^{2(2-s)}} \left( \|\partial_t w_n(t)\|^2_{H^1(\mathbb{T}^3)} + \|\nabla w_n(t)\|^2_{H^1(\mathbb{T}^3)} \right). \end{aligned} \tag{10}$$

Here the second line in (10) is needed since we need to use it to control the $L^\infty$ norm of $w_n$.

Let $F_n(t) = -\Delta v_n^{\epsilon_n} + f(v_n^{\epsilon_n} + u_L + w_n) - f(v_n^{\epsilon_n})$. From the energy estimate for the inhomogeneous linear wave equation, we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt} E_n(t) \leq&\ C n^{-(1-s)} \left\| n^{-(1-s)} \partial_t w_n(t) \right\|_{L^2(\mathbb{T}^3)} \|F_n(t)\|_{L^2(\mathbb{T}^3)} \\ &+ C n^{-(2-s)} \left\| n^{-(2-s)} \partial_t w_n(t) \right\|_{H^1(\mathbb{T}^3)} \|F_n(t)\|_{H^1(\mathbb{T}^3)}, \end{aligned}$$

and this implies that

$$\frac{d}{dt}(E_n(t))^{1/2} \leq C\left(n^{-(1-s)}\|F_n(t)\|_{L^2(\mathbb{T}^3)} + n^{-(2-s)}\|F_n(t)\|_{H^1(\mathbb{T}^3)}\right). \tag{11}$$

To simplify the notation, we denote by

$$e_n(t) := \sup_{0\leq\tau\leq t}(E_n(t))^{\frac{1}{2}}.$$

Our goal is to show that $\sup_{t\in[0,t_n]} e_n(t) \lesssim n^{-\theta}$. Write

$$G_n(t) := f\left(v_n^{\epsilon_n} + u_L + w_n\right) - f\left(v_n^{\epsilon_n}\right),$$

from Lemma 3, we have, for $t\in[0,t_n]$ that

$$\|F_n(t)\|_{L^2(\mathbb{T}^3)} \lesssim \kappa_n(\lambda_n t_n)^2 n^{2-s} + \|G_n(t)\|_{L^2(\mathbb{T}^3)}. \tag{12}$$

By the Taylor expansion,

$$|G_n| \lesssim (|u_L| + |w_n|)\left(\left|v_n^{\epsilon_n}\right|^{2\sigma} + |u_L|^{2\sigma} + |w_n|^{2\sigma}\right),$$

hence

$$\begin{aligned}&\|G_n(t)\|_{L^2(\mathbb{T}^3)}\\ &\quad\lesssim \|w_n(t)\|_{L^2(\mathbb{T}^3)}\left(1 + \left\|v_n^{\epsilon_n}(t)\right\|^{2\sigma}_{L^\infty(\mathbb{T}^3)} + \|w_n(t)\|^{2\sigma}_{L^\infty(\mathbb{T}^3)}\right) + \left\|v_n^{\epsilon_n}(t)\right\|_{L^2(\mathbb{T}^3)}\left\|v_n^{\epsilon_n}(t)\right\|^{2\sigma-1}_{L^\infty(\mathbb{T}^3)},\end{aligned}$$

where the implicit constants depend on $\|u_L(t)\|_{L^\infty(\mathbb{T}^3)}$. By writing $w_n(t,x) = \int_0^t \partial_t w_n(\tau,x)d\tau$ (since $w_n(0,\cdot)=0$), we obtain that

$$\begin{aligned}\|G_n(t)\|_{L^2(\mathbb{T}^3)} \lesssim &\int_0^t \|\partial_t w_n(\tau)\|_{L^2(\mathbb{T}^3)}\,d\tau\cdot\left(1 + \left\|v_n^{\epsilon_n}(t)\right\|^{2\sigma}_{L^\infty(\mathbb{T}^3)} + \|w_n(t)\|^{2\sigma}_{L^\infty(\mathbb{T}^3)}\right)\\ &+ \left\|v_n^{\epsilon_n}(t)\right\|_{L^2(\mathbb{T}^3)}\left\|v_n^{\epsilon_n}(t)\right\|^{2\sigma-1}_{L^\infty(\mathbb{T}^3)} + 1\\ \lesssim &\, tn^{1-s}e_n(t)\left(\lambda_n^2 + \|w_n(t)\|^{2\sigma}_{L^\infty(\mathbb{T}^3)}\right) + \kappa_n\lambda_n^{2-\frac{1}{\sigma}}n^{-s},\end{aligned} \tag{13}$$

where we have used Lemma 3 to control $\|v_n^{\epsilon_n}(t)\|_{L^\infty}$. Similarly, for $t\in[0,t_n]$, we have

$$\|\nabla F_n(t)\|_{L^2(\mathbb{T}^3)} \lesssim \kappa_n(\lambda_n t_n)^3 n^{3-s} + \|\nabla G_n(t)\|_{L^2(\mathbb{T}^3)}. \tag{14}$$

We need to estimate $\|w_n(t)\|_{L^\infty(\mathbb{T}^3)}$. From the Gagliardo–Nirenberg inequality,

$$\|w_n(t)\|_{L^\infty(\mathbb{T}^3)} \lesssim \|w_n(t)\|^{\frac{3}{4}}_{H^2(\mathbb{T}^3)}\|w_n(t)\|^{\frac{1}{4}}_{L^2(\mathbb{T}^3)} \lesssim \left(n^{2-s}e_n(t)\right)^{\frac{3}{4}}\left(te_n(t)n^{1-s}\right)^{\frac{1}{4}} = t^{\frac{1}{4}}n^{\frac{7}{4}-s}e_n(t), \tag{15}$$

where we used $w_n(t) = \int_0^t \partial_t w_n(\tau,\cdot)d\tau$ again. Since $t \leq t_n = (\log n)^{\sigma\delta_2} n^{-(\frac{3}{2}-s)\sigma}$ and $\sigma(\frac{3}{2}-s) > 1$, we have

$$\|w_n(t)\|_{L^{\infty:}(\mathbb{T}^3)} \lesssim n^{\frac{3}{2}-s}e_n(t). \tag{16}$$

Therefore,

$$\begin{aligned}n^{-(1-s)}\|F_n(t)\|_{L^2(\mathbb{T}^3)} \lesssim &\,\kappa_n(\lambda_n t_n)^2 n + \kappa_n\left(\kappa_n n^{\frac{3}{2}-s}\right)^{2\sigma-1}n^{-1} + t_n e_n(t)\left(\left(\kappa_n n^{\frac{3}{2}-s}\right)^{2\sigma} + \left(n^{\frac{3}{2}-s}e_n(t)\right)^{2\sigma}\right)\\ \lesssim &\,(\log n)^{2\sigma(\delta_2-\delta_1)-\delta_1}n + (\log n)^{-2\sigma\delta_1}n^{(2\sigma-1)(\frac{3}{2}-s)-1}\\ &+ n^{(\frac{3}{2}-s)\sigma}e_n(t)\left[(\log n)^{\sigma(\delta_2-2\delta_1)} + (\log n)^{\sigma\delta_2}(e_n(t))^{2\sigma}\right].\end{aligned}$$

Since $s > \frac{3}{2} - \frac{2}{2\sigma-1}$, we have $(2\sigma-1)(\frac{3}{2}-s) - 1 < 1$, thus

$$n^{-(1-s)}\|F_n(t)\|_{L^2(\mathbb{T}^3)} \lesssim (\log n)^{\sigma(2\delta_2-3\delta_1)}n + (\log n)^{\sigma\delta_2}n^{(\frac{3}{2}-s)\sigma}e_n(t)\left(1 + (e_n(t))^{2\sigma}\right). \tag{17}$$

Next we estimate $|\nabla G_n|$ as

$$|\nabla G_n| \lesssim \left|\nabla v_n^{\epsilon_n}\right| \left(1+\left|v_n^{\epsilon_n}\right|^{2\sigma-1}+|w_n|^{2\sigma-1}\right)(1+|w_n|) \\ +\left(1+\left|v_n^{\epsilon_n}\right|^{2\sigma}+|w_n|^{2\sigma}\right)(1+|\nabla w_n|),$$

where the implicit constants depend on $u_L, \nabla u_L$. To estimate the $L^2$ norm of $\nabla G_n$, we organize the terms as

$$\left\|\nabla v_n^{\epsilon_n}\left(1+\left|v_n^{\epsilon_n}\right|^{2\sigma-1}+|w_n|^{2\sigma-1}\right) w_n\right\|_{L^2} \le \|w_n\|_{L^2} \left\|\nabla v_n^{\epsilon_n}\right\|_{L^\infty}\left(1+\left\|v_n^{\epsilon_n}\right\|_{L^\infty}^{2\sigma-1}+\|w_n\|_{L^\infty}^{2\sigma-1}\right),$$
$$\left\|\left(1+\left|v_n^{\epsilon_n}\right|^{2\sigma}+|w_n|^{2\sigma}\right)\nabla w_n\right\|_{L^2} \le \|\nabla w_n\|_{L^2}\left(1+\left\|v_n^{\epsilon_n}\right\|_{L^\infty}^{2\sigma}+\|w_n\|_{L^\infty}^{2\sigma}\right),$$
$$\left\|\nabla v_n^{\epsilon_n}\left(1+\left|v_n^{\epsilon_n}\right|^{2\sigma-1}+|w_n|^{2\sigma-1}\right)\right\|_{L^2} \le \left\|\nabla v_n^{\epsilon_n}\right\|_{L^2}\left(1+\left\|v_n^{\epsilon_n}\right\|_{L^\infty}^{2\sigma-1}+\|w_n\|_{L^\infty}^{2\sigma-1}\right),$$
$$\left\|\left(1+\left|v_n^{\epsilon_n}\right|^{2\sigma}+|w_n|^{2\sigma}\right)\right\|_{L^2} \le \left(1+\left\|v_n^{\epsilon_n}\right\|_{L^\infty}^{2\sigma-1}\left\|v_n^{\epsilon_n}\right\|_{L^2}+\|w_n\|_{L^\infty}^{2\sigma-1}\|w_n\|_{L^2}\right).$$

Putting them together and using

$$\|w_n(t)\|_{H^k(\mathbb{T}^3)} = \left\|\int_0^t \partial_t w_n(\tau)d\tau\right\|_{H^k(\mathbb{T}^3)} \le n^{1+k-s} t e_n(t), \quad k=0,1, \tag{18}$$

we have

$$\begin{aligned} n^{-(2-s)}\|\nabla G_n(t)\|_{L^2(\mathbb{T}^3)} &\lesssim (\log n)^{\sigma\delta_2} n^{\left(\frac{3}{2}-s\right)\sigma} e_n(t)\left(1+(e_n(t))^{2\sigma}\right) \\ &\quad +(\log n)^{\sigma(\delta_2-\delta_1)} n^{(2\sigma-1)\left(\frac{3}{2}-s\right)-1}\left(1+(e_n(t))^{2\sigma-1}\right) \\ &\lesssim (\log n)^{\sigma\delta_2} n^{\left(\frac{3}{2}-s\right)\sigma} e_n(t)\left(1+(e_n(t))^{2\sigma}\right)+(\log n)^{\sigma\delta_2} n\left(1+e_n(t)^{2\sigma-1}\right). \end{aligned} \tag{19}$$

We observe that

$$\frac{de_n}{dt} \le \left|\frac{d}{dt}(E_n(t))^{1/2}\right|.$$

Therefore,

$$\frac{de_n}{dt} \le (\log n)^{3\sigma\delta_2} n + (\log n)^{\sigma\delta_2} n^{\sigma\left(\frac{3}{2}-s\right)} e_n(t)\left(1+(e_n(t))^{2\sigma}\right). \tag{20}$$

By the Grownwall type argument, we obtain

$$e_n(t) \le n^{1-\sigma\left(\frac{3}{2}-s\right)}(\log n)^{2\sigma\delta_2} e^{(\log n)^{2\sigma\delta_2}}, \quad \forall\, t \in [0, t_n].$$

Since $1 < \sigma(\frac{3}{2}-s)$, for any $0 < \theta < \frac{\sigma}{2}(\frac{3}{2}-s)-\frac{1}{2}$, we can choose $\delta_2 > 0$ sufficiently small, such that the right hand side is smaller than $n^{-\theta}$. Consequently, from (18),

$$\|w_n(t)\|_{L^2(\mathbb{T}^3)} \le n^{1-s} e_n(t) t \lesssim n^{1-s-\left(\frac{3}{2}-s\right)\sigma}(\log n)^{\delta_2\sigma} n^{-\theta} \lesssim n^{-s-\theta},\ \forall\, t \le t_n.$$

Finally, the bound for the $H^s$ norm of $w_n(t)$ follows from the interpolation. This completes the proof of Proposition 5. □

## 3. Proof of the main theorem

First we recall the following property of finite propagation speed for the wave equation.

**Lemma 6.** *Let $w_1, w_2$ be two $C^\infty$ solutions of the nonlinear wave equation*

$$\partial_t^2 w - \Delta w + |w|^{2\sigma} w = 0.$$

*If the initial data $(w_1(0), \partial_t w_1(0)), (w_2(0), \partial_t w_2(0))$ coincide on the ball $B(x_0, r_0) \subset \mathbb{R}^d$, then for $0 \le t < r_0$, $(w_1(t), \partial_t w_1(t)) = (w_2(t), \partial_t w_2(t))$ on $B(x_0, r_0 - t)$.*

**Proof.** Without loss of generality, we may assume that $x_0 = 0$. Take the difference $u = w_1 - w_2$, then

$$\partial_t^2 u - \Delta u + u = V(t,x)u,$$

where

$$V(t,x) = (2\sigma+1)\int_0^1 |(1-\lambda)w_1(t,x) + \lambda w_2(t,x)|^{2\sigma}\, d\lambda + 1 \in L^\infty_{\text{loc}}.$$

For $0 \le t_1 < t_2 < r_0$, denote by $\mathscr{C}_{t_1,t_2}(r_0) := \{(t,x) : t_1 \le t \le t_2, |x| \le r_0 - t\}$. Define the local energy density

$$e(t,x) := \frac{1}{2}\left(|\nabla u(t,x)|^2 + |\partial_t u(t,x)|^2 + |u(t,x)|^2\right).$$

Then a direct calculation yields

$$\int_{\mathscr{C}_{0,t_0}(r_0)} \partial_t u\left(\partial_t^2 - \Delta + 1\right) u dx dt = \int_0^{t_0}\int_{|x|\le r_0 - t} \frac{d}{dt} e(t,x) dx dt - \int_0^{t_0}\int_{|x| = r_0 - t} \partial_t u \partial_r u d\sigma(x) dt,$$

where $\partial_r u = \frac{x}{|x|}\cdot\nabla u$ and $r = |x|$. Notice that $\frac{d}{dt}\mathbf{1}_{|x|\le r_0-t} = -\delta_{|x|=r_0-t}$, we have

$$\begin{aligned}
&\int_{\mathscr{C}_{0,t_0}(r_0)} \partial_t u\left(\partial_t^2 - \Delta + 1\right) u dx dt \\
&\qquad = \left[\int_{|x|\le r_0-t} e(t,x)dx\right]_{t=0}^{t=t_0} + \int_0^{t_0}\int_{|x|=r_0-t} \frac{1}{2}\left[|\partial_t u - \partial_r u|^2 + |u|^2\right] d\sigma(x) dt \\
&\qquad \ge \left[\int_{|x|\le r_0-t} e(t,x)dx\right]_{t=0}^{t=t_0}.
\end{aligned}$$

Using the equation $\partial_t^2 u - \Delta u + u = Vu$, we have

$$E(t_0) \le E(0) + \left|\int_{\mathscr{C}_{0,t_0}(r_0)} Vu\cdot\partial_t u dx dt\right| \le E(0) + \|V\|_{L^\infty([0,r_0]\times B(0;r_0))}\int_0^{t_0} E(t)dt,$$

for all $0 \le t_0 < r_0$, where $E(t) = \int_{|x|\le r_0-t} e(t,x)dx$ is the local energy. Since $E(0) = 0$, from Gronwall's inequality, we deduce that $E(t) \equiv 0$ for all $0 \le t < r_0$. This completes the proof of Lemma 6. □

To prove Theorem 2, we need to do some preparations. We use the coordinate system $x = (x_1, x')$ near the origin. Let $z^k = (z_1^k, 0)$ with $z_1^k = \frac{1}{k}$. Let $n_k = e^{e^k}$, and define

$$v_{0,k}(x) := \left(\log n_k\right)^{-\delta_1} n_k^{\frac{3}{2}-s}\varphi\left(n_k\left(x_1 - z_1^k\right), n_k x'\right) = v_{n_k}\left(0, \cdot - z^k\right),$$

where $v_n(0)$ is the initial data of the ill-posed profile defined in (4). Note that there exists $k_0$, such that for all $k \ge k_0$, the supports of $v_{0,k}$ are pairwise disjoint. Moreover, for $k_0 \le k_1 < k_2$,

$$\operatorname{dist}\left(\operatorname{supp}(v_{0,k_1}), \operatorname{supp}\left(v_{0,k_2}\right)\right) \sim \frac{1}{k_1} - \frac{1}{k_2}.$$

Denote by $B_k = B(z^k, r_k)$, where $r_k = \frac{1}{k^3}$. With sufficiently large $k_0$, the balls $B_k, k \ge k_0$ are mutually disjoint. Moreover, $\operatorname{supp}(\rho_{\epsilon_{n_k}} * v_{0,k}) \subset B_k$ (recall that $\epsilon_{n_k} = \frac{n_k}{100}$). Another simple observation is that

$$\operatorname{dist}\left(\operatorname{supp}\left(\rho_{\epsilon_{n_k}} * \left(v_0 - v_{0,k}\right)\right), B_k\right) \gtrsim \frac{1}{k^2},$$

where

$$v_0 = \sum_{k\ge k_0} v_{0,k} \in H^s\left(\mathbb{T}^3\right).$$

In particular, for any $(f,g)\in C^\infty\times C^\infty$, $\rho_{\epsilon_{n_k}}*((f,g)+(v_0,0))$ coincides with $\rho_{\epsilon_{n_k}}*((f,g)+(v_{0,k},0))$ on $B_k$. Let $\widetilde{B}_k=B(z^k,r_k/3)$ be a slightly smaller ball. We observe that for $k$ large enough,

$$\operatorname{supp}\left(\rho_{\epsilon_{n_k}}*v_{0,k}\right)\subset\widetilde{B}_k.$$

Now we are able to prove Theorem 2.

**Proof of Theorem 2.** Define

$$S=C^\infty\left(\mathbb{T}^3\right)\times C^\infty\left(\mathbb{T}^3\right)+\left\{\left(\sum_{k=k_1}^{\infty}v_{0,k},0\right):k_1\geq k_0\right\}.$$

Using

$$\left\|\sum_{k=k_1}^{\infty}v_{0,k}\right\|_{H^s(\mathbb{T}^3)}\leq\sum_{k=k_1}^{\infty}\left\|v_{0,k}\right\|_{H^s(\mathbb{T}^3)}\leq\sum_{k=k_1}^{\infty}e^{-k\delta_1}\to 0\text{ as }k_1\to\infty,$$

we deduce $S$ is dense in $\mathscr{H}^s(\mathbb{T}^3)$. Now fix $(f,g)\in S$. Then by definition, there exists $(u_0,u_1)\in C^\infty\times C^\infty$ and $k_1\geq k_0$, such that

$$(f,g)=(u_0,u_1)+\left(\sum_{k=k_1}^{\infty}v_{0,k},0\right).$$

Our goal is to show that, for any $N>0$ and any $\delta>0$, there exist $\tau_N\in[0,1]$ and $0<\epsilon<\delta$, such that the solution $u^\epsilon$ to (1) with initial data $\rho_\epsilon*(f,g)$ satisfies

$$\left\|u^\epsilon(\tau_N)\right\|_{H^s(\mathbb{T}^3)}>N. \tag{21}$$

First we choose $k\geq k_1$, large enough, such that

$$\kappa_{n_k}\left(\lambda_{n_k}t_{n_k}\right)^s>N,\quad \epsilon_k=\frac{n_k}{100}<\delta.$$

This can be achieved by choosing $\delta_1<\delta_2$ such that $s\sigma(\delta_2-\delta_1)>\delta_1$. Recall that the parameters $\kappa_{n_k}=e^{-k\delta_1},\lambda_{n_k}t_{n_k}=e^{(\delta_2-\delta_1)k\sigma}$ are given by (3). Let $\widetilde{u}_k$ be the solution of (1) with the initial data $\rho_{\epsilon_{n_k}}*(u_0,u_1)+\rho_{\epsilon_{n_k}}*(v_{0,k},0)$. Let $\widetilde{v}_k$ be the solution of $\partial_t^2\widetilde{v_k}+|\widetilde{v_k}|^{2\sigma}\widetilde{v_k}=0$ with the initial data $\rho_{\epsilon_{n_k}}*(v_{0,k},0)$. We remark that $\widetilde{v}_k,\widetilde{u}_k$ are just $v_{n_k}^{\epsilon_{n_k}},u_{n_k}^{\epsilon_{n_k}}$ in Proposition 5 up to translation. In particular,

$$\left\|\widetilde{u}_k(t_{n_k})\right\|_{H^s(\mathbb{T}^3)}\gtrsim(\log n_k)^{s\sigma(\delta_2-\delta_1)-\delta_1}, \tag{22}$$

and

$$\left\|\widetilde{u}_k(t_{n_k})-S(t_{n_k})(u_0,u_1)-\widetilde{v}_k(t_{n_k})\right\|_{H^s(\mathbb{T}^3)}\lesssim n_k^{-\theta}. \tag{23}$$

We have that $\operatorname{supp}(\widetilde{v}_k(t))\subset\widetilde{B}_k$ for all $t\in[0,t_{n_k}]$. Now we apply Lemma 6 to $\widetilde{u}_k$ and $u^{\epsilon_{n_k}}$. Since at $t=0$, $(u^{\epsilon_{n_k}}(0),\partial_t u^{\epsilon_{n_k}}(0))|_{B_k}=(\widetilde{u}_k(0),\partial_t\widetilde{u}_k(0))|_{B_k}$, we deduce that

$$\left(u^{\epsilon_{n_k}}(t),\partial_t u^{\epsilon_{n_k}}(t)\right)\big|_{B(z^k,r_k-t)}=\left(\widetilde{u}_k(t),\partial_t\widetilde{u}_k(t)\right)\big|_{B(z^k,r_k-t)},\quad\forall\, 0\leq t<r_k.$$

In particular, for large $k$,

$$\left(u^{\epsilon_{n_k}}(t),\partial_t u^{\epsilon_{n_k}}(t)\right)\big|_{B(z^k,r_k/2)}=\left(\widetilde{u}_k(t),\partial_t\widetilde{u}_k(t)\right)\big|_{B(z^k,r_k/2)},\quad\forall\, t\in\left[0,t_{n_k}\right]. \tag{24}$$

**Lemma 7.** *Assume that $s_1\geq 0$. Let $u\in H^{s_1}(\mathbb{T}^3)$ and $\chi\in C_c^\infty(\mathbb{T}^3)$. Then there exists $A>0$, depending only on the function $\chi$ and $s_1$, such that for any $R\geq 1$*

$$\left\|(1-\chi(Rx))u\right\|_{H^{s_1}(\mathbb{T}^3)}+\left\|\chi(Rx)u\right\|_{H^{s_1}(\mathbb{T}^3)}\leq AR^{s_1}\|u\|_{H^{s_1}(\mathbb{T}^3)}.$$

**Proof.** First for $s_1\in\mathbb{N}$, the proof follows from the direct calculation. For general $s_1\geq 0$, the conclusion follows from the interpolation. □

Take $\chi \in C_c^\infty(\mathbb{R}^3)$, such that $\chi(x) \equiv 1$ if $|x| < \frac{1}{3}$ and $\chi \equiv 0$ if $|x| \geq \frac{1}{2}$. Define $\chi_k(x) := \chi((x - z^k)/r_k)$, hence $\chi_k|_{\widetilde{B}_k} \equiv 1$ and $\chi_k|_{(B(z^k, r_k/2))^c} \equiv 0$. Then (24) is translated to

$$\chi_k(x)\left(u^{\epsilon_{n_k}}(t), \partial_t u^{\epsilon_{n_k}}(t)\right) = \chi_k(x)\left(\widetilde{u}_k(t), \partial_t \widetilde{u}_k(t)\right), \quad \forall\, t \in \left[0, t_{n_k}\right].$$

From Lemma 7,

$$\left\| u^{\epsilon_{n_k}}\left(t_{n_k}\right)\right\|_{H^s(\mathbb{T}^3)} \gtrsim r_k^s \left\|\chi_k u^{\epsilon_{n_k}}\left(t_{n_k}\right)\right\|_{H^s(\mathbb{T}^3)} \sim (\log\log n_k)^{-3s} \left\|\chi_k(x)\widetilde{u}_k\left(t_{n_k}\right)\right\|_{H^s(\mathbb{T}^3)}.$$

Therefore,

$$\begin{aligned}\left\|\chi_k(x)\widetilde{u}_k\left(t_{n_k}\right)\right\|_{H^s(\mathbb{T}^3)} &\geq \left\|\widetilde{u}_k\left(t_{n_k}\right)\right\|_{H^s(\mathbb{T}^3)} - \left\|(1-\chi_k)\widetilde{u}_k\left(t_{n_k}\right)\right\|_{H^s(\mathbb{T}^3)} \\ &= \left\|\widetilde{u}_k\left(t_{n_k}\right)\right\|_{H^s(\mathbb{T}^3)} - \left\|(1-\chi_k)\left(\widetilde{u}_k\left(t_{n_k}\right) - \widetilde{v}_k\left(t_{n_k}\right)\right)\right\|_{H^s(\mathbb{T}^3)},\end{aligned}$$

where in the last equality, we use the fact that $(1-\chi_k)\widetilde{v}_k(t_{n_k}) = 0$, thanks to the support property of $\widetilde{v}_k$. Therefore, we have

$$\begin{aligned}&\left\|u^{\epsilon_{n_k}}\left(t_{n_k}\right)\right\|_{H^s(\mathbb{T}^3)} \\ &\quad\gtrsim \left(\log\log n_k\right)^{-3s}\left\|\widetilde{u}_k\left(t_{n_k}\right)\right\|_{H^s(\mathbb{T}^3)} - \left(\log\log n_k\right)^{-3s}\left\|\left(1-\chi_k\right)S\left(t_{n_k}\right)(u_0,u_1)\right\|_{H^s(\mathbb{T}^3)} \\ &\quad - \left(\log\log n_k\right)^{-3s}\left\|\left(1-\chi_k\right)\left(\widetilde{u}_k\left(t_{n_k}\right) - S\left(t_{n_k}\right)(u_0,u_1) - \widetilde{v}_k\left(t_{n_k}\right)\right)\right\|_{H^s(\mathbb{T}^3)}.\end{aligned} \tag{25}$$

Applying Lemma 7 again, we have

$$\left\|u^{\epsilon_{n_k}}\left(t_{n_k}\right)\right\|_{H^s(\mathbb{T}^3)} \gtrsim \left(\log\log n_k\right)^{-3s}\left(\log n_k\right)^{s\sigma(\delta_2-\delta_1)-\delta_1} - \left\|S\left(t_{n_k}\right)(u_0,u_1)\right\|_{H^s(\mathbb{T}^3)} - n_k^{-\theta}. \tag{26}$$

By choosing $\delta_1 > 0$ small such that $s\sigma(\delta_2 - \delta_1) - \delta_1 > 0$, the left hand side of (26) tends to $+\infty$ as $k \to \infty$. This completes the proof of Theorem 2. □

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