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ON SCHWARTZ'S THEOREM FOR THE MOTION GROUP

by Yitzhak WEIT

1. Introduction.

Schwartz's Theorem in the theory of mean periodic functions on the real line states that every closed, translation-invariant subspace of the space of continuous functions on $\mathbf{R}$ is spanned by the polynomial-exponential functions it contains [5]. In particular, every translation-invariant subspace contains an exponential function.

In [2] the two-sided analogue of this result was generalized to $SL_2(\mathbf{R})$. However, since [3] it is known that Schwartz's Theorem fails to hold for $\mathbf{R}^n$, $n > 1$.

Our main goal is to show that the two-sided analogue of Schwartz's Theorem holds for the motion group M(2). That is, every closed, two-sided invariant subspace of C(M(2)) contains an irreducible invariant subspace and every such subspace is spanned by a class of functions which replace the polynomial-exponentials on $\mathbf{R}$.

It seems remarkable that the analogue of Schwartz's Theorem holds for the three dimensional Lie groups $SL_2(\mathbf{R})$ and M(2) while it fails to hold for $\mathbf{R}^2$.

In section 3 we verify the two-sided Schwartz's Theorem for the motion group. In section 4 we consider the problem of one-sided spectral analysis. Finally, in section 5, we study some invariant subspaces of continuous functions on $\mathbf{R}^2$. It turns out that the one-sided Schwartz's Theorem for the motion group is intimately connected with a problem of Pompeiu type [1, 4, 7].

2. Preliminaries and Notation.

Let M(2) denote the Euclidean motion group consisting of the matrices $\begin{pmatrix} e^{i\alpha} & z \\ 0 & 1 \end{pmatrix}$, $\alpha \in \mathbf{R}$, $z \in \mathbf{C}$.

Let C(M(2)) denote the space of all continuous functions on M(2) with the usual topology of uniform convergence on compact sets. Let $\mathcal{E}(\mathbf{R}^n)$ be the space of infinitely differentiable functions on $\mathbf{R}^n$ endowed with the topology of uniform convergence of functions and their derivatives on compacta. Let $\mathcal{E}'(\mathbf{R}^n)$ be the dual of $\mathcal{E}(\mathbf{R}^n)$, the space of Schwartz distributions on $\mathbf{R}^n$ having compact support. The pairing between $\mathcal{E}(\mathbf{R}^n)$ and $\mathcal{E}'(\mathbf{R}^n)$ is denoted by $\mathrm{T}(f)$ for $f \in \mathcal{E}(\mathbf{R}^n)$ and $\mathrm{T} \in \mathcal{E}'(\mathbf{R}^n)$, and for such f and T we denote by $\mathrm{T} * f$ the convolution of T and f. For $\mathrm{T} \in \mathcal{E}'(\mathbf{R}^n)$, the Fourier transform of T is defined by $\hat{\mathrm{T}}(z) = \mathrm{T}(e^{iz \cdot x})$ where $z \in \mathbf{C}^n$, $x \in \mathbf{R}^n$ and $z \cdot x = z_1 x_1 + \ldots + z_n x_n$. By Paley-Wiener-Schwartz Theorem, the space $\hat{\mathcal{E}}'(\mathbf{R}^n)$ of Fourier transforms of elements of $\mathcal{E}'(\mathbf{R}^n)$ is identified with the space of entire functions of n complex variables of exponential type which have polynomial growth on the real subspace $\mathbf{R}^n$. The topology of $\hat{\mathcal{E}}'(\mathbf{R}^n)$ is so defined as to make the Fourier transform a topological isomorphism.

Let Π denote the group of all rotations of $\mathbf{R}^2$. We denote by $\mathcal{E}'_{(r)}(\mathbf{R}^2)$ the space of all $\mathrm{T} \in \mathcal{E}'(\mathbf{R}^2)$ which satisfy $\mathrm{T} \circ \tau = \mathrm{T}$ for every $\tau \in \Pi$. Let $\hat{\mathcal{E}}'_{(r)}(\mathbf{R}^2)$ denote the space of Fourier transforms of elements of $\mathcal{E}'_{(r)}(\mathbf{R}^2)$. We notice that each $f \in \hat{\mathcal{E}}'_{(r)}(\mathbf{R}^2)$ is a function of $z_1^2 + z_2^2$ and that for any even function $g \in \hat{\mathcal{E}}'(\mathbf{R})$ the function $\tilde{g}$ where $\tilde{g}(z_1, z_2) = g(\sqrt{z_1^2 + z_2^2})$ belongs to $\hat{\mathcal{E}}'_{(r)}(\mathbf{R}^2)$. Let $\mathcal{E}_0(\mathbf{R}^2)$ denote the space of elements of $\mathcal{E}(\mathbf{R}^2)$ having compact support and $\mathcal{E}_0^{(r)}(\mathbf{R}^2)$ the space of radial functions in $\mathcal{E}_0(\mathbf{R}^2)$.

Let $\mathrm{C}(\mathbf{R}^n)$ denote the space of continuous functions on $\mathbf{R}^n$ with the topology of uniform convergence on compacta and $\mathrm{C}^{(r)}(\mathbf{R}^2)$ the radial functions in $\mathrm{C}(\mathbf{R}^2)$. The dual of $\mathrm{C}(\mathbf{R}^n)$ is the space $\mathrm{M}_0(\mathbf{R}^n) \subset \mathcal{E}'(\mathbf{R}^n)$ of all complex-valued Radon measures having compact support. Let $\mathrm{M}_0^{(r)}(\mathbf{R}^2) = \mathrm{M}_0(\mathbf{R}^2) \cap \mathcal{E}'_{(r)}(\mathbf{R}^2)$.

Finally, for $\lambda = (\lambda_1, \lambda_2) \in \mathbf{C}^2$ and $z = x + iy \in \mathbf{C}$ let $(\lambda, z) = \lambda_1 x + \lambda_2 y$.

3. Two-sided spectral synthesis.

The two-sided analogue of Schwartz's Theorem in spectral analysis for the motion group is stated in the following:

THEOREM 1. – *Every closed, two-sided invariant subspace of* C(M(2)) *contains either a character of* M(2) *or a function* $g(e^{i\alpha}, z) = e^{i(\lambda, z)}$ *where* $\lambda = (\lambda_1, \lambda_2) \in \mathbf{C}^2$ *and* $\lambda_1^2 + \lambda_2^2 \neq 0$. *The two-sided invariant subspace generated by* $e^{i(\lambda, z)}$ *where* $\lambda = (\lambda_1, \lambda_2)$, $\lambda_1^2 + \lambda_2^2 \neq 0$, *is irreducible (minimal).*

Proof. – For $f \in C(M(2))$, $f \neq 0$, let V_f denote the closed subspace generated by the two-sided translates of f.

The subspace V_f contains all the functions g where

$$g(e^{i\alpha}, z) = f(e^{i(\alpha+\beta)}, ue^{i\alpha} + e^{i\theta} z + w) \tag{1}$$

for every $\theta, \beta \in \mathbf{R}$ and $u, w \in \mathbf{C}$. Let $u = \theta = w = 0$ in (1).

Then, for a suitable $m \in \mathbf{Z}$ the function

$$\int_0^{2\pi} f(e^{i(\alpha+\beta)}, z)\, e^{-im\beta} d\beta = e^{im\alpha} \int_0^{2\pi} f(e^{i\beta}, z)\, e^{-im\beta} d\beta = e^{im\alpha} g_1(z)$$

is non-zero and belongs to V_f. Let N denote the translation-invariant and rotation-invariant subspace of $C(\mathbf{R}^2)$ generated by g_1.

By (1) the functions $e^{im\alpha} g_1(e^{i\theta} z + w)$ belongs to V_f for every $\theta \in \mathbf{R}$ and $w \in \mathbf{C}$. That is, V_f contains all functions $e^{im\alpha}\tilde{g}(z)$ where $\tilde{g} \in N$. In [1] it was proved that every closed, translation-invariant and rotation-invariant subspace of $C(\mathbf{R}^2)$ is spanned by the polynomial-exponential functions it contains. In particular, the subspace N contains therefore an exponential function $e^{i(\lambda, z)}$, $\lambda = (\lambda_1, \lambda_2) \in \mathbf{C}^2$ and the function $h(e^{i\alpha}, z) = e^{im\alpha} e^{i(\lambda, z)}$ belongs to V_f. If $\lambda_1^2 + \lambda_2^2 = 0$ then the subspace N contains the constant functions and V_f contains therefore the character $e^{im\alpha}$. Suppose that $\lambda_1^2 + \lambda_2^2 \neq 0$.

Let $h_1 \in \mathcal{E}_0(\mathbf{R}^2)$ of the form $h_1(w) = h_2(r) e^{-i\theta m}$ where $w = re^{i\theta}$, and $h_2 \in \mathcal{E}_0^{(r)}(\mathbf{R}^2)$ such that $\hat{h}_1(\lambda_1, \lambda_2) \neq 0$.

Then the function:

$$\int_{\mathbf{R}^2} h(e^{i\alpha}, z - e^{i\alpha}w)\, h_1(w)\, dw = \hat{h}_1(\lambda_1, \lambda_2)\, e^{i(\lambda, z)} \tag{2}$$

(here dw denotes Lebesgue measure on $\mathbf{R}^2$) is non-zero and belongs to V_f. It follows, by (1) and the analyticity of the elements of $\hat{\mathcal{E}}'_{(r)}(\mathbf{R}^2)$ that V_f contains all functions $e^{i(\mu, z)}$ where $\mu = (\mu_1, \mu_2) \in \mathbf{C}^2$ such that $\mu_1^2 + \mu_2^2 = \lambda_1^2 + \lambda_2^2$. To prove the second part of the theorem, let $g(z) = e^{i(\lambda, z)}$ where $\lambda = (\lambda_1, \lambda_2) \in \mathbf{C}^2$, $\lambda_1^2 + \lambda_2^2 \neq 0$. Firstly, we will show that V_g contains no character of $M(2)$.

Suppose that $e^{im\alpha} \in V_g$ for some $m \in \mathbf{Z}$. Let $\mu \in C(M(2))$, $\mu(e^{i\alpha}, z) = e^{-im\alpha} \mu_1(z)$ where $\mu_1 \in \mathcal{E}_0^{(r)}(\mathbf{R}^2)$ such that $\hat{\mu}_1(\lambda_1, \lambda_2) = 0$ and $\hat{\mu}_1(0,0) \neq 0$. We have

$$\int_{\mathbf{R}^2} e^{i(\lambda, e^{i\theta} z)} \mu_1(z)\, dz = 0$$

for every $\theta \in \mathbf{R}$. Consequently, we deduce

$$\int_{M(2)} e^{i(\lambda, e^{i\theta} z + w e^{i\alpha})} e^{-im\alpha} \mu_1(z)\, d\alpha dz = \int_0^{2\pi} \left[\int_{\mathbf{R}^2} e^{i(\lambda, e^{i\theta} z)} \mu_1(z)\, dz \right] e^{i[(\lambda, w e^{i\alpha}) - m\alpha]}\, d\alpha = 0$$

for every $\theta \in \mathbf{R}$ and $w \in \mathbf{C}$. Namely, μ annihilates the subspace V_g. On the other hand, we have

$$\int_{M(2)} e^{im\alpha} \mu(e^{i\alpha}, z)\, d\alpha dz = \hat{\mu}_1(0,0) \neq 0, \text{ a contradiction.}$$

Suppose that $e^{i(w,z)} \in V_g$ where $w = (w_1, w_2) \in \mathbf{C}^2$. If $\lambda_1^2 + \lambda_2^2 \neq w_1^2 + w_2^2$ then for $\mu_2 \in \mathcal{E}_0^{(r)}(\mathbf{R}^2)$ where $\hat{\mu}_2(\lambda_1, \lambda_2) = 0$ and $\hat{\mu}_2(w_1, w_2) \neq 0$ we have

$$\int_0^{2\pi} \int_{\mathbf{R}^2} e^{i(\lambda, e^{i\theta} z + v e^{i\alpha})} \mu_2(z)\, dz d\alpha = \int_0^{2\pi} \left[\int_{\mathbf{R}^2} e^{i(e^{i\theta}\lambda, z)} \mu_2(z)\, dz \right] e^{i(\lambda, v e^{i\alpha})}\, d\alpha = 0$$

for each $\theta \in \mathbf{R}$ and $v \in \mathbf{C}$. However, we have

$$\int_0^{2\pi} \int_{\mathbf{R}^2} e^{i(w,z)} \mu_2(z)\, dz d\alpha \neq 0$$

which proves the irreducibility of V_g. This completes the proof.

Schwartz's Theorem in spectral synthesis is described in the following:

THEOREM 2. — *Every closed, two-sided invariant subspace of* C(M(2)) *is spanned by the functions as*

$$g(e^{i\alpha}, z) = e^{im\alpha} \mathrm{Q}(\mathrm{Re}\, z , \mathrm{Im} z) e^{i(\lambda, z)}$$

that it contains. ($\lambda \in \mathbf{C}^2$ and Q is polynomial).

Proof. — For $f \in \mathrm{C}(\mathrm{M}(2))$, $f \neq 0$ let V denote the closed subspace generated by the two-sided translates of f. Obviously, f is contained in the closed subspace generated by the functions:

$$e^{im\alpha} \mathrm{P}_m(z) = \int_0^{2\pi} f(e^{i(\alpha+\beta)}, z) e^{-im\beta} d\beta = e^{im\alpha} \int_0^{2\pi} f(e^{i\beta}, z) e^{-im\beta} d\beta$$

where $m \in \mathbf{Z}$.

By [1], each function $e^{im\alpha}\mathrm{P}_m(z)$ is contained in the closed subspace spanned by the functions $e^{im\alpha}\mathrm{Q}(\mathrm{Re}\, z , \mathrm{Im} z) e^{i(\lambda,z)}$ where $\mathrm{Q}(\mathrm{Re}\, z , \mathrm{Im}\, z) e^{i(\lambda,z)}$ is contained in the rotation-invariant and translation-invariant subspace of $\mathrm{C}(\mathbf{R}^2)$ generated by $\mathrm{P}_m(z)$, and hence in the two-sided invariant subspace generated by $\mathrm{P}_m(z)$ which completes the proof of the theorem.

4. One-sided spectral analysis.

One-sided spectral analysis of bounded functions on M(2) was studied in [6].

Notation. — Let Γ_w, $w \in \mathbf{C}$, denote the closed subspace of $\mathrm{C}(\mathbf{R}^2)$ spanned by the functions $e^{i(\lambda_1 x + \lambda_2 y)}$ (of $(x, y) \in \mathbf{R}^2$) where $\lambda_1^2 + \lambda_2^2 = w^2$. For the characterization of right-invariant subspaces of C(M(2)) we state the following:

THEOREM 3. — *Every closed, right-invariant subspace of* C(M(2)) *contains a function as*

$$g(e^{i\alpha}, z) = e^{im\alpha} g_1(z) , \; m \in \mathbf{Z} \; , \; g_1 \neq 0 .$$

Moreover, if $g_1 \notin \Gamma_0$, *then the closed right-invariant subspace generated by* g *contains a function as* $h(e^{i\alpha}, z) = g_2(z)$.

For $g_2 \in \Gamma_w$ *and* $g_1 \in \Gamma_0$ *the closed right-invariant subspaces generated by* g_2 *and by* $e^{im\alpha} g_1(z)$ *are irreducible.*

Proof. – Let $f \in \mathrm{V}$, $f \neq 0$, where V is a closed right-invariant subspace of C(M(2)). Then V contains all functions f^* such that $f^*(e^{i\alpha}, z) = f(e^{i(\alpha+\beta)}, z - e^{i\alpha}w)$ where $\beta \in \mathbf{R}$ and $w \in \mathbf{C}$. Hence, for a suitable $m \in \mathbf{Z}$ the function

$$\int_0^{2\pi} f(e^{i(\alpha+\beta)}, z)\, e^{-im\beta} d\beta = e^{im\alpha} \int_0^{2\pi} f(e^{i\beta}, z)\, e^{-im\beta} d\beta = e^{im\alpha} g_1(z)$$

is non-zero and belongs to V. Suppose that $g_1 \notin \Gamma_0$. Then if g_1 is a polynomial (in $\mathrm{Re}\, z$ and $\mathrm{I}_m z$) which is harmonic on $\mathbf{R}^2$ there exists a function $h \in \mathcal{E}_0(\mathbf{R}^2)$, $h(w) = \mu(r)\, e^{im\theta}$, $\mu \in \mathcal{E}_0^{(r)}(\mathbf{R}^2)$, $w = re^{i\theta}$, such that $g_1 * h \neq 0$.

Hence the function

$$e^{im\alpha} \int_{\mathbf{R}^2} g_1(z - e^{i\alpha}w)\, h(w)\, dw = \int_{\mathbf{R}^2} g_1(z - w)\, h(w)\, dw = g_2(z) \quad (3)$$

is non-zero and belongs to V.

Otherwise, the closed rotation-invariant and translation-invariant subspace generated by g_1 contains a function $e^{i(\lambda, z)}$ where $\lambda = (\lambda_1, \lambda_2) \in \mathbf{C}^2$, $\lambda_1^2 + \lambda_2^2 \neq 0$ [1]. Let $h_1 \in \mathcal{E}_0(\mathbf{R}^2)$, $h_1(w) = \mu_1(r)\, e^{im\theta}$ where

$$\mu_1 \in \mathcal{E}_0^{(r)}(\mathbf{R}^2), \; w = re^{i\theta}, \; \textit{such that} \; \hat{h}_1(\lambda_1, \lambda_2) \neq 0.$$

There exists $\beta \in \mathbf{R}$ such that $h_{1,\beta} * g_1 \neq 0$, where

$$h_{1,\beta}(w) = h_1(e^{i\beta}w) = e^{im\beta} h_1(w).$$

Hence, $h_1 * g_1 \neq 0$ and proceeding as in (3) we complete the proof of the first part of the theorem.

Let V_1 be the closed right-invariant subspace generated by $g_2(z)$ where $g_2 \in \Gamma_{w_0}$ for some $w_0 \in \mathbf{C}$, $w_0 \neq 0$. We may show, as in the proof of Theorem 1, that V_1 contains no functions as $e^{im\alpha} g_1(z)$ where $g_1 \in \Gamma_0$. Suppose now that $g_3 \in \mathrm{V}_1$ where $g_3 \in \Gamma_{w_1}$, $w_1 \in \mathbf{C}$. To derive the irreducibility of V_1 we will show that $g_3 = \mathrm{C} g_2$ for some $\mathrm{C} \in \mathbf{C}$. Let $\{\Phi_n\}$ be a sequence in $\mathcal{E}_0(\mathbf{R}^2)$ such that

$$\int_{\mathbf{R}^2} g_2(z - e^{i\alpha}w)\, \Phi_n(w)\, dw \xrightarrow[\mathrm{C(M(2))}]{} g_3(z).$$

Then we have

$$\frac{1}{2\pi} \int_0^{2\pi} \left[\int_{\mathbf{R}^2} g_2(z - e^{i\alpha}w)\, \Phi_n(w)\, dw \right] d\alpha \xrightarrow[\mathrm{C(M(2))}]{} g_3(z)$$

and

$$\int_{\mathbf{R}^2} g_2(z - e^{i\alpha}w)\, \Phi_n^*(|w|)dw \xrightarrow[\mathrm{C(M(2))}]{} g_3(z) \tag{5}$$

where

$$\Phi_n^*(|w|) = \frac{1}{2\pi} \int_0^{2\pi} \Phi_n(e^{-i\alpha}w)\,d\alpha\,,\ \Phi_n^* \in \mathcal{E}_0^{(r)}\,,\ n = 1\,,2,\dots \tag{6}$$

But for every n we have

$$\int_{\mathbf{R}^2} g_2(z - w)\, \Phi_n^*(|w|)dw = \hat{\Phi}_n^*(w_0)\, g_2(z)\,.$$

Consequently, $g_3 = Cg_2$, as required. Similarly, we verify the irreducibility of the closed right-invariant subspace generated by $g_1(z)\,e^{im\alpha}$ where $g_1 \in \Gamma_0$.

Remark 1. – We don't know whether Theorem 3 characterizes all the irreducible right invariant subspaces as it is not known whether the exponentials are the only functions of $C(\mathbf{R}^n)$, $n > 1$ which generate irreducible translation-invariant subspaces. Whether every translation-invariant subspace of $C(\mathbf{R}^n)$, $n > 1$ contains an irreducible subspace seems to be an open question.

Remark 2. – In view of Theorem 3 the right-sided analogue of Schwartz's Theorem in spectral analysis of continuous functions may be formulated as the following question; does every closed, right-invariant subspace of C(M(2)) contain either a funtion as $e^{im\alpha}g_1(z)$ where $g_1 \in \Gamma_0$, $g_1 \neq 0$ $m \in \mathbf{Z}$, or $g_2(z)$ where $g_2 \in \Gamma_w$, $g_2 \neq 0$, for some $w \in \mathbf{C}$?

Notation. – Let μ_R, $R \geqslant 0$, denote the normalized Lebesgue measure of the circle $\{z : |z| = R\}$. For $f \in C(\mathbf{R}^2)$ let $N_f^{(r)}$ denote the closed subspace spanned by $\{f * \mu_R : R \geqslant 0\}$ and $\tau(f)$ the closed translation-invariant subspace generated by f.

We deduce an equivalent form of the right-sided analogue of Schwartz's Theorem (as formulated in Remark 2).

It is described in

THEOREM 4. – *The following statements are equivalent:*

(i) *The right-sided analogue of Schwartz's Theorem holds for* M(2).

(ii) *Let* $f \in C(\mathbf{R}^2)$, $f \neq 0$. *Then:* (a) If $\tau(f) \cap \Gamma_0 = \{0\}$ *then there exists* $w \in \mathbf{C}$ *such that* $N_f^{(r)} \cap \Gamma_w \neq \{0\}$. (b) *If* $\tau(f) \cap \Gamma_0 \neq \{0\}$ *then, either* $N_f^{(r)} \cap \Gamma_w \neq \{0\}$ *for some* $w \in \mathbf{C}$, *or, there exist* $m \in \mathbf{Z}$, $g \in \Gamma_0$, $g \neq 0$ *and a sequence* $\psi_n \in \mathcal{E}_0^{(r)}(\mathbf{R}^2)$ *such that*

$$f * \phi_n \xrightarrow[C(\mathbf{R}^2)]{} g \tag{7}$$

where $\phi_n(r, \theta) = \psi_n(r) e^{-im\theta}$, $n = 1, 2, \ldots$. *(Here* (r, θ) *are the polar coordinates in* $\mathbf{R}^2$).

Proof. – Suppose that the right-sided analogue of Schwartz's Theorem holds for M(2). Let $f \in C(M(2))$ where $f(e^{i\alpha}, z) = f(z)$. Suppose that $\tau(f) \cap \Gamma_0 = \{0\}$. The closed right-invariant subspace W_f generated by f contains no function as $e^{im\alpha} g(z) \neq 0$ where $g \in \Gamma_0$ and $m \in \mathbf{Z}$. Since, otherwise

$$\int_{\mathbf{R}^2} f(z - e^{i\alpha} w)\, \mu_n(w)\, dw \xrightarrow[C(M(2))]{} e^{im\alpha} g(z)$$

implies for $\alpha = 0$ that: $f * \mu_n \xrightarrow[C(\mathbf{R}^2)]{} g$, a contradiction. Hence, W_f contains a function $g_1(z)$ where $g_1 \in \Gamma_w$, $g_1 \neq 0$. In other words, there exist $\Phi_n \in \mathcal{E}_0(\mathbf{R}^2)$, $n = 1, 2, \ldots$, such that

$$\int_{\mathbf{R}^2} f(z - e^{i\alpha} w)\, \Phi_n(w)\, dw \xrightarrow[C(M(2))]{} g_1(z).$$

Hence, by (5) we have:

$$\int_{\mathbf{R}^2} f(z - w)\, \Phi_n^*(|w|)\, dw \xrightarrow[C(M(2))]{} g_1(z)$$

where Φ_n^* are defined in (6). That is, $g_1 \in N_f^{(r)}$ which yields (ii) (a).

Suppose now that $\tau(f) \cap \Gamma_0 \neq \{0\}$. If $W_f \cap \Gamma_v \neq \{0\}$ for some $v \in \mathbf{C}$ then, as proved above, $N_f^{(r)} \cap \Gamma_v \neq \{0\}$ (here, the functions of Γ_v are looked upon as function on M(2)). Otherwise, the subspace W_f must contain a function as $e^{im\alpha} g_2(z)$ where $g_2 \in \Gamma_0$, $g_2 \neq 0$, and $m \in \mathbf{Z}$. Namely, there exists $\phi_n \in \mathcal{E}_0(\mathbf{R}^2)$ such that

$$\int_{\mathbf{R}^2} f(z - e^{i\alpha} w)\, \phi_n(w)\, dw \xrightarrow[C(M(2))]{} e^{im\alpha} g_2(z).$$

Hence we have

$$\frac{1}{2\pi} \int_0^{2\pi} \left[\int_{\mathbf{R}^2} f(z - \xi)\, \phi_n(e^{-i\alpha} \xi)\, d\xi \right] e^{-im\alpha}\, d\alpha \longrightarrow g_2(z)$$

which yields

$$\frac{1}{2\pi}\int_{\mathbf{R}^2} f(z-\xi)\,\widetilde{\phi}_n(\xi)\,d\xi \longrightarrow g_2(z)$$

where $\widetilde{\phi}_n(\xi) = \widetilde{\psi}_n(r)\,e^{-im\theta}$, $\widetilde{\psi}_n(r) = \int_0^{2\pi} \phi(e^{-i\eta}r)\,e^{-im\eta}d\eta$, $\xi = re^{i\theta}$, and we have shown that (i) implies (ii).

Suppose now that (ii) holds. By Theorem 3 we have to show that for every $f\in \mathrm{C(M(2))}$, $f(e^{i\alpha}, z) = f(z)$, $f \neq 0$, the subspace W_f contains either a function $g(z)$, $g\neq 0$, $g\in\Gamma_w$, or, a function $g(e^{i\alpha}, z) = e^{im\alpha}g_1(z)$ where $g_1\in\Gamma_0$, $g_1\neq 0$ and $m\in\mathbf{Z}$.

Let $f\in \mathrm{C}(\mathbf{R}^2)$, $f\neq 0$ and suppose that $\mathrm{N}_f^{(r)}\cap\Gamma_w\neq\{0\}$ for some $w\in\mathbf{C}$. Then, by definition, there exist $\psi_n\in\mathcal{E}_0^{(r)}(\mathbf{R}^2)$ $n=1,2,\ldots$, and $g\in\Gamma_w$ such that

$$\int_{\mathbf{R}^2} f(z-\xi)\,\psi_n(\xi)\,d\xi \xrightarrow[\mathrm{C}(\mathbf{R}^2)]{} g(z).$$

But we have

$$\int_{\mathbf{R}^2} f(z-e^{i\alpha}\xi)\,\psi_n(\xi)\,d\xi = \int_{\mathbf{R}^2} f(z-\xi)\,\psi_n(\xi)\,d\xi \quad \text{for} \quad n=1,2,\ldots,$$

which implies (i).

Finally, suppose that $\tau(f)\cap\Gamma_0\neq\{0\}$ and that $\mathrm{N}_f^{(r)}\cap\Gamma_w=\{0\}$ for every $w\in\mathbf{C}$. By (ii) (b) we have

$$\int_{\mathbf{R}^2} f(z-e^{i\alpha}w)\,\phi_n(w)\,dw = \int_{\mathbf{R}^2} f(z-\xi)\,\phi_n(e^{-i\alpha}\xi)\,d\xi = e^{im\alpha}\int_{\mathbf{R}^2} f(z-\xi)\,\phi_n(\xi)\,d\xi$$

for $n=1,2,\ldots$, which yields, by (7)

$$\int_{\mathbf{R}^2} f(z-e^{i\alpha}\xi)\,\psi_n(\xi)\,d\xi \xrightarrow[\mathrm{C(M(2))}]{} e^{im\alpha}g(z).$$

This completes the proof.

5. Invariant subspaces of $\mathrm{C}(\mathbf{R}^2)$.

For $f\in\mathrm{C}(\mathbf{R}^2)$ we say that $w\in Sp^{\mathrm{T.R.}}(f)$, $w\in\mathbf{C}$ if the translation-invariant and rotation-invariant subspace generated by f contains an exponential in Γ_w. Actually, the fact announced in [1] that unless $f=0$ we have $Sp^{\mathrm{T.R.}}(f)\neq\phi$ implies the main results of [1] concerning the Pompeiu problem [4, 7]. By Theorem 4, the one-sided Schwartz's Theorem for the motion group is intimately connected to the following problem:

For $f \in C(\mathbf{R}^2)$ we say that $w \in Sp^{(r)}(f)$, $w \in \mathbf{C}$, $w \neq 0$, if $\mathrm{N}_f^{(r)} \cap \Gamma_w \neq \{0\}$, and that $0 \in Sp^{(r)}(f)$ if $\mathrm{N}_f^{(r)} \cap \widetilde{\Gamma}_0 \neq \{0\}$, where $\widetilde{\Gamma}_0$ denotes the space of harmonic functions on $\mathbf{R}^2$. Suppose that $f \neq 0$. Does this imply that $Sp^{(r)}(f) \neq \phi$?

Remark 3. – We notice that for $f \in C(\mathbf{R}^2)$ we have $Sp^{(r)}(f) \subseteq Sp^{\mathrm{T.R.}}(f)$.

Remark 4. – This question is connected to the following problem of Pompeiu type:

Determine for which family $\mathrm{P} \subset \mathrm{M}_0(\mathbf{R}^2)$, the only continuous function f on $\mathbf{R}^2$ such that $\mathrm{T}(f * \mu_{\mathrm{R}}) = 0$ for all $\mathrm{T} \in \mathrm{P}$ and $\mathrm{R} \geqslant 0$, is the zero function.

Let J_n denote the nth Bessel function of the first kind. By definition, we deduce

$$\mathrm{J}_n(r)\, e^{in\theta} = \frac{1}{2\pi i^n} \int_0^{2\pi} e^{ir\cos(\phi-\theta)}\, e^{in\phi}\, d\phi\,.$$

Hence we have $\mathrm{J}_n(wr)\, e^{in\theta} \in \Gamma_w$, $Sp^{(r)}(\mathrm{J}_n(wr)\, e^{in\theta}) = \{w\}$ for $w \in \mathbf{C}$, $w \neq 0$ and $\mathrm{N}_{\mathrm{I}_n}^{(r)}$ is one-dimensional where $\mathrm{I}_n(r, \theta) = \mathrm{J}_n(wr)\, e^{in\theta}$.

A partial answer to the above question is provided by the following result:

THEOREM 5. – *Let $f \in C(\mathbf{R}^2)$, $f \neq 0$ where*

$$f(r, \theta) = \sum_{m=0}^{\mathrm{N}} g_m(r)\, e^{im\theta}\,, \quad g_m \in \mathrm{C}^{(r)}(\mathbf{R}^2) \quad (m = 0, 1, \ldots, \mathrm{N})\,.$$

Then $Sp^{(r)}(f) \neq \phi$. If $0 \notin Sp^{(r)}(f)$ there exist λ, $a_m \in \mathbf{C}$ $(m = 0, 1, \ldots, \mathrm{N})$, $\lambda \neq 0$, where $\sum_{m=0}^{\mathrm{N}} |a_m| > 0$ such that $\sum_{m=0}^{\mathrm{N}} a_m \mathrm{J}_m(\lambda r)\, e^{im\theta}$ belongs to $\mathrm{N}_f^{(r)}$. Moreover, we have

$$Sp^{(r)}(f) = \bigcup_{m=0}^{\mathrm{N}} Sp^{(r)}(g_m(r)\, e^{im\theta})\,.$$

The proof will be accomplished in several lemmas.

LEMMA 6. – Every proper closed ideal in $\hat{\mathcal{E}}'_{(r)}(\mathbf{R}^2)$ has a common zero in $\mathbf{C}^2$.

Proof. – Let J be a proper closed ideal in $\hat{\mathcal{E}}'_{(r)}(\mathbf{R}^2)$ and suppose that the functions in J have no common zeroes. Every $f \in \mathrm{J}$ is a function of $z_1^2 + z_2^2$. That is, there exists an even entire function Q_f of one complex variable such that

$$f(z_1, z_2) = \mathrm{Q}_f(\sqrt{z_1^2 + z_2^2}) \quad \text{and} \quad \mathrm{Q}_f \in \hat{\mathcal{E}}'(\mathbf{R}).$$

Let J^* be the ideal in $\hat{\mathcal{E}}'(\mathbf{R})$ generated by $\{\mathrm{Q}_f : f \in \mathrm{J}\}$.

Obviously, the functions in J^* have no common zeroes. Thus, applying Schwartz's Theorem [5] we deduce that $\mathrm{J}^* = \hat{\mathcal{E}}'(\mathbf{R})$. That is, there exists a sequence $\{\mathrm{P}_n\}$ in J^* converging to 1 in $\hat{\mathcal{E}}'(\mathbf{R})$. Each P_n must be of the form $\sum_{j=1}^{k} \mathrm{T}_j(w)\,\mathrm{S}_j(w)$ where each $\mathrm{T}_j \in \hat{\mathcal{E}}'(\mathbf{R})$ and $\mathrm{S}_j \in \mathrm{J}$. But then the function

$$\sum_{j=1}^{k} \mathrm{T}_j(w)\,\mathrm{S}_j(w) + \sum_{j=1}^{k} \mathrm{T}_j(-w)\,\mathrm{S}_j(-w) = \sum_{j=1}^{k} (\mathrm{T}_j(w) + \mathrm{T}_j(-w))\,\mathrm{S}_j(w)$$

belongs to J since each $\mathrm{T}_j(w) + \mathrm{T}_j(-w)$ belongs to $\hat{\mathcal{E}}'_{(r)}(\mathbf{R}^2)$. Hence, $\mathrm{Q}_n(w) = \dfrac{1}{2}(\mathrm{P}_n(w) + \mathrm{P}_n(-w))$ belongs to J and $\mathrm{Q}_n \longrightarrow 1$ in $\hat{\mathcal{E}}'_{(r)}(\mathbf{R}^2)$, a contradiction.

LEMMA 7. – *Let* $f \in \mathrm{C}(\mathbf{R}^2)$ *where* $f(r, \theta) = g(r)\,e^{im\theta}$, $g \in \mathrm{C}^{(r)}(\mathbf{R}^2)$, $g \neq 0$, $m \in \mathbf{Z}$. *Then* $Sp^{(r)}(f) \neq \phi$. *If* $0 \notin Sp^{(r)}(f)$ *there exists* $\lambda \in \mathbf{C}$, $\lambda \neq 0$, *such that* $\mathrm{H} \in \mathrm{N}_f^{(r)}$ *where*

$$\mathrm{H}(r, \theta) = \mathrm{J}_m(\lambda r)\,e^{im\theta}.$$

Proof. – We may assume that $f \in \mathcal{E}(\mathbf{R}^2)$. Let $\mathrm{M}_f^{(r)}$ denote the closed subspace of $\mathcal{E}(\mathbf{R}^2)$ spanned by $\{f * \mu_{\mathrm{R}} : \mathrm{R} \geqslant 0\}$. For $m \in \mathbf{Z}$ let $\mathcal{E}_m(\mathbf{R}^2)$ denote the closed subspace of functions $s \in \mathcal{E}(\mathbf{R}^2)$ sucht that $s(r, \theta) = h(r)\,e^{im\theta}$. We have $\mathrm{M}_f^{(r)} \subseteq \mathcal{E}_m(\mathbf{R}^2)$.

Let $\mathcal{E}'_m(\mathbf{R}^2) \subset \mathcal{E}'(\mathbf{R}^2)$ denote the dual of $\mathcal{E}_m(\mathbf{R}^2)$.

Let $\mathrm{M}_f^{(r)\perp} = \{\mathrm{T} \in \mathcal{E}'_m(\mathbf{R}^2) : \mathrm{T}(f) = 0\,,\ f \in \mathrm{M}_f^{(r)}\}$.

Every element of $\hat{\mathcal{E}}'_m(\mathbf{R}^2)$ is of the form $p(r)\,e^{im\theta}$ (as a function on $\mathbf{R}^2$). Let $\mathrm{P} = \{p : \hat{\mathrm{T}}(r, \theta) = p(r)\,e^{im\theta}\,,\ \mathrm{T} \in \mathrm{M}_f^{(r)\perp}\}$.

We notice that all functions of P are even or odd depending on m.

Let k be the larger integer such that 0 is a zero of order k for each $p \in \mathrm{P}$. It follows that $\dfrac{p(w)}{w^k}$, $p \in \mathrm{P}$, is an even entire function of w and by complexification of $\dfrac{p(r)}{r^k}$

$$p^*(z_1, z_2) = \frac{p(\sqrt{z_1^2 + z_2^2})}{(z_1^2 + z_2^2)^{k/2}}$$

is an entire function on $\mathbf{C}^2$. The space

$$\mathrm{J}^* = \left\{ p^* : p^*(z_1, z_2) = \frac{p(\sqrt{z_1^2 + z_2^2})}{(z_1^2 + z_2^2)^{k/2}}\,,\ p \in \mathrm{P} \right\}$$

is therefore a closed ideal in $\hat{\mathcal{E}}'_{(r)}(\mathbf{R}^2)$. If $0 \notin Sp^{(r)}(f)$ J^* is a proper ideal.

Hence, by Lemma 6, there exists

$$\lambda^* = (\lambda_1, \lambda_2) \in \mathbf{C}^2\,,\ \lambda_1^2 + \lambda_1^2 = \lambda^2 \neq 0$$

which is a common zero of J^*. Consequently, for each $\mathrm{T} \in \mathrm{M}_f^{(r)\perp}$ we have $\hat{\mathrm{T}}(w) = 0$ where $w = (w_1, w_2) \in \mathbf{C}^2$, $w_1^2 + w_2^2 = \lambda^2$. It follows that $\mathrm{T}(\mathrm{Q}) = 0$ for $\mathrm{T} \in \mathrm{M}_f^{(r)\perp}$ where

$$\mathrm{Q}(x, y) = \frac{1}{2\pi i^m} \int_0^{2\pi} e^{i\lambda_1(x\cos\phi + y\sin\phi)}\, e^{im\phi} d\phi\,.$$

But we have

$$\mathrm{Q}(r, \theta) = \frac{1}{2\pi i^m} \int_0^{2\pi} e^{i\lambda r\cos(\phi - \theta)}\, e^{im\phi} d\phi = \mathrm{J}_m(wr)\, e^{im\theta}\,.$$

Consequently, $\mathrm{Q} \in \mathrm{M}_f^{(r)} \cap \Gamma_\lambda$ which completes the proof.

Notation. – Let $\mathrm{C}(\mathbf{R}^2, \mathbf{C}^{\mathrm{N}})$ denote the space of all continuous functions on $\mathbf{R}^2$ which take values in $\mathbf{C}^{\mathrm{N}}$, with the usual topology. Let $\mathrm{M}_0(\mathbf{R}^2, \mathbf{C}^{\mathrm{N}})$ be the dual of $\mathrm{C}(\mathbf{R}^2, \mathbf{C}^{\mathrm{N}})$, the space of vector-valued measures having compact support. For $f \in \mathrm{C}(\mathbf{R}^2, \mathbf{C}^{\mathrm{N}})$, (resp. $\mu \in \mathrm{M}_0(\mathbf{R}^2, \mathbf{C}^{\mathrm{N}})$) let $(f)_n$ (resp. $(\mu)_n$) denote the nth coordinate of f (resp. μ). For $m = (m_1, m_2, \ldots, m_{\mathrm{N}}) \in \mathbf{Z}^{\mathrm{N}}$ let $\mathrm{B}_{(m)}$ denote the closed subspace of $C(\mathbf{R}^2, \mathbf{C}^N)$ which consists of all functions f where

$$(f)_n(r, \theta) = h_n(r) e^{im_n\theta} \qquad n = 1, 2, \ldots, N.$$

Let $B'_{(m)}$ be the dual of $B_{(m)}$, the space of all $\eta \in M_0(\mathbf{R}^2, \mathbf{C}^N)$ such that $(\eta)_n = \mu_n e^{-im_n\theta}$ where $\mu_n \in M_0^{(r)}(\mathbf{R}^2)$, $n = 1, 2, \ldots, N$. We will use the following equality:

$$(J_k(wr') e^{ik\theta'}) * (\mu(r') e^{im\theta'}) (r, \theta) = \phi(w) J_{k+m}(wr) e^{i(k+m)\theta} \tag{8}$$

where $\mu \in M_0^{(r)}(\mathbf{R}^2)$, $w \in \mathbf{C}$, and $\widehat{\mu(r') e^{im\theta'}}(r, \theta) = \phi(r) e^{im\theta}$. Finally, we notice that $M_0^{(r)}(\mathbf{R}^2)$ acts on $B_{(m)}$ by convolution. Namely, $f \in B_{(m)}$ and $\mu \in M_0^{(r)}(\mathbf{R}^2)$ imply that $f * \mu \in B_{(m)}$.

LEMMA 8. – *Every closed non-trivial subspace of* $B_{(m)}$, *invariant under* $M_0^{(r)}(\mathbf{R}^2)$ *contains an invariant one-dimensional subspace. Moreover, if* $f \in B_{(m)}$ *such that* $\lambda \in Sp^{(r)}((f)_n)$, $\lambda \neq 0$, *for some* n, $1 \leqslant n \leqslant N$, *then the closed subspace spanned by* $\{f * \mu_R : R \geqslant 0\}$ *contains a function* $g \neq 0$, *such that*

$$(g)_n(r, \theta) = a_n J_{m_n}(\lambda r) e^{im_n\theta} \qquad n = 1, 2, \ldots, N.$$

Proof. – By induction on N where the case $N = 1$ is provided by Lemma 7. Let $f \in B_{(m)}$ and suppose that $0 \neq \lambda \in Sp^{(r)}((f)_1)$. Let V_f denote the closed subspace of $B_{(m)}$ spanned by $\{f * \mu_R : R \geqslant 0\}$ and $V_f^{\perp} = \{\eta \in B'_{(m)} : \eta(g) = 0, g \in V_f\}$. We notice that for $\eta \in V_f^{\perp}$ we have:

$$\sum_{n=1}^{N} (g_n(r) e^{im_n\theta}) * (\mu_n e^{-im_n\theta}) = 0 \tag{9}$$

where $(\eta)_n = \mu_n e^{-im_n\theta}$ and $(f)_n = g_n(r) e^{im_n\theta}$, $n = 1, 2, \ldots, N$.

Thus we may assume that there exists $\eta \in V_f^{\perp}$ such that

$$(J_{m_N}(\lambda r) e^{im_N\theta}) * (\mu_N e^{-im_N\theta}) \neq 0. \tag{10}$$

Otherwise, the subspace V_f contains a function g^* such that $(g^*)_n = 0$ for $n = 1, 2, \ldots, N-1$, and $(g^*)_N = J_{m_N}(\lambda r) e^{im_N\theta}$ which completes the proof. To this end, let $h \in B_{(m')}$ where $(h)_n = (f)_n$ for $n = 1, 2, \ldots, N-1$, $m' = (m_1, m_2, \ldots, m_{N-1})$ and $B_{(m')} \subset C(\mathbf{R}^2, \mathbf{C}^{N-1})$. By the induction hypothesis the subspace V_h contains a function $h^* \neq 0$ such that

$$(h^*)_n = b_n \mathrm{J}_{m_n}(\lambda r)\, e^{im_n\theta} \quad \text{for} \quad n = 1,2,\ldots,\mathrm{N}-1\,.$$

That is, there exists a sequence $\{\phi_k\}$, $\phi_k \in \mathrm{M}_0^{(r)}(\mathbf{R}^2)$, such that

$$(g_n(r')\, e^{im_n\theta'} * \phi_k)(r,\theta) \xrightarrow[k\to\infty]{\mathrm{C}(\mathbf{R}^2)} b_n \mathrm{J}_{m_n}(\lambda r)\, e^{im_n\theta} \tag{11}$$

for $n = 1,2,\ldots,\mathrm{N}-1$, where $\sum_{n=1}^{\mathrm{N}-1} |b_n| > 0$. Let $\psi_k \in \mathrm{M}_0^{(r)}(\mathbf{R}^2)$ where

$$\psi_k = \phi_k * \mu_{\mathrm{N}}\, e^{-im_{\mathrm{N}}\theta} * \mu_{\mathrm{N}}\, e^{im_{\mathrm{N}}\theta} \qquad k = 1,2,\ldots,\,.$$

Then by (8), (10) and (11) we obtain:

$$g_n(r)^{im_n\theta} * \psi_k \xrightarrow[k\to\infty]{\mathrm{C}(\mathbf{R}^2)} b_n \mathrm{J}_{m_n}(\lambda r)\, e^{im_n\theta} * \mu_{\mathrm{N}}\, e^{-im_{\mathrm{N}}\theta} * \mu_{\mathrm{N}}\, e^{im_{\mathrm{N}}\theta} = b_n \mathrm{C}_1 \mathrm{J}_{m_n}(\lambda r)\, e^{im_n\theta}$$

for $n = 1,2,\ldots,\mathrm{N}-1$ where $\mathrm{C}_1 \in \mathbf{C}$, $\mathrm{C}_1 \neq 0$.

For $n = \mathrm{N}$ we have by (9) and (8):

$$g_{\mathrm{N}}(r)\, e^{im_{\mathrm{N}}\theta} * \psi_k = g_{\mathrm{N}}(r)\, e^{im_{\mathrm{N}}\theta} * \mu_{\mathrm{N}}\, e^{-im_{\mathrm{N}}\theta} * \phi_k * \mu_{\mathrm{N}}\, e^{im_{\mathrm{N}}\theta} = -\left[\sum_{n=1}^{\mathrm{N}-1} g_n(r)\, e^{im_n\theta} * \mu_n\, e^{-im_n\theta}\right] * \phi_k * \mu_{\mathrm{N}}\, e^{im_{\mathrm{N}}\theta}\,.$$

Hence we obtain

$$g_n(r)\, e^{im_{\mathrm{N}}\theta} * \psi_k \xrightarrow[k\to\infty]{\mathrm{C}(\mathbf{R}^2)} -\left[\sum_{n=1}^{\mathrm{N}-1} b_n \mathrm{J}_{m_n}(\lambda r)\, e^{im_n\theta} * \mu_n\, e^{-im_n\theta}\right] * \mu_{\mathrm{N}}\, e^{im_{\mathrm{N}}\theta} = \mathrm{C}\,\mathrm{J}_0(\lambda r) * \mu_{\mathrm{N}}\, e^{im_{\mathrm{N}}\theta} = \mathrm{C}'\,\mathrm{J}_{\mathrm{N}}(\lambda r)\, e^{im_{\mathrm{N}}\theta}\,.$$

Similarly, we may prove that if $0 \in Sp^{(r)}((f)_n)$ for some n, $1 \leqslant n \leqslant \mathrm{N}$, then V_f contains a function $g \neq 0$ such that:

$$(g)_n(r,\theta) = a_n r^{m_n} e^{im_n\theta} \qquad n = 1,2,\ldots,\mathrm{N}\,.$$

Proof of Theorem 5. – Let $h \in \mathrm{B}_{(m)}$, $\mathrm{B}_{(m)} \subset \mathrm{C}(\mathbf{R}^2, \mathbf{C}^{\mathrm{N}+1})$ where $m = (0,1,\ldots,\mathrm{N})$ and $(h)_n(r,\theta) = g_{n-1}(r)\, e^{i(n-1)\theta}$, $n = 1,2,\ldots,\mathrm{N}+1$, and suppose that $\lambda \in Sp^{(r)}((h)_{k_0})$, $\lambda \neq 0$, for some k_0, $1 \leqslant k_0 \leqslant \mathrm{N}+1$. Then by Lemma 8, there exists a sequence $\{\phi_k\}$, $\phi_k \in \mathrm{M}_0^{(r)}(\mathbf{R}^2)$ $k = 1,2,\ldots,$ such that

$$(g_{n-1}(r')\, e^{i(n-1)\theta'} * \phi_k)(r,\theta) \xrightarrow[k\to\infty]{\mathrm{C}(\mathbf{R}^2)} a_{n-1} \mathrm{J}_{n-1}(\lambda r)\, e^{i(n-1)\theta}$$

for $n = 1, 2, \ldots, \mathrm{N}+1$ where $\sum_{n=0}^{\mathrm{N}} |a_n| > 0$. Hence, we have

$$\left[\left(\sum_{n=0}^{\mathrm{N}} g_n(r')\, e^{in\theta'}\right) * \phi_k\right](r, \theta) \xrightarrow[k \to \infty]{\mathrm{C}(\mathbf{R}^2)} \sum_{n=0}^{\mathrm{N}+1} a_n \mathrm{J}_n(\lambda r)\, e^{in\theta}.$$

If $0 \in \mathrm{Sp}^{(r)}((h)_{k_0})$ then, similarly, $\mathrm{N}_f^{(r)}$ contains $g \in \widetilde{\Gamma}_0$, $g \neq 0$, where $g(r, \theta) = \sum_{n=0}^{\mathrm{N}} b_n r^n e^{in\theta}$. Finally, we may easily prove that $\mathrm{Sp}^{(r)}(f) \subseteq \bigcup_{m=0}^{\mathrm{N}} \mathrm{Sp}^{(r)}(g_m e^{im\theta})$ and the result follows.

Corollary 6. – *Let* $f \in \mathrm{C}(\mathbf{R}^2)$, $f \neq 0$ *where*

$$f(r, \theta) = \sum_{m=0}^{\mathrm{N}} g_m(r)\, e^{im\theta}, \quad g_m \in \mathrm{C}^{(r)}(\mathbf{R}^2) \quad (m = 0, 1, \ldots, \mathrm{N}).$$

Then the translation-invariant closed subspace $\tau(f)$ *generated by* f *contains an exponential function.*

Proof. – If $0 \in \mathrm{N}_f^{(r)}$ then $\tau(f)$ contains a polynomial and hence $1 \in \tau(f)$. Otherwise, by Theorem 5, $g \in \tau(f)$, $g \neq 0$ where:

$$g(r, \theta) = \sum_{m=0}^{\mathrm{N}} a_m \mathrm{J}_m(\lambda r)\, e^{im\theta}$$

for some λ, $a_m \in \mathbf{C}$, $\lambda \neq 0$, $(m = 0, 1, \ldots, \mathrm{N})$.

The subspace $\tau(f)$ contains therefore all the functions h where

$$\begin{aligned} h(x, y) &= (g * \mu)(x, y) \\ &= \mathrm{C} \sum_{m=0}^{\mathrm{N}} a_m \int_{\mathbf{R}^2} \left[\int_0^{2\pi} e^{i\lambda[(x-\alpha)\cos\phi + (y-\beta)\sin\phi]}\, e^{im\phi}\right] d\mu(\alpha, \beta) \\ &= \mathrm{C} \sum_{m=0}^{\mathrm{N}} a_m \int_0^{2\pi} \hat{\mu}(\lambda\cos\phi, \lambda\sin\phi)\, e^{i\lambda(x\cos\phi + y\sin\phi)}\, e^{im\phi} d\phi \end{aligned}$$

for every $\mu \in \mathrm{M}_0(\mathbf{R}^2)$ where $\mathrm{C} \in \mathbf{C}$, $\mathrm{C} \neq 0$.

Thus $\tau(f)$ contains all the functions u where

$$u(x, y) = \sum_{m=0}^{\mathrm{N}} a_m \int_0^{2\pi} s(\phi)\, e^{i\lambda(x\cos\phi + y\sin\phi)}\, e^{im\phi} d\phi$$

for every $s \in \mathrm{C}[0, 2\pi]$, $s(0) = s(2\pi)$. For a sequence $\{s_n\}$ converging to the Dirac mass δ_{ϕ_0} concentrated in ϕ_0 where $\sum_{m=0}^{N} a_m e^{im\phi_0} \neq 0$, we obtain, by passing to the limit, that $v \in \tau(f)$ where

$$v(x, y) = \Big(\sum_{m=0}^{N} a_m e^{im\phi_0}\Big) e^{i(x\lambda\cos\phi_0 + y\lambda\sin\phi_0)}$$

which completes the proof.

Remark 5. – To this end we may introduce the following proof to the fact that every translation-invariant and rotation-invariant closed subspace of $C(\mathbf{R}^2)$ contains an exponential function [1]. Let R_f denote the closed translation-invariant and rotation invariant subspace generated by $f \neq 0$. Then, for a suitable $m \in \mathbf{Z}$ the function g where

$$g(r, \theta) = \int_0^{2\pi} f(r, \theta + \beta) e^{-im\beta} d\beta = e^{im\theta} \int_0^{2\pi} f(r, \beta) e^{-im\beta} d\beta$$
$$= e^{im\theta} f_1(r)$$

is non-zero and belongs to R_f. Let $\mu \in M_0^{(r)}(\mathbf{R}^2)$ where $\mu(f_1) \neq 0$. Hence the function $g_1 = g * (\mu e^{-im\theta})$ is non-zero and belongs to $R_f \cap C^{(r)}(\mathbf{R}^2)$.

By Lemma 6, or by Lemma 7 for $m = 0$, there exists $\lambda \in \mathbf{C}$ such that $J_0(\lambda r) \in R_f$. Arguing as in the proof of Corollary 6, we deduce that R_f contains the exponentials $e^{i(x\lambda\cos\phi + y\lambda\sin\phi)}$ for every $\phi \in \mathbf{R}$ and the result follows.

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