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ETALE COVERINGS OF A MUMFORD CURVE

by Marius van der PUT

Introduction.

For a Riemann surface X over **C** of genus $\geqslant 2$ the finite unramified coverings $Y \longrightarrow X$ are easily obtained from the uniformization of X. Indeed, from the universal covering

$$\mathcal{H} = \{z \in \mathbf{C} \mid \mathrm{Im}(z) > 0\} \longrightarrow X$$

with group $\Gamma \cong \pi_1(X)$ one obtains all possibilities for Y by taking $\mathcal{H}/N$ where N is a subgroup of Γ of finite index.

For an algebraic curve X defined over a complete non-archimedean valued field K the situation is more complicated. In order to obtain "enough" unramified coverings $Y \longrightarrow X$ one has to suppose that X is a Mumford curve. On further distinguishes between merely unramified (or étale) coverings and analytic coverings. This is done in section 1. In the next section the abelian étale coverings of a Mumford curve over an algebraically closed field are constructed. In section 3 the base field is a local field and the abelian unramified extensions of the function field of the curve X are calculated. The result of this section is due to G. Frey. We have presented here a rigid-analytic proof of this theorem. For general background concerning analytic spaces over K we refer to [1] and [3].

1. Analytic coverings and étale coverings.

The field K is supposed to be algebraically closed and to be complete with respect to a non-archimedean valuation. A morphism

$f: Y \longrightarrow X$ of analytic spaces over K is an *étale covering* if f is surjective and if f for every point $x \in X$ there exists an affinoid subspace K of X containing x such that $f^{-1}(U)$ is a disjoint union of affinoid subspaces $V_i (i \in I)$ and such that each $f: V_i \longrightarrow U$ is a isomorphism.

Suppose that $f: Y \longrightarrow X$ is a finite morphism. This means that X has an admissible affinoid covering $(X_i)_{i \in I}$ such that each $f^{-1}(X_i)$ is a non-empty affinoid subset of Y and such that each $\mathcal{O}_X(X_i) \longrightarrow \mathcal{O}_Y(f^{-1}(X_i))$ is a finite injective map of affinoid algebra's. In case that f is finite on has: f is an étale covering if and only if for each $y \in Y$ the map $\hat{f}_y^*: \hat{\mathcal{O}}_{Y,y} \longrightarrow \hat{\mathcal{O}}_{X,f(y)}$ is an isomorphism.

Indeed, $\hat{f}_y^*$ isomorphism implies that also $f_y^*: \mathcal{O}_{Y,y} \longrightarrow \mathcal{O}_{X,f(y)}$ is an isomorphism and that there are affinoid sets V, U containing y and $f(y)$ such that $f: V \longrightarrow U$ is an isomorphism. Take $x \in X$ and put $f^{-1}(x) = \{y_1, \ldots, y_n\}$. Choose affinoid neighbourhoods V_i of y_i and U of x such that every $V_i \longrightarrow U$ is an ismorphism. After shrinking U we may suppose that the V_i are disjoint and that every point $t \in U$ has n pre-images in Y. Then clearly $f^{-1}(U) = V_1 \cup \ldots \cup V_n$, the V_i are disjoint and each $V_i \longrightarrow U$ is an isomorphism.

The morphism f is called an *analytic covering* if there exists an admissible affinoid covering $(X_i)_{i \in I}$ of X, an admissible covering $(Y_j)_{j \in J}$ of Y by affinoid subsets and a surjective map $\pi: J \longrightarrow I$ such that for all i:

(i) $f^{-1}(X_i)$ is the disjoint union of the Y_j with $\pi(j) = i$

(ii) $f: Y_j \longrightarrow X_i$ is an isomorphism for each j with $\pi(j) = i$.

An analytic covering is certainly an étale covering. The map $f: K^* \longrightarrow K^*$ given by $z \longmapsto z^n$ ($n > 1$ and n prime to char K) provides an example of an étale covering which is not an analytic covering. This is rather in contrast with the complex-analytic case where the corresponding notions coincide. In the sequel we will restrict ourselves to one-dimensional regular analytic spaces and especially to complete non-singular curves over K. It is clear however that many results will be correct for higher dimensional spaces.

LEMMA 1.1. – *Let* $f: Y \longrightarrow X$ *be an étale (resp. analytic) covering of non-singular complete irreducible algebraic curves. Then* *the minimal Galois extension* $g : Z \longrightarrow X$ *is also an étale (resp. analytic) covering.*

Proof. – For the function fields of X, Y and Z we have the inclusions $F(X) \subset F(Y) \subset F(Z)$ and $F(Z)$ is the minimal Galois-extension of $F(X)$ containing $F(Y)$. Let $Y_i \longrightarrow X$ $(i = 1, \dots, s)$ denote the morphisms corresponding to the subfields of $F(Z)$ which are conjugated with $F(Y)$. Since each $Y_i \longrightarrow X$ is an étale (resp. analytic) covering the same holds for $Y_1 \times_X \dots \times_X Y_s \longrightarrow X$. In particular $Y_1 \times_X \dots \times_X Y_s$ is non-singular and complete and every connected component is again an étale (resp. analytic) covering of X. The canonical map $Z \longrightarrow Y_1 \times_X \dots \times_X Y_s$ induces an isomorphism of Z with a connected component.

This proves the lemma.

LEMMA 1.2. – *Let* $f : Y \longrightarrow X$ *be a non-constant morphism between (non-singular, irreducible, complete) curves. The exists a unique maximal decomposition* $Y \xrightarrow{f} X = Y \xrightarrow{g_1} Y_1 \xrightarrow{f_1} X$ *where* Y_1 *is a curve and* f_1 *is an étale covering. There exists a unique maximal decomposition* $Y \xrightarrow{f} X = Y \xrightarrow{g_0} Y_0 \xrightarrow{f_0} X$ *with* Y_0 *a curve and* f_0 *an analytic covering. Moreover* $Y_1 \xrightarrow{f_1} X$ *factors as* $Y_1 \longrightarrow Y_0 \xrightarrow{f_0} X$. *If* $Y \longrightarrow X$ *is Galois then also* $Y_1 \longrightarrow X$ *and* $Y_0 \longrightarrow X$ *are Galois.*

Proof. – One has to consider subextensions of $F(X) \subset F(Y)$. For subextensions $F(Z_1)$ and $F(Z_2)$ let $F(Z_3)$ denote the least subfield containing $F(X_1)$ and $F(X_2)$. Then $Z_3 \longrightarrow X$ is an étale (resp. analytic) covering if and only if $Z_1 \longrightarrow X$ and $Z_2 \longrightarrow X$ are étale (resp. analytic) coverings.

1.3 Let now X denote the Mumford curve Ω/Γ ; Γ a Schottky group with Ω as set of ordinary points in $\mathbf{P}^1$. It is known that $\Omega \longrightarrow X$ is the universal analytic covering of X. In particular every finite analytic covering $Y \longrightarrow X$ has uniquely the form $\Omega/\Gamma_0 \longrightarrow X$ where Γ_0 is a subgroup of Γ of finite index. The étale coverings of X are hidden in Ω. We introduce the following notion: $c : \Omega_* \longrightarrow \Omega$ is a *Γ-equivariant covering* if:

(i) $c : \Omega_* \longrightarrow \Omega$ is a finite, connected, Galois, étale covering with group H.

(ii) Every automorphism $\gamma \in \Gamma$ of Ω lifts to an automorphism δ of Ω_*. (i.e. $c\delta = \gamma c$).

Let G denote the group of analytic automorphisms δ of Ω_* such that $c\delta = \gamma c$ holds for some $\gamma \in \Gamma$.

From the definitions one obtains a canonical exact sequence of groups $1 \longrightarrow H \xrightarrow{\pi} G \longrightarrow \Gamma \longrightarrow 1$. Let N denote a normal subgroup of G of finite index such that $N \cap H = \{1\}$. With the notations we can formulate the following results.

THEOREM 1.4. –

1) *Ω_*/N is a non-singular, irreducible, complete curve over K. The map $\Omega_*/N \longrightarrow \Omega/\Gamma = X$ is a Galois, étale-covering with Galois group G/N. This map decomposes uniquely into*

$$\Omega_*/N \longrightarrow \Omega/\pi(N) \longrightarrow X \quad \textit{where} \quad \Omega/\pi(N) \longrightarrow X$$

is the maximal analytic subcovering.

2) *Let Y be an irreducible non-singular complete curve and let $f : Y \longrightarrow X$ be a Galois, étale-covering. There exists a pair (Ω_*, N) (unique up to isomorphism) and an isomorphism $g : Y \longrightarrow \Omega_*/N$ such that the diagram*

$$\begin{array}{ccc} Y & \longrightarrow & X \\ \downarrow & \nearrow & \\ \Omega_*/N & & \end{array}$$

is commutative.

Proof. –

1) The construction of Ω_*/N as a 1-dimensional regular analytic space over K is very similar to the construction in [3] p. 105. One can make this construction explicit by a choice of a fundamental domain. Let $F \in \Omega$ be a good fundamental domain for the group $\pi(N)$ ([3] p. 28). Then F has the form $\mathbf{P}^1 - B_1 \cup \ldots \cup B_{2a}$ where $\pi(N) = \langle \gamma_1, \ldots, \gamma_a \rangle$ and $B_1, \ldots, B_{2a}$ are open discs such that the corresponding discs B_i^+ are still disjoint and such that γ_i is an isomorphisms of $B_i^+ - B_i$ with $B_{i+1}^+ - B_{i+a}$ $(i = 1, \ldots, a)$.

Let $\widetilde{B}_i \supset B_i^+$ denote open discs such that the closed discs $\widetilde{B}_i^+$ are still disjoint. Put $G = \mathbf{P}^1 - \widetilde{B}_1 \cup \ldots \cup \widetilde{B}_{2a}$. Then $\Omega/\pi(N)$ can be constructed by glueing the affinoid pieces G, $\widetilde{B}_1^+ - B_1, \ldots, \widetilde{B}_{2a}^+ - B_{2a}$ according to

(i) $\widetilde{B}_i^+ - B_i$ is glued to G over the subset $\widetilde{B}_i^+ - \widetilde{B}_i$.

(ii) for $1 \leqslant i \leqslant a$, $\widetilde{B}_i^+ - B_i$ is glued to $\widetilde{B}_{i+a}^+ - B_{i+a}$ by using the isomorphism $\gamma_i : B_i^+ - B_i \xrightarrow{\sim} B_{i+a}^+ - B_{i+a}$.

To obtain Ω_*/N we replace in the construction above the affinoid sets G, $\widetilde{B}_i^+ - B_i$, $B_i^+ - B_i$ by the subsets $c^{-1}(G)$, $c^{-1}(\widetilde{B}_i^+ - B_i)$, $c^{-1}(B_i^+ - B_i)$ of Ω_* and γ_i by the unique element $\widetilde{\gamma}_i \in N$ with $\pi(\widetilde{\gamma}_i) = \gamma_i$.

The only thing that one has to verify is that $c^{-1}(G)$ etc are affinoid subsets. Indeed, one can easily verify the more general statement: "Let $U \longrightarrow V$ be a finite morphism of analytic spaces over K. If V is affinoid then U is also affinoid."

Using this construction of Ω_*/N and the given affinoid covering of Ω_*/N one can calculate that $\dim_K H^1(\Omega_*/N, \mathcal{O}) < \infty$ and finally prove that Ω_*/N is actually a complete, irreducible, non-singular algebraic curve over K. (See [3] p. 106-107). The only statement that we still have to verify is the maximality of the analytic subextension $\Omega/\pi(N) \longrightarrow X$. The normal subextensions correspond to normal subgroups M of G containing N. We have to show that $\Omega_*/M \longrightarrow \Omega/\Gamma$ is an analytic covering if and only if $M \supseteq H$.

Put $M \cap H = H_1$. We replace

$$\Omega_* \xrightarrow{c} \Omega \quad \text{by} \quad \Omega'_* \doteq \Omega_*/H_1 \xrightarrow{c'} \Omega$$

and H by $H' \doteq H/H_1$; G by $G' \doteq G/H_1$ and M by $M' = M/H_1$. Again we have an exact sequence $1 \longrightarrow H' \longrightarrow G' \longrightarrow \Gamma \longrightarrow 1$ and now $M' \cap H' = \{1\}$. We have to show $\Omega'_* = \Omega$ if $\Omega'_*/M' \longrightarrow \Omega/\Gamma$ is an analytic covering. The hypothesis implies easily that $\Omega'_* \longrightarrow \Omega$ is a connected analytic covering. According to [3] p. 151, (3.4), one has $\Omega'_* \xrightarrow{\sim} \Omega$.

2) We consider the commutative diagram

$$\begin{array}{ccc} X & \xleftarrow{\ \pi\ } & \Omega \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle f'} \\ Y & \xleftarrow{\ \pi'\ } & Y \times_X \Omega = \Omega' \end{array}$$

The fibre product Ω' is as a set of points equal to

$$\{(y, \omega) \in Y \times \Omega \mid f(y) = \pi(\omega)\}.$$

One can easily give Ω' the structure of an analytic space over K since

π is an analytic covering. We denote by G_0 the Galois group of $Y \mid X$. The group $G_0 \times \Gamma$ acts as group of analytic automorphisms on Ω' in the following way: $(\sigma, \gamma)(y, \omega) = (\sigma(y), \gamma(\omega))$. Easy arguments will prove the following statements:

a) f' is an étale covering with group G_0; possibly not connected.

b) π' is an analytic covering with group Γ; possibly not connected.

c) $\Omega'/\Gamma = Y$ and $\Omega'/G_0 = \Omega$.

d) for every connected affinoid $U \subset \Omega$, the set $(f')^{-1}(U)$ is affinoid. G_0 acts transitively on the connected components and each of them is mapped surjectively to U.

e) After applying d) to a sequence $U_1 \subset U_2 \subset U_3 \subset \ldots$ of connected affinoid subsets of Ω which defines the holomorphic structure on Ω, one finds that Ω' has finitely many components $\Omega'_1, \ldots, \Omega'_s$. Each component is mapped surjectively to Ω and G_0 acts transitively on the components.

f) From $\Omega'/\Gamma = Y$ if follows that Γ acts transitively on the components and that $\Omega'_1/N = Y$ where

$$N = \{(1, \gamma) \in G_0 \times \Gamma \mid \gamma(\Omega'_1) = \Omega'_1\}.$$

Put $\Omega_* = \Omega'_1$ and let $c : \Omega_* \longrightarrow \Omega$ denote the restriction of f' to Ω_*. We make the following definitions:

$$G = \{(\sigma, \gamma) \in G_0 \times \Gamma \mid (\sigma, \gamma)\, \Omega_* = \Omega_*\}$$
$$H = \{(\sigma, 1) \in G_0 \times \Gamma \mid (\sigma, 1)\, \Omega_* = \Omega_*\}$$
$$N = \{(1, \gamma) \in G_0 \times \Gamma \mid (1, \gamma)\, \Omega_* = \Omega_*\}.$$

From c) $\Omega'/G_0 = \Omega$ it follows that $\Omega_*/H = \Omega$ and that $c : \Omega_* \longrightarrow \Omega$ is a Galois étale covering, connected, and with group H.

The sequence $1 \longrightarrow H \longrightarrow G \longrightarrow \Gamma \longrightarrow 1$ is exact since for every $\gamma \in \Gamma$ there exists a $\sigma \in G_0$ such that $\sigma(\Omega_*) = \gamma(\Omega_*)$. So $(\sigma^{-1}, \gamma) \in G$ and this element maps to γ. The group N is clearly a normal subgroup of finite index in G and $N \cap H = \{1\}$. Finally, according to f) we have $\Omega_*/N \cong Y$.

Similar methods will easily give the uniqueness (up to isomorphism) of the pair (Ω_*, N).

PROPOSITION 1.5. – *Let* Y *be a complete, non-singular, irreducible curve over* K *or a 1-dimensional, regular, connected affinoid space. Then* Y *has a universal analytic covering. The Galois group of this universal analytic covering is a finitely generated free (non-commutative) group.*

Proof of 1.5. – The analytic space Y has a reduction $r: Y \longrightarrow Z$ which is pre-stable and such that every component of Z is non-singular. (This is proved in [4].) The graph G of Z, i.e. the vertices of G are the components of Z and the edges of G are the double points of Z, is in general no a tree. Let $T \longrightarrow G$ be the universal covering of the graph. Then T is a tree and on it operates a group $\Gamma \cong \pi_1(G)$ which is a finitely generated free group such that $T/\Gamma \simeq G$. As in [3] p. 149 (3.2), one can lift the construction of T and Γ to obtain an analytic space Ω and an analytic covering $u: \Omega \longrightarrow Y$ with group Γ, such that Ω has a reduction $\overline{\Omega}$ and an induced map $\overline{u}: \overline{\Omega} \longrightarrow Z$ which is for the Zariski-topology the universal covering and such that the graph associated with $\overline{\Omega}$ is T and $\overline{\overline{u}}: T \longrightarrow G$ is the universal covering of the graph mentioned above. The proposition will follow now if we can show that Ω admits only trivial analytic coverings. It suffices to show that an affinoid space U such that its canonical reduction $\overline{U}$ consists of non-singular affine curves intersecting normally has only trivial analytic coverings. Indeed Ω is build up out of such affinoid spaces U in an acyclic way.

Let now $\varphi: V \longrightarrow U$ be an analytic covering. According to the definition $U = U_1 \cup \ldots \cup U_n$ where the U_i are affinoid subspaces of U and such that $\varphi^{-1}(U_i)$ is the disjoint union of affinoid subsets of V, each of them mapped isomorphically to U_i. After refining the covering $\{U_1, \ldots, U_n\}$ of U we may suppose that it is a pure covering such that the corresponding reduction $\overline{\overline{U}}$ of U is prestable and has non-singular components (see [4]). The reduction $\overline{V}$ of V with respect to $\{\varphi^{-1}(U_1), \ldots, \varphi^{-1}(U_n)\}$ is also prestable and the induced map $\overline{V} \longrightarrow \overline{\overline{U}}$ is a covering for the Zariski-topology. One knows that $\overline{\overline{U}}$ is obtained from $\overline{U}$ by a finite number of steps. In each step a point is replaced by a projective line over $\overline{K}$. This shows that $\overline{\overline{U}}$ has only trivial coverings for the Zariski-topology. If we assume that V is connected then also $\overline{V}$ is connected. Hence $\overline{V} = \overline{\overline{U}}$ and so $V = U$. This shows finally the existence of the universal analytic covering $u : \Omega \longrightarrow Y$. We want to show that Ω has the usual property:

"Given a morphism $f : S \longrightarrow Y$, where S is a connected analytic space which has only trivial analytic coverings, and given points $s \in S$ and $\omega \in \Omega$ with $u(\omega) = f(s)$, then there exists a unique lift $f' : S \longrightarrow \Omega$ with $uf' = f$ and $f'(s) = \omega$."

We consider the fibre-product $\Omega' = \Omega \times_Y S \longrightarrow S$. This is an analytic covering S. By assumption, every component of Ω' maps isomorphically to S. Taking the component of Ω' which contains the point (ω, s) one finds f' and one shows that f' is unique.

COROLLARY 1.6. – *Let Y, N, Ω_* be as in (1.4) and let $\Omega(Y)$ denote the universal analytic covering of Y which has group $\Gamma(Y)$. There exists a normal subgroup Γ_0 of $\Gamma(Y)$ such that $\Omega_* \cong \Omega(Y)/\Gamma_0$ and $\Gamma(Y)/\Gamma_0 \cong N$.*

Proof. – Easy consequence of (1.4) and (1.5).

Remark. – In general, Ω_* is not the universal analytic covering of Y. In section 2 we will discuss examples. The reason is that a connected, Galois, étale covering $e : \Omega_* \longrightarrow \Omega$, admits itself in general non-trivial analytic coverings.

Example 1.7. – Take

$$\Omega = \mathbf{P}^1 - \{0, \pi, 1, \infty\} \text{ where } 0 < |\pi| < 1 .$$

And let $\Omega_* = \{(x, y) \in \Omega \times K \mid y^2 = x(x - \pi)(x - 1)\}$. Assuming that the characteristic of K is unequal to two, one finds that $c : \Omega_* \longrightarrow \Omega$ is a connected étale covering with Galois group $\mathbf{Z}/2$. The elliptic curve, corresponding to the equation $y^2 = x(x - \pi)(x - 1)$ is the Tate curve $K^*/\langle q \rangle$ for a suitable q, $0 < |q| < 1$. Further $\Omega_* = K^*/\langle q \rangle - \{\pm 1, \pm q^{1/2}\}$. The Tate curve has the universal analytic covering $K^* \longrightarrow K^*/\langle q \rangle$. This easily implies that the universal analytic covering of Ω_* must be $U = K^* - \{\pm q^{n/2} \mid n \in \mathbf{Z}\}$. The resulting connected étale covering $U \longrightarrow \Omega$ is in this case Galois. Its group is generated by two elements γ, δ, defined as automorphisms of U by $\gamma(z) = qz$ and $\delta(z) = z^{-1}$. The only relations are $\delta^2 = 1$ and $\delta\gamma = \gamma^{-1}\delta$.

More examples 1.8. – Let Γ denote a finitely generated discontinuous subgroup of $PG\ell(2,K)$. Suppose that $\Gamma/[\Gamma,\Gamma]$ is a finite group. Let Ω denote the set of ordinary points for Γ. It is known that $\Omega/\Gamma \cong \mathbf{P}^1$ (see [3] Ch. VIII, (4.3)). There exists a normal subgroup $\Gamma_0 \subset \Gamma$ of finite index, which is a Schottky group. That implies that $c : \Omega \longrightarrow \Omega/\Gamma = \mathbf{P}^1$ is only ramified above a finite subset S of $\mathbf{P}^1$. Then $\Omega - c^{-1}(S) \longrightarrow \mathbf{P}^1 - S$ is a Galois étale map with group Γ. Special cases of such groups Γ are provided by Whittaker groups or by cyclic extensions of $\mathbf{P}^1$ (see [3, 6]).

Remark 1.9. – Let the Schottky group Γ and its space of ordinary points $\Omega \subset \mathbf{P}^1$ be given. It is rather difficult to construct equivariant étale coverings $\Omega_* \longrightarrow \Omega$. In the next section we will restrict our attention to abelian extensions $\Omega_* \longrightarrow \Omega$.

2. Construction of the abelian étale coverings.

We assume in this section that X is a Mumford curve over K of genus g and we fix a presentation $X = \Omega/\Gamma$ with Γ a Schottky group on g generators and in which $\Omega \subset \mathbf{P}^1$ is the subspace of ordinary points of Γ. According to (1.4) we have to construct the abelian Γ-equivariant étale morphisms $c : \Omega_* \longrightarrow \Omega$ such that in the notation of (1.3), one has $[G,G] \cap H = \{1\}$. Indeed, there must exists a normal subgroup N, of finite index, in G with abelian factor group and $N \cap H = \{1\}$. We call an abelian Γ-equivariant étale map $c : \Omega_* \longrightarrow \Omega$ *strongly abelian* if $[G,G] \cap H = \{1\}$. This condition is clearly equivalent to "G is the direct product of H and Γ". Let Θ denote the group of invertible holomorphic functions f on Ω satisfying $f(\gamma\omega)/f(\omega)$ is a constant for every $\gamma \in \Gamma$. According to [3] Ch. II, the group Θ/K^* is isomorphic to $\mathbf{Z}^g$. Elements $\theta_1, \ldots, \theta_g$ in Θ are called a basis if their images in $\mathbf{Z}^g$ form a $\mathbf{Z}$-basis. The main result of this section states that every Γ-equivariant strongly abelian covering of Ω has the form

$$\Omega_* = \{(\omega, \lambda_1, \ldots, \lambda_g) \in \Omega \times (K^*)^g \mid \lambda_i^{n_i} = \theta_i(\omega) \text{ for } i = 1, \ldots, g\}$$

where we have chosen a basis $\theta_1, \ldots, \theta_g$ of Θ and where $n_1, \ldots, n_g$ are positive integers, not divisible by char K. We start the proof by giving Ω_* the structure of an analytic space over K. Let $\{\Omega_n\}$

denote a sequence of connected affinoid subsets of Ω such that (i) $\Omega_1 \subset \Omega_2 \subset \Omega_3 \subset \dots$ and (ii) every affinoid subset of Ω is contained in some Ω_n. For each n we consider the affinoid space $\Omega_{*}n$ corresponding to the affinoid algebra

$$\mathcal{O}(\Omega_n)[X_1, \dots, X_n]/(X_1^{n_1} - \theta_1, \dots, X_g^{n_g} - \theta_g).$$

As a point set Ω_{*n} is equal to $\{(\omega, \lambda_1, \dots, \lambda_g) \in \Omega_* | \omega \in \Omega_n\}$.

The analytic space Ω_* is obtained by glueing together the affinoid spaces Ω_{*n} according to the natural inclusions $\Omega_{*n} \longrightarrow \Omega_{*m}$ (for $n \leqslant m$). The map $c : \Omega_* \longrightarrow \Omega$ is etale and finite of degree $n_1 \dots n_g$. The automorphisms of $\Omega_* \longrightarrow \Omega$ are of the form $(\omega, \lambda_1, \dots, \lambda_g) \longrightarrow (\omega, \zeta_1^{\alpha_1}\lambda_1, \dots, \zeta_g^{\alpha_g}\lambda_g)$ where ζ_i denote a primitive n_i-th root of unity and $0 \leqslant \alpha_i < n_i$. So $\Omega_* \longrightarrow \Omega$ is Galois with group $H = \mathbf{Z}/n_1 \oplus \dots \oplus \mathbf{Z}/n_g$. The function theory on Ω_* is not much more complicated than that of Ω. Indeed $\mathcal{O}(\Omega_*)$ equals $\varprojlim \mathcal{O}(\Omega_{*n})$ and turns out to be

$$\mathcal{O}(\Omega)[X_1, \dots, X_g]/(X_1^{n_1} - \theta_1, \dots, X_g^{n_g} - \theta_g).$$

As usual we write $\mathcal{M}$ for the sheaf of meromorphic functions. For any affinoid U one has $\mathcal{M}(U)$ = the total ring of fractions of $\mathcal{O}(U)$.

Again $\mathcal{M}(\Omega_*) = \varprojlim \mathcal{M}(\Omega_{*n})$ coincides with

$$\mathcal{M}(\Omega)[X_1, \dots, X_g]/(X_1^{n_1} - \theta_1, \dots, X_g^{n_g} - \theta_g).$$

The space Ω_* is connected if and only if $\mathcal{M}(\Omega_*)$ is a field. Let m denote the smallest common multiple of $n_1, \dots, n_g$. If suffices to verify that $\mathcal{M}(\Omega)[Y_1, \dots, Y_g]/(Y_i^m - \theta_i\,;\ i = 1, \dots, g)$ is a field. By Kummer-theory this is translated into: the images of $\theta_1, \dots, \theta_g$ in $\mathcal{M}(\Omega)^*/\mathcal{M}(\Omega)^{*m}$ are independent over $\mathbf{Z}/m$.

Suppose now that $\theta_1^{\alpha_1} \dots \theta_g^{\alpha_g}$, with $0 \leqslant \alpha_i < m$, equals f^m for some $f \in \mathcal{M}(\Omega)$. Then clearly $f \in \mathcal{O}(\Omega)^*$. Since $(f(\gamma\omega)/f(\omega))^m$ is constant for every $\gamma \in \Gamma$ and since Ω is connected, one finds that $f \in \Theta$. The independance of $\theta_1, \dots, \theta_g$ yields $\alpha_1 = \dots = \alpha_g = 0$. This finally shows that Ω_* is connected. Let further a_i denote the homomorphism of Γ in K^* satisfying $\theta_i(\gamma\omega) = a_i(\gamma)\,\theta_i(\omega)$. Let $b_i \in \mathrm{Hom}(\Gamma, K^*)$ be chosen such that $b_i^{n_i} = a_i$. Then we can define a Γ-action on Ω_* by

$$\gamma(\omega, \lambda_1, \dots, \lambda_g) = (\gamma(\omega), \lambda_1 b_1(\gamma), \dots, \lambda_g b_g(\gamma)).$$

This action commutes with the H-action on Ω_*. Hence $\Omega_* \longrightarrow \Omega$ is a strongly abelian Γ-equivariant étale morphism with group H. Next we want to find a presentation of Ω_* which does not depend on the choice of $\theta_1, \ldots, \theta_g, n_1, \ldots, n_g$. This is done as follows. Let G be the group of automorphisms of Ω_*, as defined in (1.3). The group acts on $\mathfrak{M}(\Omega_*)$, $\mathfrak{O}(\Omega_*)$ etc. We consider its action on $\mathfrak{O}(\Omega_*)^*/K^*$. Let $x_1, \ldots, x_g \in \mathfrak{O}(\Omega_*)^*$ be given by

$$x_i(\omega, \lambda_1, \ldots, \lambda_g) = \lambda_i .$$

A straightforward calculation shows that $H^0(G, \mathfrak{O}(\Omega_*)^*/K^*)$ is the free $\mathbf{Z}$-module generated by the images of $x_1, \ldots, x_g$. And this group is a finite extension of $H^0(\Gamma, \mathfrak{O}(\Omega)^*/K^*) = \Theta/K^*$. We obtain in this way a $\mathbf{Z}$-lattice T in $\Theta/K^* \otimes \mathbf{Q}$ containing Θ/K^*. The lattice T is uniquely determined by Ω_* and determines Ω_*. We will write $\Omega_* = \Omega(T)$ in the sequel. The group of automorphisms of $\Omega(T) \longrightarrow \Omega$ is equal to the Pontryagin dual of the cokernel of $\Theta/K^* \longrightarrow T$. We can now formulate the main result of this section, using again the notation of (1.3). We consider only lattices T such that char(K) does not divide the order of H.

THEOREM 2.1. – *For every strongly abelian Γ-equivariant map $\Omega_* \longrightarrow \Omega$ there exists a unique $\mathbf{Z}$-lattice and an isomorphism $\Omega_* \xrightarrow{\sim} \Omega(T)$.*

COROLLARY 2.2. – *Every finite abelian étale-covering of $X = \Omega/\Gamma$ has uniquely the form $\Omega(T)/N$, where T is a $\mathbf{Z}$-lattice and where N is a subgroup of G with $N \cap H = \{1\}$ and πN is a normal subgroup of Γ of finite index and with an abelian factor group.*

Proof of 2.2. – The corollary follows from (1.4), (2.1) and the fact that G is the direct product of H and Γ. A further consequence is:

COROLLARY 2.3. – *The Galois group Δ of the maximal unramified abelian extension of $\mathfrak{M}(X)$, the function field of $X = \Omega/\Gamma$, is isomorphic to:*

a) $\hat{\mathbf{Z}}^{2g}$ *if* $\operatorname{char} K = 0$

b) $\hat{\mathbf{Z}}^{g} \times \prod_{\ell \neq p} \hat{\mathbf{Z}}_\ell^g$ *if* $\operatorname{char} K = p \neq 0$.

There is further a canonical surjective homomorphism of Δ onto $\hat{\mathbf{Z}}^g$ = the Galois group of the maximal abelian analytic covering of X.

Proof of (2.1). – It suffices to show the following two statements:

a) if $\operatorname{char} K = p \neq 0$ then there does not exist an equivariant $\Omega_* \longrightarrow \Omega$ with group $\mathbf{Z}/p$.

b) if $\Omega_* \longrightarrow \Omega$ is a cyclic equivariant étale covering with group $H = \mathbf{Z}/n$ such that $\operatorname{char}(K)/n$ and $H \cap [G,G] = \{1\}$, then there is a suitable $\theta \in \Theta$ with $\Omega_* \simeq \{(\omega,\lambda) \in \Omega \times K^* \mid \lambda^n = \theta(\omega)\}$.

Proof of a).

The map $c : \Omega_* \longrightarrow \Omega$ induces a field extension $\mathcal{M}(\Omega) \subset \mathcal{M}(\Omega_*)$ which is supposed to be cyclic of degree p. By Schreier theory, $\mathcal{M}(\Omega_*)$ is obtained from $\mathcal{M}(\Omega)$ by adjoining a root of $X^p - X - f$. One can change the f in this equation by adding a meromorphic function of the form $g^p - g$ with $g \in \mathcal{M}(\Omega)$. After a suitable change of this type we may suppose that every pole (if any) of f has order $< p$. In a pole $\omega_0 \in \Omega$ of f of order $< p$ the map $\Omega_* \longrightarrow \Omega$ is ramified. So we have shown that f can be supposed to belong to $\mathcal{O}(\Omega)$.

Consider the exact sequence

$$0 \longrightarrow \mathbf{F}_p \longrightarrow \mathcal{O}(\Omega) \xrightarrow{\tau} \mathcal{O}(\Omega) \xrightarrow{\sigma} M \longrightarrow 0$$

where τ is given by $\tau(h) = h^p - h$. The extension $\mathcal{M}(\Omega_*) \mid \mathcal{M}(\Omega)$ determines uniquely the subgroup of M generated by $\tau(f)$. The action of Γ of $\mathcal{M}(\Omega)$ extends to $\mathcal{M}(\Omega_*)$. This implies that $\sigma(f \circ \gamma) = c(\gamma)\,\sigma(f)$ for a certain homomorphism $c : \Gamma \longrightarrow \mathbf{F}_p^*$. After replacing Γ by a subgroup of finite index, we may suppose that $\sigma(f)$ is invariant under Γ. We recall that $H^0(\Gamma, \mathcal{O}(\Omega)) = H^0(\Omega/\Gamma, \mathcal{O}_X)$ and $H^1(\Gamma, \mathcal{O}(\Omega)) = H^1(\Omega/\Gamma, \mathcal{O}_X)$ with $X = \Omega/\Gamma$. For the constant sheaf K_X on X with stalk K one also has $H^0(\Gamma, K) = H^0(X, K_X)$ and $H^1(\Gamma, K) = H^1(X, K_X)$. Further the canonical maps

$$H^i(X, K_X) \longrightarrow H^i(X, \mathcal{O}_X) \quad (i = 0, 1)$$

are bijective. Using the exact sequence of Γ-modules

$$0 \longrightarrow \mathbf{F}_p \longrightarrow \mathcal{O}(\Omega) \longrightarrow \mathcal{O}(\Omega)/\mathbf{F}_p \longrightarrow 0$$

one finds

$$H^0(\Gamma, \mathcal{O}(\Omega)/\mathbf{F}_p) = K/\mathbf{F}_p \quad \text{and} \quad H^1(\Gamma, \mathcal{O}(\Omega)/\mathbf{F}_p) = \operatorname{Hom}(\Gamma, K/\mathbf{F}_p).$$

The exact sequence of Γ-modules

$$0 \longrightarrow \mathcal{O}(\Omega)/\mathbf{F}_p \xrightarrow{\tau} \mathcal{O}(\Omega) \longrightarrow M \longrightarrow 0$$

induces the long exact sequence

$$0 \longrightarrow K/\mathbf{F}_p \xrightarrow{\tau} K \longrightarrow H^0(\Gamma, M) \longrightarrow \operatorname{Hom}(\Gamma, K/\mathbf{F}_p) \xrightarrow{\tau} \operatorname{Hom}(\Gamma, K) \longrightarrow \dots$$

This implies that $H^0(\Gamma, M) = 0$. Hence $\tau(f) = 0$. This contradicts the assumption that the equation $X^p - X - f$ is irreducible.

Proof of b).

The map $c: \Omega_* \longrightarrow \Omega$ induces a field extension $\mathcal{M}(\Omega) \subset \mathcal{M}(\Omega_*)$ with cyclic group $\mathbf{Z}/n$ and irreducible equation $X^n - f$, for some $f \in \mathcal{M}(\Omega)$. Since $\Omega_* \longrightarrow \Omega$ is étale one may suppose that $f \in \mathcal{O}(\Omega)^*$. We consider the exact sequence

$$1 \longrightarrow \mathcal{O}(\Omega)^*/K^* \xrightarrow{\tau} \mathcal{O}(\Omega)^*/K^* \xrightarrow{\sigma} M \longrightarrow 0$$

where τ is defined by $\tau(g) = g^n$.

The subgroup of M generated by $g = \tau$ (f mod K^*) has $\mathbf{Z}/n$ elements and is uniquely determined by the extension $\mathcal{M}(\Omega) \subset \mathcal{M}(\Omega_*)$ The action of Γ on $\mathcal{M}(\Omega)$ extends to $\mathcal{M}(\Omega_*)$. This implies that $\gamma(g) = g^{a(\gamma)}$ where $a: \Gamma \longrightarrow (\mathbf{Z}/n)^*$ is some group homomorphism. This means that $f(\gamma\omega) = f(\omega)^{a(\gamma)} b_\gamma(\omega)^n$ holds for some $b_\gamma \in \mathcal{O}(\Omega)^*$. Let x denote an element of $\mathcal{M}(\Omega_*)$ with $x^n = f$. The action of γ on $\mathcal{M}(\Omega_*)$ must have the form $\gamma(x) = x^{a(\gamma)} b_\gamma$. This action must commute with the automorphism δ of $\mathcal{M}(\Omega_*) \mid \mathcal{M}(\Omega)$ given by $\delta(x) = \zeta x$ where ζ is a primitive n-th root of unity. Since $\delta\gamma(x) = \zeta^{a(\gamma)} x^{a(\gamma)} b_\gamma$ and $\gamma\delta(x) = \zeta x^{a(\gamma)} b_\gamma$, one finds that $a(\gamma) = 1$ for all $\gamma \in \Gamma$. The map $\gamma \longmapsto b_\gamma$ is a 1-cocycle with values in $\mathcal{O}(\Omega)^*$ and its n-th power is the trivial cocycle $\gamma \longmapsto \dfrac{f \circ \gamma}{f}$. In [5] one has derived an exact sequence

$$\ldots . \operatorname{Hom}(\Gamma, K^*) \longrightarrow H^1(\Gamma, \mathcal{O}(\Omega)^*) \longrightarrow \mathbf{Z} \longrightarrow 0.$$

This implies that the image of the 1-cocycle $\{\gamma \longmapsto b_\gamma\}$ in $\mathbf{Z}$ is zero. Hence b_γ has the form $d(\gamma) . c \circ \gamma / c$ for some homomorphism $d: \Gamma \longrightarrow K^*$ and some functions $c \in \mathcal{O}(\Omega)^*$. Hence $\theta = c^{-n} f$ satisfies $\theta(\gamma\omega) = d(\gamma)^n \theta(\omega)$ and so θ belongs to Θ. The extension $\mathcal{M}(\Omega) \subset \mathcal{M}(\Omega_*)$ is then also described by the equation $X^n - \theta$. It follows easily that Ω_* is isomorphic to $\{(\omega, \lambda) \in \Omega \times K^* \mid \lambda^n = \theta(\omega)\}$. This finishes the proof of (2.1).

Example 2.4. – The special case of (2.1) and (2.2) where the genus of X is 1 is particularly simple. The statement reads:

Every finite abelian etale extension of $X = K^*/\langle q \rangle$ (where $0 < |q| < 1$) is of the form $K^*/\langle q' \rangle \xrightarrow{\varphi} K^*/\langle q \rangle$ where the map φ is induced by $z \longmapsto z^n$ from $K^* \longrightarrow K^*$ with n not divisible by char K and where q' satisfies $(q')^n \in \langle q \rangle = q^{\mathbf{Z}}$.

PROPOSITION 2.5. – *Let* $\varphi : Y \longrightarrow X$ *be a finite abelian étale of the Mumford curve* $X = \Omega/\Gamma$. *We suppose that the order of the group* H *(see (2.2)) is not divisible by* $\operatorname{char} \overline{K}$. *Let* $\mathfrak{U}$ *be a pure affinoid covering of* X *such that the reduction* $\overline{(X, \mathfrak{U})}$ *satisfies:*

(i) *every component of* $\overline{(X, \mathfrak{U})}$ *is non-singular.*

(ii) *every singular point of* $\overline{(X, \mathfrak{U})}$ *is an ordinary double point.*

Then $\varphi^{-1}(\mathfrak{U})$ *is a pure affinoid covering of* Y *and the reduction* $\overline{(Y, \varphi^{-1}(\mathfrak{U}))}$ *of* Y *with respect to* $\varphi^{-1}(\mathfrak{U})$ *also satisfies (i) and (ii). The canonical map of* $\overline{(Y, \varphi^{-1}(\mathfrak{U}))}$ *to* $\overline{(X, \mathfrak{U})}$ *is unramified outside the double points of* $\overline{(Y, \varphi^{-1}(\mathfrak{U}))}$.

Proof. – Any small enough $U \in \mathfrak{U}$ is isomorphic to an affinoid subset of $\mathbf{P}^1$. The proof of (2.5) follows from the next lemma.

LEMMA 2.6. – *Let* U *be an affinoid subset of* $\mathbf{P}^1$ *given by the inequalities:* $|\pi| \leqslant |z| \leqslant 1$; $|z - a_1| \geqslant 1, \ldots, |z - a_s| \geqslant 1$; $|z - b_1| \geqslant |\pi|, \ldots, |z - b_t| \geqslant |\pi|$ *in which* $0 < |\pi| < 1$; $|a_i| = 1$; $|a_i - a_j| = 1$ *for* $i \neq j$; $|b_i| = |\pi|$ *and* $|b_i - b_j| = |\pi|$ *for* $i \neq j$. *Let* $u_1, \ldots, u_c \in \mathcal{O}(U)^*$ *and let* $n_1, \ldots, n_c$ *denote positive integers not divisible by* char (K). *Let* V *denote the affinoid space be given by its affinoid algebra*

$$\mathcal{O}(V) = \mathcal{O}(U)\langle X_1, \ldots, X_c\rangle/(X_1^{n_1} - u_1, \ldots, X_c^{n_c} - u_c).$$

Then the canonical reduction $\overline{V}$ *of* V *has non-singular components. The only singularities of* $\overline{V}$ *are ordinary double points. The map* $\overline{V} \longrightarrow \overline{U}$ *is unramified outside the double points of* $\overline{V}$.

Proof. – We may suppose that $\mathcal{O}(V)$ is an integral domain. Let M denote the subgroup of $\mathcal{O}(U)^*$ consisting of the elements m of the form

$$m = z^{k_0}(z - a_1)^{k_1} \ldots . (z - a_s)^{k_s} \left(\frac{\pi}{z} - \frac{\pi}{b_1}\right)^{\ell_1} \ldots . \left(\frac{\pi}{z} - \frac{\pi}{b_t}\right)^{\ell_t}.$$

The k_0, k_1, etc. are integers and we write $k_0 = k_0(m)$. Then M is a free abelian group of rank $s + t + 1$. Every element of $\mathcal{O}(U)^*$ can uniquely be decomposed as $u . m$ with $m \in M$ and $u = \lambda + h$, $\lambda \in K^*$ and $h \in \mathcal{O}(U)$ such that $\|h\| < |\lambda|$. Let $d = [\mathcal{O}(V) : \mathcal{O}(U)]$ and let N denote the group of elements of $\mathcal{O}(V)^*$ having their d-th power in M. Then $N = N_0 \oplus N_1$ where N_1 is the group of the d-th roots of unity and where N_0 is a free abelian group satisfying

$[N_0:M] = d$. Take a basis $u_1, \ldots, u_{s+t+1}$ of M such that N_0 is the free group generated by $\frac{1}{n_1} u_1, \ldots, \frac{1}{n_{s+t+1}} u_{s+t+1}$ (in additive notation). With this choice one can write

$$\mathcal{O}(V) = \mathcal{O}(U)\,[X_1, \ldots, X_{s+t+1}]/(X_i^{n_i} - u_i\,;\ i = 1, \ldots, s+t+1).$$

It is possible to choose the $u_1, \ldots, u_{s+t+1}$ such that $k_0(u_1) = 1$ and $k_0(u_i) = 0$ for $i = 2, \ldots, t+s+1$.

Consider the surjective map of $\mathcal{O}(U)\langle X_1, Y_1, X_2, X_3, \ldots, X_{s+t+1}\rangle$ to $\mathcal{O}(V)$ given by $X_i \longmapsto X_i$ and $Y_1 \longmapsto \rho X_1^{-1}$ with $\rho \in K^*$ such that $\rho^{n_1} = \pi$. This map induces a norm on $\mathcal{O}(V)$ and the reduction R of $\mathcal{O}(V)$ with respect to this norm is

$$\overline{\mathcal{O}(U)}\,[X_1, Y_1, X_2, X_3, \ldots, X_{s+t+1}]$$

divided by the ideal generated by the elements $X_1^{n_1} - \overline{u}_1$, $Y_1^{n_1} - \dfrac{\overline{\pi}}{u_1}$, $X_1 Y_1$, $X_i^{n_i} - \overline{u}_i$ for $i \geqslant 2$. Further $\overline{\mathcal{O}(U)}$ is the localization of $\overline{K}[T, S]/TS$ at the element

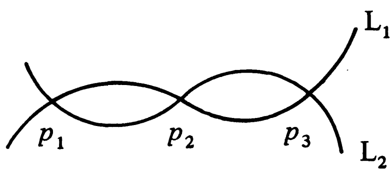
$$(T - \overline{a}_1) \ldots (T - \overline{a}_s) \left(S - \frac{\overline{\pi}}{b_1}\right) \ldots \left(S - \frac{\overline{\pi}}{b_t}\right).$$

A straightforward calculation shows that R has no nilpotents. Hence R is the reduction of $\mathcal{O}(V)$ with respect to the spectral norm. The only singular maximal ideals of R are

$$(X_1, Y_1, X_2 - c_2, \ldots, X_{s+t+1} - c_{s+t+1})$$

in which $c_i \in \overline{K}$ satisfies $c^{n_i} = \overline{u_i(\tau)}$ with $|\pi| < |\tau| < 1$. The completion of the local ring of R at such a maximal ideal is $\cong \overline{K}[[X_1, Y_1]]/(X_1 Y_1)$. Further $\overline{\mathcal{O}(U)} \longrightarrow R$ is unramified outside the ideal (S, T) of $\overline{\mathcal{O}(U)}$. This proves the lemma.

An example 2.7. – Let X be a Mumford curve of genus 2 with reduction $\overline{X}$



(Two rational curves L_1, L_2 intersecting in 3 points p_1, p_2, p_3.)

We write $r: X \longrightarrow \overline{X}$ for the reduction map. Let $\theta \in \Theta$ be a theta function for the curve X. On the affinoid part $r^{-1}(L_1 - \{p_1, p_2, p_3\})$ the function θ can be represented by a holomorphic invertible function u which is normalized by $\|u\| = 1$. The reduction $\bar{u}$ is a rational function on L_1 which is invertible and regular outside $\{p_1, p_2, p_3\}$. Let ord(θ) denote the triple $(a_1, a_2, a_3) \in \mathbf{Z}^3$ given by $a_i = \operatorname{ord}_{p_i}(\bar{u})$. This induces a group homomorphism

$$\operatorname{ord}: \Theta/K^* \longrightarrow \{(a_1, a_2, a_3) \in \mathbf{Z}^3 \mid a_1 + a_2 + a_3 = 0\}.$$

Using [5] one easily shows that it is an isomorphism. Let $\theta_1, \theta_2 \in \Theta$ be a basis for the theta functions. Put $\operatorname{ord}(\theta_1) = (a_1, a_2, a_3)$ and $\operatorname{ord} \theta_2 = (b_1, b_2, b_3)$. As in (2.2) the curve Y is given by $Y = \Omega_*/N$ in which

$$\Omega_* = \{(\omega, \lambda_1, \lambda_2) \in \Omega \times (K^*)^2 \mid \lambda_1^{n_1} = \theta_1(\omega) \text{ and } \lambda_2^{n_2} = \theta_2(\omega)\}$$

and where N maps bijectively to Γ. We assume further that char $\bar{K}$ does not divide $n_1 n_2$. The reduction of Y obtained in (2.5) in denoted by $\bar{Y}$. The étale map $\varphi: Y \longrightarrow X$ induces some $\tilde{\varphi}: \bar{Y} \longrightarrow \bar{X}$. We will use (2.5) and the proof of (2.6) in order to calculate the reduction $\bar{Y}$.

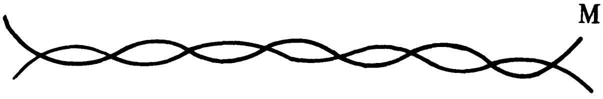
Let t be a parameter on $L_1 \cong \mathbf{P}^1$ such that $t = 0, 1, \infty$ corresponds to p_1, p_2, p_3 on L_1. Then $\tilde{\varphi}^{-1}(L_1 - \{p_1, p_2, p_3\})$ is the affine variety over $\bar{K}$ with coordinate ring

$$\bar{K}[t]_{(t-1)}[X_1, X_2]/(X_1^{n_1} - t^{a_1}(t-1)^{a_2},\ X_2^{n_2} - t^{b_1}(t-1)^{b_2}).$$

It is connected and non-singular. Its closure in $\bar{Y}$ is a curve M_1. The curve M_1 is an abelian ramified covering of $L_1 = \mathbf{P}^1$. The genus g of M_1 is given by the Riemann-Hurwitz formula

$$2g - 2 = 2n_1 n_2 + \frac{n_1 n_2}{e_1}(e_1 - 1) + \frac{n_1 n_2}{e_2}(e_2 - 1) + \frac{n_1 n_2}{e_3}(e_3 - 1).$$

In this formula e_i denotes the ramification index of a point of M_1 above p_i in L. One easily verifies that $\frac{1}{e_i}\mathbf{Z} = \frac{a_i}{n_1}\mathbf{Z} + \frac{b_i}{n_2}\mathbf{Z}$ for $i = 1, 2, 3$. One finds in the same manner that $M_2 = \tilde{\varphi}^{-1}(L_2)$ is a non-singular curve of the same genus. The two curves M_1 and M_2 meet in $\frac{n_1 n_2}{e_1} + \frac{n_1 n_2}{e_2} + \frac{n_1 n_2}{e_3}$ points (namely the $\tilde{\varphi}$-pre-images of p_1, p_2, p_3). Hence the arithmetic genus of $\bar{Y}$ is equal to $2g - 1 + n_1 n_2 \left\{\frac{1}{e_1} + \frac{1}{e_2} + \frac{1}{e_3}\right\}$.



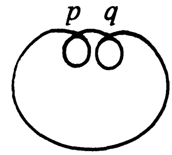
One easily computes that this number is equal to the genus of $\overline{Y}$ (as it should be).

(Picture of $\overline{Y}$)

The universal analytic covering of Y (as constructed in (1.5)) has an automorphism group $\Gamma(Y)$ which is free on $n_1 n_2 \left\{ \frac{1}{e_1} + \frac{1}{e_2} + \frac{1}{e_3} \right\}$ generators. This number is equal to $\sum_{i=1}^{3}$ g.c.d. $(n_2 a_i , n_1 b_i)$ and so $\geqslant 3$.

This shows that Ω_* cannot be the universal analytic covering of Y.

2.8 *The other examples of a Mumford curve of genus 2*



a) X *is a Mumford curve with stable reduction* $\overline{X}$:

The reduction is $\mathbf{P}^1$ parametrized by t where the two pairs of points $t = 0$, $t = \infty$ and $t = 1$, $t = d$ are identified. Again one has an isomorphism $\Theta/K^* \xrightarrow[\sim]{\text{ord}} \mathbf{Z}^2$ given as follows: $\theta \in \Theta$ lift to a function u on $r^{-1}(\overline{X} - \{p, q\})$ with constant absolute value 1. The reduction $\overline{u}$ is a rational function on the normalization $\mathbf{P}^1$ of $\overline{X}$ and we put $\text{ord}(\theta) = (\text{ord}_0 \overline{u}, \text{ord}_1 \overline{u})$. Let θ_1, θ_2 be a basis of the theta functions and put $\text{ord}(\theta_1) = (a_1, a_2)$ and $\text{ord}(\theta_2) = (b_1, b_2)$. Let Y be the curve obtained from X by (2.2) with $\Omega_* = \{(\omega, \lambda_1, \lambda_2) \mid \lambda_1^{n_1} = \theta_1(\omega), \lambda_2^{n_2} = \theta_2(\omega)\}$ and N which maps bijectively to Γ. The reduction of Y is made by using (2.5). The canonical map $\varphi\colon Y \longrightarrow X$ induces a $\widetilde{\varphi}\colon \overline{Y} \longrightarrow \overline{X}$. The pre-image $\widetilde{\varphi}^{-1}(\overline{X} - \{p, q\})$ is affine with coordinate ring

$$\overline{K}[t]_{t(t-1)(t-d)}[X, Y] / \left(X^{n_1} - t^{a_1}\left(\frac{t-1}{t-d}\right)^{a_2}, \; Y^{n_2} - t^{b_1}\left(\frac{t-1}{t-d}\right)^{b_2} \right).$$

The corresponding non-singular projective curve (i.e. the normalization of $\overline{Y}$) has genus g given by the Riemann-Hurwitz formula

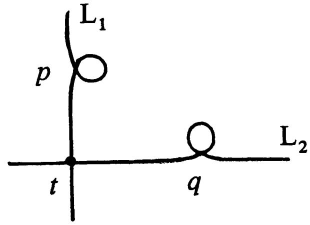
$$2g - 2 = -2n_1 n_2 + 2\,\frac{n_1 n_2}{e_1}\,(e_1 - 1) + 2\,\frac{n_1 n_2}{e_2}\,(e_2 - 1) \quad \text{and}$$

$$\frac{1}{e_1}\mathbf{Z} = \frac{a_1}{n_1}\mathbf{Z} + \frac{b_1}{n_2}\mathbf{Z} \quad \text{and} \quad \frac{1}{e_2}\mathbf{Z} = \frac{a_2}{n_1}\mathbf{Z} + \frac{b_2}{n_2}\mathbf{Z}.$$

The number of double points of $\overline{Y}$ is $\frac{n_1 n_2}{e_1} + \frac{n_1 n_2}{e_2}$. So $\overline{Y}$ is an irreducible curve with double points.

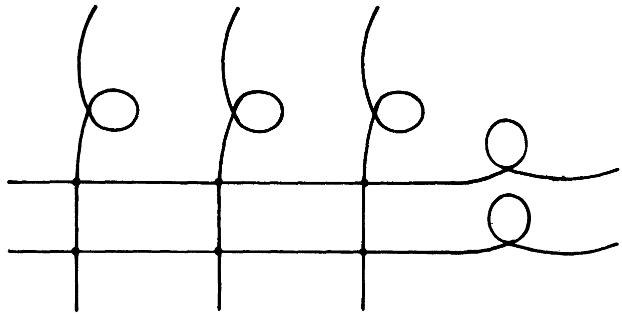
The group $\Gamma(Y)$ (see (1.5)) is free on $\frac{n_1 n_2}{e_1} + \frac{n_1 n_2}{e_2} - 1$ generates.

b) X *is Mumford curve with stable reduction* $\overline{X}$:



Let L_1 be described by a parameter t_1 where $t_1 = 1, -1$ corresponds to p and $t_1 = 0$ corresponds to r. A parameter t_2 describes L_2 in a similar way. A theta function θ for X is lifted to a function u on $r^{-1}(L_1 - \{p, r\})$. One can normalize u such that $\|u\| = 1$. Put $a_1 = \mathrm{ord}_1 \overline{u}$. In a similar way a_2 is defined. One obtains again an isomorphism $\mathrm{ord}: \Theta/K^* \longrightarrow \mathbf{Z}^2$ with $\mathrm{ord}(\theta) = (a_1, a_2)$ as given above.

Let θ_1, θ_2 be a basis of the theta functions and let $Y \xrightarrow{\varphi} X$ be defined by "$\sqrt[n_1]{\theta_1}, \sqrt[n_2]{\theta_2}$". We study now the reduction $\overline{Y}$ and the map $\widetilde{\varphi}: \overline{Y} \longrightarrow \overline{X}$. The pre-image $\widetilde{\varphi}^{-1}(L_1)$ is given by the equations $X^{n_1} - \left(\frac{t_1 - 1}{t_1 + 1}\right)^{a_1}$, $Y^{n_2} - \left(\frac{t_1 - 1}{t_1 + 1}\right)^{b_1}$. Here we have written $\mathrm{ord}(\theta_1) = (a_1, a_2)$ and $\mathrm{ord}(\theta_2) = (b_1, b_2)$. Let $e_1 \geqslant 1$ be defined by $\frac{1}{e_1}\mathbf{Z} = \frac{a_1}{n_1}\mathbf{Z} + \frac{b_1}{n_2}\mathbf{Z}$. Then $\widetilde{\varphi}^{-1}(L_1)$ turns out to be the disjoint union of $\frac{n_1 n_2}{e_1}$ curves $M_1(1), \dots, M_1\left(\frac{n_1 n_2}{e_1}\right)$. Each $M_1(i)$ is a rational curve with one double point. The $M_1(i)$ are isomorphic to each other. The map $M_1(i) \longrightarrow L_1$ has degree e_1 and is only ramified in the unique double point of $M_1(i)$. On each $M_1(i)$ lie e_1 pre-images of the point r. There is a similar description for $\widetilde{\varphi}^{-1}(L_2) = M_2(1) \cup \dots \cup M_2\left(\frac{n_1 n_2}{e_2}\right)$ with $\frac{1}{e_2}\mathbf{Z} = \frac{a_2}{n_1}\mathbf{Z} + \frac{b_2}{n_2}\mathbf{Z}$. Every $M_1(i)$ meets e_1 of the curves $M_2(j)$ and every $M_2(j)$ meets e_2 of the curves $M_1(i)$. The reduction $\overline{Y}$ is totally split and stable. The curve Y is a Mumford curve. We have made a picture of $\overline{Y}$ for the values $a_1 = 1$, $a_2 = 0$, $b_1 = 0$, $b_2 = 1$, $n_1 = e_1 = 2$ and $n_2 = e_2 = 3$.



3. Mumford curves over a local field.

In this section k denotes a local field and K will be the completion of the algebraic closure of k. Let $\Gamma \subset PG\ell(2, k)$ denote a Schottky group on g generators. Then $\mathcal{L}$ is a subset of $\mathbf{P}^1(k)$. Let Ω denote the analytic space over k, given by $\Omega = \mathbf{P}^1_k - \mathcal{L}$. The action of Γ on Ω is k-rational and one can form the quotient $X = \Omega/\Gamma$. For every (finite) extension ℓ of k the set of ℓ-rational points of $X \times_k \ell$ is equal to $\mathbf{P}^1(\ell) - \mathcal{L}/\Gamma$. In particular the set of k-rational points of X is equal to $\mathbf{P}^1(k) - \mathcal{L}/\Gamma$. For our purposes we need that X has k-rational points. So we have to assume that $\mathcal{L}$ is a proper subset of $\mathbf{P}^1(k)$. The theta functions, corresponding to Γ, are elements of $\mathcal{O}(\Omega)$ since they can be written in the form

$$\theta_\delta = \prod_{\gamma \in \Gamma} \frac{z - \gamma(a)}{z - \gamma\delta(a)}, \quad \text{where } a \in \mathbf{P}^1(k) - \mathcal{L} \text{ and } \delta \in \Gamma .$$

For every $\delta \in \Gamma$ the homomorphism $c_\delta : \Gamma \longrightarrow K^*$, given by $\theta_\delta(\gamma\omega) = c_\delta(\gamma)\, \theta_\delta(\omega)$, has also values in k^*. As in § 2 we want to calculate the abelian unramified field-extensions of $\mathcal{M}(X) = H^0(\Gamma, \mathcal{M}(\Omega))$. The field $\mathcal{M}(X)$ is a function field of genus g with precise field of constants k.

A contribution to those extensions are the abelian extensions of the field of constants k. Restrictions with respect to the extensions in § 2 are:

(i) k contains only finitely many roots of unity; let n denote their number.

(ii) For a theta function θ with $\theta(\gamma\omega) = a(\gamma)\,\theta(\omega)$, there exists in general no homomorphism $b : \Gamma \longrightarrow k^*$ with $b^n = a$.

For any lattice T (again T is a lattice in $\Theta/k^* \times \otimes_{\mathbf{Z}} \mathbf{Q}$ containing Θ/k^*) there is an analytic space $\Omega(\mathrm{T})$ over k defined by the more or less symbolic formula

$$\Omega(\mathrm{T}) = \{(\omega, \lambda_1, \ldots, \lambda_g) \in \Omega \times (k^*)^g \mid \lambda_i^{n_i} = \theta_i(\omega),\ i = 1, \ldots, g\}.$$

The function field $\mathfrak{M}(\Omega(\mathrm{T}))$ of $\Omega(\mathrm{T})$ is equal to $\mathfrak{M}(\Omega)[x_1, \ldots, x_g]$ where $x_i^{n_i} = \theta_i$. Let us write $a_i \in \mathrm{Hom}(\Gamma, k^*)$ for the homomorphism $\gamma \longmapsto \theta_i(\gamma\omega)\,\theta_i(\omega)^{-1}$. Let $b_i \in \mathrm{Hom}(\Gamma, \mathrm{K}^*)$ denote a homomorphism satisfying $b_i^{n_i} = a_i$. Let ℓ be a finite Galois extension of k containing all the values $b_i(\gamma)$. The analytic space (over k) $\Omega(\mathrm{T}) \times_k \ell$ has a group of automorphism G given by: an automorphism δ belongs to G if δ extends some automorphism $\gamma \in \Gamma$ of Ω.

From our choice of the field ℓ it follows that we have an exact sequence:

$$1 \longrightarrow \mathrm{H} \longrightarrow \mathrm{G} \xrightarrow{\pi} \Gamma \longrightarrow 1 \quad \text{with} \quad \mathrm{H} = \mathrm{Aut}(\Omega(\mathrm{T}) \times_k \ell \longrightarrow \Omega).$$

Let M denote the subgroup $\mathrm{Aut}(\Omega(\mathrm{T}) \times_k \ell \longrightarrow \Omega \times_k \ell)$ of H and let N denote the subgroup $\mathrm{Aut}(\Omega(\mathrm{T}) \times_k \ell \longrightarrow \Omega(\mathrm{T})) \cong \mathrm{Gal}(\ell \mid k)$ of H. Then M is a normal subgroup and we have an exact sequence $1 \longrightarrow \mathrm{M} \longrightarrow \mathrm{H} \longrightarrow \mathrm{Gal}(\ell \mid k) \longrightarrow 1$ and H is the semi-direct product of M and N.

According to § 2 every finite abelian unramified covering of X has the form $\Omega(\mathrm{T}) \times \ell/\mathrm{N}$ for suitable, T, ℓ and N and in which N is a normal subgroup of G and G/N is a finite abelian group.

One clearly has $[\mathrm{G}, \mathrm{G}] \cap \mathrm{H}$ is contained in N. In particular $[\mathrm{H}, \mathrm{H}]$ is contained in N. We will need the following lemma.

LEMMA 3.1. – *Let* H *denote the automorphism group of* $\Omega(\mathrm{T}) \times_k \ell \mid \Omega$ *and let* $[\mathrm{H}, \mathrm{H}]$ *denote the commutator subgroup of* H. *Then* $\Omega(\mathrm{T}) \times_k \ell/[\mathrm{H}, \mathrm{H}] \cong \Omega(\mathrm{T}') \times_k \ell'$ *where*

(i) ℓ' *is the maximal abelian subextension of* ℓ.

(ii) T' *is a sublattice of* T, *and* T' *satisfies* $n\,\mathrm{T}' \subset \Theta/k^*$.

Proof. – We choose a basis $\theta_1, \ldots, \theta_g$ of Θ such that T is the $\mathbf{Z}$-module generated by $\frac{1}{n_1}(\theta_1 \bmod k^*), \ldots, \frac{1}{n_g}(\theta_g \bmod k^*)$.

As before the function field of $\Omega(T) \times_k \mathfrak{L}$ has the form

$$\mathfrak{M}(\Omega) \otimes_k \mathfrak{L}[x_1, \ldots, x_g] \text{ with } \theta_i = X_i^{n_i}.$$

The commutator subgroup $[H, H]$ is generated by the elements $\{\sigma_1 \sigma_2 \sigma_1^{-1} \sigma_2^{-1} \mid \sigma_1, \sigma_2 \in N\}$ and $\{\sigma h \sigma^{-1} h^{-1} \mid \sigma \in N$ and $h \in M\}$. Let h_i denote the element of M given by the action $h_i(X_j) = X_j$ if $j \neq i$ and $h_i(X_i) = \zeta_i X_i$ where ζ_i is a primitive n_i-th-root of unity. An easy calculation shows that $\sigma h_i \sigma^{-1} h_i^{-1} = h_i^{a_i(\sigma)}$ where $a_i(\sigma)$ is an integer depending on i and σ. Let $e_i = \text{g.c.d.}\ (n_i,$ all $a_i(\sigma))$. One easily shows that $[H, H]$ is equal to the semi-direct product $\langle h_1^{e_1}, \ldots, h_g^{e_g} \rangle . [N, N]$. Let T' denote the sublattice of T generated by $\frac{1}{e_1}(\theta_1 \bmod k^*), \ldots, \frac{1}{e_g}(\theta_g \bmod k^*)$ and let $\mathfrak{L}'$ denote the maximal abelian extension of k contained in $\mathfrak{L}$. The function field of $\Omega(T') \times \mathfrak{L}'$ is $\mathfrak{M}(\Omega) \otimes_k \mathfrak{L}'[X_1^{d_1}, \ldots, x_g^{d_g}]$ with $d_i e_i = n_i$. The automorphism group of $\Omega(T) \times \mathfrak{L}$ over $\Omega(T') \times \mathfrak{L}'$ turns out to be $[H, H]$. Hence $\Omega(T) \times \mathfrak{L}/[H, H] = \Omega(T') \times \mathfrak{L}'$. Let us write $y_i = x_i^{d_i}$. The automorphism group of $\Omega(T') \times \mathfrak{L}' \mid \Omega$ is commutative. In particular, any

$$\sigma \in \text{Gal}(\mathfrak{L}' \mid k) = \text{Aut}(\Omega(T') \times \mathfrak{L}' \mid \Omega(T'))$$

must commute with any $h \in \text{Aut}(\Omega(T') \times \mathfrak{L}' \mid \Omega \times \mathfrak{L}')$. Take h given by the formula $h(Y_i) = \tau_i Y_i$ $(i = 1, \ldots, g)$ where τ_i is a primitive e_i-th root of unity. Then $\sigma h(Y_i) = \sigma(\tau_i) Y_i$ and $h \sigma(Y_i) = \tau_i Y_i$. So $\tau_i \in k$ and each e_i divides $n =$ the number of roots of unity of k. This finally shows that $n T' \subset \Theta/k^*$.

LEMMA 3.2. – *Let* H *denote the automorphism group of* $\Omega(T) x_k \mathfrak{L} \mid \Omega$. *Let* H_1 *be a subgroup of* H, *containing* $[H, H]$ *and such that the image of* H_1 *in* $\text{Gal}(\mathfrak{L} \mid k)$ *is contained in* $[\text{Gal}(\mathfrak{L} \mid k), \text{Gal}(\mathfrak{L} \mid k)]$. *Then* $\Omega(T) \times \mathfrak{L}/H_1 \cong \Omega(T'') \times \mathfrak{L}'$ *with*

a) $\mathfrak{L}'$ *is the maximal abelian extension of* k, *contained in* $\mathfrak{L}$.

b) T'' *is a sublattice of* T *such that* $n T'' \subset \Theta/k^*$.

Proof. – One divides first by $[H, H]$. The result $\Omega(T') \times \mathfrak{L}'$ is further divided by the group $H_1/[H, H]$ which lies by assumption in $\text{Aut}(\Omega(T') \times \mathfrak{L}' \mid \Omega \times \mathfrak{L}')$. The result is $\Omega(T'') \times \mathfrak{L}'$ where T'' is a sublattice of T'.

(3.3) We apply (3.2) to the group $H_1 = [G, G] \cap H$. Let $\varphi : \Gamma \longrightarrow G$ be a left-inverse of the canonical surjection $G \longrightarrow \Gamma$. One can define the action of $\varphi(\gamma)$ on the function field of $\Omega(T) \times \mathfrak{L}$ by: $\varphi(\gamma)(f) = f \circ \gamma$ for any $f \in \mathfrak{M}(\Omega)$; $\varphi(\gamma)\lambda = \lambda$ for any $\lambda \in \mathfrak{L}$ and $\varphi(\gamma) X_i = b_i(\gamma) X_i$.

Then $H_1 = H \cap [G, G]$ is generated by $[H, H]$ and the commutators $\varphi(\gamma) h \varphi(\gamma)^{-1} h^{-1}$ with $\gamma \in \Gamma$ and $h \in H$. This expression is 1 for any $h \in M$. For $h = \sigma \in \mathrm{Gal}(\mathfrak{L} | k) = \mathrm{Aut}(\Omega(T) \times \mathfrak{L} | \Omega(T))$ one easily sees that the commutator lies in M. This means that H_1 satisfies the condition of (3.2). Let $\Omega(T'') \times \mathfrak{L}'$ denote the quotient of $\Omega(T) \times \mathfrak{L}$ by H_1. This quotient is invariant under any $\varphi(\gamma)$. In other words, the action of Γ on Ω can be extended to action of Γ on $\Omega(T'') \times \mathfrak{L}'$.

Let us describe the function field of $\Omega(T'') \times \mathfrak{L}'$ by

$$F = \mathfrak{M}(\Omega) \otimes_k \mathfrak{L}'[Y_1, \ldots, Y_g] \text{ with } Y_i^{n_i} = \theta_i .$$

Then each n_i divides n.

The automorphism $\tilde{\gamma}$ on F which lifts the automorphism γ on $\mathfrak{M}(\Omega)$ must satisfy $\tilde{\gamma}(Y_i) = b_i(\gamma) Y_i$ for certain elements $b_i(\gamma) \in \mathfrak{L}'$. Moreover $\tilde{\gamma}$ must commute with the action of $\mathrm{Gal}(\mathfrak{L}' | k)$ on F. This implies that $b_i(\gamma) \in k$. We draw the conclusion that T'' is a sublattice of $\frac{1}{n}(\Theta / k^*)$ such that the canonical homomorphism $c : \Theta/k^* \longrightarrow \mathrm{Hom}(\Gamma, k^*)$ which is given by

$$c(\theta \bmod k^*)(\gamma) = \theta(\gamma\omega)\,\theta(\omega)^{-1},$$

extends to a group homomorphism $T'' \longrightarrow \mathrm{Hom}(\Gamma, k^*)$. This proves the main result.

THEOREM 3.3. – *Every finite abelian, unramified extension of* X *has uniquely the form* $\Omega(T) \times \mathfrak{L}/N$ *where*

(i) $\mathfrak{L}$ *is a finite abelian extension of* k

(ii) T *is a sublattice of* $\frac{1}{n}(\Theta/k^*)$ *such that the canonical homomorphism* $c : \Theta/k^* \longrightarrow \mathrm{Hom}(\Gamma, k^*)$ *extends to* T.

(iii) N *is a normal subgroup of* G *with* $N \cap H = \{1\}$. *The image* πN *of* N *in* Γ *is a normal subgroup with abelian factor group.*

COROLLARY 3.4. (G. Frey). – *The profinite Galois group* D *of the maximal abelian uramified extension of the function field* $\mathcal{M}(X)$ *of* X *is isomorphic to the direct product*

$$\mathrm{Gal}(k^{ab}/k) \oplus \hat{\mathbf{Z}}^g \oplus \mathbf{Z}/n_1 \oplus \ldots \oplus \mathbf{Z}/n_g .$$

The numbers $n_1, \ldots, n_g$ *satisfy* $n_1 | n_2 | \ldots | n_g | n$ *where* $n =$ *the number of roots of unity in* k *and they are determined by the curve* X.

Proof of (3.4). – One easily sees that there exists a largest lattice T, with $\Theta/k^* \subset \mathrm{T} \subset \frac{1}{n}\,\Theta/k^*$ such that the map

$$c : \Theta/k^* \longrightarrow \mathrm{Hom}(\Gamma, k^*)$$

extends to T. The finite group in (3.4) is the cokernel of the injection $\Theta/k^* \subset \mathrm{T}$.

Remark 3.5. – The corollary (3.4) has been proved by G. Frey [2]. His proof is quite different from the one presented here. It is based upon a detailed study of the action of the Galois group $\mathrm{Gal}(k^{ab} | k)$ on the points of finite order (or the Tate-modules) of the Jacobian variety (or a generalized Jacobian variety) of the Mumford curve $X = \Omega/\Gamma$.

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