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# THE CLASS GROUP OF A ONE-DIMENSIONAL AFFINOID SPACE

**by Marius van der PUT**

## Introduction.

The field $k$ is supposed to be complete with respect to a non-archimedean valuation. Moreover we will assume that $k$ is algebraically closed. An affinoid space Y over $k$ is the set of maximal ideals of an affinoid algebra. The standard affinoid algebra is $k\langle T_1,\ldots,T_n\rangle =$ the set of all power series $\Sigma a_\alpha T_1^{\alpha_1} \cdots T_n^{\alpha_n}$ converging on the closed polydisk

$$\{(t_1,\ldots t_n) \in k^n | \text{all} |t_i| \leqslant 1\}.$$

An affinoid algebra is a residue class ring of some $k\langle T_1,\ldots,T_n\rangle$. An algebraic variety over $k$ can be studied locally by its analytic structure over $k$, that is by means of affinoid spaces.

We show that a one-dimensional, normal, connected affinoid space Y is an affinoid subset of a non-singular, complete curve C over $k$ (Thm 1.1). If Y has a trivial classgroup then Y is in fact an affinoid subset of $\mathbf{P}^1$ (Thm 2.1). A curve is locally a unique factorization domain (U.F.D. for short) if and only the curve is a Mumford curve (i.e. can be parametrized by a Schottky group). In general the class group of Y can be expressed in terms of the Jacobi-variety of C (prop. 3.1).

Some examples show the connection between the class group of Y and the class group of the (stable) reduction of Y. For $k$-analytic spaces we refer to [2], [3]. I thank A. Escassut for bringing the problem on unique factorization on affinoid spaces to my attention. Related questions are treated in [1].

## 1. Affinoid subspaces of an algebraic curve.

A curve C (non-singular and complete) over $k$ has a natural structure as (rigid) analytic space over $k$. This structure is given by a collection of subspaces Y of C, called affinoid, and a sheaf $\mathcal{O} = \mathcal{O}_C$ with respect to the Grothendieck topology of finite coverings by affinoids. For any Y, $\mathcal{O}(Y)$ is an affinoid algebra (1-dim. and normal) over $k$ with $\mathrm{Sp}(\mathcal{O}(Y)) = Y$. We want to show :

1.1. — THEOREM. — *Every* 1-*dimensional, normal, connected affinoid space* $Y = \mathrm{sp}(A)$ *is an affinoid subspace of a non-singular complete curve.*

*Proof.* — Y is called connected and normal if the algebra A has no idempotents $\neq 0, 1$ and A is integrally closed. We use the notations $A^{o} = \{f \in A \mid \|f\| \leqslant 1\}$, $A^{oo} = \{f \in A \mid \|f\| < 1\}$ and $\bar{A} = A^{o}/A^{oo}$, where $\|f\| = \max\{|f(y)| \mid y \in Y\}$ is the spectral norm on Y. The algebra $\bar{A}$ is of finite type over $\bar{k}$ = the residue field of $k$ and the algebraic variety $\bar{Y}_c = \mathrm{Max}(\bar{A})$ is called the canonical reduction of Y. There is a natural surjective map $R : Y \to \bar{Y}_c$, also called the canonical reduction. A pure covering of an analytic space X, is an allowed covering $\mathcal{U} = (U_i)$ by affinoid spaces, such that for every $i \neq j$ with $U_i \cap U_j \neq \varnothing$, the set $U_i \cap U_j$ is the inverse image of a Zariski open set $V_{ij}$ in $\overline{(U_i)_c}$ under the map $U_i \to \overline{(U_i)_c}$. The reduction $\overline{X_{\mathcal{U}}}$ of X with respect to $\mathcal{U}$ is obtained by glueing the affine algebraic varieties $\overline{(U_i)_c}$ over the open sets $V_{ij}$. The result is an algebraic variety over $\bar{k}$. If X is separated then the $U_i \cap U_j$ are also affinoid, the $V_{ij}$ are affine and equal to $\overline{(U_i \cap U_j)_c}$ and $\overline{X_{\mathcal{U}}}$ is separated. If X is non-singular, 1-dimensional, connected and if $\overline{X_{\mathcal{U}}}$ is complete then X is a non-singular complete curve over $k$ (see [2] ch. IV 2.2).

Our proof consists of glueing affinoid spaces $Y_1, \ldots, Y_s$ to Y such that the reduction of $X = Y \cup Y_1 \cup \ldots \cup Y_s$ with respect to the pure covering $\{Y, Y_1, \ldots, Y_s\}$ is complete. Then clearly Y is an affinoid domain of the algebraic curve X. The 1-dimensional space $\overline{Y_c}$ lies in a complete 1-dimensional Z such that $F = Z - \overline{Y_c}$ is a finite set of non-singular points. Suppose that we can find for every $p \in F$ an affinoid space $Y_p$ with canonical reduction $R_p : Y_p \to \overline{(Y_p)_c} \subset Z$ where $\overline{(Y_p)_c}$ is a neighbourhood of $p$ and such that

$$Y_p \supset R_p^{-1}(\overline{(Y_p)_c} \cap \overline{Y_c}) \simeq R^{-1}(\overline{(Y_p)_c} \cap \overline{Y_c}) \subset Y.$$

Then we can glue $Y_p$ to Y. The space $X = YU \cup Y_p$ has reduction Z which is complete. So the glueing has to be done locally on Y and $\overline{Y_c}$. The component C of Z on which $p$ lies can be projected into $\mathbf{P}^2(\bar{k})$ such that (the image of) $p$ is still non-singular. A good projection onto $\mathbf{P}^1$ maps $p$ onto $o$ and $o$ is an unramified point for the projection. Replacing Y and $\overline{Y_c}$ by neighbourhoods of $p$ we may therefore suppose :

$$\overline{\mathcal{O}(Y)} = \mathcal{O}(\overline{Y_c}) = \bar{k}[t,(t,e(t))^{-1},s]/(P),$$

where

1) $e(t) = (t-\overline{a_1})\dots(t-\overline{a_s})$ with $\overline{a_1},\dots,\overline{a_s}$ different points of $\bar{k}^*$ ; they are the residues of $a_1,\dots,a_s \in k^0$.

2) P is a monic irreducible polynomial of degree $n$ with coefficients in $\bar{k}[t]$.

3) $\dfrac{dP}{ds}$ is invertible as element of $\bar{k}[t,(e(t))^{-1},s]/(P)$.

4) the point « $p$ » corresponds to $t = 0$.

Then $\mathcal{O}(Y)^0$ has the form $k^0\langle T,U,S\rangle/(TE(T)U-1,Q)$ where

$$E(T) = (T-a_1)\dots(T-a_s) \qquad \text{and} \qquad \bar{Q} = P.$$

Since Q is general with respect to the variable S, we can apply Weierstrass-division and assume that Q is a monic polynomial of degree $n$ in S with coefficients in $k^0\langle T,U\rangle/(TE(T)U-1)$. Suppose that we can find a monic polynomial $Q^*$ of degree $n$ in S and coefficients in $k^0\langle T,V\rangle/(E(T)V-1)$ such that

$$k^0\langle T,U,S\rangle/(TE(T)U-1,Q^*) \simeq \mathcal{O}(Y)^0.$$

Then $Y_p = Sp(k\langle T,V,S\rangle/(E(T)V-1,Q^*))$ has the required properties. So we have to get rid of the negative powers of T in the coefficients of

$$Q = S^n + a_{n-1}S^{n-1} + \cdots + a_0.$$

1.2. – LEMMA. – *If* $Q^* = S^n + a^*_{n-1}S^{n-1} + \cdots + a^*_0$ *has coefficients in* $A = k^0\langle T,U\rangle/(TE(T)U-1)$ *and* $\overline{Q^*} = \overline{Q} = P$, *then*

*a)* $Q^*$ *is irreducible*

*b)* $Q^*$ *has a zero in* $\mathcal{O}(Y)^0$

*c)* $k\langle T,U,S\rangle/(TE(T)U-1,Q^*) \simeq \mathcal{O}(Y)$.

*Proof.* – *a)* Let $Q^*$ be reducible over the quotient field of A. Since A is normal, $Q^*$ is a product of monic polynomials with coefficients in A. This contradicts the irreducibility of $\overline{Q^*} = P$.

*b*) First we show that $\left\{Q^*, \frac{dQ^*}{dS}\right\}$ generates the unit ideal in $A[S]$. Let $\mathfrak{m}$ be a maximal ideal containing $Q^*$ and $\frac{dQ^*}{dS}$. If $\mathfrak{m} \cap k^0 \neq 0$ then $\mathfrak{m}$ induces a maximal ideal of $\bar{k}[t,(te(t))^{-1}][S] = \bar{A}[S]$ containing $P$ and $\frac{dP}{dS}$. This contradicts our assumptions on $P$. So $\mathfrak{m}$ corresponds to a maximal ideal $\mathfrak{m}_1$, of $k\langle T,U\rangle/(TE(T)U-1)[S]$, containing $Q^*$ and $\frac{dQ^*}{dS}$.

If $\mathfrak{m}_1 \cap k\langle T,U\rangle/(TE(T)U-1) \neq 0$ then $\mathfrak{m}_1$, is the kernel of a homomorphism in $k$ given by $T \longmapsto \lambda_1 \in k$, $S \longmapsto \lambda_2 \in k$ with

$$|\lambda_1| \leqslant 1, \qquad |\lambda_1 E(\lambda_1)| = 1, \qquad |\lambda_2| \leqslant 1$$

since $Q^*(\lambda_2) = 0$. From $\left(P, \frac{dP}{dS}\right) = \bar{k}[t,(te(t))^{-1},S]$ it follows that

$$Z_1(S)Q^* + Z_2(S)\frac{dQ^*}{dS} = 1 + \sum_{i>0} a_i S^i$$

for certain $Z_1$, $Z_2 \in A[S]$ and $a_i \in A$ with $\|a_i\| < 1$. The substitution $T \longmapsto \lambda_1$ ; $S \longmapsto \lambda_2$ makes $0 = 1 + \sum_{i>0} a_i(\lambda_1)\lambda_2^i$, which is impossible. So $\mathfrak{m}$ and $\mathfrak{m}_1$ correspond to an ideal of $L[S]$ with $L$ the quotient field of $A$. Since $Q^*$ is irreducible, this means that $\frac{dQ^*}{dS} = 0$. This is obviously in contradiction with $\left(P, \frac{dP}{dS}\right) = \bar{k}[t,(te(t))^{-1}]$.

We conclude the existence of $Z_1$, $Z_2 \in A[S]$ with

$$1 = Z_1(S)Q^* + Z_2(S)\frac{dQ^*}{dS}.$$

By Newton's method we will show that $Q^*$ has a zero in $\mathcal{O}(Y)^0$. Let $\eta \in \mathcal{O}(Y)^0$ satisfy $\|Q^*(\eta)\| < 1$ (e.g. $\eta$ is the residue of $S$ mod $Q$ in $\mathcal{O}(Y)^0$). Then $1 - Z_1(\eta)Q^*(\eta) = Z_2(\eta)\frac{dQ^*}{dS}(\eta)$ and since

$\|Z_1(\eta)Q^*(\eta)\| < 1$ it follows that $\dfrac{dQ^*}{dS}(\eta)$ is invertible. Put $\eta_1 = \eta - Q^*(\eta)\left(\dfrac{dQ^*}{dS}(\eta)\right)^{-1}$. Then $\|Q^*(\eta_1)\| \leqslant \|Q^*(\eta)\|^2$. The usual procedure and the completeness of $\mathcal{O}(Y)^0$ show the existence of a root of $Q^*$ in $\mathcal{O}(Y)^0$.

*c*) The quotient field of $A[S]/Q^*$ is contained in that of $A[S]/Q$, because of (*b*). Both fields are extensions of degree $n$ of the quotient field of A. So they are equal. The rings $k\langle T,U,S\rangle/(TE(T)U-1,Q^*)$ and $\mathcal{O}(Y)$ are both the integral closure of $k\langle T,U\rangle/(TE(T)U-1)$ in that field. So they are equal.

*End of the proof of* 1.1. — We choose $Q^*$ with coefficients in $k^0\langle T,V\rangle/(VE(T)-1)$ and $Q^* = P$.

1.3. — COROLLARY. — *Let* Y *be as in* (1.1) ; *then* Y *is affinoid in a curve* X (*complete non-singular*) *such that* $\bar{X} - \bar{Y}_c$ *is a finite set of non-singular points.*

## 2. Unique factorization.

We want to show the following :

2.1. — THEOREM. — *Let* $Y = Sp\, A$ *be a* 1-*dimensional connected affinoid space. Then* A *has unique factorization if and only if* Y *is an affinoid subspace of* $\mathbf{P}^1(k)$.

*Remarks.* — 1) Since A has dimension 1 the condition « A has unique factorization » is equivalent to « A is a principal ideal domain ».

2) It seems that this theorem has also been proved by M. Raynaud.

A connected affinoid subspace Y of $\mathbf{P}^1(k)$ has clearly a U.F.D. as affinoid algebra. Before we start the proof of 2.1, we like to state its algebraic analogue. It is :

2.2. — PROPOSITION. — *Let* A *be a finitely generated algebra over an algebraically closed field* $k$. *Suppose that* A *is* 1-*dimensional and a U.F.D. Then* A *is isomorphic to the coordinate ring of a Zariski-open subset of* $\mathbf{P}^1(k)$.

*Proof.* — A is the coordinate ring of a Zariski-open subset X of some non-singular complete curve C; put $X = C - \{p_1,\ldots,p_s\}$. Let D be a divisor of degree 0 on C; since A is a U.F.D. there is a rational function $f$ on C with $D = (f)$ on X. This means that the map $\left\{ \sum_{i=1}^{s} n_i p_i | n_i \in \mathbf{Z} \text{ and } \Sigma n_i = 0 \right\} \longrightarrow J(C) =$ the Jacobi-variety of C, is surjective. If C is not a rational curve then its Jacobi variety (or better its points in $k$) is not a finitely generated group. Hence $C \simeq \mathbf{P}^1(k)$.

We prove the theorem in some steps.

2.3. — LEMMA. — *Suppose that* $\mathcal{O}(Y)$ *is a U.F.D. and that* $\bar{Y}$ *is irreducible, then* $H^1(\bar{Y}, \mathcal{O}^*_{\bar{Y}}) = 0$.

*Proof.* — $\bar{Y}$ denotes the canonical reduction of Y. An element $\xi \in H^1(\bar{Y}, \mathcal{O}^*)$ corresponds to a projective, rank 1, $\mathcal{O}(\bar{Y})$-module N; let F be a free $\mathcal{O}(\bar{Y})$-module, $\sigma : F \longrightarrow F$ an idempotent endomorphism with im $\sigma = N$. Then F, $\sigma$ lift to similar things over $\mathcal{O}(Y)^0$ since $\mathcal{O}(Y)^0$ is complete and $\mathcal{O}(\bar{Y}) = \mathcal{O}(Y)^0 \otimes \bar{k}$. So we find a projecture, rank 1, $\mathcal{O}(Y)^0$-module M with $M \otimes \bar{k} = N$.

Further $M \otimes \mathcal{O}(Y) \simeq \mathcal{O}(Y)$ since $\mathcal{O}(Y)$ is a U.F.D. There exists a Zariski-open covering of $\bar{Y}$ such that N is free on the sets of this covering. That implies the existence of $f_1, \ldots, f_s \in \mathcal{O}(Y)^0$ such that

*a*) each $\|f_i\| = 1$ and $(f_1, \ldots, f_s)\mathcal{O}(Y)^0 = \mathcal{O}(Y)^0$.

*b*) $M \otimes \mathcal{O}(X)^0\langle S\rangle/(Sf_i - 1)$ is a free $\mathcal{O}(X)^0\langle S\rangle/(Sf_i - 1)$-module.

We identify M with $M \otimes \mathcal{O}(Y)^0 \subset \mathcal{O}(Y)$ and we may suppose that $M \subset \mathcal{O}(Y)^0$; $\max\{\|m\| \,|m \in M\} = 1$ and $M \supset \lambda\mathcal{O}(Y)^0$ for certain $\lambda \in k^0$, $\lambda \neq 0$. Then

$$M \otimes \mathcal{O}(Y)^0\langle S\rangle/(Sf_i - 1) \subseteq \mathcal{O}(Y)^0\langle S\rangle/(Sf_i - 1)$$

is generated by one element $h$. This element has norm 1 and it has no zeros is $\{y \in Y | \,|f_i(Y)| = 1\} = Y_i$. So $h$ is invertible in $\mathcal{O}(Y_i)$. Its inverse $h^{-1}$ has also norm 1 since $\overline{Y_i}$ is irreducible and the norm on $\mathcal{O}(Y_i)$ is, as a consequence, multiplicative. Hence $M\mathcal{O}(Y_i)^0 = \mathcal{O}(Y_i)^0$. It follows that some power of $f_i$ lies in M. Since $(f_1, \ldots, f_s) = \mathcal{O}(Y)^0$ we find that $M = \mathcal{O}(Y)^0$. So N is free and $\xi = 0$.

2.4. — LEMMA. — *Let* L *be affine, 1-dimensional and irreducible over* $\bar{k}$. *If* $H^1(L, \mathcal{O}^*_L) = 0$ *then* L *is rational and non-singular.*

*Proof.* – Let $\pi : L_1 \longrightarrow L$ be the normalization of $L$. We have an exact sequence of sheaves on $L : 0 \longrightarrow \mathcal{O}_L^* \longrightarrow \pi_*\mathcal{O}_{L_1}^* \longrightarrow F \longrightarrow 0$ where $F$ is the skyscraper sheaf with stalks, $F_p = \tilde{\mathcal{O}}_{L,p}^*/\mathcal{O}_{L,p}^*$ and $\tilde{\mathcal{O}}_{L,p}$ is the integral closure of $\mathcal{O}_{L,p}$.

One finds an exact sequence

$$0 \longrightarrow \mathcal{O}(L)^* \longrightarrow \mathcal{O}(L_1)^* \longrightarrow H^0(F) \longrightarrow H^1(L,\mathcal{O}_L^*) \longrightarrow H^1(L_1,\mathcal{O}_{L_1}^*) \longrightarrow 0.$$

So clearly (by 2.2) $L_1 = \mathbf{P}^1(\bar{k}) - \{p_1,\ldots,p_s\}$ and the group $\mathcal{O}(L_1)^*$ is isomorphic to $\bar{k}^* \oplus N$ where $N$ is a subgroup of $\mathbf{Z}^{s-1}$.

So we find that $H^0(F)$ is a finitely generated $\mathbf{Z}$-module.

If $L$ has a singular point $p$ then $H^0(F)$ has $\tilde{\mathcal{O}}_{L,p}^*/\mathcal{O}_{L,p}^*$ as component. The last group has $\bar{k}$ or $\bar{k}^*$ as quotient group. It is not finitely generated. So we conclude that $L$ is non-singular, and hence a Zariski-open subset of $\mathbf{P}^1(\bar{k})$.

2.5. – *Continuation of the proof of* 2.1.

We have to consider the case where $\bar{Y}$, the canonical reduction of $Y$, has more than one component. Let $L$ be a component and $L_1 = L - \{$the intersection of $L$ with the other components$\}$ ; $Y_1 = R^{-1}(L_1)$. Then $Y_1$ is affinoid, also a U.F.D. and with canonical reduction $L_1$. We know by 2.3 and 2.4 that $L_1$ is Zariski-open in $\mathbf{P}^1(\bar{k})$ and so $Y_1$ must be an affinoid subset of $\mathbf{P}^1(k)$ of the form

$$\{z \in k \mid |z| \leqslant 1, \qquad |z - a_i| \geqslant 1 \qquad (i=1,\ldots,s)\}.$$

Let $a_{d+1},\ldots,a_s$ correspond to the points of intersection of $L$ with the other components of $\bar{Y}$. Let $Y_2 = \{z \in k \mid |z| \leqslant 1$ and $|z - a_i| \geqslant 1$ for $i = d + 1, \ldots, s\}$. Then we glue $Y_2$ to $Y$ over the open subset $Y_1$. The resulting analytic space $Y \cup Y_2$ has as reduction with respect to the covering $\{Y,Y_2\}$ the space $\bar{Y} \cup \bar{Y}_2$. From [2] ch. IV (2.2) it follows that $Z = Y \cup Y_2$ is also affinoid and its canonical reduction is obtained by contracting the complete one of $\bar{Y} \cup \bar{Y}_2$ to a point. If we can show that $Z$ is also a U.F.D., then (2.1) follows by induction on the number of components of $\bar{Y}$. Since

$$H^1(Y,\mathcal{O}_Y^*) = H^1(Y_1,\mathcal{O}_{Y_1}^*) = H^1(Y_2,\mathcal{O}_{Y_2}^*) = 0$$

we can calculate $H^1(Z,\mathcal{O}_Z^*) =$ the class group of $Z$, with respect to the covering $\{Y_2,Y\}$. That $Z$ is a U.F.D. is equivalent with $H^1(Z,\mathcal{O}_Z^*) = 0$ and will follow from the following

2.6. — LEMMA. — *The map* $\mathcal{O}(Y)^* \oplus \mathcal{O}(Y_2)^* \longrightarrow \mathcal{O}(Y_1)^*$, *given by* $(f_1,f_2) \longrightarrow f_1 f_2^{-1}$, *is surjective.*

*Proof.* — The norm on $\mathcal{O}(Y_1)$ is multiplicative. So any $f \in \mathcal{O}(Y_1)^*$ has the form $f = cg$ with $c \in k^*$ and $g \in (\mathcal{O}(Y_1)^0)^*$. Further the analoguous map $\mathcal{O}(\bar{Y})^* \oplus \mathcal{O}(\bar{Y}_2)^* \longrightarrow \mathcal{O}(\bar{Y}_1)^*$ is clearly surjective. So $\bar{g} = \bar{f}_1 \bar{f}_2^{-1}$ for certain $f_1 \in (\mathcal{O}(Y)^0)^*$ and $f_2 \in (\mathcal{O}(Y_2)^0)^*$. We are reduced to consider $f \in \mathcal{O}(Y_1)^*$ of the form $1 + h$ with $h \in \mathcal{O}(Y_1)$, $\|h\| < 1$. We want to write $f$ as $(1+h_1)(1+h_2)^{-1}$ with $h_1 \in \mathcal{O}(Y)$, $h_2 \in \mathcal{O}(Y_2)$ and $\|h_1\| < 1$, $\|h_2\| < 1$. This amounts to showing that $\beta : \mathcal{O}(Y)^0 \oplus \mathcal{O}(Y_2)^0 \longmapsto \mathcal{O}(Y_1)^0$, given by $(h_1,h_2) \longmapsto h_1 - h_2$, is surjective. By [2], ch. IV (2.2), we know that the cokernel of $\beta$ is a finitely generated $k^0$-module M. Moreover $M \otimes \bar{k} = 0$ since $\mathcal{O}(\bar{Y}) \oplus \mathcal{O}(\bar{Y}_2) \longrightarrow \mathcal{O}(\bar{Y}_1)$ is surjective. So $M = 0$, $\beta$ is surjective and the Lemma is proved.

2.7. — COROLLARY. — *Let* X *be a complete non-singular curve over* $k$. *Then* X *is a Mumford curve* (*i.e. can be parametrized by a Schottky group*) *if and only if* X *is locally a U.F.D.*

*Proof.* — Locally a U.F.D. means that X has an affinoid covering $(X_i)_{i=1}^s$ such that each $\mathcal{O}(X_i)$ is a unique factorization domain. According to (2.1) this implies $X_i \subset \mathbf{P}^1(k)$. According to [2], ch. IV (5.1), this is equivalent with X is a Mumford curve.

## 3. Class groups.

X will denote a normal, connected, 1-dimensional affinoid space. The class group of X (i.e. the group of isomorphy-classes of projective, rank 1, $\mathcal{O}(X)$-modules) is equal to the analytic cohomology group $H^1(X,\mathcal{O}_X^*)$. This follows from the bijective correspondance between projective, rank 1, $\mathcal{O}(X)$-modules and invertible sheaves on X.

3.1. — PROPOSITION. — *Let* X *be embedded in a complete non-singular curve* C. *Then* $H^1(X,\mathcal{O}_X^*) \simeq J(C)/H$ *where* J(C) *is the Jacobi-variety of* C *and* H *is the subgroup consisting of the images of the divisors of degree zero on* C *with support in* C − X. *The group* H *is an open subgroup in the topology of* J(C) *induced by the topology of* $k$.

*Proof.* — The restriction map $\mathrm{Div}_0(C) \longrightarrow \mathrm{Div}(X)$ induces a surjective homomorphism $\mathrm{Div}_0(C)/P(C) \longrightarrow \mathrm{Div}(X)/P(X)$ where P(C) denotes the principal divisors on C and $P(X) = \{(f)$ on $X | f$

meromorphic on X}. It is easily seen that $H^1(X,\mathcal{O}_X^*) = \mathrm{Div}(X)/P(X)$. Let $D \in \mathrm{Div}_0(C)$ have image 0 in $H^1(X,\mathcal{O}_X^*)$, then there exists a meromorphic function $f$ on X with $(f) = D$ on X. As one can calculate (see [2], ch. III (1.18.5) and on) any divisor of a holomorphic (or meromorphic) function on X is the divisor of a rational function on C restricted to X. So there is a rational function $g$ on C with $(g) = D$ on X. Then $D - (g)$ is a divisor of degree 0 with support in $C - X$. This proves the first assertion. The map $C \times \ldots \times C \longrightarrow J(C)$ given by $(x_1,\ldots,x_g) \longmapsto \sum_{i=1}^{g} x_i - gx_0$ (where $x_0 \in C - X$ is fixed) is surjective and induces the algebraic structure and topology on J(C). The map is almost bijective and open. So the image of $(C-X) \times \ldots \times (C-X)$ is open and H is open.

*Remark.* – In general it seems to be rather difficult to calculate explicitely $H^1(X,\mathcal{O}_X^*)$. However using (3.1) one can work out the following special cases.

3.2. – *Example.* – Let the curve C have a reduction $R : C \longrightarrow \bar{C}$ such that $\bar{C}$ is rational and has one ordinary double point $p$. Take $p_1, \ldots, p_s$ points in $\bar{C} - \{p\}$ and put $X = R^{-1}(\bar{C} - \{p_1,\ldots,p_s\})$. Then X is affinoid and its canonical reduction is $C - \{p_1,\ldots,p_s\}$. The curve C is a Tate-curve and $\simeq k^*/\langle q\rangle$ with $0 < |q| < 1$. The points $p_1, \ldots, p_s$ correspond to open discs of radii 1 around points $1 = a_1, a_2, \ldots, a_s \in k$ with all $|a_i| = 1$ and $|a_i - a_j| = 1$ if $i \neq j$. Using (3.1) one finds an exact sequence :

$$1 \longrightarrow \bar{k}^*/\langle \overline{a_2}, \ldots, \overline{a_s}\rangle \longrightarrow H^1(X,\mathcal{O}_X^*) \longrightarrow |k^*|/\langle |q|\rangle \longrightarrow 1$$

where $\langle \overline{a_2},\ldots,\overline{a_s}\rangle$ is the subgroup of $\bar{k}^*$ generated by $\overline{a_2}, \ldots, \overline{a_s}$ ; $|k^*|$ is the value group of $k$ and $\langle |q|\rangle$ its subgroup generated by $|q|$. Note further that $\bar{k}^*/\langle \overline{a_2},\ldots,\overline{a_s}\rangle = H^1(\bar{X},\mathcal{O}_{\bar{X}}^*)$.

3.3. – *Example.* – Let C be a Mumford curve of genus $g \geqslant 1$ and let $R : C \longrightarrow \bar{C}$ be its stable reduction. (The components of C are rational, the only singularities are ordinary double points.) The Jacobi-variety of C is a holomorphic torus $(k^*)^g/\Lambda$ where $\Lambda$ is a lattice in $(k^*)^g$. Take ordinary points $p_1, \ldots, p_s \in \bar{C}$ and put $X = R^{-1}(\bar{C} - \{p_1,\ldots,p_s\})$. Then X is affinoid and using (3.1) one calculates an exact sequence :

$$1 \longrightarrow (\bar{k}^*)^g/S \longrightarrow H^1(X,\mathcal{O}_X^*) \longrightarrow |k^*|^g/|\Lambda| \longrightarrow 1$$

where

$$|\Lambda| = \{(|\lambda_1|, |\lambda_2|, \ldots, |\lambda_g|) | (\lambda_1, \ldots, \lambda_g) \in \Lambda\}$$

and S is a finitely generated subgroup of $(\bar{k}^*)^g$. The group $(\bar{k}^*)^g$ is in fact the Jacobi-variety of $\bar{C}$ and the subgroup S is the subgroup of the divisors of degree 0 on $\bar{C}$ with support in $\{p_1, \ldots, p_s\}$. So $(\bar{k}^*)^g/S$ is again $H^1(\bar{X}_S, \mathcal{O}^*)$ where $\bar{X}_S$ denotes the stable reduction of X.

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