Université Grenoble Alpes

ANNALES DE L'INSTITUT FOURIER

Bo Yang & Fangyang Zheng

Hirzebruch manifolds and positive holomorphic sectional curvature

Tome 69, nº 6 (2019), p. 2589-2634.

<http://aif.centre-mersenne.org/item/AIF_2019__69_6_2589_0>

Ann. Inst. Fourier, Grenoble
69, 6 (2019) 2589-2634

HIRZEBRUCH MANIFOLDS AND POSITIVE HOLOMORPHIC SECTIONAL CURVATURE

by Bo YANG & Fangyang ZHENG (*)

Abstract. — This paper is the first step in a systematic project to study examples of Kähler manifolds with positive holomorphic sectional curvature ($H > 0$). Hitchin proved that any compact Kähler surface with $H > 0$ must be rational and he constructed such examples on Hirzebruch surfaces $M_{2,k} = \mathbb{P}(H^k \oplus 1_{\mathbb{CP}^1})$. We generalize Hitchin's construction and prove that any Hirzebruch manifold $M_{n,k} = \mathbb{P}(H^k \oplus 1_{\mathbb{CP}^{n-1}})$ admits a Kähler metric of $H > 0$ in each of its Kähler classes. We demonstrate that pinching behaviors of holomorphic sectional curvatures of new examples differ from those of Hitchin's which were studied in the recent work of Alvarez–Chaturvedi–Heier. Some connections to previous works on the Kähler–Ricci flow on Hirzebruch manifolds are also discussed.

It seems interesting to study the space of all Kähler metrics of $H > 0$ on a given Kähler manifold. We give higher dimensional examples such that some Kähler classes admit Kähler metrics with $H > 0$ and some do not.

Résumé. — Cet article est la première étape d'un projet d'étude systématique d'exemples de variétés kähleriennes à courbure sectionnelle holomorphe positive ($H > 0$). Hitchin a prouvé que tout surface kählerienne compacte avec $H > 0$ doit être rationnelle et il a construit de tels exemples sur les surfaces de Hirzebruch. Nous généralisons la construction de Hitchin et prouvons que toute variété de Hirzebruch admet une métrique kählerienne avec $H > 0$ dans chacune de ses classes kähleriennes.

Il semble intéressant d'étudier l'espace de toutes les métriques kähleriennes avec $H > 0$ sur une variété kählerienne donnée. Nous donnons des exemples de dimension supérieure tels que certaines classes kähleriennes admettent des métriques kähleriennes avec $H > 0$ et d'autres non.

Keywords: Kähler manifolds, holomorphic sectional curvature, Kähler–Ricci flow.
2010 *Mathematics Subject Classification:* 53C55, 32Q15.
(*) Bo Yang is partially supported by an AMS-Simons Travel Grant and a grant from Xiamen University (0020-ZK1088).
Fangyang Zheng is partially supported by a Simons Collaboration Grant 355557.

1. Introduction

Let (M, J, g) be a Kähler manifold, then one can define the holomorphic sectional curvature of any J-invariant real 2-plane $\pi = \mathrm{Span}\{X, JX\}$ by

$$H(\pi) = \frac{R(X, JX, JX, X)}{\|X\|^4}.$$

It is the Riemannian sectional curvature restricted on any J-invariant real 2-plane ([29, p.165]). Compact Kähler manifolds with positive holomorphic sectional ($H > 0$) form an interesting class of complex manifolds. For example, these manifolds are simply-connected ([42]), and $H > 0$ implies positive scalar curvature ([6]), which further leads to the vanishing of its pluri-canonical ring [30]. In 1975 Hitchin [27] proved that any Hirzebruch surface $M_{2,k} = \mathbb{P}(H^k \oplus 1_{\mathbb{CP}^1})$ admits a Kähler metrics with $H > 0$. Moreover, it is proved in [27] that any rational surface admits a Kähler metric with positive scalar curvature.

There is another positive curvature condition much studied on Kähler manifolds: the so-called (holomorphic) bisectional curvature (See Definition 2.1). Any compact Kähler manifold with positive holomorphic bisectional curvature is biholormophic to $\mathbb{CP}^n$ by Mori [35] and Siu–Yau [39]. Motivated by these results and Hitchin's example, Yau [46] asked if the positivity of holomorphic sectional curvature can be used to characterize the rationality of algebraic manifolds. More precisely, the following question was asked.

Conjecture 1.1 (Yau [47, Problems 67 and 68]). — *Consider a compact Kähler manifold with positive holomorphic sectional curvature, is it unirational? Is it projective? If a projective manifold is obtained by blowing up a compact manifold with positive holomorphic sectional curvature along a subvariety, does it still carry a metric with positive holomorphic sectional curvature? In general, can we find a geometric criterion to distinguish the concept of unirationality and rationality?*

There is some progress on Question 1.1 in recent years. For example, an important criterion on non-uniruledness of projective manifolds in terms of pseudoeffective canonical line bundles was established by Boucksom–Demailly–Păun–Peternell [9]. Heier–Wong [25] applied this result to show that any projective manifold with a Kähler metric with positive total scalar curvature is uniruled. Later, they [26] proved that any projective manifold with a Kähler metric of $H > 0$ is rationally connected. We also refer to [26] and [45] for more results on Kähler or Hermitian manifolds with $H \geqslant 0$.

In this paper, we focus on examples of Kähler metrics with $H > 0$. More specifically, we want to carry out a detailed study of such metrics on any Hirzebruch manifold $M_{n,k} = \mathbb{P}(H^k \oplus 1_{\mathbb{CP}^{n-1}})$. We are partly motivated by the work of Chen–Tian ([17] and [18]), where they proved that the space of all Kähler metrics with positive bisectional curvature on $\mathbb{CP}^n$ is path-connected.

Conjecture 1.2. — *Given any Hirzebruch manifold $M_{n,k} = \mathbb{P}(H^k \oplus 1_{\mathbb{CP}^{n-1}})$, what can we say about the space of all Kähler metrics with $H > 0$? Is it path-connected?*

As a first step to answer Conjecture 1.2, we prove the following result.

Theorem 1.3. — *Given any Hirzebruch manifold $M_{n,k}$, there exists a Kähler metric of $H > 0$ in each of its Kähler classes.*

We refer interested readers to Chapter 4 of Griffiths–Harris [22] for a detailed discussion on rational surfaces. In general dimensions, a Hirzebruch manifold $M_{n,k}$ is defined to be the projective bundle associated to the rank-2 vector bundle $H^k \oplus 1_{\mathbb{CP}^{n-1}}$, where H being the hyperplane bundle and $1_{\mathbb{CP}^{n-1}}$ the trivial bundle over $\mathbb{CP}^{n-1}$. Therefore the natural projection $\pi : M_{n,k} \to \mathbb{CP}^{n-1}$ realizes $M_{n,k}$ as the total space of $\mathbb{CP}^1$-bundle over $\mathbb{CP}^{n-1}$. Note that $M_{n,k}$ can also be described as $\mathbb{P}(H^{-k} \oplus 1_{\mathbb{CP}^{n-1}})$. Let E_0 denote the divisor in $M_{n,k}$ corresponding to the section $(0,1)$ of $H^{-k} \oplus 1_{\mathbb{CP}^{n-1}}$, E_∞ the divisor in $M_{n,k}$ corresponding to the section $(0,1)$ of $H^k \oplus 1_{\mathbb{CP}^{n-1}}$, and F the divisor corresponding to the pull-back line bundle $\pi^* H$ over $M_{n.k}$[(1)]. Then the Picard group of $M_{n,k}$ is generated by the divisors E_0 and F, while $E_\infty = E_0 + kF$. The integral cohomology ring of $M_{n,k}$ is

$$\mathbb{Z}[F, E_0]/\langle F^n, E_0^2 + kE_0 F\rangle.$$

The anti-canonical class of $M_{n,k}$ can be expressed as

$$K^{-1}_{M_{n,k}} = 2E_\infty - (k-n)F = \frac{n+k}{k} E_\infty - \frac{n-k}{k} E_0,$$

and every class α in the Kähler cone of $M_{n,k}$ can be expressed as

$$\alpha = \frac{b}{k}[E_\infty] - \frac{a}{k}[E_0] \tag{1.1}$$

for any $b > a > 0$. We refer readers to [11] and [40] for these standard facts on Hirzebruch manifolds.

(1) Note that we follow the notations of E_0 and E_∞ as in [11, p. 278] (see also the introduction of [40]), which are different with those in [22, p. 517].

Hitchin's examples of Kähler metrics with $H > 0$ on $M_{2,k}$ ([27]) were motivated by a natural choice of Kähler metrics on any projective vector bundles over compact Kähler manifolds. Namely let $\pi : (E, h) \to (M, g)$ be any Hermitian vector bundle over a compact Kähler manifold. The Chern curvature form $\Theta(\mathcal{O}_{\mathbb{P}(E)}(1))$ of $\mathcal{O}_{\mathbb{P}(E)(1)}$ over $\mathbb{P}(E)$ has the fiber direction components given by the Fubini–Study form, hence is positive. Therefore

$$\tilde{\omega} = \pi^*\omega_g + s\sqrt{-1}\Theta(\mathcal{O}_{\mathbb{P}(E)}(1)) \tag{1.2}$$

is a well-defined Kähler metric on $P(E)$ when $s > 0$ is sufficiently small. Given any Hirzebruch surface $M_{2,k}$, one picks $(E, h) = (H^k \oplus 1_{\mathbb{CP}^1}, h)$ and (M, g) as $(\mathbb{CP}^1, g_{FS})$ where g_{FS} is the standard Fubini–Study metric and h the induced metric. It was proved in [27] and [1] that the resulting Kähler metric satisfies $H > 0$ if and only if $0 < s < \frac{1}{k^2}$. We note that Hitchin's examples can only exist on a proper open set of Kähler cone of $M_{2,k}$. For example, on $M_{2,1}$, by a scaling his examples lie in Kähler class $b[E_\infty] - aE_0$ where $a < b < 2a$. Our Theorem 1.3 establishes the existence of Kähler metric of $H > 0$ in each of Kähler class of any $M_{n,k}$.

To prove Theorem 1.3, we follow Calabi's ansatz ([10, 11]). The crucial observation (pointed out in [11, p. 279]) that the group of holomorphic transformations of $M_{n,k}$ contains $U(n)/Z_k$ as its maximal compact group. Therefore it is natural to study Kähler metrics with $U(n)$-symmetry. Calabi's ansatz has found wide applications on the study of special Kähler metrics, including Kähler–Einstein metrics, Kähler–Ricci solitons, Kähler metrics with constant scalar curvature, extremal Kähler metrics, etc.. See for example [3, 10, 11, 12, 19, 20, 28, 31, 32, 33, 38] (this list is by no means exhaustive). Precisely we follow the method of Koiso–Sakane [32] to prove Theorem 1.3. It turns out that for Hirzebruch manifolds $M_{n,k}$ their approach is equivalent to Calabi's.

In general, given any $\mathbb{C}^*$ bundle $\pi : L^* \to M$ obtained by a Hermitian line bundle (L, h), we may consider the following metric on the total space of L^*:

$$\tilde{g} = \pi^* g_t + \mathrm{d}t^2 + (\mathrm{d}t \circ \tilde{J})^2, \tag{1.3}$$

where g_t is a continuous family of Kähler metrics on the base (M, J), t is a function which only depends on the norm of Hermitian metric h, and $\tilde{J}$ the complex structure on the total space of L. Koiso–Sakane [32] gave a sufficient condition such that the resulting metric $\tilde{g}$ is Kähler and studied the compactification of such metrics. They applied this method to construct new examples of non-homogeneous Kähler–Einstein metrics in [32].

Let us choose $H^{-k} \to \mathbb{CP}^{n-1}$ as $L \to M$. After a suitable reparametrization, one can show that $\tilde{g}$ defined in (1.3) can be compactified to produce smooth Kähler metrics on $M_{n,k}$ if one can find a single-variable function ϕ with suitable boundary conditions. Let $(M_{n,k}, \tilde{g})$ denote the resulting metric for notational simplicity and the corresponding ϕ a generating function. It can be shown that the curvature tensors of $(M_{n,k}, \tilde{g})$ are completely determined by three components in terms of ϕ, ϕ', and ϕ''. Here we make a crucial use of the $U(n)$-isometric action on $(M_{n,k}, \tilde{g})$.

(1) A which is the holomorphic sectional curvature along the fiber direction F,
(2) B which is the bisectional curvature along the fiber and any direction in the base E_0,
(3) C which is the holomorphic sectional curvature along E_0.

We observe that Hitchin's example is canonical among all Kähler metrics with $H > 0$ on $M_{n,k}$ in the following sense:

Proposition 1.4. — *Hitchin's examples can be uniquely characterized as $U(n)$-invariant Kähler metrics on $M_{n,k}$ which have the constant curvature component A.*

In the level of the generating function $\phi(U)$ each of Hitchin's example corresponds to some quadratic even function defined on $[-c, c]$ with $0 < c < \frac{n}{k(2k+1)}$. Indeed, the boundary conditions of the generating function ϕ reflects the Kähler class of the resulting metric $\tilde{g}$. To construct Kähler metrics with $H > 0$ in each of the Kähler class of $M_{n,k}$ is equivalent to find examples of generating functions $\phi(U)$ defined on $[-c, c]$ for any $c \in (0, \frac{n}{k})$ and yet satisfying some differential inequalities and suitable boundary behaviors. We refer to Proposition 3.7 for a precise statement. To prove Theorem 1.3 we have to construct such $\phi(U)$ by establishing some quite delicate estimates on even polynomials with large degrees (See Proposition 3.17).

It is natural to study pinching behaviors of H on a compact Kähler manifolds with $H > 0$. We refer to Definition 2.2 for local and global pinching constants for H.

Proposition 1.5 (Alvarez–Chaturvedi–Heier [1]). — *The local and global pinching constants of holomorphic sectional curvature are the same for any of the Hitchin's examples on $M_{2,k}$. The maximum among them is $\frac{1}{(2k+1)^2}$ and the ray of the corresponding Kähler classes is $b[E_\infty] - aE_0$ of the slope $\frac{b}{a} = \frac{2k+2}{2k+1}$.*

We show that the conclusion of Theorem 1.5 is not always true for other Kähler metrics with $U(n)$-symmetry and with $H > 0$, which again reflects the specialness of Hitchin's examples.

Proposition 1.6. — *There exist Kähler metrics with $H > 0$ on $M_{n,k}$ whose local and global pinching constants for holomorphic sectional curvature are not equal.*

In general, the local holomorphic pinching constant of any $U(n)$-invariant Kähler metric on $M_{n,k}$ is bounded from above by $\frac{1}{k^2}$.

We also show that the same conclusion as in Proposition 1.5 on $M_{2,k}$ holds on $M_{n,k}$. In other words, the optimal pinching constant $\frac{1}{(2k+1)^2}$ for Hitchin's examples is dimension free. It seems quite complicated to determine the optimal holomorphic pinching constant among all $U(n)$-invariant Kähler metrics on $M_{n,k}$, though we believe it is achieved exactly by Hitchin's examples. In any case, we would like to propose the following question:

Conjecture 1.7. — *If a compact Kähler surface with $H > 0$ has its local pinching constant $\lambda > \frac{1}{9}$, then it must be biholomorphic to $\mathbb{CP}^2$ or $\mathbb{CP}^1 \times \mathbb{CP}^1$.*

As a partial evidence on Conjecture 1.7, we give a complete classification of compact Kähler manifolds with local holomorphic pinching constant $\lambda \geqslant \frac{1}{2}$. We refer to Proposition 2.6 for a complete statement, and just point out that in the case of Kähler surfaces, they are biholomorphic to $\mathbb{CP}^2$ or $\mathbb{CP}^1 \times \mathbb{CP}^1$.

Getting back to Conjecture 1.2. we have the following result as a corollary of the proof of Theorem 1.3.

Corollary 1.8. — *The space of all $U(n)$-invariant Kähler metrics of $H > 0$ on $M_{n,k}$ is path-connected.*

Without the assumption of $U(n)$-symmetry, a general answer to Conjecture 1.2 is still open. In the meantime it seems impossible to make use of the path constructed in Corollary 1.8 to study the optimal holomorphic pinching constants. In the other direction, motivated by [17, 18], one may wonder if the Kähler–Ricci flow can be used to study the space of all Kähler metrics of $H > 0$. To that end, we study the holomorphic sectional curvatures for the Kähler–Ricci shrinking solitons on Fano Hirzebuch manifold $M_{n,k}(k < n)$ constructed by Koiso [31] and Cao [12], and the complete noncompact Feldman–Ilmanen–Knopf (F-I-K) shrinking solitons on the total space of $H^{-k} \to \mathbb{CP}^{n-1}$ with $k < n$ in [20].

Our calculation suggests that the Cao–Koiso shrinking soliton on $M_{n,k}$ admits $H>0$ as the ratio $\frac{n}{k}$ is sufficiently large. However, in the noncompact case, we find that the F-I-K shrinking solitons do not have $H>0$ if $k<n\leqslant k^2+2k$. In fact we expect that none of F-I-K shrinking solitons satisfy $H>0$. A more ambitious question is that whether any complete Kähler–Ricci soliton with $H>0$ must be compact, in view of [36, 37]. Combined with the previous works on Kähler–Ricci flow on Hirzebruch manifolds ([21, 24, 40, 48]), we conclude:

Corollary 1.9. — *$H>0$ is not preserved under the Kähler–Ricci flow.*

Another generalization of Hitchin's examples was studied in a recent work [2], it was proved that the projectivization $\mathbb{P}(E)$ of any Hermitian vector bundle E over a compact Kähler manifold with $H>0$ also admits a Kähler metric of $H>0$. The resulting metric on $\mathbb{P}(E)$ is of the form (1.2) for s sufficiently small. Instead of working with the line bundle H^{-k} on $\mathbb{CP}^{n-1}$, it is possible to apply the method of Koiso–Sakane developed in the proof of Theorem 1.3 to get more examples of Kähler metrics of $H>0$ on some $\mathbb{CP}^k$ bundles. For example, consider $M=\mathbb{CP}^{n_1-1}\times\mathbb{CP}^{n_2}$ and $L=\pi_1^*H_1^{-1}\otimes\pi_1^*H_2^{-k_2}$ where H_1 and H_2 are hyperplanes bundles on $\mathbb{CP}^{n_1-1}$ and $\mathbb{CP}^{n_2}$, π_1 and π_2 are projections to its factors. Then we can produce a $\mathbb{CP}^{n_1}$-bundle over $\mathbb{CP}^{n_2}$ as a suitable compactification of $L^*\to M$. It seems interesting to study the space of all Kähler metrics of $H>0$ on these manifolds.

As a further study of Conjecture 1.2 and Conjecture 1.7, we also prove:

Proposition 1.10. — *Let M be the hypersurface in $\mathbb{CP}^n\times\mathbb{CP}^n$ defined by $\sum_{i=1}^{n+1} z_iw_i=0$ equipped with the restriction of the product of the Fubini–Study metric, where z, w are the homogeneous coordinates. Then the holomorphic pinching constant of M is $\frac{1}{4}$.*

Consider N which is a smooth bidegree $(p,1)$ hypersurface in $\mathbb{CP}^r\times\mathbb{CP}^s$ where $r,s\geqslant 2$, $p\geqslant 1$, and $p>r+1$, then some Kähler classes of N admit Kähler metrics of $H>0$ and some do not.

Note that when $n=2$, the bidegree $(1,1)$ hypersurface in the first part of Proposition 1.10 is exactly the flag 3-fold with the its canonical Kähler–Einstein metric. One may wonder:

Conjecture 1.11. — *Is it true that if a compact Kähler 3-fold with $H>0$ has its local holomorphic pinching constant $\lambda>\frac{1}{4}$, then it must be biholomorphic to a compact Hermitian symmetric space?*

This paper is organized as follows: In Section 2, we prove the classification theorem of compact Kähler manifolds with local holomorphic pinching constant $\lambda \geqslant \frac{1}{2}$. In Section 3, we prove the main Theorem 1.3 and discuss the relation between Kähler–Ricci flow and $H > 0$ on $M_{n,k}$. In Section 4, we consider the canonical Kähler–Einstein metric on the flag 3-fold and prove Proposition 1.10. We end the paper with some general discussions on $H > 0$ in the higher dimensions the submanifold point of view.

Acknowledgments. We would like to thank Gordon Heier for helpful suggestions and a few corrections on some inaccuracies on a preliminary version of this paper, and Bennett Chow for his interests. Bo Yang is grateful to Xiaodong Cao for his encouragement, the Math Department at Cornell University for the excellent working condition, as well as Bin Guo, Zhan Li, and Shijin Zhang for helpful discussions. We are very thankful to the anonymous referee for carefully reading an earlier version of the paper and making many constructive comments. Their patience and professionalism is much appreciated.

2. Holormophic sectional curvature: preliminary results

Let us begin the definition of various curvatures on a Kähler manifold.

Definition 2.1. — *Let (M, g, J) be a Kähler manifold of complex dimension $n \geqslant 2$ with it Riemannian curvature tensor R.*

(1) *Sectional curvature for any real 2-plane $\pi \subset T_p(M)$ is defined by $K(\pi) = \frac{R(X,Y,Y,X)}{|X|^2|Y|^2 - g(X,Y)^2}$ where $\pi = \operatorname{span}\{X, Y\}$.*
(2) *Holomorphic sectional curvature (H) for any J-invariant real 2-plane $\pi \subset T_p(M)$ is defined by $H(\pi) = \frac{R(X,JX,JX,X)}{|X|^4}$ where $\pi = \operatorname{span}\{X, JX\}$. For the sake of conveinience, we freely use $H(X)$, $H(X - \sqrt{-1}JX)$ or $H(\pi)$ for its holomorphic sectional curvature.*
(3) *(Holomorphic) bisectional curvature for any two J-invariant real 2-planes $\pi, \pi' \subset T_p(M)$ is defined by $B(\pi, \pi') = \frac{R(X,JX,JY,Y)}{|X|^2|Y|^2}$ where $\pi = \operatorname{span}\{X, JX\}$ and $\pi' = \operatorname{span}\{Y, JY\}$.*

In the study of Kähler manifolds with positive curvature, it is useful to consider various curvature pinching conditions in either a local or a global sense.

Definition 2.2 (Local pinching and global pinching). — *Let $\lambda, \delta \in (0,1)$, we define the following pinching conditions on a Kähler manifold (M, g).*

(1) *$\lambda \leqslant H \leqslant 1$ in the local sense if for any $p \in M$, $0 < \lambda H(\pi') \leqslant H(\pi) \leqslant \frac{1}{\lambda} H(\pi')$ for any J-invariant real 2-planes $\pi, \pi' \subset T_p(M)$. In other words, there exists a function $\varphi(p) > 0$ on M^n such that $0 < \lambda\varphi(p) \leqslant H(p, \pi) \leqslant \frac{1}{\lambda}\varphi(p)$ for any p and any holomorphic plane $\pi \subset T_p(M)$.*

(2) *$\delta \leqslant K \leqslant 1$ in the local sense if for any $p \in M$, $0 < \delta K(\pi') \leqslant K(\pi) \leqslant \frac{1}{\delta} K(\pi')$ for any two real 2-planes $\pi, \pi' \subset T_p(M)$.*

(3) *$\lambda \leqslant H \leqslant 1$ in the global sense if $\lambda \leqslant H(\pi) \leqslant 1$ for any $p \in M$ and any J-invariant real 2-plane $\pi \subset T_p(M)$. $\delta \leqslant K \leqslant 1$ in the global sense is defined similarly.*

Compact Kähler manifolds with $H > 0$ are less understood and somewhat mysterious. For example, if one works with linear algebra aspects of curvature tensors, then $H > 0$ alone does not give much information on the Ricci curvature. In fact, any Hirzebruch surface $M_{2,k}$ ($k \geqslant 2$) are not Fano, thus do not admit any Kähler metric with positive Ricci curvature. Nonetheless one may study Kähler manifolds with $H > 0$ pinched by a large constant. In this regard, the following results of Berger [4] and Bishop–Goldberg [7] are very interesting.

Proposition 2.3 (Berger [4]). — *Let (M^n, g) be Kähler, then $0 < \lambda \leqslant H \leqslant 1$ in the local sense implies $\frac{7\lambda - 5}{8} \leqslant K \leqslant \frac{4-\lambda}{3}$ in the local sense.*

Proposition 2.4 (Bishop–Goldberg [7]). — *If (M^n, g) is Kähler, then $0 < \lambda \leqslant H(p) \leqslant 1$ implies*

$$\frac{1}{4}[3(1 + \cos^2\theta)\lambda - 2] \leqslant K(X, Y) \leqslant 1 - \frac{3}{4}\lambda \sin^2\theta$$

for any unit tangent vectors X, Y at p with $g(X, Y) = 0$ and $g(X, JY) = \cos\theta$. In particular, λ-holomorphic pinching implies $\frac{1}{4}(3\lambda - 2)$-pinching on sectional curvatures.

In the proof of the above Proposition 2.3, Berger discovered an interesting inequality.

Lemma 2.5 (Berger [4, 5]). — *Let (M^n, g) be a Kähler manifold and $0 < \lambda \leqslant H \leqslant 1$ in the local sense, then for any unit vector X, Y with $g(X, Y) = 0$ and $g(X, JY) = \cos\theta$, we have*

$$\lambda - \frac{1}{2} + \frac{\lambda}{2}\cos^2\theta \leqslant R(X, JX, JY, Y) \leqslant 1 - \frac{\lambda}{2} + \frac{1}{2}\cos^2\theta. \tag{2.1}$$

For the convenience of the readers, we sketch Berger's proof of Lemma 2.5, as it will be crucial in the proof of Proposition 2.6 below.

Berger's proof of Lemma 2.5. — Given any unit vector X, Y with $g(X,Y) = 0$ and $g(X, JY) = \cos\theta$, consider

$$\lambda \leqslant \frac{1}{2}[H(aX+bY) + H(aX-bY)] \leqslant 1 \tag{2.2}$$

By the left half of inequality (2.2), we conclude that

$$\begin{aligned}(H(X)-\lambda)a^4 + (R(X,JX,JY,Y)\\ + 2R(X,JY,JY,X) - \lambda)2a^2b^2 + (H(Y)-\lambda)b^4 \geqslant 0\end{aligned} \tag{2.3}$$

holds for any real numbers a, b. Apply $H(X), H(Y) \leqslant 1$, it follows from (2.3) that

$$R(X,JX,JY,Y) + 2R(X,JY,JY,X) \geqslant 2\lambda - 1. \tag{2.4}$$

Next consider

$$\lambda \leqslant \frac{1}{2}[H(aX+bJY) + H(aX-bJY)] \leqslant 1. \tag{2.5}$$

Since

$$\frac{1}{2}\{(a^2+b^2+2ab\cos\theta)^2 + (a^2+b^2-2ab\cos\theta)^2\} = (a^2+b^2)^2 + 4a^2b^2\cos^2\theta,$$

a similar argument as in (2.3) and (2.4) leads to

$$3R(X,JX,JY,Y) - 2R(X,JY,JY,X) \geqslant 2\lambda + 2\lambda\cos^2\theta - 1. \tag{2.6}$$

By adding (2.4) and (2.6) we have

$$R(X,JX,JY,Y) \geqslant \lambda + \frac{\lambda}{2}\cos^2\theta - \frac{1}{2}. \tag{2.7}$$

The right half of inequality (2.1) can be proved similarly if we work on the right halves of inequalities in both (2.2) and (2.5). □

It is possible to get some characterization of Kähler manifolds with a large holomorphic pinching constant λ. For example, Bishop–Goldberg [7] proved that if $\frac{4}{5} < \lambda \leqslant H \leqslant 1$ holds in the local sense on a compact Kähler manifold (M,g), then M has the homotopy type of $\mathbb{CP}^n$. They also proved in [8] that $\lambda > \frac{1}{2}$ implies $b_2(M) = 1$. Note that a direct calculation shows that $\mathbb{CP}^k \times \mathbb{CP}^l$ with the product of Fubini–Study metric has exactly $\frac{1}{2} \leqslant H \leqslant 1$ (see [1] for a general result on holomorphic pinching of product metrics). In light of these results, it is natural to ask if $\frac{1}{2} < \lambda \leqslant H \leqslant 1$ in the local sense implies that M^n is biholomorphic to $\mathbb{CP}^n$. This is indeed the case and we have the following:

Proposition 2.6. — *Let (M^n, g) be a compact Kähler manifold with $0 < \lambda \leqslant H \leqslant 1$ in the local sense, then the following holds:*

(1) *If $\lambda > \frac{1}{2}$, then M^n is bibolomorphic to $\mathbb{CP}^n$.*
(2) *If $\lambda = \frac{1}{2}$, then M^n is one of the following:*
   (a) *M^n is biholomorphic to $\mathbb{CP}^n$,*
   (b) *M^n is holomorphically isometric to $\mathbb{CP}^k \times \mathbb{CP}^{n-k}$ with a product of Fubini–Study metrics. Moreover, each factor must have the same constant H,*
   (c) *M^n is holomorphically isometric to an irreducible compact Hermitian symmeric space of rank 2 with its canonical Kähler–Einstein metric.*

Proof of Proposition 2.6. — Let us consider $n \geqslant 2$, the crucial observation is that $\frac{1}{2} \leqslant \lambda \leqslant 1$ in the local sense implies that (M^n, g) has nonnegative orthogonal holomorphic bisectional curvature. Namely for any two J-invariant planes $\pi = \operatorname{span}\{X, JX\}$ and $\pi' = \operatorname{span}\{Y, JY\}$ in $T_p(M)$ which are orthogonal in the sense that $g(X, Y) = g(X, JY) = 0$, then

$$R(X, JX, JY, Y) \geqslant 0.$$

This follows from Berger's inequality (2.1).

Nonnegative and positive orthogonal bisectional curvature is well studied in [14, 16, 23, 43]. Now if $\lambda > \frac{1}{2}$ then (M^n, g) has positive orthogonal bisectional curvature, it is proved in [16, 23, 43] that the Kähler–Ricci flow evolves such a metric to positive bisectional curvature, which is biholomorphic to $\mathbb{CP}^n$ by [35] and Siu–Yau [39].

If $\lambda = \frac{1}{2}$ then (M^n, g) has nonnegative orthogonal bisectional curvature, according to a classification result due to Gu–Zhang ([23, Theorem 1.3]), noting that $H > 0$ implying simply-connectedness, then (M, g) is holomorphically isometric to

$$(\mathbb{CP}^{k_1}, g_{k_1}) \times \cdots \times (\mathbb{CP}^{k_r}, g_{k_r}) \times (N^{l_1}, h_{l_1}) \times \dots (N^{k_r}, h_{l_s}). \tag{2.8}$$

Next we consider (N^{l_i}, h_{l_i}) each of which is a compact irreducible Hermitian symmetric spaces of rank $\geqslant 2$ with its canonical Kähler–Einstein metric, the holomorphic pinching constant of such a metric was well-studied and it is exactly the reciprocal of its rank, see for example [15]. We conclude that if there are only one factor in the decomposition (2.8), then M is either holomorphic to $\mathbb{CP}^n$ or holomorphically isometric to an irreducible Hermitian symmetric space of rank 2 with its Kähler–Einstein metric.

The remaining case where there are more than one factor in the decomposition (2.8) will follow from the holomorphic pinching of product Kähler metrics. Indeed, fix a point $(p_1, p_2) \in M = M_1 \times M_2$, if we consider a special case that each factor metric of M_1 and M_2 is normalized so that $\lambda_i \leqslant H(p_i)(g_i) \leqslant 1$ for $i = 1$ and 2, it is proved in [1] that the local pinching constant at (p_1, p_2) of a product Kähler metric $(M_1 \times M_2, g_1 \times g_2)$ is $\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}$. It is clear that $\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2} = \frac{1}{2}$ is equivalent to $\lambda_1 = \lambda_2 = 1$. In the general case of $\frac{1}{2}$-pinching in the local sense, a similar argument as in [1] can be used to show there have to be exactly two factors and each factor must have the same constant H value at p_1 and p_2. Since $p_1 \in M_1$ and $p_2 \in M_2$ can be chosen arbitrarily, we conclude that both of them are holomorphically isometric to complex projective spaces with their Fubini–Study metrics having the same H value. □

A natural question following Proposition 2.6 is what is the next threshold, if any, for the holomorphic pinching constants for Kähler manifolds with $H > 0$. In general, the situation might be complicated. Note that the canonical Kähler–Einstein metric on a compact Hermitian symmetric space has holomorphic pinching constant determined by its rank ([15]). The Kähler–Einstein metrics on a lot of the Kähler C-spaces also satisfy $H > 0$, and in general one has to work with the corresponding Lie algebra carefully to determine its holomorphic pinching constant. Nonetheless, in this paper we focus on the case of dimension 2 and 3, we will see in Section 3 and 4 that Hirzebruch surfaces and the flag 3-space might be the right objects to provide the next interesting threshold for the holomorphic pinching.

3. Kähler metrics with $H > 0$ on Hirzebruch manifolds

In this section we first review Hitchin's examples on Hirzebruch surfaces $M_{2,k}$, then we prove the main Theorem 1.3 and study the relation between the Kähler–Ricci flow and $H > 0$.

3.1. A review of Hitchin's construction

Hitchin [27] proved that any compact Kähler surface with positive sectional curvature is rational. Any rational surface can be obtained by blowing up points on $\mathbb{CP}^2$, $\mathbb{CP}^1 \times \mathbb{CP}^1$, and Hirzebruch surfaces $M_{2,k}$. The natural question is which rational surface admits Kähler metric with $H > 0$. In this regard, Hitchin proved that any Hirzebruch surface $M_{2,k}$ admits a Hodge metric of $H > 0$. Moreover, it is proved ([27, Corollary 5.18]) that the blow up of any compact Kähler manifold with positive scalar curvature admits a Kähler metric with positive scalar curvature when the complex dimension $n \geqslant 2$. As a corollary, any rational surface admits a Kähler metric with positive scalar curvature. However his construction [27] can not be generalized to produce Kähler metrics with $H > 0$ in a direct way. Therefore, it is an interesting open question whether there is a Kähler metric with $H > 0$ on $\mathbb{CP}^2$ with two points blown up.

In general, given any Hermitian vector bundle $(E, h) \to (M, g)$ where (M, g) is a compact Kähler manifold, the Chern curvature form $\Theta(\mathcal{O}_{\mathbb{P}(E)}(1))$ of $\mathcal{O}_{\mathbb{P}(E)(1)}$ over $\mathbb{P}(E)$ has the fiber direction component given by the Fubini–Study form, hence is positive. Therefore

$$\tilde{\omega} = \pi^*\omega_g + s\sqrt{-1}\Theta(\mathcal{O}_{\mathbb{P}(E)}(1)) \tag{3.1}$$

is a well-defined Kähler metric on $P(E)$ when $s > 0$ is sufficiently small.

Hitchin [27] studied Kähler metrics of the from (3.1) on Hirzebruch surfaces $M_{2,k}$, Here we pick $(E, h) = (H^k \oplus 1_{\mathbb{CP}^1}, h)$ and (M, g) as $(\mathbb{CP}^1, g_{FS})$ where g_{FS} is the standard Fubini–Study metric and h the induced metric. If we use the local parametrization $(z_1, (dz_1)^{-\frac{k}{2}}, z_2)$ and write down the metric locally

$$\tilde{\omega} = \sqrt{-1}\partial\bar{\partial}\log(1 + |z_1|^2) + s\sqrt{-1}\partial\bar{\partial}\log[(1 + |z_1|^2)^k + |z_2|^2]. \tag{3.2}$$

In this case, since the vector bundle $H^k \oplus 1_{\mathbb{CP}^1}$ has nonnegative curvature, the component of the Chern curvature form $\Theta(\mathcal{O}_{\mathbb{P}(E)}(1))$ along the base direction is nonnegative, so $\tilde{\omega}$ is in fact a Kähler metric for all $s > 0$.

It is proved in [1, 27] that $\tilde{\omega}$ has $H > 0$ as long as $s < \frac{1}{k^2}$. Let us define the optimal local (global) holomorphic pinching constant to be the maximum value among all the local (global) pinching constants in Definition 2.2. It was shown in [1] that the optimal local and global holomorphic pinching constants of Hitchin's examples are the same and equal to $\frac{1}{(2k+1)^2}$ and the corresponding $s = \frac{1}{2k^2+k}$. Note that this s value corresponds to Kähler class $b[E_\infty] - aE_0$ where $b = \frac{2k+2}{2k+1}a > 0$. In particular, if $k = 1$, then $s = \frac{1}{3}$, the corresponding Kähler metric $\tilde{\omega}$ is not in the anti-canonical class of $2\pi c_1(M_{2,1})$.

Let us rephrase the question we proposed in Section 1 of this paper.

Question. — *Hitchin's examples produce a family of Kähler metrics with $H > 0$ whose Kähler classes only stay in a subset of the Kähler cone of $M_{2,k}$. Indeed, this subset does not approach a piece of the essential* *boundary of the Kähler cone ($b[E_\infty] - aE_0$ where $a \to 0+$). Here by "essential" we mean that here the boundary of Kähler cone with its vertex excluded.*

Are there Kähler metric with $H > 0$ from each of the Kähler classes of $M_{2,k}$? In particular, since $c_1(M_{2,1}) > 0$, it would be interesting to know there is any metric with $H > 0$ from the anti-canonical class of $M_{2,1}$.

What is the best holomorphic pinching constant λ_k among all Kähler metrics of $H > 0$ on the Hirzebruch surfaces $M_{2,k}$? Note that Hitchin's examples are of $U(2)$-symmetry, it seems reasonable to expect the optimal holomorphic pinching constant λ_k to be realized by some Kähler metric with a large symmetry.

Let λ_k denote the optimal holomorphic pinching constant among all Kähler metrics of $H > 0$ on $M_{2,k}$. Is it true that any compact Kähler surface with pinching constant strictly greater λ_1 must be biholomorphic to $\mathbb{CP}^2$ or $\mathbb{CP}^1 \times \mathbb{CP}^1$?

3.2. Hirzebruch manifolds by Calabi's ansatz

Let us recall a powerful method to construct canonical metrics pioneered by Calabi (Calabi's ansatz). Our exposition follows more closely from Koiso–Sakane [32]. As we shall see later, for Hirzebruch manifolds $M_{n,k}$, Calabi's ansatz can be applied to produce $U(n)-$invariant Kähler metrics which include Hitchin's examples as a special case.

3.2.1. Kähler metrics on $\mathbb{C}^*$-bundles reviewed

First we review some facts on the construction of a Kähler metric on a $\mathbb{C}^*$-bundle over a compact Kähler manifold where $\mathbb{C}^* = \mathbb{C} - \{0\}$. Given a holomorphic line bundle $L \to M$ on a complex manifold M, where π is the natural projection, we consider the $\mathbb{C}^*$-action on $L^* = L \setminus L_0$, where L_0 is the zero section of L. Denote by V and S the two holomorphic vector fields generated by the $\mathbb{R}^+$ and $\mathbb{S}^1$ action, respectively.

Let $\pi : (L, h) \to (M, g)$ be a Hermitian line bundle over a compact Kähler manifold (M, g). Denote by $\tilde{J}$ the complex structure on L. Assume t is a smooth function on L^* which only depends on the norm of Hermitian metric h on L. By this we mean, fix a point $v \in L^*$, under a local trivialization of L, we may write $v = \xi e_L$ with $\xi \neq 0$ and $h(v) = |\xi|^2 h(e_L)$, then t is a single-variable function of $\sqrt{h(v)}$. Moreover we assume t is strictly increasing with respect to $\sqrt{h(v)}$.

Consider a Hermitian metric on L^* of the form

$$\tilde{g} = \pi^* g_t + \mathrm{d}t^2 + (\mathrm{d}t \circ \tilde{J})^2, \tag{3.3}$$

where g_t is a family of Riemannian metrics on M to be decided. Let $u(t)^2 = \tilde{g}(V, V)$ and it can be checked that u depends only on t.

The following results were proved in [32].

Lemma 3.1 ([32]). — *The Hermitian metric $\tilde{g}$ defined by (3.3) is Kähler on L^* if and only if each g_t is Kähler on M, and $g_t = g_0 - U\Theta(L)$, where $U'(t) = u(t)$ and $\Theta(L)$ is the curvature form of (L, h). Morever, we assume the range of t includes 0 and $U(0) = 0$, then U is determined by $U = \int_0^t u(\tau)\mathrm{d}\tau$.*

From now on, we always assume the eigenvalues of the curvature $\Theta(L)$ with respect to g_0 are constant on M.

Let $z_1 \ldots z_{n-1}$ be local holomorphic coordinates on M and $z_0 \ldots z_{n-1}$ be local coordinates on L^* such that $\frac{\partial}{\partial z_0} = V - \sqrt{-1}S$.

Lemma 3.2 ([32]). — $\tilde{g}_{0\bar{0}} = 2u^2$, $\tilde{g}_{\alpha\bar{0}} = 2u\partial_\alpha t$, $\tilde{g}_{\alpha\bar{\beta}} = g_{t\alpha\bar{\beta}} + 2\partial_\alpha t \partial_{\bar{\beta}} t$. *Define $p = \det(g_0^{-1} \cdot g_t)$, then $\det(\tilde{g}) = 2u^2 \cdot p \cdot \det(g_0)$.*

Lemma 3.3 ([32]). — *If we assume that $\partial_\alpha t = \partial_{\bar{\alpha}} t = 0$ $(1 \leqslant \alpha \leqslant n-1)$ on a fiber, and if a function f on L^* depends only on t, then $\partial_0\partial_{\bar{0}} f = u\frac{\mathrm{d}}{\mathrm{d}t}(u\frac{\mathrm{d}f}{\mathrm{d}t})$, $\partial_\alpha\partial_{\bar{0}} f = 0$, $\partial_\alpha\partial_{\bar{\beta}} f = -\frac{1}{2}u\frac{\mathrm{d}f}{\mathrm{d}t}\Theta(L)_{\alpha\bar{\beta}}$. Moreover, the Ricci curvature of $\tilde{g}$ becomes: $\tilde{R}_{0\bar{0}} = -u \cdot \frac{\mathrm{d}}{\mathrm{d}t}(u \cdot \frac{\mathrm{d}}{\mathrm{d}t}(log(u^2 p)))$, $\tilde{R}_{\alpha\bar{0}} = 0$, $\tilde{R}_{\alpha\bar{\beta}} = \mathrm{Ric}(g_0)_{\alpha\bar{\beta}} + \frac{1}{2}u \cdot \frac{\mathrm{d}}{\mathrm{d}t}(log(u^2 p)) \cdot \Theta(L)_{\alpha\beta}$.*

It is convenient to reparametrize and introduce two functions $\phi(U) = u^2(t)$ and $Q(U) = p$. The following lemma characterizes any $\phi(U)$ which corresponds to a smooth (maybe incomplete) Kähler metric in the form of (3.3) on the total space of the $\mathbb{C}^*$-bundle L^*.

Lemma 3.4. — *Given any hermitian line bundle (L, h) over a compact Kähler manifold (M, g_0) such that the eigenvalues of the curvature $\Theta(L)$ with respect to g_0 are constant on M, fix $-\infty < U_{\min} < U_{\max} \leqslant +\infty$ such that $g_t \doteq g_0 - U\Theta(L)$ remains positive on $(U_{\min}, U_{\max})$.*

Let $\phi(U)$ be a smooth positive function on $(U_{\min}, U_{\max})$, if we assume that

$$\int_{U_{\min}}^{U} \frac{\mathrm{d}U}{\phi(U)} = +\infty, \int_{U}^{U_{\max}} \frac{\mathrm{d}U}{\phi(U)} = +\infty, \tag{3.4}$$

$$\int_{U_{\min}}^{U} \frac{\mathrm{d}U}{\sqrt{\phi(U)}} < \infty \quad \forall\ U \in (U_{\min}, U_{\max}). \tag{3.5}$$

Then we can solve for t as a strictly increasing function of $\sqrt{h}$ which is defined on $(0,\infty)$. Moreover, we may choose $t_{\min} > -\infty$ where $(t_{\min}, t_{\max})$ is the range of t. The corresponding $\tilde{g}$ in the form of (3.3) is a smooth Kähler metric on the total space of L^*.

Proof of Lemma 3.4. — This is motivated from Lemma 2.1 in [44] and we include a proof for the sake of self-containedness.

Recall that $\sqrt{h}$ is the norm of Hermitian metric on L. According to the definition of $\mathbb{C}^*$-action on L^*, we have:

$$V = \sqrt{h}\frac{\mathrm{d}}{\mathrm{d}\sqrt{h}}, \text{ which leads to } u = \sqrt{\phi(U)} = \sqrt{h}\frac{\mathrm{d}}{\mathrm{d}\sqrt{h}}. \tag{3.6}$$

It follows:

$$\frac{\mathrm{d}U}{\sqrt{\phi(U)}} = \mathrm{d}t, \quad \text{and} \quad \frac{\mathrm{d}U}{\phi(U)} = \frac{\mathrm{d}\sqrt{h}}{\sqrt{h}}.$$

Therefore (3.5) is a necessary condition so that the range of $\sqrt{h}$ is $[0,\infty)$ and $t_{\min}$ is finite.

On the other hand, for any given $\phi(U)$ satisfying (3.5), we can solve $t = t(U)$ and $U = U(\sqrt{h})$, hence t as function of $\sqrt{h}$. Note that $t = t(\sqrt{h})$ is defined on $(0,+\infty)$. The resulting metric $\tilde{g}$ of the form (3.3) is a smooth metric defined on the total space of L^*. □

3.2.2. Kähler metrics on $\mathbb{P}(L \oplus 1)$

It is discussed ([32, p.169]) how to extend the Kähler metric $\tilde{g}$ on L^* onto $\mathbb{P}(L \oplus 1)$. We summarize their results below.

Lemma 3.5 ([32]). — *Assume t extends to $\mathbb{P}(L \oplus 1)$ with the range $[t_{\min}, t_{\max}]$ with both $t_{\min}$ and $t_{\max}$ are finite, and the subset E_0 (or E_∞) defined by $t = t_{\min}$ (or $t = t_{\max}$) is a complex submanifold with codimension $D_{\min}$ or $D_{\max}$. Moreover, assume the Kähler metric $\tilde{g}$ extends to $\mathbb{P}(L \oplus 1)$, which is also denoted by $\tilde{g}$.*

Then it implies that near $U = U_{\min}$ the Taylor expansion of $\phi(U)$ has the first term $2(U - U_{\min})$, and near $U = U_{\max}$, it has the first term $2(U_{\max} - U)$.

Indeed, once we assume $\phi(U)$ is a smooth function defined on $[U_{\min}, U_{\max}]$ and satisfies the endpoint condition in Lemma 3.5, one may check (3.5) holds automatically and Kähler metric $\tilde{g}$ extends to $\mathbb{P}(L \oplus 1)$. In this case, t extends to a smooth function on $[0,\infty]$ with its range $[t_{\min}, t_{\max}]$. Furthermore, because $\tilde{g}$ is in the particular form (3.3), $t - t_{\min}$ measures the distance from E_0 to points in $\mathbb{P}(L \oplus 1)$ and $t_{\max} - t$ from E_∞ in $\mathbb{P}(L \oplus 1)$.

A standard example which satisfies the assumptions of Lemma 3.5 is Hirzebruch manifold $M_{n,k}$. Indeed we may view any $M_{n,k}$ as the compactification of the total space of $\mathbb{C}^*$-bundle induced from k-th power of the tautological bundle $H^{-k} \to \mathbb{CP}^{n-1}$. Here we assume the base $\mathbb{CP}^{n-1}$ is endowed with the Fubini–Study metric $\mathrm{Ric}(g_0) = g_0$, hence

$$R_{i\bar{j}k\bar{l}}(g_0) = \frac{1}{n}((g_0)_{i\bar{j}}(g_0)_{k\bar{l}} + (g_0)_{k\bar{j}}(g_0)_{i\bar{l}}). \tag{3.7}$$

It follows from Lemma 3.1 to Lemma 3.5 that

$$\tilde{g}_{i\bar{j}} = \left(1 + \frac{k}{n}U\right)(g_0)_{i\bar{j}}, \quad \Theta(H^{-k}) = -\frac{k}{n}(g_0)_{i\bar{j}}, \quad t_{i\bar{j}} = \frac{k}{2n}u(g_0)_{i\bar{j}}. \tag{3.8}$$

Here we need to pick $[U_{\min}, U_{\max}]$ such that $U_{\min} > -\frac{n}{k}$ and $U_{\max} < \infty$. It follows from (3.8) that these conditions on $U_{\min}$ and $U_{\max}$ are necessary so that the extension metric $\tilde{g}$ is well-defined and positive definite on E_0 and E_∞.

Let us remark that there are other types of compactification covered in Lemma 3.5. For example, consider the tautological bundle $H^{-1} \to \mathbb{CP}^{n-1}$, if we pick $U_{\min} = -n$ and $U_{\max} < \infty$, the corresponding compactification will produce a Kähler metric on $\mathbb{CP}^n$. Note that in this case, it follows from (3.8) that the complex submanifold $E_{\min}$ becomes a single point and $E_{\max}$ is $\mathbb{CP}^{n-1}$. The compactification is obtained by adding a copy of $\mathbb{CP}^{n-1}$ along the infinity of $\mathbb{C}^n$, resulting in $\mathbb{CP}^n$. We refer to [19, p. 281] for a more general discussion on various types of compatification obtained from $\mathbb{C}^*$-bundles over products of Kähler–Einstein manifolds.

From now on, we consider Hirzebruch manifolds $M_{n,k}$. From dicussions following Lemma 3.5, for any given $-\frac{n}{k} < U_{\min} < U_{\max} < \infty$, and any smooth function $\phi : (U_{\min}, U_{\max}) \to R^+$. Assume $\phi(U)$ which can be smoothly extended to $[U_{\min}, U_{\max}]$ with the asymptotic conditions

$$\begin{aligned} \phi(U) &= 2(U - U_{\min}) + O((U - U_{\min})^2), \\ \phi(U) &= 2(U_{\max} - U) + O((U_{\max} - U)^2). \end{aligned}$$

It follows from Lemma 3.4 that we can solve $t = t(\sqrt{h})$ as an increasing function of the norm of Hermitian metric of $H^{-k} \to \mathbb{CP}^{n-1}$. Therefore the Kähler metric of the form (3.3) is well-defined on $M_{n,k}$. The following proposition gives the formulas of curvature tensors of such a metric on $M_{n,k}$.

Proposition 3.6 (Curvature tensors of $\tilde{g}$ on $M_{n,k}$). — *Consider $\pi : M_{n,k} \setminus (E_0 \cup E_\infty) \to \mathbb{CP}^{n-1}$ and any given point $p \in \mathbb{CP}^{n-1}$. Let us assume that $\partial_\alpha t = \partial_{\overline{\alpha}} t = 0$ $(1 \leqslant \alpha \leqslant n-1)$ on the fiber $\pi^{-1}(p)$ as in Lemma 3.3.*

Consider the unitary frame $\{e_0, e_1, \dots, e_{n-1}\}$ *along this fiber:*

$$e_0 = \frac{1}{\sqrt{2\phi}}\frac{\partial}{\partial z_0},\ e_i = \frac{1}{\sqrt{(1+\frac{k}{n}U)(g_0)_{i\bar{i}}}}\frac{\partial}{\partial z_i}\ (1 \leqslant i \leqslant n-1). \tag{3.9}$$

Then the only nonzero curvature components of $\tilde{g}$ *on* $M_{n,k}$ *are:*

$$A \doteq \tilde{R}_{0\bar{0}0\bar{0}} = -\frac{1}{2}\frac{\mathrm{d}^2\phi}{\mathrm{d}U^2}, \tag{3.10}$$

$$B \doteq \tilde{R}_{0\bar{0}i\bar{i}} = \frac{k^2\phi - k(n+kU)\frac{\mathrm{d}\phi}{\mathrm{d}U}}{2(n+kU)^2}, \tag{3.11}$$

$$C \doteq \tilde{R}_{i\bar{i}i\bar{i}} = 2\tilde{R}_{i\bar{i}j\bar{j}} = \frac{[2(n+kU)-k^2\phi]}{(n+kU)^2}. \tag{3.12}$$

Here $1 \leqslant i, j \leqslant n-1$ *and* $i \neq j$*.*

Proof of Proposition 3.6. — Given any point $p \in \mathbb{CP}^{n-1}$, assume $\partial_\alpha t = \partial_{\bar{\alpha}} t = 0$ as in Lemma 3.3, we could do a local calculation along a fiber $\pi^{-1}(p) \subset M_{n,k}$ to solve curvature components of $(M_{n,k}, \tilde{g})$.

Recall the formula of curvature tensors of a Kähler manifold

$$\tilde{R}_{i\bar{j}k\bar{l}} = -\frac{\partial^2 \tilde{g}_{k\bar{l}}}{\partial z_i \partial \overline{z_j}} + \tilde{g}^{\lambda\bar{\mu}}\frac{\partial \tilde{g}_{k\bar{\mu}}}{\partial z_i}\frac{\partial \tilde{g}_{\lambda\bar{l}}}{\partial \overline{z_j}}. \tag{3.13}$$

Combined with Lemma 3.3 and (3.8), one can solve

$$\begin{aligned}\tilde{R}&\left(\frac{\partial}{\partial z_0}, \frac{\partial}{\partial \overline{z_0}}, \frac{\partial}{\partial z_0}, \frac{\partial}{\partial \overline{z_0}}\right)\\ &= -\phi\frac{\mathrm{d}(2\phi\frac{\mathrm{d}\phi}{\mathrm{d}U})}{\mathrm{d}U} + \frac{1}{2\phi}\phi\frac{\mathrm{d}\phi}{\mathrm{d}U}\phi\frac{\mathrm{d}\phi}{dU} = -2\phi^2\frac{\mathrm{d}^2\phi}{\mathrm{d}U^2}.\end{aligned} \tag{3.14}$$

Here we apply $\frac{\partial}{\partial z_0} = V - \sqrt{-1}S = \phi(\frac{\mathrm{d}}{\mathrm{d}U} - \sqrt{-1}\tilde{J}(\frac{\mathrm{d}}{\mathrm{d}U}))$ because of (3.6). All other curvature components can be calculated in a similar way.

Next we note that the transitive $U(n)$-action on the base $\mathbb{CP}^{n-1}$ can be lifted to the total space of $H^{-k} \oplus 1_{\mathbb{CP}^{n-1}}$, This action preserves the fiber metric and we get a natural $U(n)$-isometric action on $(M_{n,k}, \tilde{g})$. Therefore, the above calculation on the fiber $\pi^{-1}(p) \subset M_{n,k}$ can be carried out on any other fiber. So we get the right hand sides of (3.10), (3.11), and (3.12) are well-defined on $M_{n,k} \setminus (E_0 \cup E_\infty)$.

It remains to check (3.10), (3.11), and (3.12) is also well-defined on E_0 and E_∞. By Lemma (3.5) we can solve $t(\sqrt{h}) = t_{\min} + c\sqrt{h} + O(h)$ for some $c > 0$ near E_0. It follows from (3.6) that

$$\frac{1}{\sqrt{2\phi}}\frac{\partial}{\partial z_0} = \frac{1}{\sqrt{2}}\frac{1}{\frac{\mathrm{d}t}{\mathrm{d}\sqrt{r}}}\left(\frac{\mathrm{d}}{\mathrm{d}\sqrt{h}} - \sqrt{-1}\tilde{J}\left(\frac{\mathrm{d}}{\mathrm{d}\sqrt{h}}\right)\right) = \frac{1}{\sqrt{2}}\left(\frac{\mathrm{d}}{\mathrm{d}t} - \sqrt{-1}\tilde{J}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\right).$$

Hence $e_0 = \frac{1}{\sqrt{2\phi}}\frac{\partial}{\partial z_0}$, as the radial vector field in terms of geodesic coordinates, is not well-defined on E_0. Instead, we can always pick a $(1,0)$ vector f_0 well-defined on a neighborhood of p which is unit along the fiber, so that $\{f_0, e_1, \dots, e_{n-1}\}$ is a unitary frame along the fiber $\pi^{-1}(p)$ including p. Note that along the fiber away from p, we have $f_0 = e^{\sqrt{-1}\theta}e_0$ for some θ. Obviously the right hand sides of (3.10), (3.11), and (3.12) remain the same under the new frame $\{f_0, e_1, \dots, e_{n-1}\}$ along the fiber. After taking the limit to p, we see the curvature formulas for A, B, and C holds true on E_0, By a similar argument they also hold on E_∞, hence on the whole $M_{n,k}$. □

Proposition 3.6 leads to the following characterization of $U(n)$-invariant Kähler metrics with $H > 0$ on $M_{n,k}$.

Proposition 3.7 (The generating function ϕ). — *Any $U(n)$-invariant Kähler metric on $M_{n,k}$ has positive holomorphic sectional curvature if and only if*

$$A > 0, \quad C > 0, \quad 2B > -\sqrt{AC}. \tag{3.15}$$

In other words, it is characterized by a smooth concave function $\phi(U)$ where $-\infty < U_{\min} \leqslant U \leqslant U_{\max} < +\infty$ with $1+\frac{k}{n}U_{\min} > 0$ such that the following conditions hold:

(1) $\phi > 0$ *on* $(U_{\min}, U_{\max})$, $\phi(U_{\min}) = \phi(U_{\max}) = 0$, $\phi'(U_{\min}) = 2$, *and* $\phi'(U_{\max}) = -2$.

(2) $\phi(U) < \frac{2}{k^2}(n+kU), \quad \frac{k\phi}{n+kU} - \phi' > -\sqrt{\phi''\left[\frac{\phi}{2} - \frac{1}{k^2}(n+kU)\right]}$ *for any* $U \in [U_{\min}, U_{\max}]$.

Proof of Proposition 3.7. — We have pointed out that the metric $\tilde{g}$ in the form of (3.3) is $U(n)$-invariant, It remains to see any $U(n)$-invariant Kähler metric on $M_{n,k}$ can be written in the form (3.3). Recall that in the standard version of Calabi's ansatz ([11, p. 278–282]), any $U(n)$-invariant Kähler metric on $M_{n,k}$ can be expressed as a compactification of a Kähler metric on $\mathbb{C}^n \setminus \{0\}$ in the form $\sqrt{-1}\partial\bar{\partial}u(|z|)$. Here some suitable asymptotic conditions at $|z| = 0$ and $|z| = \infty$ are imposed to ensure a smooth extension onto $M_{n,k}$, It follows from a straightforward check (see [20, (19), p. 186] for example) that the above metric corresponds to the metric (3.3) we considered. Therefore on Hirzebruch manifolds $M_{n,k}$, the class of $U(n)$-invariant Kähler metrics is the same as the class of Kähler metrics satisfying (3.3) and (3.8).

Given a $(1,0)$ vector X at any point $p \in M_{n,k}$, after a $U(n)$ which preserves the point p, we may choose $X = x_0 e_0 + x_1 e_1$ with $|x_0|^2 + |x_1|^2 = 1$. Then the expression of the holomorphic sectional curvature is

$$H(X) = A|x_0|^4 + 4B|x_0 x_1|^2 + C|x_1|^4.$$

It is direct to check that $H(X) > 0$ is equivalent to $A > 0$, $C > 0$, and $2B > -\sqrt{AC}$.

Note that ϕ is a concave function since $A > 0$, condition (1) is the endpoint condition on ϕ in order to get the smooth compactification, and condition (2) simply means $C > 0$ and $2B > -\sqrt{AC}$. $\square$

As a corollary of Proposition 3.6 and 3.7, we have the following rough estimates on holomorphic pinching constants of $U(n)$-invariant Kähler metrics on $M_{n,k}$.

Corollary 3.8 (A rough estimate on holomorphic pinching constants). *Any $U(n)$-invariant Kähler metric on $M_{n,k}$ with $H > 0$ have its local holomorphic pinching constant bounded from above by $\frac{1}{k^2}$, Fix $0 < c < \frac{n}{k}$, if we consider any $U(n)$-invariant Kähler metric with $H > 0$ whose corresponding Kähler class lies in the following ray S in the Kähler cone*

$$S = \left\{ b[E_\infty] - a[E_0] \,\middle|\, \textit{where} \;\; a = b\,\frac{n-kc}{n+kc} \right\}.$$

Then its holomorphic pinching constant is bounded from above by $\frac{2c}{n-kc}$.

Proof of Corollary 3.8. — For the first part, we simply evaluate curvature tensors at $U = U_{\min}$.

$$A(U_{\min}) = -\frac{1}{2}\phi''(U_{\min}), \quad B(U_{\min}) = -\frac{k}{n+kU_{\min}},$$
$$C(U_{\min}) = \frac{2}{n+kU_{\min}}.$$

Note that condition (2) in Proposition 3.7 implies $-\phi''(U_{\min}) > \frac{4k^2}{n+kU_{\min}}$. Therefore the local holomorphic pinching constant has to be bounded from above by $\frac{C(U_{\min})}{A(U_{\min}}} < \frac{1}{k^2}$.

For the second part, it suffices to consider the case $[U_{\min}, U_{\max}] = [-c, c]$ for any fixed $c \in (0, \frac{n}{k})$. On one hand, we have

$$\max_{U\in[-c,c]} C(U) \leqslant \max \frac{2}{n-kc} - \frac{k^2\phi}{(n-kc)^2} \leqslant \frac{2}{n-kc}$$

On the other hand, from $\int_{-c}^{c} -\phi'' \mathrm{d}U = 4$ we conclude there must be a point $U_0 \in [-c, c]$ such that $A(U_0) = \frac{1}{c}$. These lead to an upper bound for the local pinching constant:

$$\frac{C(U_0)}{A(U_0)} \leqslant \frac{2c}{n-kc}. \qquad \square$$

Another consequence of Proposition 3.6 and 3.7 is the path-connectedness of Kähler metrics of $H>0$ in the same Kähler class on $M_{n,k}$.

Corollary 3.9 (A convexity property on $H>0$ in a fixed Kähler class). — *If ϕ_1 and ϕ_2 are two generating functions of two Kähler metrics of $H>0$ in the same Kähler class on $M_{n,k}$, so is any convex combination $t\phi_1+(1-t)\phi_2$ with $0<t<1$.*

Proof of Corollary 3.9. — It suffices to check $t\phi_1+(1-t)\phi_2$ satisfies the assumptions in Proposition 3.7. This is a straightforward calculation once we apply the elementary inequality:

$$-t\sqrt{A_1B_1}-(1-t)\sqrt{A_2B_2} \geqslant -\sqrt{\Big(tA_1+(1-t)A_2\Big)\Big(tB_1+(1-t)B_2\Big)}$$

for any nonnegative real numbers A_1, A_2, B_1, B_2 and $t\in[0,1]$. $\square$

Remark 3.10. — It would be interesting know if a similar conclusion as in Corollary 3.9 holds without the assumption on $U(n)$-symmetry.

It is interesting to note that another version of Calabi's ansatz [10] can be used to study Kähler metrics on the projective compactfication of some holomorphic vector bundle over a compact Kähler manifold. In more details, assume $\pi:(E,h)\to M$ is a Hermitian vector bundle over a compact Kähler manifold (M,ω_g), consider the Kähler form on the total space of E defined by

$$\hat{\omega}=\pi^*\omega_g+\sqrt{-1}\partial\bar{\partial}u(\sqrt{h}). \tag{3.16}$$

Here u is some suitable function on E which only depends on the hermitian metric h of E. The corresponding Kähler metric $\hat{g}$ is the one considered in [10, p. 274], which is similar as $\tilde{g}$ in (3.3). Note that the restriction of $\hat{g}$ on each fiber has a $U(r)$-invariant Hermitian structure. By considering a suitable asymptotic behavior of $u(\sqrt{h})$ along $\sqrt{h}=0$ and $\sqrt{h}=\infty$, we could get an smooth extension of $\hat{g}$ on the projectivization $\mathbb{P}(E\oplus 1)$. It is natural to expect new examples of compact Kähler metrics with of $H>0$ can be constructed in the form of (3.16), and this viewpoint seems to be more direct compared to the remark after Corollary 1.9. In any case, we focus on Hirzebruch manifolds $M_{n,k}$ in this paper.

3.2.3. Hitchin's examples reformulated

Hitchin's construction gives a family of Kähler metrics with $U(2)$ symmetry on $M_{2,k}$. We observe a similar construction works for $M_{n,k}$.

Example 3.11 (Hitchin's examples on $M_{n,k}$). — Given $s > 0$, $U_{\min} = 0$, $U_{\max} = ns$, define $\phi_s(U) = -\frac{2}{ns}U^2 + 2U$. Now

$$A = \frac{2}{ns}, \quad B = \frac{\frac{k^2}{ns}U^2 + \frac{4k}{s}U - kn}{(n+kU)^2}, \quad C = \frac{\frac{2k^2}{ns}U^2 + (2k-2k^2)U + 2n}{(n+kU)^2}.$$

It is a direct calculation to show the above example is exactly the same as $\tilde{\omega}$ in (3.2). In fact, we observe that Hitchin's example is canonical in the following sense.

Proposition 3.12. — *Hitchin's examples can be uniquely characterized as $U(n)$-invariant Kähler metrics on $M_{n,k}$ with the constant radial curvature A. In particular, the following example gives the unique form of $\phi(U)$ up to a scaling and a translation of $[U_{\min}, U_{\max}]$.*

Example 3.13 (Hitchin's example in a canonical form). — Let $c > 0$, $U_{\min} = -c$, $U_{\max} = c$, define $\phi_c(x) = c - \frac{x^2}{c}$ on $[-c, c]$. Since $\phi'(-c) = 2$, $\phi'(c) = -2$, and $\phi(\pm c) = 0$, we have a Kähler metric on $M_{2,1}$. Now

$$A = \frac{1}{c}, \quad B = \frac{\frac{1}{c}U^2 + \frac{4}{c}U + c}{2(U+2)^2}, \quad C = \frac{\frac{1}{c}U^2 + 2U + 4 - c}{(U+2)^2}.$$

If we assume $0 < c < 2$, then obviously $1 - \frac{1}{2}U > 0$, $A > 0$, and $C > 0$ on $[-c, c]$. Consider

$$D \doteq 2B + \sqrt{AC} = \frac{\frac{1}{c}U^2 + \frac{4}{c}U + c + \frac{1}{c}\sqrt{U^2 + 2cU + c(4-c)} \cdot (U+2)}{(U+2)^2}.$$

Then $D(-c) > 0$ is equivalent to $c < \frac{2}{3}$. Moreover, one can check that the numerator of $D(U)$ is increasing on $U \in (-c, c)$, hence $D(U) > 0$ for any $-c < U < c$ and $0 < c < \frac{2}{3}$. Therefore ϕ_c provides a family of Kähler metrics of $H > 0$ for any $0 < c < \frac{2}{3}$.

Next let us find the pinching constant for $\phi_c(x) = c - \frac{x^2}{c}$. For any given ϕ_c, the expression of the holomorphic sectional curvature is

$$H(X) = A|x_0|^4 + 4B|x_0x_1|^2 + C|x_1|^4,$$

where $X = x_0e_0 + x_1e_1$ with $|x_0|^2 + |x_1|^2 = 1$.

If we set $t = |x_0|^2$, then $H(X) = (A + C - 4B)t^2 + t(4B - 2C) + C$ with $t \in [0, 1]$. it is elementary to discuss its extremal values. In particular, we will show that for any $c\in(0,\frac{2}{3})$, the pinching constant $\inf_{U\in(-c,c)}\frac{\min H}{\max H}(U)$ is always attained at $U=-c$, i.e. along the zero section of $M_{2,1}$.

Indeed

$$\min_{\|v\|=1} H(U,v)=\frac{AC-4B^2}{(A+C-4B)},\ \max_{\|v\|=1} H(U,v)=A.$$

Therefore, the local pinching constant equals

$$\frac{AC-4B^2}{A(A+C-4B)}=\frac{2U^3+(6-3c)U^2-12cU-c^3-2c^2-8c}{(2U-2-3c)(U+2)^2}.$$

It is direct to check that the above expression is increasing on $U\in[-c,c]$. If $U=-c$, it becomes $\frac{2c(c-3c)}{(2-c)(5c+2)^2}$, When $c=\frac{2}{7}\approx 0.2857$, it attains the maximum $\frac{1}{9}$. The optimal pinching constant among the family ϕ_c agrees with the result in [1], and the corresponding optimal Kähler metrics are just multiples of those in [1].

It is also straightforward to solve the optimal holomorphic pinching constant of Hitchin's examples on any Hirzebruch manifold $M_{n,k}$. Given any $1\leqslant k<n$, pick $c>0$, $U_{\min}=-c$, $U_{\max}=c$, define $\phi_c(U)=c-\frac{x^2}{c}$ on $[-c,c]$. We claim that ϕ_c gives a Kähler metric on $M_{n,k}$ with $H>0$ as long as $c<\frac{n}{k(2k+1)}$. Note that

$$A=\frac{1}{c},\quad B=\frac{\frac{k^2}{c}U^2+\frac{2nk}{c}U+ck^2}{2(n+kU)^2},\quad C=\frac{\frac{k^2}{c}U^2+2kU+2n-ck^2}{(n+kU)^2}.$$

Similarly

$$D\doteq 2B+\sqrt{AC}=\frac{\frac{k^2}{c}U^2+\frac{2nk}{c}U+ck^2+\frac{n+kU}{c}\sqrt{k^2U+2kcU+(2n-ck^2)c}}{(n+kU)^2}$$

First note that $0<c<\frac{n}{k(2k+1)}$ is equivalent to $D(-c)>0$, then similarly we could show under this condition on c the numerator of D is strictly increasing on $U\in(-c,c)$, therefore $D(U)>0$ holds on $[-c,c]$.

Note that when $n\geqslant 4$, for any positive integer k satisfying $k(2k+1)<n$, we may pick $c=1$. In this case the Kähler class of the resulting metric is proportional to the anti-canonical class.

For the above metric on $M_{n,k}$ given by $\phi_c(x)=c-\frac{x^2}{c}$ on $[-c,c]$, where $0<c<\frac{n}{k(2k+1)}$ is a constant, one can carry out a similar calculation to conclude that the local pinching constant achieves its maximum at $U=-c$, which is

$$\frac{2c(n-c(2k^2+k))}{(n-ck)((3k+2)c+n)}.$$

It can be shown that it obtains its maximum value at $c = \frac{n}{4k^2+3k}$, and the optimal pinching constant is $\frac{1}{(2k+1)^2}$ which is the same as in [1]. Note that the optimal pinching constant is dimension free.

3.3. New examples and the proof of Theorem 1.3

Let us consider $M_{2,1}$ which is the only Fano Hirzebruch surface. Note that Kähler metrics of Hitchin's examples on $M_{2,1}$ can not be proportional to the anti-cananical class. A natural question is whether there exists a Kähler metric with $H > 0$ in $2\pi c_1(M_{2,1})$. Note that the corresponding $\phi(U)$ of such a metric must satisfy $(2 + U_{\max}) = 3(2 + U_{\min})$ besides $\phi(U_{\min}) = \phi(U_{\max}) = 0$, $\phi'(U_{\min}) = 2$, and $\phi'(U_{\max}) = -2$ required by the smooth compactification.

In the following we exhibit such an example with different global and local holomorphic pinching constants.

Proposition 3.14 (A new family of Kähler metrics of $H > 0$ on $M_{2,1}$). *Given any real number $0 < c < \frac{6}{5}$, pick a real number $\mu \in (\frac{1}{2}c, c)$, define $\phi_{c,\mu} : [-c, c] \to \mathbb{R}$ by*

$$\phi_{c,\mu}(U) = \mu - \left(\frac{1}{c^3} - \frac{\mu}{c^4}\right) U^4 - \left(\frac{2\mu}{c^2} - \frac{1}{c}\right) U^2$$

Thus $\phi_{c,\mu}$ determines a family of Kähler metrics on $M_{2,1}$, and in particular when $c = 1$, the Kähler class of $\phi_{c,\mu}$ is proportional to the anti-canonical class of $M_{2,1}$.

There exists some $\delta \in (0, \frac{1}{2})$ which depends on c such that for any $\frac{1}{2}c < \mu < (\frac{1}{2} + \delta)c$, $\phi_{c,\mu}(U)$ defines a Kähler metric on $M_{2,1}$ with $H > 0$.

Proof of Proposition 3.14. — We begin with the formulas of curvature tensors of $\phi_{c,\mu}(U)$:

$$A = 6\left(\frac{1}{c^3} - \frac{\mu}{c^4}\right) U^2 + \left(\frac{2\mu}{c^2} - \frac{1}{c}\right),$$

$$B = \frac{3(\frac{1}{c^3} - \frac{\mu}{c^4})U^4 + 8(\frac{1}{c^3} - \frac{\mu}{c^4})U^3 + (\frac{2\mu}{c^2} - \frac{1}{c})U^2 + 4(\frac{2\mu}{c^2} - \frac{1}{c})U + \mu}{2(U+2)^2},$$

$$C = \frac{(\frac{1}{c^3} - \frac{\mu}{c^4})U^4 + (\frac{2\mu}{c^2} - \frac{1}{c})U^2 + 2(U+2) - \mu}{(U+2)^2}.$$

First note that for any $c \in (0, 2)$, $\mu \in (\frac{1}{2}c, c)$, we have $A > 0$ for any $U \in [-c, c]$, then since $C(-c) = \frac{2}{2-c}$ and $C(U+2)^2$ is increasing on $[-c, c]$, we also have $C > 0$.

Next one can check that

$$2B+\sqrt{AC}|_{U=-c}=-\frac{2}{2-c}+\frac{1}{c}\sqrt{\frac{2(5c-4\mu)}{2-c}},$$

From here, it is direct to see that given any $c\in(0,\frac{6}{5})$, there exists some $\delta>0$ such that for any $\frac{1}{2}c<\mu<(\frac{1}{2}+\delta)c$, $2B+\sqrt{AC}>0$ at $U=-c$.

From now on let us consider $T_\mu(U)=(U+2)^2(2B+\sqrt{AC})$, it suffices to show that $T_\mu(U)>0$ for any $U\in[-c,c]$ if μ is sufficiently close to $\frac{1}{2}c$. Note that

$$T_\mu(U)=P_\mu(U)+(U+2)\sqrt{Q_\mu(U)}$$

where

$$P_\mu(U)=3\left(\frac{1}{c^3}-\frac{\mu}{c^4}\right)U^4+8\left(\frac{1}{c^3}-\frac{\mu}{c^4}\right)U^3+\left(\frac{2\mu}{c^2}-\frac{1}{c}\right)U^2+4\left(\frac{2\mu}{c^2}-\frac{1}{c}\right)U+\mu$$

and

$$\begin{aligned}Q_\mu(U)=6&\left(\frac{1}{c^3}-\frac{\mu}{c^4}\right)^2U^6+7\left(\frac{1}{c^3}-\frac{\mu}{c^4}\right)\left(\frac{2\mu}{c^2}-\frac{1}{c}\right)U^4+12\left(\frac{1}{c^3}-\frac{\mu}{c^4}\right)U^3\\&+\left[\left(\frac{2\mu}{c^2}-\frac{1}{c}\right)^2+6(4-\mu)\left(\frac{1}{c^3}-\frac{\mu}{c^4}\right)\right]U^2+2\left(\frac{2\mu}{c^2}-\frac{1}{c}\right)U\\&+\left(-\frac{2}{c^2}\mu^2+\frac{8+c}{c^2}\mu-\frac{4}{c}\right).\end{aligned}$$

Claim. — *$T_{\frac{c}{2}}(U)>0$ on $[-c,c]$ and in particular it has a positive lower bound at* 0.

Proof of the Claim. — To see it is true, note that:

$$T_{\frac{c}{2}}(U)=\left(\frac{3}{2c^3}U^4+\frac{4}{c^3}U^3+\frac{c}{2}\right)+(U+2)\sqrt{\frac{3}{2c^6}U^6+\frac{6}{c^3}U^3+\frac{3(8-c)}{2c^3}U^2}.$$

It suffices to consider the interval $[-c,0]$, we will show that $T_{\frac{c}{2}}(U)$ is strictly concave on $[-c,0]$, thus it attains its minimum either at $U=-c$ or at $U=0$, which are both positive.

$$\frac{\mathrm{d}^2T_{\frac{c}{2}}(U)}{\mathrm{d}U^2}=\frac{6}{c^3}U(3U+4)+\frac{\frac{9}{2c^{12}}U^3\cdot R(U)}{\left(\sqrt{\frac{3}{2c^6}U^6+\frac{6}{c^3}U^3+\frac{3(8-c)}{2c^3}U^2}\right)^3},$$

where

$$\begin{aligned}R(U)=6U^8+6U^7+36c^3U^5+(108c^3-9c^4)U^4+(80c^3-10c^4)U^3\\+30c^6U^2+(108c^6-12c^7)U+c^8-20c^7+96c^6.\end{aligned}$$

Note that $R(-c) = 4c^6(2-c)^2 > 0$, we will prove that $\frac{\mathrm{d}R(U)}{\mathrm{d}U} > 0$ on $[-c, 0]$, which leads to $R(U) > 0$ for any $0 < c < \frac{6}{5}$. Indeed,

$$\begin{aligned}\frac{1}{6}\frac{\mathrm{d}R(U)}{\mathrm{d}U} &= 8U^7 + 7U^6 + 30c^3U^4 + 6(12c^3 - c^4)U^3 \\ &\qquad + 5(8c^3 - c^4)U^2 + 10c^6U + (18c^6 - 2c^7) \\ &= I_1 + I_2 + I_3.\end{aligned}$$

where we have

$$\begin{aligned}I_1 &= 10c^6U + \frac{72}{5}c^6 - 2c^7 \geqslant \frac{72}{5}c^6 - 12c^7 = 12\left(\frac{6}{5} - c\right) \geqslant 0, \\ I_2 &= 8U^7 + 7U^6 + \frac{13}{6}c^3U^4 \geqslant U^6(8U + 7 + \frac{13}{6}c) \geqslant 7U^6\left(1 - \frac{5}{6}c\right) \geqslant 0, \\ I_3 &= \frac{167}{6}c^4U^4 + \frac{18}{5}c^6 + 6(12c^3 - c^4)U^3 + 5(8c^3 - c^4)U^2,\end{aligned}$$

for any $U \in [-c, 0]$ where $0 < c < \frac{6}{5}$.

Next we prove $I_3 > 0$ on $[-c, 0]$.

$$I_3 \geqslant U^2\left[\frac{167}{6}c^3U^2 + 6(12c^3 - c^4)U + 5(8c^3 - c^4) + \frac{18}{5}c^4\right].$$

Let $S(U)$ denote the quadratic function inside the bracket:

$$S(U) = \frac{167}{6}c^3U^2 + 6(12c^3 - c^4)U + 5(8c^3 - c^4) + \frac{18}{5}c^4.$$

Now it is straightforward to see that under the assumption $0 < c < \frac{6}{5}$, $S(U)$ attains its minimum at $U = -c$, and

$$S(-c) = c^3\left(\frac{203}{6}c^2 - \frac{367}{5}c + 40\right) > 0$$

Putting these together, we have proved that $\frac{\mathrm{d}^2 T_{\frac{c}{2}}(U)}{\mathrm{d}U^2}$ is strictly negative on $[-c, 0]$, and therefore $T_{\frac{c}{2}}(U) > 0$ on $[-c, c]$. $\square$

Now let us continue with the proof of Proposition 3.14, note that as $\mu \to (\frac{c}{2})^+$, $T_\mu(U)$ converges to $T_{\frac{c}{2}}(U)$ uniformly on $[-1, -U_0] \cup [U_0, 1]$ for some fixed small number $U_0 > 0$,

$$\begin{aligned}\frac{\partial T_\mu(U)}{\partial\mu} &= \frac{\partial P}{\partial \mu} + \frac{(U+2)}{2\sqrt{Q}}\frac{\partial Q}{\partial\mu}\\ &= \left(-\frac{3}{c^4}U^4 - \frac{8}{c^4}U^3 + \frac{2}{c^2}U^2 + \frac{8}{c^2}U + 1\right)\\ &\quad + \frac{(U+2)}{2\sqrt{Q}}\left\{-\frac{12}{c^4}\left(\frac{1}{c^3}-\frac{\mu}{c^4}\right)U^6\right.\\ &\qquad + 7\left[\frac{2}{c^2}\left(\frac{1}{c^3}-\frac{\mu}{c^4}\right) - \frac{1}{c^4}\left(\frac{2\mu}{c^2}-\frac{1}{c}\right)\right]U^4 - \frac{12}{c^4}U^3\\ &\qquad - \left[\frac{4}{c^2}\left(\frac{2\mu}{c^2}-\frac{1}{c}\right) - 6\left(\frac{1}{c^3}-\frac{\mu}{c^4}\right) - \frac{6}{c^4}(4-\mu)\right]U^2\\ &\qquad \left.+ \frac{4}{c^2}U + \frac{8+c-4\mu}{c^2}\right\}\end{aligned}$$

Take $U_0 = \min\{\frac{1}{100}, \frac{1}{100}c^2\}$, we see that $\frac{\partial T_\mu(U)}{\partial\mu}$ is strictly positive for any $|U| < U_0$ as long as $\mu - \frac{c}{2}$ is small enough. In other words, we can find some $\delta > 0$ such that $T_\mu(U) > 0$ on $[-U_0, U_0]$ for any $\frac{c}{2} < \mu < (\frac{1}{2}+\delta)c$. Moreover, $T_\mu(U)$ converges to $T_{\frac{c}{2}}(U)$ outside $[-U_0, U_0]$, hence we get $T_\mu(U) > 0$ for $\frac{1}{2}c < \mu < (\frac{1}{2}+\delta)c$. □

Example 3.15 (Pinching constants in the anti-canonical class). — Now we focus on the example in the anti-canonical class constructed in Proposition 3.14, namely, with $c = 1$. We expect that Proposition 3.14 is still true for any $\frac{1}{2} < \mu < \frac{3}{4}$, as numerical tests suggest that $2B + \sqrt{AC}$ is indeed positive on $U \in [-1, 1]$ for any $\frac{1}{2} < \mu < \frac{3}{4}$. However, it seems rather tedious to prove it, as $T_\mu(U)$ is not always increasing on $[-1, 1]$. We need some better estimates on the critical points of $T_\mu(U)$ which lie in $[-1, 0)$. Let us we only consider the case when μ is close to $\frac{1}{2}$. The following table lists the minimum and maximum of H for any given $U \in [-1, 1]$.

Intervals of U	$[-1, U_1]$	$[U_1, U_2]$	$[U_2, U_3]$	$[U_3, U_4]$	$[U_4, 1]$
$\min_{\lVert v\rVert=1} H(U,v)$	$\frac{AC-4B^2}{A+C-4B}$	$\frac{AC-4B^2}{A+C-4B}$	A	$\frac{AC-4B^2}{A+C-4B}$	$\frac{AC-4B^2}{A+C-4B}$
$\max_{\lVert v\rVert=1} H(U,v)$	A	C	C	C	A

In the above, $U_1 < U_4$ are values which corresponds to $A = C$, and $U_2 < U_3$ are values which corresponds to $A = 2B$.

For example along the zero section $U = -1$, $\min H = \frac{AC-B^2}{A+C-4B} = \frac{6-8\mu}{11-4\mu}$ and $\max H = A(-1) = 5 - 4\mu$. Therefore the pinching constant along zero section is $\frac{6-8\mu}{(5-4\mu)(11-4\mu)}$, which is close to $\frac{2}{27}$ as μ goes close to $\frac{1}{2}$. It is clear that the global maximum of holomorphic sectional curvature is attained at $U = -1$ by $A = 5 - 4\mu$ while the global minimum is attained at $U = 0$

by $A=2\mu-1$. Therefore, we conclude that the local pinching constant of Kähler metric generated by ϕ_μ is obtained at $U=0$:

$$\min_{U\in[-1,1]} \frac{\min_{\|v\|=1} H(U,v)}{\max_{\|v\|=1} H(U,v)} = \frac{A(0)}{C(0)} = \frac{4(2\mu-1)}{4-\mu}.$$

And the global holomorphic pinching constant is

$$\frac{\min_{U,\|v\|=1} H(U,v)}{\max_{U,\|v\|=1} H(U,v)} = \frac{A(0)}{A(1)} = \frac{2\mu-1}{5-4\mu}.$$

Therefore, we see the global pinching constant is strictly smaller than the local one for the generating function $\phi_{1,\mu}$ defined in Proposition 3.14 when μ is close to $\frac{1}{2}$.

Now we are ready to prove our main theorem.

THEOREM 3.16. — *Given any Hirzebruch manifold $M_{n,k}=\mathbb{P}(H^k\oplus 1_{\mathbb{CP}^{n-1}})$, there exists a Kähler metric of $H>0$ in each of its Kähler classes.*

As we mentioned the paragraph following Proposition 1.4 in Section 1, the boundary conditions of the generating function ϕ defined on $[-c,c]$ reflects the Kähler class of the resulting metric $\tilde{g}$. For example, the volume of the zero section E_0 is $(1-\frac{k}{n}c)V_{FS}$, where V_{FS} denote the volume of $\mathbb{CP}^{n-1}$ endowed with $\mathrm{Ric}(g_{FS})=g_{FS}$, and the volume of the infinity section E_∞ is $(1+\frac{k}{n}c)V_{FS}$. Therefore, to produce examples in each of the Kähler class of $M_{n,k}$, it suffices to find examples of generating functions $\phi(U)$ defined on $[-c,c]$ for any $c\in(0,\frac{n}{k})$ which satisfy Proposition 3.7. It is the purpose of the following proposition to prove the existence of such $\phi(U)$, hence prove Theorem 3.16.

PROPOSITION 3.17. — *Let $n\geqslant 2$ and $k\geqslant 1$ be any two integers, there exists some $p_0(n,k)\in\mathbb{N}$ and a sequence of positive real numbers $\{\epsilon_p\}_{p\geqslant p_0}$ with $\lim_{p\to\infty}\epsilon_p=0$, such that for any $c\in(0,\frac{n}{k}-2\epsilon_p]$, there exists $p_1(n,k,c)>p_0$ with the following property:*

Given any $p\geqslant p_1$ there exists some $\delta_1>0$ and $\delta_2>0$ such that for any $\alpha_2\in(0,\delta_1)$ and $\mu=\frac{c}{p}+\delta_2$, $\phi(x)$ defined on $[-c,c]$ by

$$\phi(x)=\mu-\alpha_2x^2-\alpha_{2p-2}x^{2p-2}-\alpha_{2p}x^{2p}. \tag{3.17}$$

where

$$\alpha_{2p-2}=\frac{p\mu-c-(p-1)\alpha_2c^2}{c^{2p-2}},\quad \alpha_{2p}=\frac{c-(p-1)\mu+(p-2)\alpha_2c^2}{c^{2p}},$$

generates a Kähler metric with $H>0$ on $M_{n,k}$.

Proof of Proposition 3.17. — Let $\epsilon_p = \frac{2n}{2p+2k-1}$ and pick any $c < \frac{n}{k} - 2\epsilon_p$, we will determine constants δ_1 and δ_2 step by step. A quick observation is that in order to make sure both α_{2p-2} and α_{2p} positive, we need

$$\delta_2 < \frac{c}{p(p-1)}, \text{ and } (p-1)\delta_1 c^2 < p\,\delta_2. \tag{3.18}$$

Note that

$$A = p(2p-1)\alpha_{2p}U^{2p-2} + (p-1)(2p-3)\alpha_{2p-2}U^{2p-4} + \alpha_2, \tag{3.19}$$

$$B = \frac{(2p-1)\alpha_{2p}U^{2p} + 2p\,nk\,\alpha_{2p}\,U^{2p-1} + (2p-3)k^2\alpha_{2p-2}U^{2p-2}}{2(n+kU)^2} + \frac{(2p-2)nk\alpha_{2p-2}U^{2p-3} + k^2\alpha_2 U^2 + 2nk\alpha_2 U + k^2\mu}{2(n+kU)^2}, \tag{3.20}$$

$$C = \frac{k^2\alpha_{2p}U^{2p} + k^2\alpha_{2p-2}U^{2p-2} + k^2\alpha_2 U^2 + 2kU + 2n - k^2\mu}{(n+kU)^2}. \tag{3.21}$$

Obviously $A > 0$ on $[-c, c]$, note that

$$C > \frac{2n - k^2\mu - 2kc}{(n+kU)^2} \quad \text{for any } U \in [-c, c]. \tag{3.22}$$

By plugging $c \leqslant \frac{n}{k} - 2\epsilon_p$ and $\mu = \frac{c}{p} + \delta_2$ into the right hand side of (3.22) we find that a sufficient condition for $C > 0$ is

$$\delta_2 < \frac{n(2p+2k+1)}{kp(2p+2k-1)}. \tag{3.23}$$

Note that $2B + \sqrt{AC}|_{U=-c} > 0$ is equivalent to $\phi''(-c) < -\frac{4k^2}{n-kc}$. By a direct calculation we see that it is further equivalent to

$$\frac{k(2p-1) + 2k^2}{c(n-kc)}\left[\frac{n}{k} - \epsilon_p - c\right] > \frac{2p(p-1)}{c^2}\delta_2 - \alpha_2(2p^2 - 6p + 4). \tag{3.24}$$

In order that (3.24) holds, it suffices to pick δ_2 so that the following holds:

$$\frac{k(2p-1) + 2k^2}{(n-kc)}\left[\frac{n}{k} - \epsilon_p - c\right] > \frac{2p(p-1)}{c}\delta_2. \tag{3.25}$$

In other words, for any $c \in (0, \frac{n}{k} - 2\epsilon_p]$, it is easy to pick δ_1 and δ_2 such that all the inequalities in (3.18), (3.23), and (3.25) are satisfied.

It remains to show $2B + \sqrt{AC}$ is positive on $[-c, c]$. Motivated by the proof of Proposition 3.14 let us introduce

$$T(\mu, \alpha_2, U) = P(\mu, \alpha_2, U) + (n + kU)\sqrt{Q(\mu, \alpha_2, U)},$$

where

$$P(\mu,\alpha_2,U) = 2(n+kU)^2B(U),$$
$$\text{and}\quad Q(\mu,\alpha_2,U) = (n+kU)^2A(U)\cdot C(U).$$

where A, B, C are given in (3.19), (3.20), and (3.21).

We need to show that $T(\mu,\alpha_2,U) > 0$ for any $U\in[-c,c]$ under the assumption $c\in(0,\frac{n}{k}-2\epsilon_p]$, $\alpha_2\in(0,\delta_1)$, and $\mu=\frac{c}{p}+\delta_2$, where δ_1 and δ_2 satisfy (3.18), (3.23), and (3.24). By a similar argument as in the proof of Proposition 3.14, one checks that for α_2, δ_2 small, there exists U_0 sufficiently small.

$$\frac{\partial T(\mu,\alpha_2,U)}{\partial\mu} > 0,\ \text{and}\ \frac{\partial T(\mu,\alpha_2,U)}{\partial\alpha_2} > 0$$

for any $|U|<U_0$.

For example, note that

$$\frac{\partial\alpha_{2p-2}}{\partial\mu} = \frac{p}{c^{2p-2}},\frac{\partial\alpha_{2p}}{\partial\mu} = -\frac{p-1}{c^{2p}}$$

Therefore,

$$\begin{aligned}
\frac{\partial P}{\partial\mu} &= -\frac{(p-1)(2p-1)k^2}{c^{2p}}U^{2p} + (2p-3)\frac{k^2p}{c^{2p-2}}U^{2p-2}\\
&\quad + (2p-2)\frac{nkp}{c^{2p-2}}U^{2p-3} + k^2,\\
\frac{\partial Q}{\partial\mu} &= \frac{\partial A}{\partial\mu}C + A\frac{\partial C}{\partial\mu}\\
&= \left(-\frac{p(p-1)(2p-1)}{c^{2p}}U^{2p} + \frac{p(p-1)(2p-3)}{c^{2p-2}}U^{2p-4}\right)\cdot C\\
&\quad + \left(-\frac{k^2(p-1)}{c^{2p}}U^{2p} + \frac{k^2p}{c^{2p-2}}U^{2p-2} - k^2\right)\cdot A,\\
&\geqslant \frac{p(p-1)(2p-3)}{2c^{2p-2}}U^{2p-4}(2n-k^2\mu)\\
&\quad - 2k^2\left(\alpha_2 + (p-1)(2p-3)\alpha_{2p-2}U^{2p-4}\right).
\end{aligned}$$

It follows that when $|U|$, α_2, δ_2 are small enough,

$$\frac{\partial T(\mu,\alpha_2,U)}{\partial\mu} = \frac{\partial P}{\partial\mu} + \frac{(n+kU)}{2\sqrt{Q}}\frac{\partial Q}{\partial\mu} \geqslant \frac{k^2}{2} - \frac{4nk^2\alpha_2}{2\sqrt{\alpha_2\cdot n}} > 0. \tag{3.26}$$

Therefore to prove Theorem 3.17 it suffices to show that $T(\frac{c}{p},0,U)>0$ for $U\in[-c,c]$. Now we have

$$\alpha_{2p-2} = 0,\ \text{and}\ \alpha_{2p} = \frac{1}{p\,c^{2p-1}}.$$

Therefore,

$$P\left(\frac{c}{p},0,U\right)=\frac{(2p-1)k^2}{pc^{2p-1}}U^{2p}+\frac{2nk}{c^{2p-1}}U^{2p-1}+\frac{c}{p}k^2, \tag{3.27}$$

$$Q\left(\frac{c}{p},0,U\right)=\frac{2p-1}{c^{2p-1}}U^{2p-2}\left(\frac{k^2}{pc^{2p-1}}U^{2p}+2kU+2n-\frac{c}{p}k^2\right). \tag{3.28}$$

Let us reparametrize $x=\frac{U}{c}$, then $T(x)$ is defined on $[-1,1]$.

$$T\left(\frac{c}{p},0,x\right)=P(x)+(n+kcx)\sqrt{Q(x)} \tag{3.29}$$

$$\begin{aligned}&=\frac{(2p-1)k^2c}{p}x^{2p}+2nkx^{2p-1}+\frac{c}{p}k^2\\&\quad+(n+kc\,x)\sqrt{(2p-1)x^{2p-2}\left(\frac{k^2}{p}x^{2p}+2kx+\frac{2n}{c}-\frac{k^2}{p}\right)}.\end{aligned} \tag{3.30}$$

Note that $P(x)$ is increasing on $[-1,0]$ and has a unique zero $x_0\in(-1,0)$. We already have $T(-1)>0$ and $T(x_0)>0$ from (3.18), (3.23), and (3.24). It suffices to show $T(x)>0$ on $[-1,x_0]$. To that end, let us introduce

$$W(x)=-(P(x))^2+(n+kxc)^2Q(x).$$

Obviously we also have $W(-1)>0$ and $W(x_0)>0$. By Lemma 3.18 below we conclude that $W(x)>0$ on $[-1,x_0]$, which implies $T(x)>0$ for any $x\in[-1,0]$, thus completing the proof of Proposition 3.17. □

LEMMA 3.18. — *Given any $n\geqslant 2$ and k positive integers and $c\in(0,\frac{n}{k}-2\epsilon_p]$ for $p\geqslant p_0(n,k)$, there exists $p_1(n,k,c)$ such that for any $p>p_1$, either there exists $-1<x_1<x_0$ such that $W(x)$ is increasing on $[-1,x_1]$ and decreasing on $[x_1,x_0]$, or $W(x)$ is increasing on $[-1,x_0]$.*

Proof of Lemma 3.18. — A straightforward calculation shows that

$$\begin{aligned}\frac{\mathrm{d}W}{\mathrm{d}x}&=-2P(x)P'(x)+2(n+kcx)kcQ(x)+(n+kcx)^2Q'(x)\\&=(2p-1)(n+kcx)x^{2p-3}J(x),\end{aligned}$$

where

$$\begin{aligned}J(x)=&-4k^3c\frac{p-1}{p}x^{2p+1}-2nk^2\frac{2p+1}{p}x^{2p}+2k^2c(2p+1)x^2\\&+\left(8npk-2kn-\frac{4k^3c}{p}-2k^3c\right)x+\left(\frac{4pn^2}{c}-\frac{4n^2}{c}-2k^2n+\frac{2nk^2}{p}\right).\end{aligned}$$

We add a brief remark on $J(-1)$ when $c = \frac{n}{k} - 2\epsilon_p$. It follows that

$$\begin{aligned} J(-1) &= \frac{4p}{c}(n-kc)^2 + 6k^3c - 6nk^2 + 2k^2c + 2kn - \frac{4n^2}{c} \\ &= \frac{4pk^2}{\frac{n}{k} - 2\epsilon_p} 4\epsilon_p^2 + \epsilon_p\left(-12k^3 - 4k^2 - \frac{8nk}{\frac{n}{k} - 2\epsilon_p}\right) \\ &= \epsilon_p\left(-12k^3 - 4k^2 - \frac{8nk}{\frac{n}{k} - 2\epsilon_p} + \frac{16pk^2\epsilon_p}{\frac{n}{k} - 2\epsilon_p}\right). \end{aligned} \tag{3.31}$$

Recall that $\epsilon_p = \frac{2n}{2p+2k-1}$. A tedious calculation will lead to the fact that for k suitably large ($k \geqslant 5$), $J(-1) > 0$ for $p = p(n,k)$ large enough. On the other hand, if k is small ($k \leqslant 2$ for example), then $J(-1) < 0$.

The crucial observation which leads to the proof of the lemma is that, for any $0 < c \leqslant \frac{n}{k} - 2\epsilon_p$, we can find $p_1 = p(n,k,c)$ so that $J'(x) > 0$ for any $x \in [-1, x_0]$.

First let us estimate x_0, the unique zero of $P(x)$ on $(-1,0)$. We have

$$P(x_0) = \frac{(2p-1)k^2c}{p} x_0^{2p} + 2nkx_0^{2p-1} + \frac{c}{p}k^2 \tag{3.32}$$

$$= x_0^{2p-1}\left[\left(2 - \frac{1}{p}\right)k^2cx + 2nk\right] + \frac{c}{p}k^2. \tag{3.33}$$

Note that (3.32) implies that $2nk(-x_0)^{2p-1} > \frac{c}{p}k^2$, while (3.33) implies that

$$(-x_0)^{2p-1}\left[2nk - \left(2 - \frac{1}{p}\right)k^2c\right] < \frac{c}{p}k^2.$$

To sum up, we have the following estimates on x_0

$$\left(\frac{ck}{2pn}\right)^{\frac{1}{2p-1}} < -x_0 < \left(\frac{ck}{n}\right)^{\frac{1}{2p-1}}. \tag{3.34}$$

Next we compute $J'(x)$:

$$\begin{aligned} \frac{1}{p}J'(x) = -4k^3c\left(1 - \frac{1}{p}\right)\left(2 + \frac{1}{p}\right)x^{2p} - 4nk^2\left(2 + \frac{1}{p}\right)x^{2p-1} \\ + 4k^2c\left(2 + \frac{1}{p}\right)x + \left(8nk - \frac{2kn}{p} - \frac{4k^3c}{p^2} - \frac{2k^3c}{p}\right). \end{aligned} \tag{3.35}$$

In order to do calculation in the $O(\frac{1}{p})$ order, we note that when $p = p(n,k)$ is large enough, we have

$$\frac{2n}{3p} < \epsilon_p < \frac{n}{p}. \tag{3.36}$$

Plugging into $c = \frac{n}{k} - s$ where $2\epsilon_p \leqslant s < \frac{n}{k}$, it follows from (3.35) that

$$\begin{aligned}\frac{1}{p}J'(x) = &-4k^3\left(\frac{n}{k} - s\right)\left(2 - \frac{1}{p} - \frac{1}{p^2}\right)x^{2p} - 4nk^2\left(2 + \frac{1}{p}\right)x^{2p-1}\\ &+ 4k^2\left(\frac{n}{k} - s\right)\left(2 + \frac{1}{p}\right)x\\ &+ \left(8nk - \frac{2kn}{p} - \frac{4nk^2}{p^2} + \frac{4k^3 s}{p^2} - \frac{2nk^2}{p} + \frac{2k^3 s}{p}\right)\\ \geqslant &-8nk^2x^{2p-1}(x+1) + 8nk(1+x)\\ &+ 4k^3\frac{n}{kp}x^{2p} - 4\frac{nk^2}{p}x^{2p-1} + 4k^2\left(-2s + \frac{n}{kp}\right)x\\ &- \frac{2kn}{p} - \frac{2nk^2}{p} + O\left(\frac{1}{p^2}\right)\\ \geqslant &\, 4k^2\left(2s - \frac{n}{kp}\right)(-x) - \frac{2kn}{p} - \frac{2nk^2}{p} + O\left(\frac{1}{p^2}\right).\end{aligned}$$

It follows from (3.34), (3.36), and the above estimate on $J'(x)$ that

$$\begin{aligned}\frac{1}{p}J'(x) \geqslant\ &4k^2\,\frac{5n}{3p}\left[\frac{ck}{2pn}\right]^{\frac{1}{2p-1}} - \frac{2kn}{p} - \frac{2nk^2}{p} + O\left(\frac{1}{p^2}\right)\\ &> \frac{2kn(2k-1)}{p} > 0\end{aligned} \tag{3.37}$$

for any $p > p_1(n,k,c)$ large enough and any $-1 \leqslant x \leqslant x_0$. Note that in (3.37), we have used

$$\lim_{p\to+\infty}\left[\frac{ck}{2pn}\right]^{\frac{1}{2p-1}} = 1. \tag{3.38}$$

This completes the proof of Lemma 3.18. □

Let us remark that if we consider $M_{2,1}$, the conclusion of Theorem 3.16 is not necessarily stronger than that of Proposition 3.14. The point is that the degree of the generating polynomial $\phi(U)$ in Proposition 3.17 might depend on the c where $[-c, c]$ is the domain of $\phi(U)$. In particular, the degree p goes to infinity as c approaches to 0, while the generating function in Proposition 3.14 is a quartic polynomial. However, we are able to show that the proof of Proposition 3.17 can be used to establish the path-connectedness of all $U(n)$-invariant Kähler metrics of $H > 0$ on any Hirzebruch manifold $M_{n,k}$.

Corollary 3.19. — *The space of all $U(n)$-invariant Kähler metrics of $H > 0$ on $M_{n,k}$ is path-connected.*

Proof. — In view of Corollary 3.9, it suffices to show that given any $0 < c_1 < c_2 < \frac{n}{k}$, we can construct a continuous family of generating functions $\phi_c(U)$ where $U \in [-c, c]$ for any $c_1 \leqslant c \leqslant c_2$.

Such a family can be constructed following the proof of Proposition 3.17. In particular, if we examine (3.18), (3.23), (3.25), (3.26), and (3.38), we conclude by the continuous dependence of parameters that for any $c \in [c_1, c_2]$ there exists a sufficiently large integer $p = p(n, k)$ which is independent of the choice of c, $\delta_1 = \delta_1(p, n, k, c)$, and $\delta_2 = \delta_2(p, n, k, c)$ such that $\phi_c(U)$ with $c \in [c_1, c_2]$ defined by (3.17) is a continuous path of Kähler metrics with $H > 0$. □

3.4. Complete Kähler–Ricci solitons revisited

First let us recall the definition of gradient Kähler–Ricci solitons.

Definition 3.20 (Shrinking gradient Kähler–Ricci solitons). — *Let (M^n, J) be a complex manifold with a Kähler metric g and f a real-valued function. The quadruple (M^n, J, g, f) is called a shrinking gradient Kähler–Ricci soliton if the following two conditions hold:*

(1) $\mathrm{Ric}(g)_{i\bar{j}} - g_{i\bar{j}} - f_{i\bar{j}} = 0$,

(2) *After expressing ∇f in terms of holomorphic coordinates, its* $(1,0)$-*part is a holomorphic vector field.*

Shrinking Kähler–Ricci solitons are important objects in the study of finite type singularity of Kähler–Ricci flow. We refer to [13] for a survey of recent developments on (Kähler-)Ricci solitons.

In this subsection, we are interested in complete Kähler–Ricci solitons in the following two cases.

(1) Compact shrinking Kähler–Ricci soliton on Fano Hirzebuch manifolds $M_{n,k}(k < n)$ constructed by Cao [12] and Koiso [31],

(2) Complete noncompact shrinking Kähler–Ricci soliton on the total space of $H^{-k} \to \mathbb{CP}^{n-1}$ when $k < n$. These were constructed by Feldman–Ilmanen–Knopf (F-I-K) in [20].

We are motivated by the following general question:

Question. — *Are there are any characterization of compact shrinking Kähler–Ricci soliton with $H > 0$, and it is true that any complete noncompact shrinking Kähler–Ricci soliton with $H > 0$ must be compact?*

Let us review the example constructed in [12, 20, 31]. We follow Koiso's approach in the compact case and also treat the complete noncompact case in a similar way (See also [44]). Recall the Kähler metric $\tilde{g}$ on the compactification of the $\mathbb{C}^*$-bundle obtained by $H^{-k} \to \mathbb{CP}^{n-1}$ discussed in Subsection 3.2.1. In the compact case, we consider the smooth compatification which gives Hirzebruch manifold $M_{n,k}$, as discussed in Subsection 3.2.2. While in the noncompact case, we consider the compatification near E_0 and requires the metric to be complete along infinity. Therefore it results a complete Kähler metric on the total space of $H^{-k} \to \mathbb{CP}^{n-1}$.

Lemma 3.21 ([32, p. 169–172]). — *Given any Kähler metric $\tilde{g}$ on $M_{n,k}(k < n)$ in the form* (3.3) *and satisfy the assumption of Lemma 3.5 and* (3.8). *If we assume in addition that $\tilde{g}$ is in $2\pi c_1(M_{n,k})$, this is to say, there exists a function f on $M_{n,k}$ such that*

$$\mathrm{Ric}(\tilde{g}) - \tilde{\omega_g} = \sqrt{-1}\partial\bar{\partial} f. \tag{3.39}$$

If we further set $U_{\min} = -1$ after a normalization, then we conclude that $U_{\max} = 1$, then f defined in (3.39) *only depends on U, and it must satisfy*

$$\frac{\mathrm{d}\phi}{\mathrm{d}U} + \frac{\phi}{Q}\frac{\mathrm{d}Q}{\mathrm{d}U} + 2U - \phi\frac{\mathrm{d}f}{\mathrm{d}U} = 0. \tag{3.40}$$

If we further assume that f in (3.39) is given by $f = -\alpha U$ for some constant α, then it follows that $\nabla f = -\frac{\alpha}{2}V$ is a holomorphic vector field. Equation (3.40) becomes

$$\frac{\mathrm{d}\phi}{\mathrm{d}U} + \frac{\phi}{Q}\frac{\mathrm{d}Q}{\mathrm{d}U} + 2U - \alpha\phi = 0. \tag{3.41}$$

Assume the soliton metric is of the form (3.3), one may apply Lemma 3.2, Lemma 3.3, and the proof of Lemma 3.21 to conclude that (3.41) is exactly the reduction of shrinking soliton equation in Definition 3.20. See also [31, 32] for more details.

Note that $Q = (1 + \frac{k}{n}U)^{n-1}$, so Equation (3.41) takes the form

$$\frac{\mathrm{d}\phi}{\mathrm{d}U} + \frac{k(n-1)}{n+kU}\phi + 2U - \alpha\phi = 0. \tag{3.42}$$

Equation (3.42) can be solved explicitly:

$$\phi(U) = \frac{2\eta(U,\alpha)}{Q(U)} - \frac{2e^{\alpha(U+1)}}{Q(U)}\eta(-1,\alpha), \tag{3.43}$$

where $\eta(U,\alpha)$ is a polynomial of degree n defined by

$$\int xe^{-\alpha x}Q(x)\mathrm{d}x = -e^{-\alpha x}\eta(x,\alpha). \tag{3.44}$$

Theorem 3.22 (Koiso [31], Cao [12], Feldman–Ilmanen–Knopf [20]). *For any integer $1 \leqslant k < n$, consider $(\mathbb{CP}^{n-1}, g_0)$ where g_0 is the Fubini–Study metric with* $\mathrm{Ric}(g_0) = g_0$.

(1) *There exists a unique shrinking Kähler–Ricci soliton of the form* (3.3) *on each $M_{n,k}$. The corresponding generating function $\phi : [-1,1] \to \mathbb{R}$ is strictly positive on $(-1,1)$ with $\phi(1) = \phi(-1) = 0$, $\phi'(-1) = -\phi'(1) = 2$, and of the form* (3.43). *It is unique in the sense that the value $\alpha > 0$ is determined by the unique solution of $\phi(1) = 0$.*

It is proved in [12] *that the Ricci curvature of the soliton metric is positive on $M_{n,k}$ if and only if $k = 1$.*

(2) *There exists a unique complete shrinking Kähler–Ricci soliton of the form* (3.3) *on the total space of $L^k \to \mathbb{CP}^{n-1}$. The corresponding generating function $\phi : [-1,+\infty] \to \mathbb{R}$ is strictly positive on $(-1,+\infty)$ with $\phi(-1) = 0$, $\phi'(-1) = 2$, and of the form* (3.43). *Here the value $\alpha > 0$ is determined by the unique solution to $\eta(-1,\alpha) = 0$.*

It follows from (3.42) and Proposition 3.6 that the curvature components for shrinking solitons are:

$$
\begin{aligned}
A &= -\frac{1}{2}\left[\alpha^2 - \frac{2\alpha k(n-1)}{n+kU} - \frac{k^2 n(n-1)}{(n+kU)^2}\right]\phi + \left(\alpha - \frac{k(n-1)}{n+kU}\right)U + 1\\
B &= \frac{[k^2 n - k(n+kU)\alpha]\phi + 2k(n+kU)U}{2(n+kU)^2},\\
C &= \frac{2(n+kU) - k^2\phi}{(n+kU)^2}.
\end{aligned}
$$

It is easy to see that along the zero section $U = -1$, $2B + \sqrt{AC} > 0$ implies that

$$
\alpha < \alpha_0(n,k) \doteq \frac{(n-2k)(k+1)}{n-k} \tag{3.45}
$$

For example, when $n = 2, k = 1$, the necessary condition for $H > 0$ is $\alpha < \alpha_0(2,1) = 0$, however the corresponding α on $M_{2,1}$ is the unique positive root of the equation $e^{2\alpha}(-\alpha^2 + 2) - 3\alpha^2 - 4\alpha - 2 = 0$ ($\alpha \simeq 0.5276195199$). Therefore the Cao–Koiso shrinking soliton on $M_{2,1}$ does not satisfy $H > 0$.

Next we analyze the noncompact case in more details. Consider the following polynomial which is of degree n in terms of α.

$$
\alpha^{n+1}\eta(x,\alpha) = -\alpha^{n+1}e^{\alpha x}\int x e^{-\alpha x}\left(1 + \frac{k}{n}x\right)^{n-1}\mathrm{d}x
$$

Making use of the integration formula

$$\int z^n e^z \mathrm{d}z = \left(\sum_{l=0}^{n} (-1)^{n-l} \frac{n!}{l!} z^l \right) e^z + C,$$

Some routine calculation leads to

$$\alpha^{n+1} \eta(U, \alpha) = \left(\frac{k}{n} \right)^{n-1} \left(\sum_{l=1}^{n} \frac{n!}{l!} \frac{(n+kU)^{l-1}(n+kU-l)}{k^l} \alpha^l + n! \right).$$

Therefore the value of α_* which solves the shrinking soliton equation, namely, the root of the polynomial $\eta(-1, \alpha)$, is reduced to root of the following polynomial of degree n

$$\chi(\alpha) = \sum_{l=1}^{n} \frac{n!}{l!} \frac{(n-k)^{l-1}(n-k-l)}{k^l} \alpha^l + n!.$$

Similarly as in (36) in [20, p. 197], since $\chi(\infty) < 0$ and $\chi(0) > 0$, Descartes' rule of signs implies there exists a unique positive root α_* for $\chi(\alpha)$. With this choice $\alpha = \alpha_*$, $\phi(U)$ has the asymptotical behavior $\phi \sim \frac{1}{\alpha_*} U$ as $U \to \infty$, and it shows that the soliton metric is complete along infinity.

Now we are interested in a more precise estimate of α_*. First note that $\alpha_* > k$. To see this, we check

$$\begin{aligned} \chi(\alpha) &= \sum_{l=1}^{n} \frac{n!}{l!} \frac{(n-k)^{l-1}(n-k-l)}{k^l} \alpha^l + n! \\ &= \left(\frac{\alpha(n-k)}{k} \right)^n + \sum_{l=0}^{n-1} \frac{n!}{l!} \left(\frac{\alpha(n-k)}{k} \right)^l \left(1 - \frac{\alpha}{k} \right). \end{aligned} \tag{3.46}$$

We propose the following conjecture on a precise estimate on α_*.

Conjecture 3.23. — *For any $1 \leqslant k < n$, α_*, which is the unique positive root of the polynomial $\chi(\alpha)$, satisfies $\alpha_0(n, k) < \alpha_* < k+1$. If so, then none of F-I-K shrinking solitons satisfy $H > 0$.*

Argue similarly as in (3.46), we can show that Conjecture 3.23 is indeed true if $k < n \leqslant k^2 + 2k$. It is very likely that it is true in general, as numerical experiments suggest.

On the other hand, compact shrinking solitons on $M_{n,k}$ could have positive holomorphic sectional curvature as the ratio $\frac{n}{k}$ grows larger.

Proposition 3.24 (Some shrinking solitons on $M_{n,k}$ have $H > 0$). — *If we fix $k = 1$, then the lowest dimensional example of Cao–Koiso shrinking* *solitons which satisfies $H > 0$ is on $M_{3,1}$. On $M_{3,1}$ its local pinching constant is $\frac{1-\alpha}{(2-\alpha)(5-\alpha)} \simeq 0.05587$ where $\alpha \simeq 0.6820161326$. If $k = 2$, the lowest dimensional example with $H > 0$ is on $M_{7,2}$, where $\alpha \simeq 1.742423694$.*

Let us explain the calculation in the case of $M_{3,1}$. In this case the corresponding α is the unique positive solution of the following equation

$$16\alpha^3 + 24\alpha^2 + 18\alpha + 6 - e^{2\alpha}(-4\alpha^3 + 6\alpha + 6) = 0.$$

In particular, $\alpha \simeq 0.6820161326 < \alpha_0(3,1) = 1.5$ where $\alpha_0(n,k)$ defined in (3.45). One can show that $(2B + \sqrt{AC})(3 + U)^2$ is increasing on $U \in [-1,1]$ by a direct calculation, therefore we have $H > 0$. In general, for any $M_{n,1}$ with $n \geqslant 3$, it is likely that $\alpha < \alpha_0(n,k)$ and $(2B+\sqrt{AC})(n+kU)^2$ is increasing on $U \in [-1,1]$. It seems tedious to verify it and we do not have an affirmative answer. In any case, we expect that the Cao–Koiso shrinking soliton on Hirzebruch manifold $M_{n,1}$ have $H > 0$ for any $n \geqslant 3$.

Let us calculate the local holomorphic pinching constant of Cao–Koiso soliton on $M_{3,1}$. A similar argument as in Example 3.14 shows that

$$\frac{\min_{\|v\|=1} H(U,v)}{\max_{\|v\|=1} H(U,v)} = \begin{cases} \frac{AC-4B^2}{A(A+C-4B)}, & U \in [-1, U_*] \\ \frac{C}{A}, & U \in [U_*, 1] \end{cases}$$

Here U_* is the solution of $2B = C$ on $[-1,1]$, whose numerical value is about -0.573003. Therefore the pinching constant is obtained at $U = -1$, which is $\frac{1-E}{(2-E)(5-E)} \simeq 0.05587$.

Let us remark that if we drop the assumption of the shrinking Kähler–Ricci soliton, it is easy to write down examples of complete Kähler metrics of $H > 0$ on the total space of $H^{-k} \to \mathbb{CP}^1$. For instance, we have

Example 3.25. — Define a function $\psi(U)$ on $[-\frac{1}{2}, +\infty)$ as follows:

$$\psi(U) = \begin{cases} \frac{1}{2} - 2U^2, & U \in [-\frac{1}{2}, -\frac{1}{4}) \\ c(U+2) + \frac{1}{2}\ln(U+2), & U \in [-\frac{1}{4}, \infty) \end{cases}$$

where $c = \frac{3}{14} - \frac{2}{7}\ln(\frac{7}{4}) \sim 0.05439$ so that ψ is continuous at $U = -\frac{1}{4}$. Choose a small number $\delta > 0$ so that ψ admits a convex smoothing ϕ which equals to ψ except inside $(-\frac{1}{4} - \delta, -\frac{1}{4} + \delta)$, Note that for ψ, both $C > 0$ and $2B + \sqrt{AC}$ have positive lower bounds in $(-\frac{1}{4} - \delta, -\frac{1}{4})$ and $(-\frac{1}{4}, -\frac{1}{4} + \delta)$. It guarantees the existence of a convex smoothing ϕ which in turns gives a complete Kähler metric with $H > 0$ on the total space of $L^{-1} \to \mathbb{CP}^1$. The metric actually has positive bisectional curvature outside a compact subset. Moreover, its bisectional curvatures decay quadratically along the infinity. Asymptotically such a metric has a conical end and admits holomorphic functions with polynomial growth.

3.5. $H > 0$ is not preserved along the Kähler–Ricci flow

Recently there has been much progress on Kähler–Ricci flow with $U(n)$-symmetry on Hirzebruch manifolds $M_{n,k}$. See for example Zhu [48], Song–Weinkove [40], Fong [21, 34], and Guo–Song [24]. In this subsection, we apply Hitchin's example and new examples constructed in Theorem 3.16 to show that in general $H > 0$ is not preserved by the Kähler–Ricci flow.

These results imply that if the initial metric g_0 is $U(n)$-invariant and in the Kähler class $\frac{b_0}{k}[E_\infty] - \frac{a_0}{k}[E_0]$ where $b_0 > a_0 > 0$, then the flow always develops a singularity in finite time. Let $T < \infty$ denote the maximal existence time. Their results can be summarized as:

(1) Suppose that the initial metric g_0 satisfies $\frac{|E_\infty|}{|E_0|} = \frac{b_0}{a_0} = \frac{n+k}{n-k}$, where $|E_0|$ denotes the volume of the divisor E_0 with respect to g_0. In this case the Kähler class of g_0 is proportional to the anti-canonical class of $M_{n,k}$. Then the Kähler–Ricci flow shrinks the fiber and the base uniformly and collapses to a point. The rescaled flow converges to the Cao–Koiso soliton on $M_{n,k}$ ([48]).

(2) If g_0 satisfies $\frac{b_0}{a_0} < \frac{n+k}{n-k}$, then the Kähler–Ricci flow shrinks the fiber first, and the flow collapses to the base $\mathbb{CP}^{n-1}$ ([40]). The rescaled flow converges to $\mathbb{C}^{n-1} \times \mathbb{CP}^1$ ([21]).

(3) If g_0 satisfies $\frac{b_0}{a_0} > \frac{n+k}{n-k}$, then the Kähler–Ricci flow shrinks the zero section E_0 first and hence "contracts the exceptional divisor" ([40]). The flow, after a parabolic rescaling converges to the F-I-K shrinking soliton on the total space of $H^{-k} \to \mathbb{CP}^{n-1}$ ([24]).

(4) If $n \leqslant k$, then the Kähler–Ricci flow shrinks the fiber first, and the flow converges to the base $\mathbb{CP}^{n-1}$ in the Gromov–Hausdorff sense as $t \to T$ ([40]).

Based on Hitchin's examples reformulated in Example 3.13 and new examples we constructed in Proposition 3.17, we have the following examples of the $U(n)$-invariant Kähler–Ricci flow.

On $M_{2,1}$, let us take the metric in the anti-canonical class constructed in Example 3.14 as the initial metric. Then the normalized flow converges to the Cao–Koiso soliton. Unfortunately, the positivity of H breaks down. Therefore, in general, $H > 0$ is not preserved along the Kähler–Ricci flow. If instead we start from the initial metric corresponding to $\phi_c(U) = c - \frac{x^2}{c}$ on $[-c, c]$ s in Example 3.13, where $0 < c < \frac{2}{3}$, then the limiting metric of the unnormalized flow is $(\mathbb{CP}^1, cg_{FS})$. In this case, it is not clear whether $H > 0$ is preserved, or how the holomorphic pinching constant of $g(t)$ evolves.

On $M_{3,1}$, if we pick initial metric as the Cao–Koiso shrinking soliton, it follows from Proposition 3.24 that it satisfies $H > 0$. Since it is a fixed point of the normalized flow, the holomorphic pinching constant remains constant.

On $M_{4,1}$, if we pick initial metric by the examples $\phi_c(U) = c - \frac{x^2}{c}$ on $[-c, c]$ for any $0 < c < \frac{4}{3}$. Then each of the first three cases mentioned above could occur. For $c = 1$, the normalized flow evolves $\phi_1(U)$ to the Cao–Koiso soliton. While the initial metric has the holomorphic pinching constant $2/27 \simeq 0.074$, the limit metric has the holomorphic pinching constant $\simeq 0.095$. The pinching constant indeed improves after a long time, though we do not know the short time behavior of pinching constants along the flow. If $1 < c < \frac{4}{3}$, then the rescaled flow converges to the F-I-K shrinking soliton on the total space of $L^{-1} \to \mathbb{CP}^3$. However this F-I-K soliton no longer has $H > 0$, and once again we see that $H > 0$ is not preserved under Kähler–Ricci flow.

Therefore a natural question arises: is there an effective way to construct a one-parameter family of deformation of Kähler metrics with $H > 0$? It would be ideal if the holomorphic pinching constant could enjoy some monotonicity properties along this deformation.

4. Kähler metrics of $H > 0$ from the submanifold point of view

In this section, we discuss holomorphic pinching of the canonical Kähler–Einstein metrics on some Kähler C-spaces. Furthermore, we make some general remarks on the question of constructing $H > 0$ metric from the submanifold point of view.

Proposition 4.1. — *Consider the flag threefold, or more generally, let M be the hypersurface in $\mathbb{CP}^n \times \mathbb{CP}^n$ defined by*

$$\sum_{i=1}^{n+1} z_i w_i = 0, \tag{4.1}$$

where $n \geqslant 2$ and $([z], [w])$ are the homogeneous coordinates. Let g be the restriction on M of the product of the Fubini–Study metrics (each of which has $H = 2$). Then the holomorphic sectional curvature of g is between 2 and $\frac{1}{2}$. So the holomorphic pinching constant is $\frac{1}{4}$, which is dimension free.

Proof of Proposition 4.1. — Let us work on the case $n = 2$ and in the inhomogeneous coordinates $[1, z_1, z_2]$ and $[w_1, 1, w_2]$. The hypersurface $M^3 \subset \mathbb{CP}^2 \times \mathbb{CP}^2$ is defined by $w_1 + z_1 + z_2 w_2 = 0$ and can be parametrized by

$$(4.2) \qquad (t_1, t_2, t_3) \to [1, t_1, t_2] \times [-t_1 - t_2 t_3, 1, t_3].$$

It follows that

$$\frac{\partial}{\partial t_1} = \frac{\partial}{\partial z_1} - \frac{\partial}{\partial w_1}, \ \frac{\partial}{\partial t_2} = \frac{\partial}{\partial z_2} - t_3 \frac{\partial}{\partial w_1}, \ \frac{\partial}{\partial t_3} = -t_2 \frac{\partial}{\partial w_1} + \frac{\partial}{\partial w_2}.$$

Therefore under $\{\frac{\partial}{\partial t_1}, \frac{\partial}{\partial t_2}, \frac{\partial}{\partial t_3}\}$ the induced metric $\widetilde{g}$ has the form:

$$\begin{pmatrix} g_{1\bar{1}} + h_{1\bar{1}} & g_{1\bar{2}} + \overline{t_3} h_{1\bar{1}} & \overline{t_2} h_{1\bar{1}} + h_{1\bar{2}} \\ g_{2\bar{1}} + \overline{t_3} h_{1\bar{1}} & g_{2\bar{2}} + |t_3|^2 h_{1\bar{1}} & t_3 \overline{t_2} h_{1\bar{1}} - t_3 h_{1\bar{2}} \\ t_2 h_{1\bar{1}} - h_{2\bar{1}} & t_2 \overline{t_3} h_{1\bar{1}} - \overline{t_3} h_{2\bar{1}} & |t_2|^2 h_{1\bar{1}} + h_{2\bar{2}} \end{pmatrix}$$

where $g_{i\bar{j}} = g(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \overline{z_j}})$ and $h_{i\bar{j}} = g(\frac{\partial}{\partial w_i}, \frac{\partial}{\partial \overline{w_j}})$ where $1 \leqslant i, j \leqslant 2$ are Fubini–Study metrics on two factors $\mathbb{CP}^2$ respectively:

$$g_{i\bar{j}} = \frac{\delta_{ij}}{1 + |z|^2} - \frac{z_j \overline{z_i}}{(1 + |z|^2)^2}, \quad h_{i\bar{j}} = \frac{\delta_{ij}}{1 + |w|^2} - \frac{w_j \overline{w_i}}{(1 + |w|^2)^2}.$$

Recall that the curvature tensor of $\widetilde{g}$ is given by the formula (3.13). Moreover, we observe that there is a natural $U(3)$-action on M which acts transitively because of the defining equation (4.1). Therefore it suffices to calculate the curvature tensor of the induced metric $\widetilde{g}$ at the point $(t_1, t_2, t_3) = (0, 0, 0)$. Now pick an orthonormal frame $\{e_1, e_2, e_3\} = \{\frac{1}{\sqrt{2}} \frac{\partial}{\partial t_1}, \frac{\partial}{\partial t_2}, \frac{\partial}{\partial t_3}\}$. Then a straight forward calculation shows that the only non vanishing curvature components under $\{e_1, e_2, e_3\}$ are the following:

$$R_{1\bar{1}1\bar{1}} = 1, \quad R_{2\bar{2}2\bar{2}} = 2, \quad R_{3\bar{3}3\bar{3}} = 2,$$
$$R_{1\bar{1}2\bar{2}} = R_{1\bar{1}3\bar{3}} = \frac{1}{2}, \quad R_{2\bar{2}3\bar{3}} = -\frac{1}{2}.$$

From this, we get that $R_{i\bar{j}} = 2\delta_{ij}$ for any $1 \leqslant i \neq j \leqslant 3$, so $\widetilde{g}$ is Kähler–Einstein. Once we have all the curvature components, it is direct to see that $\min_{\|X\|=1} H(X) = H(\frac{e_2+e_3}{\sqrt{2}}) = \frac{1}{2}$ and $\max_{\|X\|=1} H(X) = 2$.

The same argument works for the hypersurface defined by $\sum_{i=1}^{n+1} z_i w_i = 0$ in $\mathbb{P}^n \times \mathbb{P}^n$. It gives the same pinching constant $\frac{1}{4}$ for any $n \geqslant 2$. $\square$

However, if we try a similar calculation on other types of bidegree (p, q) hypersurfaces in $\mathbb{CP}^n \times \mathbb{CP}^n$, the argument breaks down.

As a simple example, consider the bidegree $(2, 1)$ hypersurface defined by $\sum_{i=0}^{2} z_i^2 w_i = 0$ in $\mathbb{CP}^2 \times \mathbb{CP}^2$ given by homogeneous coordinates $([z], [w])$. Consider the following parametrization of the hypersurface:

$$(t_1, t_2, t_3) \to [1, t_1, t_2,] \times [-t_1^2 - t_2^2 t_3, 1, t_3].$$

and

$$\frac{\partial}{\partial t_1}=\frac{\partial}{\partial z_1}-2t_1\frac{\partial}{\partial w_1},\ \frac{\partial}{\partial t_2}=\frac{\partial}{\partial z_2}-2t_2t_3\frac{\partial}{\partial w_1},\ \frac{\partial}{\partial t_3}=-t_2^2\frac{\partial}{\partial w_1}+\frac{\partial}{\partial w_2}.$$

The corresponding $\widehat{g}$ induced from the product of Fubini–Study metric on $\mathbb{CP}^2\times\mathbb{CP}^2$ is

$$\begin{pmatrix} g_{1\bar{1}}+4|t_1|^2h_{1\bar{1}} & g_{1\bar{2}}+4t_1\overline{t_2t_3}h_{1\bar{1}} & 2t_1\overline{t_2}^2h_{1\bar{1}}-2t_1h_{1\bar{2}} \\ g_{2\bar{1}}+4\overline{t_1}t_2t_3h_{1\bar{1}} & g_{2\bar{2}}+4|t_2|^2|t_3|^2h_{1\bar{1}} & 2|t_2|^2\overline{t_2}t_3\overline{t_2}h_{1\bar{1}}-2t_2t_3h_{1\bar{2}} \\ t_2^2\overline{t_1}h_{1\bar{1}}-2\overline{t_1}h_{2\bar{1}} & 2|t_2|^2t_2\overline{t_2t_3}h_{1\bar{1}}-2\overline{t_2t_3t_3}h_{2\bar{1}} & |t_2|^4h_{1\bar{1}}+h_{2\bar{2}} \end{pmatrix}$$

If we only calculate the curvature at $(t_1,t_2,t_3)=(0,0,0)$, we already encounter some negativity of H. First note that $(\frac{\partial}{\partial t_1},\frac{\partial}{\partial t_2},\frac{\partial}{\partial t_3})$ is orthonormal at $(0,0,0)$, then it follows that

$$R_{1\bar{1}1\bar{1}}=-\frac{\partial^2\widehat{g}_{1\bar{1}}}{\partial t_1\overline{t_1}}=-4h_{1\bar{1}}-\frac{\partial^2 g_{1\bar{1}}}{\partial z_1\overline{z_1}}=-2.$$

The same problem occurs for a general bidegree $(2,1)$ hypersurface in $\mathbb{CP}^n\times\mathbb{CP}^n$. Of course, this just means that for a bidegree $(2,1)$ hypersurface in $\mathbb{CP}^n\times\mathbb{CP}^n$, the restriction of the product of the Fubini–Study metric on the hypersurface does not have $H>0$. But presumably there could be other metrics on it with $H>0$. This is indeed the case and we have the following result:

Proposition 4.2. — *Let M^n be any smooth bidegree $(p,1)$ hypersurface in $\mathbb{CP}^r\times\mathbb{CP}^s$, where $n=r+s-1$, $p\geqslant 1$, and r, $s\geqslant 2$. Then M^n admits a Kähler metric with $H>0$. Morever, when $p>r+1$, the Kähler classes of all the Kähler metrics with $H>0$ form a proper subset of the Kähler cone of M^n.*

Proof of Proposition 4.2. — Let $[z]$ and $[w]$ be the homogeneous coordinates of $\mathbb{CP}^r$ and $\mathbb{CP}^s$, respectively. Let $\pi:\mathbb{CP}^r\times\mathbb{CP}^s\to\mathbb{CP}^r$ be the projection map. Suppose that M^n is defined by

$$\sum_{i=1}^{s+1} f_i(z_1,\dots,z_{r+1})w_i=0,$$

where each f_i is a homogeneous polynomial of degree p. Consider the sheaf map $h:\mathcal{O}^{\oplus(s+1)}\to\mathcal{O}(p)$ on $\mathbb{CP}^r$ defined by

$$h(e_i)=f_i(z),\quad 1\leqslant i\leqslant s+1,$$

where $e_i=(0,\dots,0,1,0,\dots,0)$ has 1 at the i-th position. Clearly, h is surjective, and its kernel sheaf E is locally free. Since $M^n=\mathbb{P}(E)$ over $\mathbb{CP}^r$, by the result of [2], we know that M^n admits Kähler metrics with $H>0$.

To see the second part of the statement, let us denote by H_1, H_2 the hyperplane section from the two factors restricted on M, then we have $c_1(M) = (r+1-p)H_1 + sH_2$. Clearly, $H_1^{r+1} = 0$, $H_2^{s+1} = 0$, and since $M^n \sim pH_1 + H_2$, we have $H_1^r H_2^{s-1} = 1$ and $H_1^{r-1} H_2^s = p$ on M^n. For any Kähler class $[\omega] = aH_1 + bH_2$ where $a > 0$ and $b > 0$, we have

$$c_1(M) \cdot [\omega]^{n-1} = a^{r-2} b^{s-2} \left[\binom{n-1}{r} sa^2 + \binom{n-1}{r-1} (r+1+sp-p)ab \right. \\ \left. + \binom{n-1}{r-2} (r+1-p)b^2 \right].$$

So when $p > r+1$ and $b >> a$, we know that the total scalar curvature of (M^n, ω) is negative. Thus the Kähler classes of metrics with $H > 0$ can not fill in the entire Kähler cone. □

For a smooth bidegree $(p,2)$ hypersurface M^n in $\mathbb{CP}^r \times \mathbb{CP}^s$, where $n = r+s-1$ and $p \geqslant 2$, one may raise the question of whether M^n admits Kähler metrics with $H > 0$? The answer might be yes in view of Proposition 4.2. Note that if we project to $\mathbb{CP}^r$, then M becomes a holomorphic fibration over $\mathbb{CP}^r$ whose generic fiber are smooth quadrics.

Conjecture 4.3. — *If M^3 is a compact Kähler manifold with local holomorphic pinching constant strictly greater $\frac{1}{4}$, then is it biholomorphic to a compact Hermitian symmetric space, i.e. $\mathbb{CP}^3$, $\mathbb{CP}^2 \times \mathbb{CP}^1$, $\mathbb{CP}^1 \times \mathbb{CP}^1 \times \mathbb{CP}^1$, or Q^3 which is the smooth quadric in $\mathbb{CP}^4$?*

Let us conclude the discussion here by a couple of general remarks. First, if we want to construct metrics with $H > 0$ from the submanifold point of view, in particular, as complete intersections. Then it seems difficult to find examples other than those already known (such as Hermitian symmetric spaces or Kähler C-spaces, or projectivized vector bundles covered in [2]). For instance, if we consider a cubic hypersurface $M^n \subset \mathbb{CP}^{n+1}$ and g be the restriction on M of the Fubini–Study metric. Then it is unclear if (M^n, g) can have $H > 0$, though we expect the answer is no. As another example, if we consider the restriction of the ambient Fubini–Study metric onto a complete intersection, where typically we need to restrict to degree 1 or 2. Let us consider M^n as the intersection of two quadrics, in the case of $n = 2$, it is the del Pezzo surface of degree 4, or $\mathbb{CP}^2$ blowing up 5 points. It is unlikely that the induced metric on M^n admits $H > 0$.

Secondly, it is a general belief that the existence of a Kähler metric of $H > 0$ is a "large open" condition, as illustrated in this paper on any Hirzebruch manifold. Therefore, it is reasonable to expect if a projective manifold M admits a Kähler metric g_0 of $H > 0$, then there exsits a small deformation of such a metric g_1 which lies in a Hodge class and still has $H > 0$. hence one can conclude from a theorem of Tian ([41]) that g_1 can be approximated by pull backs of Fubini–Study metrics by a sequence of projective embeddings $\phi_k : M \to \mathbb{CP}^{N_k}$. However, it seems difficult to construct examples of Kähler metrics of $H > 0$ in this way because of the implicit nature of ϕ_k and N_k.

In a sequel of this paper, we will study examples of Kähler metrics with $H > 0$ on other rational surfaces, other Kähler C-spaces, and higher dimensional projective manifolds.

BIBLIOGRAPHY

[1] A. Alvarez, A. Chaturvedi & G. Heier, "Optimal pinching for the holomorphic sectional curvature of Hitchin's metrics on Hirzebruch surfaces", in *Rational points, rational curves, and entire holomorphic curves on projective varieties*, Contemporary Mathematics, vol. 654, American Mathematical Society, 2015, p. 133-142.

[2] A. Alvarez, G. Heier & F. Zheng, "On projectivized vector bundles and positive holomorphic sectional curvature", *Proc. Am. Math. Soc.* **146** (2018), no. 7, p. 2877-2882.

[3] V. Apostolov, D. M. J. Calderbank, P. Gauduchon & C. W. Tønnesen-Friedman, "Extremal Kähler metrics on ruled manifolds and stability", in *Géométrie différentielle, physique mathématique, mathématiques et société. II*, Astérisque, vol. 322, Société Mathématique de France, 2008, p. 93-150.

[4] M. Berger, "Pincement riemannien et pincement holomorphe", *Ann. Sc. Norm. Super. Pisa, Cl. Sci.* **14** (1960), p. 151-159.

[5] ———, "Sur quelques variétés riemanniennes suffisamment pincées", *Bull. Soc. Math. Fr.* **88** (1960), p. 57-71.

[6] ———, "Sur les variétés d'Einstein compactes", in *Comptes Rendus de la IIIe Réunion du Groupement des Mathématiciens d'Expression Latine (Namur, 1965)*, Librairie Universitaire, Louvain, 1966, p. 35-55.

[7] R. L. Bishop & S. I. Goldberg, "On the topology of positively curved Kaehler manifolds", *Tôhoku Math. J.* **15** (1963), p. 359-364.

[8] ———, "On the topology of positively curved Kaehler manifolds. II", *Tôhoku Math. J.* **17** (1965), p. 310-318.

[9] S. Boucksom, J.-P. Demailly, M. Păun & T. Peternell, "The pseudo-effective cone of a compact Kähler manifold and varieties of negative Kodaira dimension", *J. Algebr. Geom.* **22** (2013), no. 2, p. 201-248.

[10] E. Calabi, "Métriques kählériennes et fibrés holomorphes", *Ann. Sci. Éc. Norm. Supér.* **12** (1979), no. 2, p. 269-294.

[11] ———, "Extremal Kähler metrics", in *Seminar on Differential Geometry*, Annals of Mathematics Studies, vol. 102, Princeton University Press, 1982, p. 259-290.

[12] H.-D. Cao, "Existence of gradient Kähler–Ricci solitons", in *Elliptic and parabolic methods in geometry (Minneapolis, 1994)*, A K Peters, 1996, p. 1-16.

[13] ———, "Geometry of complete gradient shrinking Ricci solitons", in *Geometry and analysis. No. 1*, Advanced Lectures in Mathematics (ALM), vol. 17, International Press., 2011, p. 227-246.

[14] H.-D. Cao & R. Hamilton, unpublished, 1992.

[15] B.-Y. Chen, "Extrinsic spheres in Kähler manifolds", *Mich. Math. J.* **23** (1976), no. 4, p. 327-330.

[16] X. Chen, "On Kähler manifolds with positive orthogonal bisectional curvature", *Adv. Math.* **215** (2007), no. 2, p. 427-445.

[17] X. Chen & G. Tian, "Ricci flow on Kähler–Einstein surfaces", *Invent. Math.* **147** (2002), no. 3, p. 487-544.

[18] ———, "Ricci flow on Kähler-Einstein manifolds", *Duke Math. J.* **131** (2006), no. 1, p. 17-73.

[19] A. S. Dancer & M. Y. Wang, "On Ricci solitons of cohomogeneity one", *Ann. Global Anal. Geom.* **39** (2011), no. 3, p. 259-292.

[20] M. Feldman, T. Ilmanen & D. Knopf, "Rotationally symmetric shrinking and expanding gradient Kähler-Ricci solitons", *J. Differ. Geom.* **65** (2003), no. 2, p. 169-209.

[21] F. T.-H. Fong, "Kähler-Ricci flow on projective bundles over Kähler-Einstein manifolds", *Trans. Am. Math. Soc.* **366** (2014), no. 2, p. 563-589.

[22] P. Griffiths & J. Harris, *Principles of algebraic geometry*, Wiley Classics Library, John Wiley & Sons, 1994, xiv+813 pages.

[23] H. Gu & Z. Zhang, "An extension of Mok's theorem on the generalized Frankel conjecture", *Sci. China, Math.* **53** (2010), no. 5, p. 1253-1264.

[24] B. Guo & J. Song, "On Feldman-Ilmanen-Knopf's conjecture for the blow-up behavior of the Kähler Ricci flow", *Math. Res. Lett.* **23** (2016), no. 6, p. 1681-1719.

[25] G. Heier & B. Wong, "Scalar curvature and uniruledness on projective manifolds", *Commun. Anal. Geom.* **20** (2012), no. 4, p. 751-764.

[26] ———, "On projective Kähler manifolds of partially positive curvature and rational connectedness.", `https://arxiv.org/abs/1509.02149`, 2015.

[27] N. Hitchin, "On the curvature of rational surfaces", in *Differential geometry (Proc. Sympos. Pure Math., Vol. XXVII, Part 2, Stanford Univ., Stanford, 1973)*, American Mathematical Society, 1975, p. 65-80.

[28] A. D. Hwang & M. A. Singer, "A momentum construction for circle-invariant Kähler metrics", *Trans. Am. Math. Soc.* **354** (2002), no. 6, p. 2285-2325.

[29] S. Kobayashi & K. Nomizu, *Foundations of differential geometry. Vol. I*, Wiley Classics Library, John Wiley & Sons, 1996, xii+329 pages.

[30] S. Kobayashi & H.-H. Wu, "On holomorphic sections of certain hermitian vector bundles", *Math. Ann.* **189** (1970), p. 1-4.

[31] N. Koiso, "On rotationally symmetric Hamilton's equation for Kähler-Einstein metrics", in *Recent topics in differential and analytic geometry*, Advanced Studies in Pure Mathematics, vol. 18, Academic Press Inc., 1990, p. 327-337.

[32] N. Koiso & Y. Sakane, "Nonhomogeneous Kähler-Einstein metrics on compact complex manifolds", in *Curvature and topology of Riemannian manifolds (Katata, 1985)*, Lecture Notes in Mathematics, vol. 1201, Springer, 1986, p. 165-179.

[33] C. LeBrun, "Counter-examples to the generalized positive action conjecture", *Commun. Math. Phys.* **118** (1988), no. 4, p. 591-596.

[34] D. Máximo, "On the blow-up of four-dimensional Ricci flow singularities", *J. Reine Angew. Math.* **692** (2014), p. 153-171.

[35] S. Mori, "Projective manifolds with ample tangent bundles", *Ann. Math.* **110** (1979), no. 3, p. 593-606.

[36] O. Munteanu & J. Wang, "Positively curved shrinking Ricci solitons are compact", *J. Differ. Geom.* **106** (2017), no. 3, p. 499-505.

[37] L. Ni, "Ancient solutions to Kähler-Ricci flow", *Math. Res. Lett.* **12** (2005), no. 5-6, p. 633-653.

[38] S. R. Simanca, "Kähler metrics of constant scalar curvature on bundles over $\mathbf{C}\mathrm{P}_{n-1}$", *Math. Ann.* **291** (1991), no. 2, p. 239-246.

[39] Y. T. Siu & S. T. Yau, "Compact Kähler manifolds of positive bisectional curvature", *Invent. Math.* **59** (1980), no. 2, p. 189-204.

[40] J. Song & B. Weinkove, "The Kähler-Ricci flow on Hirzebruch surfaces", *J. Reine Angew. Math.* **659** (2011), p. 141-168.

[41] G. Tian, "On a set of polarized Kähler metrics on algebraic manifolds", *J. Differ. Geom.* **32** (1990), no. 1, p. 99-130.

[42] Y. Tsukamoto, "On Kählerian manifolds with positive holomorphic sectional curvature", *Proc. Japan Acad.* **33** (1957), p. 333-335.

[43] B. Wilking, "A Lie algebraic approach to Ricci flow invariant curvature conditions and Harnack inequalities", *J. Reine Angew. Math.* **679** (2013), p. 223-247.

[44] B. Yang, "A characterization of noncompact Koiso-type solitons", *Int. J. Math.* **23** (2012), no. 5, article ID 1250054 (13 pages).

[45] X. Yang, "Hermitian manifolds with semi-positive holomorphic sectional curvature", *Math. Res. Lett.* **23** (2016), no. 3, p. 939-952.

[46] S. T. Yau, "A review of complex differential geometry", in *Several complex variables and complex geometry, Part 2 (Santa Cruz, CA, 1989)*, Proceedings of Symposia in Pure Mathematics, vol. 52, American Mathematical Society, 1991, p. 619-625.

[47] ———, "Open problems in geometry", in *Differential geometry: partial differential equations on manifolds (Los Angeles, 1990)*, Proceedings of Symposia in Pure Mathematics, vol. 54, American Mathematical Society, 1993, p. 1-28.

[48] X. Zhu, "Kähler-Ricci flow on a toric manifold with positive first Chern class", in *Differential geometry*, Advanced Lectures in Mathematics (ALM), vol. 22, International Press., 2012, p. 323-336.

Manuscrit reçu le 14 décembre 2016,
révisé le 30 juin 2018,
accepté le 17 janvier 2019.

Bo YANG
School of Mathematical Sciences
Xiamen University
Xiamen, Fujian, 361005 (China)
boyang@xmu.edu.cn

Fangyang ZHENG
Department of Mathematics
The Ohio State University
231 West 18th Avenue
Columbus, OH 43210 (USA)
zheng.31@osu.edu