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FIRST AND SECOND K-GROUPS OF AN ELLIPTIC CURVE OVER A GLOBAL FIELD OF POSITIVE CHARACTERISTIC

by Satoshi KONDO & Seidai YASUDA (*)

Abstract. — In this paper, we show that the maximal divisible subgroup of groups K_1 and K_2 of an elliptic curve E over a function field is uniquely divisible. Further those K-groups modulo this uniquely divisible subgroup are explicitly computed. We also calculate the motivic cohomology groups of the minimal regular model of E, which is an elliptic surface over a finite field.

Résumé. — On démontre que les plus grands sous-groupes divisibles des-groupes K_1 et K_2 d'une courbe elliptique E sur un corps global de caractéristique positive sont uniquement divisibles et on décrit explicitement les K-groupes modulo leurs plus grands sous-groupes divisibles. On calcule également la cohomologie motivique du modèle minimal de E qui est une surface elliptique sur un corps fini.

1. Introduction

In this paper, we study the kernel and cokernel of boundary maps in the localization sequence of G-theory of the triple: an elliptic curve E over a global function field k of positive characteristic, the regular minimal model $\mathcal{E}$ of E that is proper flat over the curve C associated with k, and the fibers of $\mathcal{E} \to C$. The aim of this initial section is to provide our motivation, describe some necessary background, and present known results more generally. Precise mathematical statements regarding our results are presented

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in Section 1.5. Among these statements, we suggest that our most interesting point is represented in assertion (2) of Theorem 1.2, which relates the kernel of a boundary map to the special value $L(E,0)$ of the L-function of E.

1.1. Background and conjectures

From Grothendieck's theory of motives, Beilinson envisioned the abelian category of mixed motives over a base (see [35]). The existence of such a category remains a conjecture, but we now have various constructions of the triangulated category of (mixed) motives, which serves as the derived category of the sought-after category. Here, motivic cohomology groups are extension groups in the category of mixed motives, or, unconditionally, homomorphism groups in the triangulated category of motives. Algebraic K-theory also enters the picture here via the Atiyah–Hirzebruch-type spectral sequence (AHSS) of Grayson–Suslin or Levine [19], [36], [57] (see also [14, (1.8)], [18]), i.e.,

$$E_2^{s,t} = H_{\mathcal{M}}^{s-t}(X, \mathbb{Z}(-t)) \Rightarrow K'_{-s-t}(X)$$

or via formula [5, 9.1] (see also [18, p. 59])

$$K'_n(X) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \bigoplus_i H_{\mathcal{M}}^{2i-n}(X, \mathbb{Q}(i))$$

of Bloch. Also applicable here are the Riemann–Roch theorem for higher K-theory and the motivic cohomology theory by Gillet [17], Levine [33], Riou [54], and Kondo–Yasuda [31].

Arithmetic geometers are individuals who study motives (or varieties) over number fields, over the function fields of curves C over finite fields (denoted function fields for short), or over finite fields. For such a variety, the L-function, also known as the zeta function, one of the fundamental invariants in arithmetic geometry, is defined. The motivation for our work stems from a range of conjectures that describe motivic cohomology in terms of special values of these L-functions. Conjectures over $\mathbb{Q}$ were formulated by Lichtenbaum, Beilinson, and Bloch–Kato (see [27] for a summary of these conjectures). Using the analogy between number fields and function fields, parts of these conjectures can be translated into conjectures over function fields.

The case over function fields and over finite fields are related as follows. Given a variety over a function field, take a model, i.e., a variety over the base finite field fibered over C whose generic fiber is the given variety. Then, the motivic cohomology groups and the L-functions (over finite fields, i.e., congruence zeta functions) are related, thus, we obtain some conjectural statements regarding motivic cohomology groups of the model and its congruence zeta function from these conjectures over function fields concerning L-functions and vice versa. Note that Geisser [13] presents a description of K-theory and motivic cohomology groups of varieties over finite fields assuming both Parshin's conjecture and Tate's conjecture.

Finally, let us mention the finite generation conjecture of Bass [27, Conjecture 36], which states that the K-groups of a regular scheme of finite type over Spec $\mathbb{Z}$ are finitely generated.

1.2. Known results: one-dimensional cases

Unconditionally verifying the conjectures above is difficult, as is verifying the consequences of the conjectures. The key theorem here is the Rost–Voevodsky theorem (i.e., the Bloch–Kato conjecture) and, as a consequence, the Geisser–Levine theorem (i.e., a part of the Beilinson–Lichtenbaum conjecture) [16, Theorem 1.1, Corollary 1.2]. These theorems state that the motivic cohomology with torsion coefficients and étale cohomology are isomorphic in certain range of bidegrees.

These theorems and the aforementioned AHSS can be used as the primary ingredients for computing the K-groups and motivic cohomology groups of the ring of integers of global fields. In these "one-dimensional" cases, the Bass conjecture on finite generation is known to hold based on work by Quillen [53] and Grayson–Quillen [18]. In positive characteristic, these ingredients are sufficient for computing the motivic cohomology groups, and a substantial amount of computation can also be performed in the number field case. We refer to [62] for details on these unconditional results.

Unfortunately, much less is known regarding higher dimensions. Our results may be regarded as the next step in determining this in that we treat elliptic curves over a function field and elliptic surfaces over a finite field. The theorems of Rost–Voevodsky and Geisser–Levine do indeed cover many of the bidegrees, but there remain lower bidegrees that require additional work.

1.3. Computing motivic cohomology groups

To compute motivic cohomology with $\mathbb{Z}$-coefficient, one first divides the problem into $\mathbb{Q}$-coefficient, torsion coefficient, and the divisible part in the cohomology with $\mathbb{Z}$-coefficient. Then, for the prime-to-the-characteristic coefficient, one uses the étale realization, i.e., a map to the étale cohomology. For the p-part, one uses the map to the cohomology of de Rham–Witt complexes. Here, the target groups are presumably easier to compute than the motivic cohomology groups, and these realization maps are now known to be isomorphisms in many cases, due to Rost–Voevodsky and Geisser–Levine; however, no general method for computing the $\mathbb{Q}$-coefficient and divisible parts is known. In one-dimensional cases, we have finite generation theorems, which then imply that the divisible part is zero and the $\mathbb{Q}$-coefficient part is finite dimensional, and the $\mathbb{Q}$-coefficient part is zero except for bidegrees $(0,0)$, $(1,1)$, and $(2,1)$. For varieties over finite fields, the divisible part is conjecturally always zero, but this has not been shown in higher dimensions.

In this paper, we therefore study the weaker problem as to whether the divisible part is uniquely divisible for a smooth surface X over a finite field. In Section 2, we prove that the divisible part of motivic cohomology groups, which may still contain torsion, is actually uniquely divisible except for bidegree $(3,2)$, and we explicitly compute the quotient group. Further, by using the main result of our work in [30], we achieve a similar statement for the exceptional bidegree when X is a model of an elliptic curve over a function field.

When the coefficient in the motivic cohomology groups is the second Tate twist, especially when the bidegree is $(3,2)$, many results closely related to the computation of the torsion coefficients were obtained in a series of papers by Raskind, Colliot-Thélène, Sansuc, Soulé, Gros, and Suwa [8], [9], [20]. The general strategy is described by Colliot-Thélène–Sansuc–Soulé in [9], in which the primary focus was on the prime-to-p part. The p-part was treated by Gros–Suwa [20], though we need a supplementary statement on the p-part, which we provide in the Appendix. We follow their outline closely in Section 2 below.

We note here that the series of results above are written in terms of $\mathcal{K}$-cohomology groups and the associated realization maps. The $\mathcal{K}$-cohomology groups were used as a substitute for motivic cohomology groups and are now known to be isomorphic to motivic cohomology groups in particular bidegrees, as shown by Bloch, Jannsen, Landsburg, Müller–Stach, and Rost (see [45, Bloch's formula, Corollary 5.3, Theorem 5.4, p. 297]). Some earlier results do imply some of the results in our paper via this comparison isomorphism; however we present our exposition independent of earlier results and ensure that our work is self-contained for two reasons.

First, if we used earlier results, in possible applications of our result, some compatibility checks would have become necessary. For example, in earlier papers, a certain map from $\mathcal{K}$-cohomology to étale cohomology groups was used. Conversely, we use a cycle map from higher Chow groups given by Geisser–Levine [16]. It is nontrivial to check if these two maps are compatible under the comparison isomorphism. Therefore, while we do refer to earlier papers, we use only the non-motivic statements, reproducing the motivic statements in our paper where applicable. Second, some earlier results (e.g., those of [20]) are usually stated for projective schemes. We also need similar results for not necessarily projective schemes, since we treat a curve over a function field as the limit of surfaces fibered over (affine) curves over a finite field. When we remove the condition that the scheme considered is projective, it becomes much more difficult to show that the divisible part of motivic cohomology is uniquely divisible.

1.4. Motivic cohomology and K-theory

In our paper, we use Chern classes to relate K-theory and motivic cohomology. We use the Riemann–Roch theorem without denominators, and we perform some direct computations for singular curves over finite fields. We further explain these two independent issues below. Finally, in the last paragraph of this subsection, we describe the application.

The first issue is the comparison isomorphism between Levine's motivic cohomology and Bloch's higher Chow groups, an issue we detail in Section 3. The general problem is summarized as follows. We have various constructions of motivic cohomology theory (or groups), i.e., Suslin–Voevodsky–Friedlander [61], Hanamura [23], Levine [33], Bloch [5] (higher Chow groups), Morel–Voevodsky [44], and the graded pieces of rational K-groups. The motivic cohomology groups defined in various ways are generally known to be isomorphic, however, it is nontrivial to verify whether the comparison isomorphisms respect other structures, such as functoriality, Chern classes (characters), localization sequences, product structures, etc. The Riemann–Roch theorem without denominators is known to hold in the following three cases: by Gillet for cohomology theories satisfying his axioms [17]; by Levine for his motivic cohomology theory [33]; and by Kondo–Yasuda in the context of motivic homotopy theory of Morel and Voevodsky [31]. None of these directly apply to the motivic cohomology groups in our paper, namely the higher Chow groups of Bloch, hence we use the comparison isomorphism between higher Chow groups and Levine's motivic cohomology groups, importing the Riemann–Roch theorem of Levine into our setting by checking various compatibilities. The compatibility proof for Gysin maps and localization sequences under the comparison isomorphism are thus the focus of Section 3.

The second issue is the integral construction of Chern characters for singular curves over finite fields, which we cover more fully in Section 4. The Chern characters from K-theory (or G-theory) to motivic cohomology are usually defined for nonsingular varieties and come with denominators in higher degrees. In our paper, we focus on elliptic fibrations; singular curves over a finite field appear naturally as fibers, and we must study their G-theory. We do so by constructing a generalization of Chern characters in an ad hoc manner, generalizing those in the smooth case. To be able to use such an approach, we make substantial use of the fact that we consider only curves (however singular) over finite fields. We use the known computation of motivic cohomology and K-groups of nonsingular curves over finite fields, and, as one additional ingredient, we use a result from our previous work (Lemma 4.1). The principal result of Section 4 is Proposition 4.3 which states that the 0-th and 1-st G-groups are integrally isomorphic via these Chern characters to the direct sum of relevant motivic cohomology groups with a $\mathbb{Z}$-coefficient.

Finally, in Section 5, we apply results from Sections 2, 3, and 4, to compute the lower K-groups of a curve over a function field in terms of motivic cohomology groups. An additional ingredient here is the computation of K_3 of fields given by Nesterenko–Suslin [47] and Totaro [59].

1.5. Statement of results on K-groups

The aim of our paper is to explicitly compute the K_1 and K_2 groups, the motivic cohomology groups of an elliptic curve over a function field, and the motivic cohomology groups of an elliptic surface over a finite field. In this subsection, we provide the precise statements of our results (i.e., Theorems 1.1 and 1.2) concerning K_1 and K_2 groups of an elliptic curve. We refer to Theorems 6.2, 6.3, 7.1, and 7.2 below for more detailed results concerning motivic cohomology groups.

Let k be a global field of positive characteristic p, and let E be an elliptic curve over Spec k. Let C be the proper smooth irreducible curve over a finite field whose function field is k. We regard a place $\wp$ of k as a closed point of C, and vice versa. Let $\kappa(\wp)$ denote the residue field at $\wp$ of C. Let $f : \mathcal{E} \to C$ denote the minimal regular model of the elliptic curve

$E \to \operatorname{Spec} k$. This f is a proper, flat, generically smooth morphism such that for almost all closed points $\wp$ of C, the fiber $\mathcal{E}_\wp = \mathcal{E} \times_C \operatorname{Spec} \kappa(\wp)$ at $\wp$ is a genus one curve, and such that the generic fiber is the elliptic curve $E \to \operatorname{Spec} k$.

Let us identify the K-theory and G-theory of regular Noetherian schemes. First, there is a localization sequence of G-theory, i.e.,

$$K_i(\mathcal{E}) \longrightarrow K_i(E) \xrightarrow{\oplus\partial^i_\wp} \bigoplus_\wp G_{i-1}(\mathcal{E}_\wp) \longrightarrow K_{i-1}(\mathcal{E}),$$

in which $\wp$ runs over all primes of k.

For a scheme X, we let X_0 be the set of closed points of X. Let us use ∂_2 to denote the boundary map $\oplus\partial^2_\wp$, $\partial : K_2(E)^{\mathrm{red}} \to \bigoplus_{\wp\in C_0} G_0(\mathcal{E}_\wp)$ to denote the boundary map induced by ∂_2, and $\partial_1 : K_1(E)^{\mathrm{red}} \to \bigoplus_{\wp\in C_0} G_0(\mathcal{E}_\wp)$ to denote the boundary map induced by $\oplus\partial^1_\wp$. Here, for an abelian group M, we let M_{div} denote the maximal divisible subgroup of M, and set $M^{\mathrm{red}} = M/M_{\mathrm{div}}$.

In Theorems 1.1 and 1.2 below, we use the following notation. We use subscript $-_{\mathbb{Q}}$ to mean $- \otimes_{\mathbb{Z}} \mathbb{Q}$. For a scheme X, let $\operatorname{Irr}(X)$ be the set of irreducible components of X. Let $\mathbb{F}_q$ be the field of constants of C. For a scheme X of finite type over $\operatorname{Spec} \mathbb{F}_q$ and for $i \in \mathbb{Z}$, choose a prime number $\ell \neq p$ and set

$$L(h^i(X), s) = \det(1 - \mathrm{Frob} \cdot q^{-s}; H^i_{\mathrm{et}}(X \times_{\operatorname{Spec} \mathbb{F}_q} \operatorname{Spec} \overline{\mathbb{F}}_q, \mathbb{Q}_\ell)),$$

where $\overline{\mathbb{F}}_q$ is an algebraic closure of $\mathbb{F}_q$ and $\mathrm{Frob} \in \operatorname{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ is the geometric Frobenius element. In all cases considered in Theorems 1.1 and 1.2, the function $L(h^i(X), s)$ does not depend on the choice of ℓ. Let $T'_{(1)}$ denote what we call the twisted Mordell–Weil group

$$T'_{(1)} = \bigoplus_{\ell\neq p} (E(k \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_q)_{\mathrm{tors}} \otimes_{\mathbb{Z}} \mathbb{Z}_\ell(1))^{\operatorname{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)}.$$

We write S_0 (resp. S_2) for the set of primes of k at which E has split multiplicative (resp. bad) reduction; we also regard it as a closed subscheme of C with the reduced structure; note that the set of primes S_1 will be introduced later. Further, for a set M, we denote its cardinality by $|M|$. Finally, we let $r = |S_0|$.

Theorem 1.1 (see Theorem 6.1(1)(2) and Theorem 7.2 (2)). — *Suppose that S_2 is non-empty, or equivalently that f is not smooth.*

(1) *The dimension of the $\mathbb{Q}$-vector space $(K_2(E)^{\mathrm{red}})_{\mathbb{Q}}$ is r.*

(2) *The cokernel of the boundary map* $\partial_2 : K_2(E) \to \bigoplus_{\wp \in C_0} G_1(\mathcal{E}_\wp)$ *is a finite group of order*
$$\frac{(q-1)^2 |L(h^0(\mathrm{Irr}(\mathcal{E}_{S_2})), -1)|}{|T'_{(1)}| \cdot |L(h^0(S_2), -1)|}.$$

(3) *The group* $K_2(E)_{\mathrm{div}}$ *is uniquely divisible, and the kernel of the boundary map* $\partial : K_2(E)^{\mathrm{red}} \to \bigoplus_{\wp \in C_0} G_1(\mathcal{E}_\wp)$ *is a finite group of order* $|L(h^1(C), -1)|^2$

Let $L(E, s)$ denote the L-function of E (see Section 6.3). We write $\mathrm{Jac}(C)$ for the Jacobian of C.

Theorem 1.2 (see Theorem 6.1(3)(4)). — *Suppose that* S_2 *is nonempty, or equivalently that* f *is not smooth.*

(1) *The group* $K_1(E)_{\mathrm{div}}$ *is uniquely divisible.*
(2) *The kernel of the boundary map* $\partial_1 : K_1(E)^{\mathrm{red}} \to \bigoplus_{\wp \in C_0} G_0(\mathcal{E}_\wp)$ *is a finite group of order* $(q-1)^2 |T'_{(1)}| \cdot |L(E, 0)|$*. The cokernel of* ∂_1 *is a finitely generated abelian group of rank* $2 + |\mathrm{Irr}(\mathcal{E}_{S_2})| - |S_2|$ *whose torsion subgroup is isomorphic to* $\mathrm{Jac}(C)(\mathbb{F}_q)^{\oplus 2}$.

1.6. Similarities with the Birch–Tate conjecture

In this subsection, we describe similarities between our statements and the Birch–Tate conjecture [58, p. 206–207]. Although there is no direct connection between them, we hope this subsection provides some justification for the way our statements are formulated. The similarities were pointed out by Takao Yamazaki.

Our results for K-theory are stated in terms of the boundary map in the localization sequence. For example, in Theorem 1.2 (2), in which we describe the K_1 group of an elliptic curve over a function field, we consider the boundary map to the G_0-groups of the fibers. Since the G-groups of curves over finite fields (i.e., the fibers) are known, the statement gives a description of the K_1 group. This type of description is not directly related to the conjectures of Lichtenbaum and Beilinson.

The Birch–Tate conjecture focuses on the K_2 group of a global field in any characteristic. They study the boundary map, which they denote λ, from the K_2 group to the direct sum of the K_1 group of the residue fields. A part of their conjecture is that the order of the kernel of the boundary map λ is expressed using the special value $\zeta_F(-1)$ and the invariant w_F, which is expressed in terms of the number of roots of unity.

In our theorem, we consider the K_1 group instead of the K_2 group, an elliptic curve over a function field instead of the function field (i.e., a global field with positive characteristic), and the Hasse–Weil L-function $L(E, s)$ instead of the zeta function of the global field. Note that the value $|T'_{(1)}|$ in our setting plays the role of w_F.

Next, we note that there is no counterpart for the factor $(q-1)^2$ in their conjecture, because there is more than one degree for which the cohomology groups are conjecturally nonzero in our setting, in turn because the variety is one-dimensional as opposed to zero-dimensional; the global field itself is regarded as a zero-dimensional variety over the global field.

Finally, while it may be interesting to degenerate the elliptic curve E to compare our results with the conjecture, we have not pursued this point.

1.7. Higher genus speculation

In this subsection, we speculate on higher genus cases, which in turn points to the reason why we restrict our efforts to elliptic curves and to where the difficulty lies in our results. More specifically, the results of Sections 2, 3, 4, and 5 are valid and proved for curves of arbitrary genus. Only in Sections 6 and 7 do we use the fact that E is of genus 1, which we use in two ways, one being through Theorem 1.3, the other through Lemma 6.11. The former is more motivic, whereas the second is primarily used in the computation of the étale cohomology of elliptic surfaces and is not so motivic in nature.

Note that we separately treat cases $j \geqslant 3$ and $j \leqslant 2$ in Sections 7 and 6, respectively. The former case is less involved in that it does not use Theorem 1.3. We use some consequences of Lemma 6.11 that express the cohomology of an elliptic surface in terms of the base curve, hence our results appear as if there are little contributions from the cohomology of $\mathcal{E}$. The second author suspects that a result analogous to Lemma 6.11 should also hold true in higher genus cases, thus we may formulate statements that look similar to those given in our paper.

The results of Section 6, i.e., for the $j \leqslant 2$ case, use the following theorem as an additional ingredient:

THEOREM 1.3 ([30, Theorem 1.1]). — *Let the notations be as in Section 1.5. For an arbitrary set S of closed points of C, the homomorphism induced by the boundary map ∂_2, i.e.,*

$$K_2(E)_{\mathbb{Q}} \xrightarrow{\oplus \partial_{\wp \mathbb{Q}}} \bigoplus_{\wp \in S} G_1(\mathcal{E}_{\wp})_{\mathbb{Q}}$$

is surjective.

Theorem 1.3 is a consequence of Parshin's conjecture, hence a similar statement is expected to hold even if we replace E by a curve of higher genus, but the proof is not known. In [7], Chida et al. constructed curves other than elliptic curves for which surjectivity similar to that of Theorem 1.3 holds true. We might be able to draw some consequences similar to the theorems in our present paper for those curves.

1.8. Organization

Finally, in this subsection, we provide a short explanation of the content of each section. As seen above, the motivation and brief explanation regarding the results of Section 2 and Sections 3-5 are given in Sections 1.3 and 1.4, respectively. Each of these sections is fairly independent.

In Section 2, we compute the motivic cohomology groups of an arbitrary smooth surface X over finite fields. The difficult case is that of $H^3_{\mathcal{M}}(X, \mathbb{Z}(2))$, which is therefore the focus of Section 2.2. In Section 3, we prove the compatibility of Chern characters and the localization sequence of motivic cohomology. In Section 4, we define Chern characters for singular curves over finite fields, though the treatment is quite ad hoc. In Section 5, we then give the relation via the Chern class map between the K_1 and K_2 groups of curves over function fields and motivic cohomology groups. We present our main result in Section 6. Then, applying results of Sections 2, 4, and 5, and using special features of elliptic surfaces, including Theorem 1.3, we explicitly compute the orders of certain torsion groups. We treat the p-part separately in Appendix 7.3; see its introduction for more technical details. In Section 7, we use the Bloch–Kato conjecture, as proved by Rost and Voevodsky (see Theorem 2.1), and generalize our results in Section 6.

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2. Motivic cohomology groups of smooth surfaces

We begin this section by referring to Section 1.3, which provides a general overview of the contents of this section.

Aside from the uniquely divisible part, we understand the motivic cohomology groups of smooth surfaces over finite fields fairly well. It follows from the Bass conjecture that the divisible part is zero [27, Conjecture 37].

The main goal of this section is to prove Theorem 2.3, therefore let us provide a brief description of the statement. Let X be a smooth surface over a finite field. We find that $H^i_{\mathcal{M}}(X, \mathbb{Z}(j))$ is an extension of a finitely generated abelian group by a uniquely divisible group except when $(i, j) = (3, 2)$. Aside from cases $(i, j) = (0, 0), (1, 1), (3, 2)$, and $(4, 2)$, the finitely generated abelian group is a finite group, which is either zero or is written in terms of étale cohomology groups of X. We refer to Theorem 2.3, the following table, and the following paragraph for further details. Note that for a prime number ℓ, we let $|\ |_\ell : \mathbb{Q}_\ell \to \mathbb{Q}$ denote the ℓ-adic absolute value normalized such that $|\ell|_\ell = \ell^{-1}$.

2.1. Motivic cohomology of surfaces over a finite field

Let $\mathbb{F}_q$ be a field of cardinal q of characteristic p. For a separated scheme X which is essentially of finite type over $\operatorname{Spec} \mathbb{F}_q$, we define the motivic cohomology group $H^i_{\mathcal{M}}(X, \mathbb{Z}(j))$ as the homology group $H^i_{\mathcal{M}}(X, \mathbb{Z}(j)) = H_{2j-i}(z^j(X, \bullet))$ of Bloch's cycle complex $z^j(X, \bullet)$ [5, Introduction] (see also [16, 2.5] to remove the condition that X is quasi-projective). When X is quasi-projective, we have $H^i_{\mathcal{M}}(X, \mathbb{Z}(j)) = \mathrm{CH}^j(X, 2j-i)$, where the right-hand side is the higher Chow group of Bloch [5]. We say that a scheme X is essentially smooth over a field k if X is a localization of a smooth scheme of finite type over k. When X is essentially smooth over $\operatorname{Spec} \mathbb{F}_q$, it coincides with the motivic cohomology group defined in [33, Part I, Chapter I, 2.2.7] or [61] (cf. [34, Theorem 1.2], [60, Corollary 2]). For a discrete abelian group M, we set $H^i_{\mathcal{M}}(X, M(j)) = H_{2j-i}(z^j(X, \bullet) \otimes_{\mathbb{Z}} M)$.

We note here that this notation is inappropriate if X is not essentially smooth, for in that case it would be a Borel–Moore homology group. A reason for using this notation is that in Section 4.2, we define Chern classes for higher Chow groups of low degrees as if higher Chow groups were forming a cohomology theory.

First, let us recall that the groups $H^i_{\mathcal{M}}(X, \mathbb{Z}(j))$ have been known for $j \leqslant 1$. By definition, $H^i_{\mathcal{M}}(X, \mathbb{Z}(j)) = 0$ for $j \leqslant 0$ and $(i, j) \neq (0, 0)$, and

$H^0_{\mathcal{M}}(X,\mathbb{Z}(0)) = H^0_{\mathrm{Zar}}(X,\mathbb{Z})$. We have $H^i_{\mathcal{M}}(X,\mathbb{Z}(1)) = 0$ for $i \neq 1,2$. By [5, Theorem 6.1], we have $H^1_{\mathcal{M}}(X,\mathbb{Z}(1)) = H^0_{\mathrm{Zar}}(X,\mathbb{G}_m)$, and $H^2_{\mathcal{M}}(X,\mathbb{Z}(1)) = \mathrm{Pic}(X)$.

Below is a conjecture of Bloch–Kato [28, §1, Conjecture 1], which has been proved by Rost, Voevodsky, Haesemeyer, and Weibel.

Theorem 2.1 (Rost–Voevodsky; the Bloch–Kato conjecture). — *Let $j \geqslant 1$ be an integer. Then, for any finitely generated field K over $\mathbb{F}_q$ and any positive integer $\ell \neq p$, the symbol map $K^M_j(K) \to H^j_{\mathrm{et}}(\mathrm{Spec}\ K, \mathbb{Z}/\ell(j))$ is surjective.*

Definition 2.2. — *Let M be an abelian group. We say that M is* finite modulo a uniquely divisible subgroup *(resp.* finitely generated modulo a uniquely divisible subgroup*) if M_{div} is uniquely divisible and M^{red} is finite (resp. M_{div} is uniquely divisible and M^{red} is finitely generated).*

We note that if M is finite modulo a uniquely divisible subgroup, then M_{tors} is a finite group and $M = M_{\mathrm{div}} \oplus M_{\mathrm{tors}}$.

Recall that for a scheme X, we let $\mathrm{Irr}(X)$ denote the set of irreducible components of X. The aim of Section 2.1 is to prove the below theorem.

Theorem 2.3. — *Let X be a smooth surface over $\mathbb{F}_q$. Let $R \subset \mathrm{Irr}(X)$ denote the subset of irreducible components of X that are projective over $\mathrm{Spec}\ \mathbb{F}_q$. For $X' \in \mathrm{Irr}(X)$, let $q_{X'}$ denote the cardinality of the field of constants of X'.*

(1) *$H^i_{\mathcal{M}}(X,\mathbb{Z}(2))$ is finitely generated modulo a uniquely divisible subgroup if $i \neq 3$ or if X is projective. More precisely,*
   (a) *$H^i_{\mathcal{M}}(X,\mathbb{Z}(2))$ is zero for $i \geqslant 5$.*
   (b) *$H^4_{\mathcal{M}}(X,\mathbb{Z}(2))$ is a finitely generated abelian group of rank $|R|$.*
   (c) *If $i \leqslant 1$ or if X is projective and $i \leqslant 3$, the group $H^i_{\mathcal{M}}(X,\mathbb{Z}(2))$ is finite modulo a uniquely divisible subgroup.*
   (d) *$H^2_{\mathcal{M}}(X,\mathbb{Z}(2))$ is finitely generated modulo a uniquely divisible subgroup.*
   (e) *For $i \leqslant 2$, the group $H^i_{\mathcal{M}}(X,\mathbb{Z}(2))_{\mathrm{tors}}$ is canonically isomorphic to the direct sum $\bigoplus_{\ell \neq p} H^{i-1}_{\mathrm{et}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))$. In particular, $H^i_{\mathcal{M}}(X,\mathbb{Z}(2))$ is uniquely divisible for $i \leqslant 0$.*
   (f) *If X is projective, then $H^3_{\mathcal{M}}(X,\mathbb{Z}(2))_{\mathrm{tors}}$ is isomorphic to the direct sum of the group $\bigoplus_{\ell \neq p} H^2_{\mathrm{et}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))$ and a finite p-group of order $|\mathrm{Hom}(\mathrm{Pic}^o_{X/\mathbb{F}_q}, \mathbb{G}_m)| \cdot |L(h^2(X),0)|_p^{-1}$. Here, we let $\mathrm{Hom}(\mathrm{Pic}^o_{X/\mathbb{F}_q}, \mathbb{G}_m)$ denote the set of morphisms $\mathrm{Pic}^o_{X/\mathbb{F}_q} \to \mathbb{G}_m$ of group schemes over $\mathrm{Spec}\ \mathbb{F}_q$.*

(2) *Let $j \geqslant 3$ be an integer. Then, for any integer i, the group $H^i_{\mathcal{M}}(X, \mathbb{Z}(j))$ is finite modulo a uniquely divisible subgroup. More precisely,*

  (a) *$H^i_{\mathcal{M}}(X, \mathbb{Z}(j))$ is zero for $i \geqslant \max(6, j+1)$, is isomorphic to $\bigoplus_{X' \in R} \mathbb{Z}/(q_{X'}^{j-2} - 1)$ for $(i,j) = (5,3)$, $(5,4)$, and is finite for $(i,j) = (4,3)$.*

  (b) *$H^i_{\mathcal{M}}(X, \mathbb{Z}(j))_{\mathrm{tors}}$ is canonically isomorphic to the direct sum $\bigoplus_{\ell \neq p} H^{i-1}_{\mathrm{et}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(j))$. In particular, $H^i_{\mathcal{M}}(X, \mathbb{Z}(j))$ is uniquely divisible for $i \leqslant 0$ or $6 \leqslant i \leqslant j$, and $H^1_{\mathcal{M}}(X, \mathbb{Z}(j))_{\mathrm{tors}}$ is isomorphic to the direct sum $\bigoplus_{X' \in \mathrm{Irr}(X)} \mathbb{Z}/(q_{X'}^{j} - 1)$.*

In the table below, we summarize the description of the groups $H^i_{\mathcal{M}}(X, \mathbb{Z}(j))$ stated in Theorem 2.3. Here, we write u.d, f./u.d., f.g./u.d., f., and f.g. for uniquely divisible, finite modulo a uniquely divisible subgroup, finite generated modulo a uniquely divisible subgroup, finite, and finitely generated, respectively.

<table>
<tr><th>$j \setminus i$</th><th>< 0</th><th>0</th><th>$0 < i < j$</th><th>$j (\neq 0)$</th><th>$j+1$</th><th>$j+2$</th><th>$\geqslant j+3$</th></tr>
<tr><td>0</td><td>0</td><td>$H^0(\mathbb{Z})$</td><td colspan="2">-</td><td colspan="3">0</td></tr>
<tr><td>1</td><td colspan="2">0</td><td>-</td><td>$H^0(\mathbb{G}_m)$</td><td>$\mathrm{Pic}(X)$</td><td colspan="2">0</td></tr>
<tr><td rowspan="2">2</td><td colspan="2" rowspan="2">u.d.</td><td rowspan="2">f./u.d.</td><td>f.g./u.d.</td><td>?</td><td rowspan="2">f.g.</td><td rowspan="2">0</td></tr>
<tr><td colspan="2">f./u. d. if projective</td></tr>
<tr><td>3</td><td colspan="2">u.d.</td><td colspan="2">f./u.d.</td><td colspan="2">f.</td><td>0</td></tr>
<tr><td>4</td><td colspan="2">u.d.</td><td colspan="2">f./u.d.</td><td>f.</td><td colspan="2">0</td></tr>
<tr><td rowspan="2">$\geqslant 5$</td><td colspan="2" rowspan="2">u.d.</td><td colspan="2">f./u.d.</td><td colspan="3" rowspan="2">0</td></tr>
<tr><td colspan="2">u.d. if $6 \leqslant i \leqslant j$</td></tr>
</table>

Lemma 2.4. — *Let X be a separated scheme essentially of finite type over $\mathrm{Spec}\,\mathbb{F}_q$. Let i and j be integers. If both $H^{i-1}_{\mathcal{M}}(X, \mathbb{Q}/\mathbb{Z}(j))$ and $\varprojlim_m H^i_{\mathcal{M}}(X, \mathbb{Z}/m(j))$ are finite, then $H^i_{\mathcal{M}}(X, \mathbb{Z}(j))$ is finite modulo a uniquely divisible subgroup and its torsion subgroup is isomorphic to $H^{i-1}_{\mathcal{M}}(X, \mathbb{Q}/\mathbb{Z}(j))$.*

Proof. — Let us consider the exact sequence

$$(2.1)\quad 0 \longrightarrow H^{i-1}_{\mathcal{M}}(X, \mathbb{Z}(j)) \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} \longrightarrow H^{i-1}_{\mathcal{M}}(X, \mathbb{Q}/\mathbb{Z}(j)) \longrightarrow H^i_{\mathcal{M}}(X, \mathbb{Z}(j))_{\mathrm{tors}} \longrightarrow 0.$$

Since $H^{i-1}_{\mathcal{M}}(X, \mathbb{Q}/\mathbb{Z}(j))$ is a finite group, all groups in the above exact sequence are finite groups. Then, $H^{i-1}_{\mathcal{M}}(X, \mathbb{Z}(j)) \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}$ must be zero, since it is finite and divisible, hence we have a canonical isomorphism $H^{i-1}_{\mathcal{M}}(X, \mathbb{Q}/\mathbb{Z}(j)) \to H^i_{\mathcal{M}}(X, \mathbb{Z}(j))_{\mathrm{tors}}$. The finiteness of $H^i_{\mathcal{M}}(X, \mathbb{Z}(j))_{\mathrm{tors}}$

implies that the divisible group $H^i_{\mathcal{M}}(X,\mathbb{Z}(j))_{\mathrm{div}}$ is uniquely divisible and the canonical homomorphism

$$H^i_{\mathcal{M}}(X,\mathbb{Z}(j))^{\mathrm{red}} \longrightarrow \varprojlim_m H^i_{\mathcal{M}}(X,\mathbb{Z}(j))/m$$

is injective. The latter group $\varprojlim_m H^i_{\mathcal{M}}(X,\mathbb{Z}(j))/m$ is canonically embedded in the finite group $\varprojlim_m H^i_{\mathcal{M}}(X,\mathbb{Z}/m(j))$, hence we conclude that $H^i_{\mathcal{M}}(X,\mathbb{Z}(j))^{\mathrm{red}}$ is finite. This proves the claim. □

Lemma 2.5. — *Let X be a smooth projective surface over $\mathbb{F}_q$. Let j be an integer. Then, $H^i_{\mathcal{M}}(X,\mathbb{Q}/\mathbb{Z}(j))$ and $\varprojlim_m H^i_{\mathcal{M}}(X,\mathbb{Z}/m(j))$ are finite if $i \neq 2j$ or $j \geqslant 3$.*

Proof. — The claim for $j \leqslant 1$ is clear. Suppose that $j = 2$. Then, the claim for $i \geqslant 5$ holds since the groups are zero by dimension reasons. If $p \nmid m$, from the theorem of Geisser and Levine [16, Corollary 1.2] (see also [16, Corollary 1.4]) and the theorem of Merkurjev and Suslin [39, (11.5), Theorem], it follows that the cycle class map $H^i_{\mathcal{M}}(X,\mathbb{Z}/m(2)) \to H^i_{\mathrm{et}}(X,\mathbb{Z}/m(2))$ is an isomorphism for $i \leqslant 2$ and is an embedding for $i = 3$. By [9, Théorème 2] and the exact sequence [9, 2.1, (29)], for $i \leqslant 3$, the group $\varinjlim_{m,\,p\nmid m} H^i_{\mathrm{et}}(X,\mathbb{Z}/m(2))$ and the group $\varprojlim_{m,\,p\nmid m} H^i_{\mathrm{et}}(X,\mathbb{Z}/m(2))$ are finite.

Let $W_n\Omega^\bullet_{X,\log}$ denote the logarithmic de Rham–Witt sheaf (cf. [25, Part I, 5.7]), which was introduced by Milne in [41]. There is an isomorphism $H^i_{\mathcal{M}}(X,\mathbb{Z}/p^n(2)) \cong H^{i-2}_{\mathrm{Zar}}(X,W_n\Omega^2_{X,\log})$ (cf. [15, Theorem 8.4]). In particular, we have $H^i_{\mathcal{M}}(X,\mathbb{Q}_p/\mathbb{Z}_p(2)) = 0$ for $i \leqslant 1$.

By [9, §2, Théorème 3], $\varinjlim_n H^i_{\mathrm{et}}(X,W_n\Omega^2_{X,\log})$ is a finite group for $i = 0, 1$. By [9, Pas n°1, p. 783], the projective system $\{H^i_{\mathrm{et}}(X,W_n\Omega^2_{X,\log})\}_n$ satisfies the Mittag–Leffler condition. Using the same argument used in [9, Pas n°4, p. 784], we obtain an exact sequence

$$0 \longrightarrow \varprojlim_n H^i_{\mathrm{et}}(X,W_n\Omega^2_{X,\log}) \otimes_{\mathbb{Z}} \mathbb{Q}_p/\mathbb{Z}_p \longrightarrow \varinjlim_n H^i_{\mathrm{et}}(X,W_n\Omega^2_{X,\log}) \longrightarrow \varprojlim_n H^{i+1}_{\mathrm{et}}(X,W_n\Omega^2_{X,\log})_{\mathrm{tors}} \longrightarrow 0.$$

We then see that $\varprojlim_n H^i_{\mathrm{et}}(X,W_n\Omega^2_{X,\log})$ is also finite for $i = 0, 1$ and is isomorphic to $\varinjlim_n H^{i-1}_{\mathrm{et}}(X,W_n\Omega^2_{X,\log})$. Since the homomorphism

$$H^i_{\mathrm{Zar}}(X,W_n\Omega^2_{X,\log}) \longrightarrow H^i_{\mathrm{et}}(X,W_n\Omega^2_{X,\log})$$

induced by the change of topology $\varepsilon : X_{\mathrm{et}} \to X_{\mathrm{Zar}}$, is an isomorphism for $i = 0$ and is injective for $i = 1$, we observe that $\varprojlim_n H^2_{\mathcal{M}}(X,\mathbb{Z}/p^n(2))$ is zero and that both $H^2_{\mathcal{M}}(X, \mathbb{Q}_p/\mathbb{Z}_p(2))$ and $\varprojlim_n H^3_{\mathcal{M}}(X, \mathbb{Z}/p^n(2))$ are finite groups. This proves the claim for $j = 2$.

Suppose $j \geqslant 3$. When $i \geqslant 2j$, since X is a surface, the cycle complex defining the higher Chow groups are zero in the negative degrees, hence the claim holds true. Since $j \geqslant 3$, we have $H^i_{\mathcal{M}}(X, \mathbb{Z}/p^n(j)) \cong H^{i-j}_{\mathrm{Zar}}(X, W_n\Omega^j_{X,\log}) = 0$. Using the theorem of Rost and Voevodsky (Theorem 2.1), by the theorem of Geisser–Levine [16, Theorem 1.1], the group $H^i_{\mathcal{M}}(X, \mathbb{Z}/m(j))$ is isomorphic to $H^i_{\mathrm{Zar}}(X, \tau_{\leqslant j} R\varepsilon_* \mathbb{Z}/m(j))$ if $p \nmid m$. Since any affine surface over $\mathbb{F}_q$ has ℓ-cohomological dimension three for any $\ell \neq p$, it follows that $H^i_{\mathrm{Zar}}(X, \tau_{\leqslant j} R\varepsilon_* \mathbb{Z}/m(j)) \cong H^i_{\mathrm{et}}(X, \mathbb{Z}/m(j))$ for all i. Hence, by [9, Théorème 2] and the exact sequence [9, §2.1, (29)], for $i \leqslant 2j - 1$, the groups $H^i_{\mathcal{M}}(X, \mathbb{Q}/\mathbb{Z}(j))$ and $\varprojlim_m H^i_{\mathcal{M}}(X, \mathbb{Z}/m(j))$ are finite. This proves the claim for $j \geqslant 3$. □

Lemma 2.6. — *Let Y be a scheme of dimension $d \leqslant 1$ of finite type over* Spec $\mathbb{F}_q$. *Then $H^i_{\mathcal{M}}(Y, \mathbb{Z}(j))$ is a torsion group unless $0 \leqslant j \leqslant d$ and $j \leqslant i \leqslant 2j$.*

Proof. — By taking a smooth affine open subscheme of Y_{red} whose complement is of dimension zero, and using the localization sequence of motivic cohomology, we are reduced to the case in which Y is connected, affine, and smooth over Spec $\mathbb{F}_q$. When $d = 0$ (resp. $d = 1$), the claim follows from the results of Quillen [52, Theorem 8 (i)] (resp. Harder [24, Korollar 3.2.3] (see [18, Theorem 0.5] for the correct interpretation of his results)) on the structure of the K-groups of Y, combined with the Riemann–Roch theorem for higher Chow groups [5, Theorem 9.1]. □

Lemma 2.7. — *Let $\varphi : M \to M'$ be a homomorphism of abelian groups such that* Ker φ *is finite and* (Coker $\varphi)_{\mathrm{div}} = 0$. *If M_{div} or M'_{div} is uniquely divisible, then φ induces an isomorphism $\varphi_{\mathrm{div}} : M_{\mathrm{div}} \xrightarrow{\cong} M'_{\mathrm{div}}$.*

Proof. — First, we show that φ_{div} is surjective. Since (Coker $\varphi)_{\mathrm{div}} = 0$, for any $a \in M'_{\mathrm{div}}$, we have $\varphi(a)^{-1} \neq \emptyset$. Let N be a positive integer such that $N(\mathrm{Ker}\ \varphi) = 0$. Set $M_0 = \varphi^{-1}(M'_{\mathrm{div}})$ and $M_1 = NM_0$. Since φ induces a surjection $M_1 \to M'_{\mathrm{div}}$, it suffices to show that M_1 is divisible. Let $x \in M_1$ and $n \in \mathbb{Z}_{\geqslant 1}$. Take $y \in M_0$ such that $x = Ny$; $z \in M'_{\mathrm{div}}$ such that $nz = \varphi(y)$; $y' \in M_0$ such that $\varphi(y') = z$. Set $x' = Ny' \in M_1$. Then $\varphi(y - ny') = 0$ implies $x - nx' = Ny - nNy' = 0$. This shows that x is divisible, hence proving the surjectivity.

Next, let us prove the injectivity. Suppose M_{div} is uniquely divisible. Then, since Ker φ is torsion, Ker $\varphi \cap M_{\mathrm{div}} = 0$, and φ_{div} is injective. Suppose M'_{div} is uniquely divisible. Then, the torsion subgroup of M_{div} is contained in Ker φ. Since a nonzero torsion divisible group is infinite, M_{div} must be uniquely divisible, hence we are reduced to the case above and φ_{div} is injective. □

Proof of Theorem 2.3. — Without loss of generality, we may assume that X is connected. We first prove the claims assuming X is projective. It is clear that the group $H^i_{\mathcal{M}}(X, \mathbb{Z}(j))$ is zero for $i \geqslant \min(j+3, 2j+1)$. It follows from [9, Proposition 4] that the degree map $H^4_{\mathcal{M}}(X, \mathbb{Z}(2)) = \mathrm{CH}_0(X) \to \mathbb{Z}$ has finite kernel and cokernel, proving the claim for $i \geqslant \min(j+3, 2j)$. Next, we fix $j \geqslant 2$. For $i \leqslant 2j-1$, the group $H^i_{\mathcal{M}}(X, \mathbb{Z}(j))$ is finite modulo a uniquely divisible subgroup by Lemmas 2.4 and 2.5. The claim on the identification of $H^i_{\mathcal{M}}(X, \mathbb{Z}(j))_{\mathrm{tors}}$ with the étale cohomology follows immediately from the argument in the proof of Lemma 2.5 except for the p-primary part of $H^3_{\mathcal{M}}(X, \mathbb{Z}(2))$, which follows from Proposition A.5.

To finish the proof, it remains to prove that $H^i_{\mathcal{M}}(X, \mathbb{Z}(j))_{\mathrm{div}}$ is zero for $j \geqslant 3$ and $i = j+1, j+2$. It suffices to prove that $H^i_{\mathcal{M}}(X, \mathbb{Z}(j))$ is a torsion group for $j \geqslant 3$ and $i \geqslant j+1$. Consider the limit

$$\varinjlim_{Y} H^{i-2}_{\mathcal{M}}(Y, \mathbb{Z}(j-1)) \longrightarrow H^i_{\mathcal{M}}(X, \mathbb{Z}(j)) \longrightarrow \varinjlim_{Y} H^i_{\mathcal{M}}(X \setminus Y, \mathbb{Z}(j))$$

of the localization sequence in which Y runs over the reduced closed subschemes of X of pure codimension one. $H^{i-2}_{\mathcal{M}}(Y, \mathbb{Z}(j-1))$ is torsion by Lemma 2.6 and we have $\varinjlim_Y H^{i-2}_{\mathcal{M}}(X \setminus Y, \mathbb{Z}(j-1)) = 0$ for dimension reasons, hence the claim follows. This completes the proof in the case where X is projective.

For a general connected surface X, take an embedding $X \hookrightarrow X'$ of X into a smooth projective surface X' over $\mathbb{F}_q$ such that $Y = X' \setminus X$ is of pure codimension one in X'. We can show that such an X' exists via [46] and a resolution of singularities [1, p. 111], [37, p. 151]. Then, the claims, except for that of the identification of $H^i_{\mathcal{M}}(X, \mathbb{Z}(j))_{\mathrm{tors}}$ with the étale cohomology, easily follow from Lemma 2.7 and the localization sequence

$$\cdots \longrightarrow H^{i-2}_{\mathcal{M}}(Y, \mathbb{Z}(j-1)) \longrightarrow H^i_{\mathcal{M}}(X', \mathbb{Z}(j)) \longrightarrow H^i_{\mathcal{M}}(X, \mathbb{Z}(j)) \longrightarrow \cdots.$$

The claim on the identification of $H^i_{\mathcal{M}}(X, \mathbb{Z}(j))_{\mathrm{tors}}$ with the étale cohomology can be obtained using a similar approach to that used in the proof of Lemma 2.5. This completes the proof. □

2.2. A criterion for the finiteness of $H^3_{\mathcal{M}}(X,\mathbb{Z}(2))_{\mathrm{tors}}$

Proposition 2.8. — *Let X be a smooth surface over $\mathbb{F}_q$. Let $X \hookrightarrow X'$ be an open immersion such that X' is smooth projective over $\mathbb{F}_q$ and $Y = X' \setminus X$ is of pure codimension one in X'. Then, the following conditions are equivalent.*

(1) *$H^3_{\mathcal{M}}(X,\mathbb{Z}(2))$ is finitely generated modulo a uniquely divisible subgroup.*
(2) *$H^3_{\mathcal{M}}(X,\mathbb{Z}(2))_{\mathrm{tors}}$ is finite.*
(3) *The pull-back map $H^3_{\mathcal{M}}(X',\mathbb{Z}(2)) \to H^3_{\mathcal{M}}(X,\mathbb{Z}(2))$ induces an isomorphism $H^3_{\mathcal{M}}(X',\mathbb{Z}(2))_{\mathrm{div}} \xrightarrow{\cong} H^3_{\mathcal{M}}(X,\mathbb{Z}(2))_{\mathrm{div}}$.*
(4) *The kernel of the pull-back map $H^3_{\mathcal{M}}(X',\mathbb{Z}(2)) \to H^3_{\mathcal{M}}(X,\mathbb{Z}(2))$ is finite.*
(5) *The cokernel of the boundary map $\partial : H^2_{\mathcal{M}}(X,\mathbb{Z}(2)) \to H^1_{\mathcal{M}}(Y,\mathbb{Z}(1))$ is finite.*

Further, if the above equivalent conditions are satisfied, then the torsion group $H^3_{\mathcal{M}}(X,\mathbb{Z}(2))_{\mathrm{tors}}$ is isomorphic to the direct sum of a finite group of p-power order and the group $\bigoplus_{\ell\neq p} H^2_{\mathrm{et}}(X,\mathbb{Q}_\ell/\mathbb{Z}_\ell(2))^{\mathrm{red}}$, and the localization sequence induces a long exact sequence

$$\cdots \longrightarrow H^{i-2}_{\mathcal{M}}(Y,\mathbb{Z}(1)) \longrightarrow H^i_{\mathcal{M}}(X',\mathbb{Z}(2))^{\mathrm{red}} \longrightarrow H^i_{\mathcal{M}}(X,\mathbb{Z}(2))^{\mathrm{red}} \longrightarrow \cdots \tag{2.2}$$

of finitely generated abelian groups.

Proof. — Condition (1) clearly implies condition (2). The localization sequence shows that conditions (4) and (5) are equivalent and that condition (3) implies condition (1). From the localization sequence and Lemma 2.7, condition (4) implies condition (3).

We claim that condition (2) implies condition (4). We assume condition (2) and suppose that condition (4) is not satisfied. We set $M = \mathrm{Ker}[H^3_{\mathcal{M}}(X',\mathbb{Z}(2)) \to H^3_{\mathcal{M}}(X,\mathbb{Z}(2))]$. The localization sequence shows that M is finitely generated. By assumption, M is not torsion. Since $H^3_{\mathcal{M}}(X',\mathbb{Z}(2))$ is finite modulo a uniquely divisible subgroup, the intersection $H^3_{\mathcal{M}}(X',\mathbb{Z}(2))_{\mathrm{div}} \cap M$ is a nontrivial free abelian group of finite rank, hence $H^3_{\mathcal{M}}(X,\mathbb{Z}(2))$ contains a group isomorphic to $H^3_{\mathcal{M}}(X',\mathbb{Z}(2))_{\mathrm{div}}/(H^3_{\mathcal{M}}(X',\mathbb{Z}(2))_{\mathrm{div}}\cap M)$, which contradicts condition (2), hence condition (2) implies condition (4). This completes the proof of the equivalence of conditions (1)–(5).

Suppose that conditions (1)–(5) are satisfied. The localization sequence shows that the kernel (resp. the cokernel) of the pull-back $H^i_{\mathcal{M}}(X',\mathbb{Z}(2)) \to H^i_{\mathcal{M}}(X,\mathbb{Z}(2))$ is a torsion group (resp. has no nontrivial divisible subgroup) for any $i \in \mathbb{Z}$, hence by Lemma 2.7, $H^i_{\mathcal{M}}(X, \mathbb{Z}(2))_{\mathrm{div}}$ is uniquely divisible and sequence (2.2) is exact. Condition (2) and exact sequence (2.1) for $(i,j) = (3,2)$ yield an isomorphism $H^3_{\mathcal{M}}(X, \mathbb{Z}(2))_{\mathrm{tors}} \cong H^2_{\mathcal{M}}(X, \mathbb{Q}/\mathbb{Z}(2))^{\mathrm{red}}$. Then, the claim on the structure of $H^3_{\mathcal{M}}(X, \mathbb{Z}(2))_{\mathrm{tors}}$ follows from the theorem of Geisser and Levine [16, Corollary 1.2. See also Corollary 1.4] and the theorem of Merkurjev and Suslin [39, (11.5), Theorem]. This completes the proof. $\square$

Let X be a smooth projective surface over $\mathbb{F}_q$. Suppose that X admits a flat, surjective, and generically smooth morphism $f : X \to C$ to a connected smooth projective curve C over $\mathbb{F}_q$. For each point $\wp \in C$, let $X_\wp = X \times_C \wp$ denote the fiber of f at $\wp$.

Corollary 2.9. — *Let the notations be as above. Let $\eta \in C$ denote the generic point. Suppose that the cokernel of the homomorphism $\partial : H^2_{\mathcal{M}}(X_\eta, \mathbb{Z}(2)) \to \bigoplus_{\wp \in C_0} H^1_{\mathcal{M}}(X_\wp, \mathbb{Z}(1))$, which is the inductive limit of the boundary maps of the localization sequences, is a torsion group. Then, the group $H^i_{\mathcal{M}}(X_\eta, \mathbb{Z}(2))_{\mathrm{div}}$ is uniquely divisible for all $i \in \mathbb{Z}$ and the inductive limit of localization sequences induces a long exact sequence*

$$\cdots \longrightarrow \bigoplus_{\wp \in C_0} H^{i-2}_{\mathcal{M}}(X_\wp, \mathbb{Z}(1)) \longrightarrow H^i_{\mathcal{M}}(X, \mathbb{Z}(2))^{\mathrm{red}} \longrightarrow H^i_{\mathcal{M}}(X_\eta, \mathbb{Z}(2))^{\mathrm{red}} \longrightarrow \cdots .$$

Proof. — Since $\bigoplus_{\wp \in C_0} H^i_{\mathcal{M}}(X_\wp, \mathbb{Z}(1))$ has no nontrivial divisible subgroup for all $i \in \mathbb{Z}$ and is torsion for $i \neq 1$ by Lemma 2.6, the claim follows from Lemma 2.7. $\square$

3. The compatibility of Chern characters and the localization sequence

We first refer the reader to the first half of Section 1.4 for a general overview of the contents of this section.

The aim of this section is to prove Lemma 3.1. Only Lemma 3.1 and Remark 3.4 will be used beyond this section.

In this paper, for the Chern class, we use the Chern class for motivic cohomology of Levine (see below) combined with the comparison isomorphism between Levine's motivic cohomology groups and higher Chow groups. We also use the Riemann–Roch theorem for such Chern classes by checking the compatibility of the localization sequences with the comparison isomorphisms (i.e., Lemmas 3.2 and 3.3).

3.1. Main statement

Given an essentially smooth scheme X (see Section 2.1 for the definition of "essentially smooth") over $\operatorname{Spec}\mathbb{F}_q$ and integers $i, j \geqslant 0$, let $c_{i,j} : K_i(X) \to H^{2j-i}_{\mathcal{M}}(X, \mathbb{Z}(j))$ be the Chern class map. Several approaches to constructing the map $c_{i,j}$ have been proposed, including [5, p. 293], [33, Part I, Chapter III, 1.4.8. Examples, (i)], [51, Definition 5]; all of these approaches are based on Gillet's work [17, Definition 2.22]. In this paper, we adopt the definition of Levine [33, Part I, Chapter III, 1.4.8. Examples, (i)] in which $c_{i,j}$ is denoted by $c^{j,2j-i}_X$, given in [33, Part I, Chapter I, 2.2.7]. The definition of the target $H^{2j-i}(X, \mathbb{Z}(j)) = H^{2j-i}_X(X, \mathbb{Z}(j))$ of $c^{j,2j-i}_X$, which we denote by $H^{2j-i}_{\mathcal{L}}(X, \mathbb{Z}(j))$, is different from the definition of the group $H^{2j-i}_{\mathcal{M}}(X, \mathbb{Z}(j))$; however, by combining (ii) and (iii) of [33, Part I, Chapter II, 3.6.6. Theorem], we obtain a canonical isomorphism

$$\beta^i_j : H^i_{\mathcal{M}}(X, \mathbb{Z}(j)) \xrightarrow{\cong} H^i_{\mathcal{L}}(X, \mathbb{Z}(j)), \tag{3.1}$$

which is compatible with the product structures. The precise definition of our Chern class map $c_{i,j}$ is the composition $c_{i,j} = (\beta^i_j)^{-1} \circ c^{j,2j-i}_X$.

The map $c_{i,j}$ is a group homomorphism if $i \geqslant 1$ or $(i, j) = (0, 1)$. Let $\mathrm{ch}_{0,0} : K_0(X) \to H^0_{\mathcal{M}}(X, \mathbb{Z}(0)) \cong H^0_{\mathrm{Zar}}(X, \mathbb{Z})$ denote the homomorphism that sends the class of locally free $\mathcal{O}_X$-module $\mathcal{F}$ to the rank of $\mathcal{F}$. For $i \geqslant 1$ and $a \in K_i(X)$, we put formally $\mathrm{ch}_{i,0}(a) = 0$.

Lemma 3.1. — *Let X be a scheme which is a localization of a smooth quasi-projective scheme over* $\operatorname{Spec}\mathbb{F}_q$. *Let $Y \subset X$ be a closed subscheme of pure codimension d, which is essentially smooth over* $\operatorname{Spec}\mathbb{F}_q$. *Then for $i, j \geqslant 1$ or $(i, j) = (0, 1)$, the diagram*

$$\begin{tikzcd}
K_i(Y) \arrow[r, "\alpha_{i,j}"] \arrow[d] & H^{2j-i-2d}_{\mathcal{M}}(Y, \mathbb{Z}(j-d)) \arrow[d] \\
K_i(X) \arrow[r, "c_{i,j}"] \arrow[d] & H^{2j-i}_{\mathcal{M}}(X, \mathbb{Z}(j)) \arrow[d] \\
K_i(X \setminus Y) \arrow[r, "c_{i,j}"] \arrow[d] & H^{2j-i}_{\mathcal{M}}(X \setminus Y, \mathbb{Z}(j)) \arrow[d] \\
K_{i-1}(Y) \arrow[r, "\alpha_{i-1,j}"] & H^{2j-i-2d+1}_{\mathcal{M}}(Y, \mathbb{Z}(j-d))
\end{tikzcd} \tag{3.2}$$

is commutative. Here, the homomorphism $\alpha_{i,j}$ is defined as follows: for $a \in K_i(Y)$, the element $\alpha_{i,j}(a)$ equals

$$G_{d,j-d}(\mathrm{ch}_{i,0}(a), c_{i,1}(a), \dots, c_{i,j-d}(a); c_{0,1}(\mathcal{N}), \dots, c_{0,j-d}(\mathcal{N})),$$

where $G_{d,j-d}$ is the universal polynomial in [4, Exposé 0, Appendice, Proposition 1.5]*, $\mathcal{N}$ is the conormal sheaf of Y in X, and the left (resp. the right) vertical sequence is the localization sequence of K-theory (resp. of higher Chow groups established in* [6, Corollary (0.2)]*).*

Proof. — We may assume that X is quasi-projective and smooth over $\operatorname{Spec} \mathbb{F}_q$. It follows from [33, Part I, Chapter III, 1.5.2] and the Riemann–Roch theorem without denominators [33, Part I, Chapter III, 3.4.7. Theorem] that diagram (3.2) is commutative if we replace the right vertical sequence by Gysin sequence

$$(3.3)\quad H^{2j-i-2d}_{\mathcal{L}}(Y, \mathbb{Z}(j-d)) \longrightarrow H^{2j-i}_{\mathcal{L}}(X, \mathbb{Z}(j)) \\ \longrightarrow H^{2j-i}_{\mathcal{L}}(X \setminus Y, \mathbb{Z}(j)) \longrightarrow H^{2j-i-2d+1}_{\mathcal{L}}(Y, \mathbb{Z}(j-d))$$

in [33, Part I, Chapter III, 2.1]. It suffices to show that Gysin sequence (3.3) is identified with the localization sequence of higher Chow groups. Here, we use the notations from [33, Part I, Chapter I, II]. Let $S = \operatorname{Spec} \mathbb{F}_q$ and $\mathcal{V}$ denote the category of schemes that is essentially smooth over $\operatorname{Spec} \mathbb{F}_q$. Let $\mathcal{A}_{\mathrm{mot}}(\mathcal{V})$ be the DG category defined in [33, Part I, Chapter I, 1.4.10 Definition]. For an object Z in $\mathcal{V}$ and a morphism $f : Z' \to Z$ in $\mathcal{V}$ that admits a smooth section, and for $j \in \mathbb{Z}$, we have the object $\mathbb{Z}_Z(j)_f$ in $\mathcal{A}_{\mathrm{mot}}(\mathcal{V})$. When $f = \mathrm{id}_Z$ is the identity, we abbreviate $\mathbb{Z}_Z(j)_{\mathrm{id}_Z}$ by $\mathbb{Z}_Z(j)$. For a closed subset $W \subset Z$, let $\mathbb{Z}_{Z,W}(j)$ be the object introduced in [33, Part I, Chapter I, (2.1.3.1)]; this is an object in the DG category $\mathbf{C}^b_{\mathrm{mot}}(\mathcal{V})$ of bounded complexes in $\mathcal{A}_{\mathrm{mot}}(\mathcal{V})$. The object $\mathbb{Z}_Z(j)_f$ belongs to the full subcategory $\mathcal{A}_{\mathrm{mot}}(\mathcal{V})^*$ of $\mathcal{A}_{\mathrm{mot}}(\mathcal{V})$ introduced in [33, Part I, Chapter I, 3.1.5], and the object $\mathbb{Z}_{Z,W}(j)$ belongs to the DG category $\mathbf{C}^b_{\mathrm{mot}}(\mathcal{V})^*$ of bounded complexes in $\mathcal{A}_{\mathrm{mot}}(\mathcal{V})^*$. For $i \in \mathbb{Z}$, we set $H^i_{\mathcal{L},W}(Z, \mathbb{Z}(j)) = \operatorname{Hom}_{\mathbf{D}^b_{\mathrm{mot}}(\mathcal{V})}(1, \mathbb{Z}_{Z,W}(j)[i])$ where 1 denotes the object $\mathbb{Z}_{\operatorname{Spec} \mathbb{F}_q}(0)$ and $\mathbf{D}^b_{\mathrm{mot}}(\mathcal{V})$ denotes the category introduced in [33, Part I, Chapter I, 2.1.4 Definition].

Let X, Y be as in the statement of Lemma 3.1. Let $\mathbf{K}^b_{\mathrm{mot}}(\mathcal{V})$ be the homotopy category of $\mathbf{C}^b_{\mathrm{mot}}(\mathcal{V})$. Then, we have a distinguished triangle

$$\mathbb{Z}_{X,Y}(j) \longrightarrow \mathbb{Z}_X(j) \longrightarrow \mathbb{Z}_{X \setminus Y}(j) \xrightarrow{+1}$$

in $\mathbf{K}^b_{\mathrm{mot}}(\mathcal{V})$. This distinguished triangle yields a long exact sequence

$$\cdots \longrightarrow H^i_{\mathcal{L},Y}(X,\mathbb{Z}(j)) \longrightarrow H^i_{\mathcal{L}}(X,\mathbb{Z}(j)) \longrightarrow H^i_{\mathcal{L}}(X\setminus Y,\mathbb{Z}(j)) \longrightarrow H^{i+1}_{\mathcal{L},Y}(X,\mathbb{Z}(j)) \longrightarrow \cdots . \tag{3.4}$$

In [33, Part I, Chapter III, (2.1.2.2)], Levine constructs an isomorphism $\iota_* : \mathbb{Z}_Y(j-d)[-2d] \to \mathbb{Z}_{X,Y}(j)$ in $\mathbf{D}^b_{\mathrm{mot}}(\mathcal{V})$. This isomorphism induces an isomorphism $\iota_* : H^{i-2d}_{\mathcal{L}}(Y,\mathbb{Z}(j-d)) \xrightarrow{\cong} H^i_{\mathcal{L},Y}(X,\mathbb{Z}(j))$. This latter isomorphism, together with long exact sequence (3.4) yields the Gysin sequence (3.3).

We set $z^j_Y(X,-\bullet) = \mathrm{Cone}(z^j(X,-\bullet) \to z^j(X\setminus Y,-\bullet))[-1]$ and define cohomology with support $H^i_{\mathcal{M},Y}(X,\mathbb{Z}(j)) = H^{i-2j}(z^j_Y(X,-\bullet))$. The distinguished triangle

$$z^j_Y(X,-\bullet) \longrightarrow z^j(X,-\bullet) \longrightarrow z^j(X\setminus Y,-\bullet) \xrightarrow{+1}$$

in the derived category of abelian groups induces a long exact sequence

$$\cdots \longrightarrow H^i_{\mathcal{M},Y}(X,\mathbb{Z}(j)) \longrightarrow H^i_{\mathcal{M}}(X,\mathbb{Z}(j)) \longrightarrow H^i_{\mathcal{M}}(X\setminus Y,\mathbb{Z}(j)) \longrightarrow H^{i+1}_{\mathcal{M},Y}(X,\mathbb{Z}(j)) \longrightarrow \cdots . \tag{3.5}$$

The push-forward map $z^{j-d}(Y,-\bullet) \to z^j(X,-\bullet)$ of cycles gives a homomorphism $z^{j-d}(Y,-\bullet) \to z^j_Y(X,-\bullet)$ of complexes of abelian groups, which is known to be a quasi-isomorphism by [6, Theorem (0.1)], hence it induces an isomorphism $\iota_* : H^{2j-i-2d}_{\mathcal{M}}(Y,\mathbb{Z}(j-d)) \xrightarrow{\cong} H^{2j-i}_{\mathcal{M},Y}(X,\mathbb{Z}(j))$. Then, the claim follows from Lemmas 3.2 and 3.3 below. □

3.2. Compatibility of localization sequences

LEMMA 3.2. — *Let X and Y be as in Lemma 3.1. For each $i,j\in\mathbb{Z}$, there exists a canonical isomorphism*

$$\beta^i_{Y,j} : H^i_{\mathcal{M},Y}(X,\mathbb{Z}(j)) \xrightarrow{\cong} H^i_{\mathcal{L},Y}(X,\mathbb{Z}(j))$$

such that the long exact sequence (3.4) *is identified with the long exact sequence* (3.5) *via this isomorphism and the isomorphism* (3.1).

Proof. — First, let us recall the relation between the group

$$H^i_{\mathcal{M}}(X,\mathbb{Z}(j)) \xrightarrow[\beta^i_j]{\cong} H^i_{\mathcal{L}}(X,\mathbb{Z}(j))$$

and the naive higher Chow group introduced by Levine [33, 2.3.1. Definition]. For an object Γ of $\mathbf{C}^b_{\mathrm{mot}}(\mathcal{V})$, the naive higher Chow group $\mathrm{CH}_{naif}(\Gamma,p)$

is by definition the cohomology group $H^{-p}(\mathcal{Z}_{\mathrm{mot}}(\Gamma,*))$. Here, $\mathcal{Z}_{\mathrm{mot}}(\ ,*)$ is as in [33, Part I, Chapter II, 2.2.4. Definition], which is a DG functor from the category $\mathbf{C}^b_{\mathrm{mot}}(\mathcal{V})^*$ to the category of complexes of abelian groups bounded from below. Since X is a localization of a smooth quasi-projective scheme over k, it follows from [33, 2.4.1.] that we have a natural isomorphism $\mathrm{CH}_{naif}(\mathbb{Z}_X(j)[2j], 2j-i) \cong H^i_{\mathcal{L}}(X,\mathbb{Z}(j))$.

Observe that the functor $\mathcal{Z}_{\mathrm{mot}}$ [33, Part I, Chapter I, (3.3.1.2)] from the category $\mathbf{C}^b_{\mathrm{mot}}(\mathcal{V})$ to the category of bounded complexes of abelian groups is compatible with taking cones, hence the DG functor $\mathcal{Z}_{\mathrm{mot}}(\ ,*)$ is also compatible with taking cones. Since $\mathcal{Z}_{\mathrm{mot}}(\mathbb{Z}_X(j)_{\mathrm{id}_X},*)$ is canonically isomorphic to the cycle complex $z^j(X,-\bullet)$, the complex $\mathcal{Z}_{\mathrm{mot}}(\mathbb{Z}_{X,Y}(j)_{\mathrm{id}_X},*)$ is canonically isomorphic to $z^j_Y(X,-\bullet)$. For an object Γ in $\mathbf{C}^b_{\mathrm{mot}}(\mathcal{V})^*$, let $\mathcal{CH}(\Gamma,p)$ be the higher Chow group defined in [33, Part I, Chapter II, 2.5.2. Definition]. From the definition of $\mathcal{CH}(\Gamma,p)$, we obtain canonical homomorphisms $H^{2j-i}_{\mathcal{M}}(X,\mathbb{Z}(j)) \to \mathcal{CH}(\mathbb{Z}_X(j),i)$, $H^{2j-i}_{\mathcal{M}}(X\setminus Y,\mathbb{Z}(j)) \to \mathcal{CH}(\mathbb{Z}_{X\setminus Y}(j),i)$, and $H^{2j-i}_{\mathcal{M},Y}(X,\mathbb{Z}(j)) \to \mathcal{CH}(\mathbb{Z}_{X,Y}(j),i)$ such that the diagram

$$\begin{tikzcd}
H^{2j-i}_{\mathcal{M},Y}(X,\mathbb{Z}(j)) \arrow[r] \arrow[d] & \mathcal{CH}(\mathbb{Z}_{X,Y}(j),i) \arrow[d] \\
H^{2j-i}_{\mathcal{M}}(X,\mathbb{Z}(j)) \arrow[r] \arrow[d] & \mathcal{CH}(\mathbb{Z}_X(j),i) \arrow[d] \\
H^{2j-i}_{\mathcal{M}}(X\setminus Y,\mathbb{Z}(j)) \arrow[r] \arrow[d] & \mathcal{CH}(\mathbb{Z}_{X\setminus Y}(j),i) \arrow[d] \\
H^{2j-i+1}_{\mathcal{M},Y}(X,\mathbb{Z}(j)) \arrow[r] & \mathcal{CH}(\mathbb{Z}_{X,Y}(j),i-1)
\end{tikzcd} \tag{3.6}$$

is commutative.

Recall the definition of the cycle class map $\mathrm{cl}(\Gamma) : \mathcal{CH}(\Gamma) = \mathcal{CH}(\Gamma,0) \to \mathrm{Hom}_{\mathbf{D}^b_{\mathrm{mot}}(\mathcal{V})}(1,\Gamma)$ [33, p. 76] for an object $\Gamma \in \mathbf{C}^b_{\mathrm{mot}}(\mathcal{V})^*$. Also recall that

$$\mathcal{CH}(\Gamma) = \varinjlim_{\Gamma\to\Gamma_{\widetilde{\mathcal{U}}}} H^0(\mathcal{Z}_{\mathrm{mot}}(\mathrm{Tot}\,\Gamma_{\widetilde{\mathcal{U}}},*))$$

where $\Gamma \to \Gamma_{\widetilde{\mathcal{U}}}$ runs over the hyper-resolutions of Γ [33, Part I, Chapter II, 1.4.1. Definition] and $\mathrm{Tot} : \mathbf{C}^b(\mathbf{C}^b_{\mathrm{mot}}(\mathcal{V})^*) \to \mathbf{C}^b_{\mathrm{mot}}(\mathcal{V})^*$ denotes the total complex functor in [33, Part I, Chapter II, 1.3.2]. The homomorphism $\mathrm{cl}(\Gamma)$

is defined as the inductive limit of the composition

$$\begin{aligned}\mathrm{cl}_{\mathrm{naif}}(\mathrm{Tot}\,\Gamma_{\widetilde{\mathcal{U}}}) : H^0(\mathcal{Z}_{\mathrm{mot}}(\mathrm{Tot}\,\Gamma_{\widetilde{\mathcal{U}}}, *)) &\longrightarrow \mathrm{Hom}_{\mathbf{D}^b_{\mathrm{mot}}(\mathcal{V})}(1, \mathrm{Tot}\,\Gamma_{\widetilde{\mathcal{U}}}) \\ &\xleftarrow{\cong} \mathrm{Hom}_{\mathbf{D}^b_{\mathrm{mot}}(\mathcal{V})}(1, \Gamma).\end{aligned}$$

For an object Γ in $\mathbf{C}^b_{\mathrm{mot}}(\mathcal{V})^*$, the homomorphism $\mathrm{cl}_{\mathrm{naif}}(\Gamma)$ is, by definition [33, Part I, Chapter II, (2.3.6.1)], equal to the composition

$$\begin{aligned}H^0(\mathcal{Z}_{\mathrm{mot}}(\Gamma, *)) &\xleftarrow{\cong} H^0(\mathcal{Z}_{\mathrm{mot}}(\Sigma^N(\Gamma)[N])) \\ &\xleftarrow{\cong} \mathrm{Hom}_{\mathbf{K}^b_{\mathrm{mot}}(\mathcal{V})}(\mathfrak{e}^{\otimes a} \otimes 1, \Sigma^N(\Gamma)[N]) \\ &\longrightarrow \mathrm{Hom}_{\mathbf{D}^b_{\mathrm{mot}}(\mathcal{V})}(\mathfrak{e}^{\otimes a} \otimes 1, \Sigma^N(\Gamma)[N]) \\ &\xleftarrow{\cong} \mathrm{Hom}_{\mathbf{D}^b_{\mathrm{mot}}(\mathcal{V})}(1, \Gamma)\end{aligned}$$

for sufficiently large integers $N, a \geqslant 0$. Here, Σ^N is the suspension functor in [33, Part I, Chapter II, 2.2.2. Definition], and $\mathfrak{e}$ is the object in [33, Part I, Chapter I, 1.4.5] that we regard as an object in $\mathcal{A}_{\mathrm{mot}}(V)$. Let $\Gamma \to \Gamma'$ be a morphism in $\mathbf{C}^b_{\mathrm{mot}}(\mathcal{V})$ and set $\Gamma'' = \mathrm{Cone}(\Gamma \to \Gamma')[-1]$. Since the functor Σ^N is compatible with taking cones, the diagram

$$\begin{tikzcd} \mathcal{CH}(\Gamma'', i) \arrow[r, "\mathrm{cl}(\Gamma''[-i])"] \arrow[d] & \mathrm{Hom}_{\mathbf{D}^b_{\mathrm{mot}}(\mathcal{V})}(1, \Gamma''[-i]) \arrow[d] \\ \mathcal{CH}(\Gamma, i) \arrow[r, "\mathrm{cl}(\Gamma[-i])"] \arrow[d] & \mathrm{Hom}_{\mathbf{D}^b_{\mathrm{mot}}(\mathcal{V})}(1, \Gamma[-i]) \arrow[d] \\ \mathcal{CH}(\Gamma', i) \arrow[r, "\mathrm{cl}(\Gamma'[-i])"] \arrow[d] & \mathrm{Hom}_{\mathbf{D}^b_{\mathrm{mot}}(\mathcal{V})}(1, \Gamma'[-i]) \arrow[d] \\ \mathcal{CH}(\Gamma'', i-1) \arrow[r, "\mathrm{cl}(\Gamma''[-i+1])"] & \mathrm{Hom}_{\mathbf{D}^b_{\mathrm{mot}}(\mathcal{V})}(1, \Gamma''[-i+1]) \end{tikzcd} \tag{3.7}$$

is commutative.

The homomorphism $\beta^i_j : H^i_{\mathcal{M}}(X, \mathbb{Z}(j)) \to H^{2j-i}_{\mathcal{L}}(X, \mathbb{Z}(j))$ is, by definition, equal to the composition

$$\begin{aligned}H^i_{\mathcal{M}}(X, \mathbb{Z}(j)) &\longrightarrow \mathcal{CH}(\mathbb{Z}_X(j), 2j-i) \\ &\xrightarrow{\mathrm{cl}(\mathbb{Z}_X(j)[i-2j])} \mathrm{Hom}_{\mathbf{D}^b_{\mathrm{mot}}(\mathcal{V})}(1, \mathbb{Z}_X(j)[i]) = H^i_{\mathcal{L}}(X, \mathbb{Z}(j)).\end{aligned}$$

We define $\beta^i_{Y,j} : H^i_{\mathcal{M},Y}(X,\mathbb{Z}(j)) \to H^{2j-i}_{\mathcal{L},Y}(X,\mathbb{Z}(j))$ to be the composition

$$\begin{aligned} &H^i_{\mathcal{M},Y}(X,\mathbb{Z}(j)) \longrightarrow \mathcal{CH}(\mathbb{Z}_{X,Y}(j), 2j-i) \\ &\qquad \xrightarrow{\mathrm{cl}(\mathbb{Z}_{X,Y}(j)[i-2j])} \mathrm{Hom}_{\mathbf{D}^b_{\mathrm{mot}}(\mathcal{V})}(1, \mathbb{Z}_{X,Y}(j)[i]) = H^i_{\mathcal{L},Y}(X,\mathbb{Z}(j)). \end{aligned}$$

By (3.6) and (3.7), we have a commutative diagram

$$\begin{tikzcd}
H^{2j-i}_{\mathcal{M},Y}(X,\mathbb{Z}(j)) \arrow[r, "\beta^{2j-i}_{Y,j}"] \arrow[d] & H^{2j-i}_{\mathcal{L},Y}(X,\mathbb{Z}(j)) \arrow[d] \\
H^{2j-i}_{\mathcal{M}}(X,\mathbb{Z}(j)) \arrow[r, "\beta^{2j-i}_{j}", "\cong"'] \arrow[d] & H^{2j-i}_{\mathcal{L}}(X,\mathbb{Z}(j)) \arrow[d] \\
H^{2j-i}_{\mathcal{M}}(X\setminus Y,\mathbb{Z}(j)) \arrow[r, "\beta^{2j-i}_{j}", "\cong"'] \arrow[d] & H^{2j-i}_{\mathcal{L}}(X\setminus Y,\mathbb{Z}(j)) \arrow[d] \\
H^{2j-i+1}_{\mathcal{M},Y}(X,\mathbb{Z}(j)) \arrow[r, "\beta^{2j-i+1}_{j}"] & H^{2j-i+1}_{\mathcal{L},Y}(X,\mathbb{Z}(j)),
\end{tikzcd}$$

where the right vertical arrow is the long exact sequence (3.4), hence $\beta^{2j-i}_{Y,j}$ is an isomorphism and the claim follows. □

3.3. Compatibility of Gysin maps

Lemma 3.3. — *The diagram*

$$\begin{tikzcd}
H^{2j-i-2d}_{\mathcal{M}}(Y,\mathbb{Z}(j-d)) \arrow[r, "\iota_*", "\cong"'] \arrow[d, "\beta^{2j-i-2d}_{j-d}"', "\cong"] & H^{2j-i}_{\mathcal{M}.Y}(X,\mathbb{Z}(j)) \arrow[d, "\beta^{2j-i}_{Y,j}"', "\cong"] \\
H^{2j-i-2d}_{\mathcal{L}}(Y,\mathbb{Z}(j-d)) \arrow[r, "\iota_*", "\cong"'] & H^{2j-i}_{\mathcal{L},Y}(X,\mathbb{Z}(j))
\end{tikzcd}$$

is commutative.

Proof. — Recall the construction of the upper horizontal isomorphism ι_* in [33, Part I, Chapter III, (2.1.2.2)]. Let Z be the blow-up of $X \times_{\mathrm{Spec}\,\mathbb{F}_q} \mathbb{A}^1_{\mathbb{F}_q}$ along $Y \times_{\mathrm{Spec}\,\mathbb{F}_q} \{0\}$. Let W be the proper transform of $Y \times_{\mathrm{Spec}\,\mathbb{F}_q} \mathbb{A}^1_{\mathbb{F}_q}$ to Z. Then, W is canonically isomorphic to $Y \times_{\mathrm{Spec}\,\mathbb{F}_q} \mathbb{A}^1_{\mathbb{F}_q}$. Let P be the inverse image of $Y \times_{\mathrm{Spec}\,\mathbb{F}_q} \{0\}$ under the map $Z \to X \times_{\mathrm{Spec}\,\mathbb{F}_q} \mathbb{A}^1_{\mathbb{F}_q}$ and let

$Q = P \times_Z W$. We set $Z' = Z \amalg (X \times_{\mathrm{Spec}\, \mathbb{F}_q} \{1\}) \amalg P$ and let $f : Z' \to Z$ denote the canonical morphism. We then have canonical morphisms

$$\mathbb{Z}_{P,Q}(j) \leftarrow \mathbb{Z}_{Z,W}(j)_f \longrightarrow \mathbb{Z}_{X \times_{\mathrm{Spec}\, \mathbb{F}_q}\{1\}, Y \times_{\mathrm{Spec}\, \mathbb{F}_q}\{1\}}(j) = \mathbb{Z}_{X,Y}(j)$$

in $\mathbf{C}^b_{\mathrm{mot}}(\mathcal{V})^*$, which become isomorphisms in the category $\mathbf{D}^b_{\mathrm{mot}}(\mathcal{V})$.

Let $g : P \to Y \times_{\mathrm{Spec}\, \mathbb{F}_q} \{0\} \cong Y$ be the canonical morphism. The restriction of g to $Q \subset P$ is an isomorphism, hence giving a section $s : Y \to P$ to g. The cycle class $\mathrm{cl}^d_{P,Q}(Q) \in H^{2d}_Q(P, \mathbb{Z}(d))$ in [33, Part I, Chapter I, (3.5.2.7)] comes from the map $[Q]_Q : \mathfrak{e} \otimes 1 \to \mathbb{Z}_{P,Q}(d)[2d]$ in $\mathbf{C}^b_{\mathrm{mot}}(\mathcal{V})$, as defined in [33, Part I, Chapter I, (2.1.3.3)]. We then have morphisms

$$\begin{aligned}\mathfrak{e} \otimes \mathbb{Z}_P(j-d)[-2d] &\longrightarrow \mathbb{Z}_{P,Q}(d) \otimes \mathbb{Z}_P(j-d) \\ &\xrightarrow{\gamma} \mathbb{Z}_{P \times_{\mathrm{Spec}\, \mathbb{F}_q} P, Q \times_{\mathrm{Spec}\, \mathbb{F}_q} P}(j) \\ &\longleftarrow \mathbb{Z}_{P \times_{\mathrm{Spec}\, \mathbb{F}_q} P, Q \times_{\mathrm{Spec}\, \mathbb{F}_q} P}(j)_{f'} \xrightarrow{\Delta_P^*} \mathbb{Z}_{P,Q}(j)\end{aligned} \tag{3.8}$$

in $\mathbf{C}^b_{\mathrm{mot}}(\mathcal{V})$. Here, γ is the map induced from the external products, i.e., $\boxtimes_{P,P} : \mathbb{Z}_P(d) \otimes \mathbb{Z}_P(j-d) \to \mathbb{Z}_{P \times_{\mathrm{Spec}\, \mathbb{F}_q} P}(j)$ and $\boxtimes_{Q,P} : \mathbb{Z}_Q(d) \otimes \mathbb{Z}_P(j-d) \to \mathbb{Z}_{Q \times_{\mathrm{Spec}\, \mathbb{F}_q} P}(j)$; further, $\Delta_P : P \to P \times_{\mathrm{Spec}\, \mathbb{F}_q} P$ denotes the diagonal embedding and f' is the morphism

$$f' = \mathrm{id}_{P \times_{\mathrm{Spec}\, \mathbb{F}_q} P} \amalg \Delta_P : P \times_{\mathrm{Spec}\, \mathbb{F}_q} P \amalg P \longrightarrow P \times_{\mathrm{Spec}\, \mathbb{F}_q} P.$$

The morphisms in (3.8) above induce morphism $\delta : \mathbb{Z}_P(j-d)[-2d] \to \mathbb{Z}_{P,Q}(j)$ in $\mathbf{D}^b_{\mathrm{mot}}(\mathcal{V})$. The composite morphism $\mathbb{Z}_Y(j-d)[-2d] \xrightarrow{q^*} \mathbb{Z}_P(j-d)[-2d] \xrightarrow{\delta} \mathbb{Z}_{P,Q}(j)$ induces a homomorphism $\delta_* : H^{i-2d}_{\mathcal{L}}(Y, \mathbb{Z}(j-d)) \to H^i_{\mathcal{L},Q}(P, \mathbb{Z}(j))$ for each $i \in \mathbb{Z}$.

From the construction of δ_*, we observe that the diagram

$$\begin{array}{ccc} H^{i-2d}_{\mathcal{M}}(Y, \mathbb{Z}(j-d)) & \xrightarrow{s_*} & H^i_{\mathcal{M},Q}(P, \mathbb{Z}(j)) \\ {\scriptstyle \beta^{i-2d}_{j-d}} \downarrow & & \downarrow {\scriptstyle \beta^i_{Q,j}} \\ H^{i-2d}_{\mathcal{L}}(Y, \mathbb{Z}(j-d)) & \xrightarrow{\delta_*} & H^i_{\mathcal{L},Q}(P, \mathbb{Z}(j)) \end{array}$$

is commutative. Here, the upper horizontal arrow s_* is the homomorphism that sends the class of a cycle $V \in z^{j-d}(Y, 2j-i)$ to the class in $H^i_{\mathcal{M},Q}(P, \mathbb{Z}(j))$ of the cycle $s(V)$, which belongs to the kernel of $z^j(P, 2j-i) \to z^j(P \setminus Q, 2j-i)$. Next, the isomorphism $\iota_* : H^{2j-i-2d}_{\mathcal{L}}(Y, \mathbb{Z}(j-d)) \xrightarrow{\cong}$

$H^{2j-i}_{\mathcal{L},Y}(X,\mathbb{Z}(j))$ equals the composition

$$\begin{aligned} &H^{2j-i-2d}_{\mathcal{L}}(Y,\mathbb{Z}(j-d)) \xrightarrow{\delta_*} H^{2j-i}_{\mathcal{L},Q}(P,\mathbb{Z}(j)) \leftarrow H^{2j-i}_{\mathcal{L},W}(Z,\mathbb{Z}(j)) \\ &\qquad\longrightarrow H^{2j-i}_{\mathcal{L},Y\times_{\mathrm{Spec}\,\mathbb{F}_q}\{1\}}(X\times_{\mathrm{Spec}\,\mathbb{F}_q}\{1\},\mathbb{Z}(j)) = H^{2j-i}_{\mathcal{L},Y}(X,\mathbb{Z}(j)). \end{aligned}$$

We can easily verify that the isomorphism $\iota_* : H^{2j-i-2d}_{\mathcal{M}}(Y,\mathbb{Z}(j-d)) \xrightarrow{\cong} H^{2j-i}_{\mathcal{M},Y}(X,\mathbb{Z}(j))$ equals the composition

$$\begin{aligned} &H^{2j-i-2d}_{\mathcal{M}}(Y,\mathbb{Z}(j-d)) \xrightarrow{s_*} H^{2j-i}_{\mathcal{M},Q}(P,\mathbb{Z}(j)) \leftarrow H^{2j-i}_{\mathcal{M},W}(Z,\mathbb{Z}(j)) \\ &\qquad\longrightarrow H^{2j-i}_{\mathcal{M},Y\times_{\mathrm{Spec}\,\mathbb{F}_q}\{1\}}(X\times_{\mathrm{Spec}\,\mathbb{F}_q}\{1\},\mathbb{Z}(j)) = H^{2j-i}_{\mathcal{M},Y}(X,\mathbb{Z}(j)). \end{aligned}$$

Given this, the claim follows. □

Remark 3.4. — For $j = d$, we have $\alpha_{i,d} = (-1)^{d-1}(d-1)! \cdot \mathrm{ch}_{i,0}$. For $i \geqslant 1$ and $j = d+1$, we have $\alpha_{i,d+1} = (-1)^d d! \cdot c_{i,1}$.

Suppose that $d = 1$ and $\mathcal{N} \cong \mathcal{O}_Y$. Then, we have $\alpha_{i,1} = \mathrm{ch}_{i,0}$ and $\alpha_{i,j}(a) = (-1)^{j-1}Q_{j-1}(c_{i,1}(a),\dots,c_{i,j-1}(a))$ for $i \geqslant 0$, $j \geqslant 2$, where Q_{j-1} denotes the $(j-1)$-st Newton polynomial, which expresses the $(j-1)$-st power sum polynomial in terms of the elementary symmetric polynomials. In particular, we have $\alpha_{i,2} = -c_{i,1}$ for $i \geqslant 0$ and $\alpha_{i,j} = -(j-1)c_{i,j-1}$ for $i \geqslant 1$, $j \geqslant 2$.

4. Motivic Chern characters for singular curves over finite fields

Before embarking on this section, we refer to the latter half of Section 1.4 for a general overview. Note that the output of this section consists of Lemma 4.1 and Proposition 4.3.

In this section, we construct Chern characters of low degrees for singular curves over finite fields with values in the higher Chow groups in an ad hoc manner. Bloch defines Chern characters with values in the higher Chow groups tensored with $\mathbb{Q}$ in [5, (7.4)]. We restrict ourselves to one-dimensional varieties over finite fields, but the target group lies with coefficients in $\mathbb{Z}$.

4.1. A lemma on a curve over a finite field

Below, we first state a lemma to be used in this section and later in Lemma 5.4. For a scheme X, we let $\mathcal{O}(X) = H^0(X,\mathcal{O}_X)$ denote the coordinate ring of X.

LEMMA 4.1. — *Let X be a connected scheme of pure dimension one, separated and of finite type over* Spec $\mathbb{F}_q$. *Then, the push-forward map*

$$\alpha_X : H^3_{\mathcal{M}}(X, \mathbb{Z}(2)) \longrightarrow H^1_{\mathcal{M}}(\mathrm{Spec}\ \mathcal{O}(X), \mathbb{Z}(1))$$

is an isomorphism if X is proper, and $H^3_{\mathcal{M}}(X, \mathbb{Z}(2))$ is zero if X is not proper.

Proof. — This follows from Theorem 1.1 of [32]. □

4.2. Integral Chern characters in low degrees

Let Z be a scheme over Spec $\mathbb{F}_q$ of pure dimension one, separated and of finite type over Spec $\mathbb{F}_q$. We construct a canonical homomorphism $\mathrm{ch}'_{i,j}$: $G_i(Z) \to H^{2j-i}_{\mathcal{M}}(Z, \mathbb{Z}(j))$ for $(i,j) = (0,0)$, $(0,1)$, $(1,1)$, and $(1,2)$. Next, we show in Proposition 4.3 that the homomorphism

$$(\mathrm{ch}'_{i,i}, \mathrm{ch}'_{i,i+1}) : G_i(Z) \longrightarrow H^i_{\mathcal{M}}(Z, \mathbb{Z}(i)) \oplus H^{i+2}_{\mathcal{M}}(Z, \mathbb{Z}(i+1)) \tag{4.1}$$

is an isomorphism for $i = 0, 1$. Since the G-theory of Z and the G-theory of Z_{red} are isomorphic, and the same holds for the motivic cohomology, it suffices to treat the case where Z is reduced.

Consider a dense affine open smooth subscheme $Z_{(0)} \subset Z$, and let $Z_{(1)} = Z \setminus Z_{(0)}$ be the complement of $Z_{(0)}$ with the reduced scheme structure. We define $\mathrm{ch}'_{0,0}$ to be the composition

$$G_0(Z) \longrightarrow K_0(Z_{(0)}) \xrightarrow{\mathrm{ch}_{0,0}} H^0_{\mathcal{M}}(Z_{(0)}, \mathbb{Z}(0)) \cong H^0_{\mathcal{M}}(Z, \mathbb{Z}(0)).$$

We then use the following lemma.

LEMMA 4.2. — *For $i = 0$ (resp. $i = 1$), the diagram*

$$\begin{tikzcd} K_{i+1}(Z_{(0)}) \arrow[r] \arrow[d, "c_{i+1,i+1}"'] & K_i(Z_{(1)}) \arrow[d, "\mathrm{ch}_{0,0}\ (\textit{resp. } c_{1,1})"] \\ H^{i+1}_{\mathcal{M}}(Z_{(0)}, \mathbb{Z}(i+1)) \arrow[r] & H^i_{\mathcal{M}}(Z_{(1)}, \mathbb{Z}(i)), \end{tikzcd}$$

where each horizontal arrow is a part of the localization sequence, is commutative.

Proof. — Let $\widetilde{Z}$ denote the normalization of Z. We write $\widetilde{Z}_{(0)} = Z_{(0)} \times_Z \widetilde{Z} (\cong Z_{(0)})$ and $\widetilde{Z}_{(1)} = (Z_{(1)} \times_Z \widetilde{Z})_{\mathrm{red}}$. Comparing diagrams

$$\begin{tikzcd} K_{i+1}(\widetilde{Z}_{(0)}) \arrow[r] \arrow[d] & K_i(\widetilde{Z}_{(1)}) \arrow[d] \\ K_{i+1}(Z_{(0)}) \arrow[r] & K_i(Z_{(1)}) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} H^{i+1}_{\mathcal{M}}(\widetilde{Z}_{(0)}, \mathbb{Z}(i+1)) \arrow[r] \arrow[d] & H^i_{\mathcal{M}}(\widetilde{Z}_{(1)}, \mathbb{Z}(i)) \arrow[d] \\ H^{i+1}_{\mathcal{M}}(Z_{(0)}, \mathbb{Z}(i+1)) \arrow[r] & H^i_{\mathcal{M}}(Z_{(1)}, \mathbb{Z}(i)) \end{tikzcd}$$

reduces us to proving the same claim for $\widetilde{Z}_{(0)}$ and $\widetilde{Z}_{(1)}$. This then follows from Lemma 3.1. □

We define $\mathrm{ch}'_{1,1}$ to be the opposite of the composition

$$\begin{aligned} G_1(Z) &\longrightarrow \mathrm{Ker}[K_1(Z_{(0)}) \longrightarrow K_0(Z_{(1)})] \\ &\xrightarrow{c_{1,1}} \mathrm{Ker}[H^1_{\mathcal{M}}(Z_{(0)}, \mathbb{Z}(1)) \longrightarrow H^0_{\mathcal{M}}(Z_{(1)}, \mathbb{Z}(0))] \cong H^1_{\mathcal{M}}(Z, \mathbb{Z}(1)). \end{aligned}$$

Next, we define $\mathrm{ch}'_{1,2}$ when Z is connected. If Z is not proper, then $H^3_{\mathcal{M}}(Z, \mathbb{Z}(2))$ is zero by Lemma 4.1. We set $\mathrm{ch}'_{1,2} = 0$ in this case. If Z is proper, then the push-forward map

$$H^3_{\mathcal{M}}(Z, \mathbb{Z}(2)) \longrightarrow H^1_{\mathcal{M}}(\mathrm{Spec}\ H^0(Z, \mathcal{O}_Z), \mathbb{Z}(1)) \cong K_1(\mathrm{Spec}\ H^0(Z, \mathcal{O}_Z))$$

is an isomorphism by Lemma 4.1. We define $\mathrm{ch}'_{1,2}$ to be (-1)-times the composition

$$G_1(Z) \longrightarrow K_1(\mathrm{Spec}\ H^0(Z, \mathcal{O}_Z)) \cong H^3_{\mathcal{M}}(Z, \mathbb{Z}(2)).$$

Next, we define $\mathrm{ch}'_{1,2}$ for non-connected Z to be the direct sum of $\mathrm{ch}'_{1,2}$ for each connected component of Z.

Observe that the group $G_0(Z)$ is generated by the two subgroups

$$M_1 = \mathrm{Im}[K_0(Z_{(1)}) \longrightarrow G_0(Z)] \text{ and } M_2 = \mathrm{Im}[K_0(\widetilde{Z}) \longrightarrow G_0(Z)].$$

Using Lemma 4.2 and the localization sequences, the isomorphism $\mathrm{ch}_{0,0} : K_0(Z_{(1)}) \xrightarrow{\cong} H^0_{\mathcal{M}}(Z_{(1)}, \mathbb{Z}(0))$ induces a homomorphism $\mathrm{ch}'_{0,1} : M_1 \to H^2_{\mathcal{M}}(Z, \mathbb{Z}(1))$. The kernel of $K_0(\widetilde{Z}) \to G_0(Z)$ is contained in the image of $K_0(\widetilde{Z}_{(1)}) \to K_0(\widetilde{Z})$. It can then be easily verified that the composition

$$K_0(\widetilde{Z}_{(1)}) \longrightarrow K_0(\widetilde{Z}) \xrightarrow{c_{0,1}} H^2_{\mathcal{M}}(\widetilde{Z}, \mathbb{Z}(1)) \longrightarrow H^2_{\mathcal{M}}(Z, \mathbb{Z}(1))$$

equals the composition

$$K_0(\widetilde{Z}_{(1)}) \longrightarrow K_0(Z_{(1)}) \twoheadrightarrow M_1 \xrightarrow{\mathrm{ch}'_{0,1}} H^2_{\mathcal{M}}(Z, \mathbb{Z}(1)).$$

Given the above, the homomorphism $c_{0,1} : K_0(\widetilde{Z}) \to H^2_{\mathcal{M}}(\widetilde{Z}, \mathbb{Z}(1))$ induces a homomorphism $\mathrm{ch}'_{0,1} : M_2 \to H^2_{\mathcal{M}}(Z, \mathbb{Z}(1))$ such that the two homomorphisms $\mathrm{ch}'_{0,1} : M_i \to H^2_{\mathcal{M}}(Z, \mathbb{Z}(1))$, $i = 1, 2$, coincide on $M_1 \cap M_2$. Thus, we obtain a homomorphism $\mathrm{ch}'_{0,1} : G_0(Z) \to H^2_{\mathcal{M}}(Z, \mathbb{Z}(1))$.

Next, we observe that the four homomorphisms $\mathrm{ch}'_{0,0}$, $\mathrm{ch}'_{0,1}$, $\mathrm{ch}'_{1,1}$, and $\mathrm{ch}'_{1,2}$ do not depend on the choice of $Z_{(0)}$.

Proposition 4.3. — *The homomorphism* (4.1) *for* $i = 0, 1$ *is an isomorphism.*

Proof. — It follows from [2, Corollary 4.3] that the Chern class map $c_{1,1} : K_1(Z_{(0)}) \to H^1_{\mathcal{M}}(Z_{(0)}, \mathbb{Z}(1))$ is an isomorphism, hence by construction, $\mathrm{ch}'_{1,1}$ is surjective and its kernel equals the image of $K_1(Z_{(1)}) \to G_1(Z)$. It follows from the vanishing of K_2 groups of finite fields that the homomorphism $c_{2,2} : K_2(Z_{(0)}) \to H^2_{\mathcal{M}}(Z_{(0)}, \mathbb{Z}(2))$ is an isomorphism. We then have isomorphisms

$$\begin{aligned}&\mathrm{Im}[K_1(Z_{(1)}) \longrightarrow G_1(Z)] \\ &\qquad \cong \mathrm{Im}[H^1_{\mathcal{M}}(Z_{(1)}, \mathbb{Z}(1)) \longrightarrow H^3_{\mathcal{M}}(Z, \mathbb{Z}(2))] \cong H^3_{\mathcal{M}}(Z, \mathbb{Z}(2)),\end{aligned}$$

the first of which is by Lemma 4.2, the second by [2, Corollary 4.3]. Therefore, the composition $\mathrm{Ker}\ \mathrm{ch}'_{1,1} \hookrightarrow G_1(Z) \xrightarrow{\mathrm{ch}'_{1,2}} H^3_{\mathcal{M}}(Z, \mathbb{Z}(2))$ is an isomorphism. This proves the claim for $G_1(Z)$.

By the construction of $\mathrm{ch}'_{0,1}$, the image of $\mathrm{ch}'_{0,1}$ contains the image of $H^0_{\mathcal{M}}(Z_{(1)}, \mathbb{Z}(0)) \to H^2_{\mathcal{M}}(Z, \mathbb{Z}(1))$, and the composition

$$K_0(\widetilde{Z}) \longrightarrow G_0(Z) \xrightarrow{\mathrm{ch}'_{0,1}} H^2_{\mathcal{M}}(Z, \mathbb{Z}(1)) \longrightarrow H^2_{\mathcal{M}}(Z_{(0)}, \mathbb{Z}(1))$$

equals the composition

$$K_0(\widetilde{Z}) \longrightarrow K_0(\widetilde{Z}_{(0)}) \cong K_0(Z_{(0)}) \xrightarrow{c_{0,1}} H^2_{\mathcal{M}}(Z_{(0)}, \mathbb{Z}(1)).$$

The above implies that $\mathrm{ch}'_{0,1}$ is surjective and the homomorphism

$$\mathrm{Ker}\ \mathrm{ch}'_{0,1} \longrightarrow \mathrm{Ker}[K_0(Z_{(0)}) \xrightarrow{c_{0,1}} H^2_{\mathcal{M}}(Z_{(0)}, \mathbb{Z}(1))]$$

is an isomorphism. This proves the claim for $G_0(Z)$. □

5. K-groups and motivic cohomology of curves over a function field

Please note that the last paragraph of Section 1.4 presents an overview of the contents of this section.

In this section, we focus on the following setup. Let C be a smooth projective geometrically connected curve over a finite field $\mathbb{F}_q$. Let k denote the function field of C. Let X be a smooth projective geometrically connected curve over k. Let $\mathcal{X}$ be a regular model of X, which is proper and flat over C.

From the computations of the motivic cohomology of a surface with a fibration (e.g., $\mathcal{X}$ or $\mathcal{X}$ with some fibers removed), we deduce some results concerning the K-groups of low degrees of the generic fiber X. We relate the two using the Chern class maps and by taking the limit.

Lemma 5.1. — *The map*

$$K_1(X) \xrightarrow{(c_{1,1},c_{1,2})} k^\times \oplus H^3_{\mathcal{M}}(X,\mathbb{Z}(2))$$

is an isomorphism. The group $H^4_{\mathcal{M}}(X,\mathbb{Z}(3))$ is a torsion group and there exists a canonical short exact sequence

$$0 \longrightarrow H^4_{\mathcal{M}}(X,\mathbb{Z}(3)) \xrightarrow{\beta} K_2(X) \xrightarrow{c_{2,2}} H^2_{\mathcal{M}}(X,\mathbb{Z}(2)) \longrightarrow 0,$$

such that the composition $c_{2,3} \circ \beta$ equals the multiplication-by-2 map.

Proof. — Let X_0 denote the set of closed points of X. Let $\kappa(x)$ denote the residue field at $x \in X_0$. We construct a commutative diagram by connecting the localization sequence

$$\bigoplus_{x\in X_0} K_2(\kappa(x)) \longrightarrow K_2(X) \longrightarrow K_2(k(X))$$
$$\longrightarrow \bigoplus_{x\in X_0} K_1(\kappa(x)) \longrightarrow K_1(X) \longrightarrow K_1(k(X)) \longrightarrow \bigoplus_{x\in X_0} K_0(x)$$

with the localization sequence

$$\bigoplus_{x\in X_0} H^0_{\mathcal{M}}(\mathrm{Spec}\ \kappa(x),\mathbb{Z}(1)) \longrightarrow H^2_{\mathcal{M}}(X,\mathbb{Z}(2)) \longrightarrow H^2_{\mathcal{M}}(\mathrm{Spec}\ k(X),\mathbb{Z}(2))$$
$$\longrightarrow \bigoplus_{x\in X_0} H^1_{\mathcal{M}}(\mathrm{Spec}\ \kappa(x),\mathbb{Z}(1)) \longrightarrow H^3_{\mathcal{M}}(X,\mathbb{Z}(2)) \longrightarrow H^3_{\mathcal{M}}(\mathrm{Spec}\ k(X),\mathbb{Z}(2)),$$

using the Chern class maps. Since $H^0_{\mathcal{M}}(\mathrm{Spec}\ \kappa(x),\mathbb{Z}(1)) = 0$ and the K-groups and motivic cohomology groups of fields agree in low degrees, the claim for $K_1(X)$ follows from diagram chasing.

It also follows from diagram chasing that

$$K_3(k(X)) \longrightarrow \bigoplus_{x\in X_0} K_2(\kappa(x)) \longrightarrow K_2(X) \xrightarrow{c_{2,2}} H^2_{\mathcal{M}}(X,\mathbb{Z}(2)) \longrightarrow 0$$

is exact. By [47, Theorem 4.9] and [59, Theorem 1], the groups $H^3_{\mathcal{M}}(\mathrm{Spec}\ k(X), \mathbb{Z}(3))$ and $H^2_{\mathcal{M}}(\mathrm{Spec}\ \kappa(x), \mathbb{Z}(2))$ for each $x \in X_0$ are isomorphic to the Milnor K-groups $K_3^M(k(X))$ and $K_2^M(\kappa(x))$, respectively. From the definition of these isomorphisms in [59], the boundary map $H^3_{\mathcal{M}}(k(X), \mathbb{Z}(3)) \to H^2_{\mathcal{M}}(\mathrm{Spec}\ \kappa(x), \mathbb{Z}(2))$ is identified under these isomorphisms with the boundary map $K_3^M(k(X)) \to K_2^M(\kappa(x))$. Given the above, by [40, Proposition 11.11], we obtain the first of the two isomorphisms

$$\begin{aligned}
&\mathrm{Coker}[K_3(k(X)) \longrightarrow \textstyle\bigoplus_{x\in X_0} K_2(\kappa(x))] \\
&\qquad \xrightarrow{\cong} \mathrm{Coker}[H^3_{\mathcal{M}}(k(X), \mathbb{Z}(3)) \longrightarrow \textstyle\bigoplus_{x\in X_0} H^2_{\mathcal{M}}(\mathrm{Spec}\ \kappa(x), \mathbb{Z}(2))] \\
&\qquad \xrightarrow{\cong} H^4_{\mathcal{M}}(X, \mathbb{Z}(3)).
\end{aligned}$$

This gives us the desired short exact sequence. The identity $c_{2,3} \circ \beta = 2$ follows from Remark 3.4. Since $H^2_{\mathcal{M}}(\mathrm{Spec}\ \kappa(x), \mathbb{Z}(2))$ is a torsion group for each $x \in X_0$, the group $H^4_{\mathcal{M}}(X, \mathbb{Z}(3))$ is also a torsion group. This completes the proof. □

Lemma 5.2. — *Let $U \subset C$ be a non-empty open subscheme. We use $\mathcal{X}^U$ to denote the complement $\mathcal{X} \setminus \mathcal{X} \times_C U$ with the reduced scheme structure. Then, for $(i,j) = (0,0)$, $(0,1)$ or $(1,1)$, the diagram*

$$\begin{tikzcd}
K_{i+1}(X) \arrow[rr] \arrow[d, "c_{i+1,j+1}"'] & & G_i(\mathcal{X}^U) \arrow[d, "(-1)^j \mathrm{ch}'_{i,j}"] \\
H^{2j-i+1}_{\mathcal{M}}(X, \mathbb{Z}(j+1)) \arrow[rr] & & H^{2j-i}_{\mathcal{M}}(\mathcal{X}^U, \mathbb{Z}(j)),
\end{tikzcd} \tag{5.1}$$

where each horizontal arrow is a part of the localization sequence, is commutative.

Proof. — Let $\mathcal{X}^U_{\mathrm{sm}} \subset \mathcal{X}^U$ denote the smooth locus. The commutativity of diagram (5.1) for $(i,j) = (1,1)$ (resp. for $(i,j) = (0,0)$) follows from the commutativity, which follows from Lemma 3.1, of the diagram

$$\begin{tikzcd}
K_{i+1}(X) \arrow[rr] \arrow[d, "c_{i+1,j+1}"'] & & K_i(\mathcal{X}^U_{\mathrm{sm}}) \arrow[d, "\substack{-c_{1,1} \\ (\text{resp. } \mathrm{ch}_{0,0})}"] \\
H^{2j-i+1}_{\mathcal{M}}(X, \mathbb{Z}(j+1)) \arrow[rr] & & H^{2j-i}_{\mathcal{M}}(\mathcal{X}^U_{\mathrm{sm}}, \mathbb{Z}(j))
\end{tikzcd}$$

and the injectivity of $H^{2j-i}_{\mathcal{M}}(\mathcal{X}^U, \mathbb{Z}(j)) \to H^{2j-i}_{\mathcal{M}}(\mathcal{X}^U_{\mathrm{sm}}, \mathbb{Z}(j))$.

By Lemma 5.1, the group $K_1(X)$ is generated by the image of the push-forward $\bigoplus_{x\in X_0} K_1(\kappa(x)) \to K_1(X)$ and the image of the pull-back

$K_1(k) \to K_1(X)$. Then, the commutativity of diagram (5.1) for $(i,j) = (0,1)$ follows from the commutativity of the diagram

$$\begin{tikzcd} K_0(Y) \arrow[r] \arrow[d, "\mathrm{ch}_{0,0}"'] & G_0(\mathcal{X}^U) \arrow[d, "\mathrm{ch}'_{0,1}"] \\ H^0_{\mathcal{M}}(Y, \mathbb{Z}(0)) \arrow[r] & H^2_{\mathcal{M}}(\mathcal{X}^U, \mathbb{Z}(1)) \end{tikzcd}$$

for any reduced closed subscheme $Y \subset \mathcal{X}^U$ of dimension zero, where the horizontal arrows are the push-forward maps by closed immersion, and the fact that the composition $K_0(C \setminus U) \xrightarrow{f^{U*}} G_0(\mathcal{X}^U) \xrightarrow{\mathrm{ch}'_{0,1}} H^2_{\mathcal{M}}(\mathcal{X}^U, \mathbb{Z}(1))$ is zero. Here, $f^U : \mathcal{X}^U \to C \setminus U$ denotes the morphism induced by the morphism $\mathcal{X} \to C$. $\square$

Lemma 5.3. — *The diagram*

(5.2)
$$\begin{tikzcd} 0 \arrow[r] & H^4_{\mathcal{M}}(X, \mathbb{Z}(3)) \arrow[r] \arrow[d] & K_2(X) \arrow[r, "c_{2,2}"] \arrow[d] & H^2_{\mathcal{M}}(X, \mathbb{Z}(2)) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \bigoplus_{\wp \in C_0} H^3_{\mathcal{M}}(\mathcal{X}_\wp, \mathbb{Z}(2)) \arrow[r] & \bigoplus_{\wp \in C_0} G_1(\mathcal{X}_\wp) \arrow[r, "-\mathrm{ch}'_{1,1}"] & \bigoplus_{\wp \in C_0} H^1_{\mathcal{M}}(\mathcal{X}_\wp, \mathbb{Z}(1)) \arrow[r] & 0, \end{tikzcd}$$

where the first row is as in Lemma 5.1, the second row is obtained from Proposition 4.3, and the vertical maps are the boundary maps in the localization sequences, is commutative.

Proof. — It follows from Lemma 5.2 that the right square is commutative. For each closed point $x \in X_0$, let D_x denote the closure of x in $\mathcal{X}$ and write $D_{x,\wp} = D_x \times_C \operatorname{Spec} \kappa(\wp)$ for $\wp \in C_0$. Then, the commutativity of the left square in (5.2) follows from the commutativity of the diagram

$$\begin{tikzcd} H^4_{\mathcal{M}}(X, \mathbb{Z}(3)) \arrow[d] & H^2_{\mathcal{M}}(\operatorname{Spec} \kappa(x), \mathbb{Z}(2)) \arrow[l] \arrow[d] \arrow[r, phantom, "\cong"] & K_2(\kappa(x)) \arrow[r] \arrow[d] & K_2(X) \arrow[d] \\ \bigoplus_{\wp \in C_0} H^3_{\mathcal{M}}(\mathcal{X}_\wp, \mathbb{Z}(2)) & \bigoplus_{\wp \in C_0} H^1_{\mathcal{M}}(D_{x,\wp}, \mathbb{Z}(1)) \arrow[l] \arrow[r, phantom, "\cong"] & \bigoplus_{\wp \in C_0} G_1(D_{x,\wp}) \arrow[r] & \bigoplus_{\wp \in C_0} G_1(\mathcal{X}_\wp). \end{tikzcd}$$

Here, each vertical map is a boundary map, the middle horizontal arrows are Chern classes, and the left and right horizontal arrows are the push-forward maps by the closed immersion. $\square$

Lemma 5.4. — *Let U be an open subscheme of C such that $U \neq C$. Let $\partial : H^4_{\mathcal{M}}(\mathcal{X} \times_C U, \mathbb{Z}(3)) \to H^3_{\mathcal{M}}(\mathcal{X}_U, \mathbb{Z}(2))$ denote the boundary map of the localization sequence. Then, the following composition is an isomorphism:*

$$\alpha : \mathrm{Coker}\ \partial \hookrightarrow H^5_{\mathcal{M}}(\mathcal{X}, \mathbb{Z}(3)) \longrightarrow H^1_{\mathcal{M}}(\mathrm{Spec}\ \mathbb{F}_q, \mathbb{Z}(1)) \cong \mathbb{F}_q^{\times}.$$

Here the first map is induced by the push-forward map by the closed immersion, and the second map is the push-forward map by the structure morphism.

Proof. — For each closed point $x \in X_0$, let D_x denote the closure of x in $\mathcal{X}$. We set $D_{x,U} = D_x \times_C U$. Let D_x^U denote the complement $D_x \setminus D_{x,U}$ with the reduced scheme structure. Let $\iota_x : D_x \hookrightarrow \mathcal{X}$, $\iota_{x,U} : D_{x,U} \hookrightarrow \mathcal{X}_U$, $\iota_x^U : D_x^U \hookrightarrow \mathcal{X}^U$ denote the canonical inclusions. Let us consider the commutative diagram

$$\begin{tikzcd}
H^2_{\mathcal{M}}(D_{x,U}, \mathbb{Z}(2)) \arrow[r] \arrow[d, "\iota_{x,U\,*}"'] & H^1_{\mathcal{M}}(D_x^U, \mathbb{Z}(1)) \arrow[r, "\beta"] \arrow[d, "\iota^U_{x\,*}"'] & H^3_{\mathcal{M}}(D_x, \mathbb{Z}(2)) & \\
H^4_{\mathcal{M}}(\mathcal{X} \times_C U, \mathbb{Z}(3)) \arrow[r, "\partial"] & H^3_{\mathcal{M}}(\mathcal{X}^U, \mathbb{Z}(2)) \arrow[r] & \mathrm{Coker}\ \partial \arrow[r] & 0,
\end{tikzcd}$$

where the first row is the localization sequence. Since X is geometrically connected, it follows from [21, Corollaire (4.3.12)] that each fiber of $\mathcal{X} \to C$ is connected. In particular, D_x intersects every connected component of $\mathcal{X}^U$, which implies that the homomorphism $\iota^U_{x\,*}$ in the above diagram is surjective, hence we have a surjective homomorphism $\mathrm{Im}\ \beta \twoheadrightarrow \mathrm{Coker}\ \partial$. Let $\mathbb{F}(x)$ denote the finite field $H^0(D_x, \mathcal{O}_{D_x})$. Then, the isomorphism $H^3_{\mathcal{M}}(D_x, \mathbb{Z}(2)) \to H^1_{\mathcal{M}}(\mathrm{Spec}\ \mathbb{F}(x), \mathbb{Z}(1)) \cong \mathbb{F}(x)^{\times}$ (see Lemma 4.1) gives an isomorphism $\mathrm{Im}\ \beta \cong \mathbb{F}(x)^{\times}$. Hence $|\mathrm{Coker}\,\partial|$ divides $\gcd_{x \in X_0}(|\mathbb{F}(x)^{\times}|) = q-1$, where the equality follows from [56, 1.5.3 Lemme 1]. We can easily verify that the composition

$$\mathbb{F}(x)^{\times} \cong \mathrm{Im}\ \beta \twoheadrightarrow \mathrm{Coker}\ \partial \xrightarrow{\alpha} \mathbb{F}_q^{\times}$$

equals the norm map $\mathbb{F}(x)^{\times} \to \mathbb{F}_q^{\times}$, which implies $|\mathrm{Coker}\ \partial| \geqslant q-1$, hence $|\mathrm{Coker}\,\partial| = q-1$ and the homomorphism α is an isomorphism. The claim is proved. □

6. Main results for $j \leqslant 2$

The reader is referred to Sections 1.6 and 1.7 for some remarks concerning the contents of this section.

The objective is to prove Theorems 6.1, 6.2, and 6.3. The statements give some information on the structures of K-groups and motivic cohomology groups of elliptic curves over global fields and of the (open) complements of some fibers of an elliptic surface over finite fields. We compute the orders of some torsion groups, in terms of the special values of L-functions, the torsion subgroup of (twisted) Mordell–Weil groups, and some invariants of the base curve. We remark that Milne [42] expresses the special values of zeta functions in terms of the order of arithmetic étale cohomology groups. Our result is similar in spirit.

Let us give the ingredients of the proof. Using Theorem 1.3, we deduce that the torsion subgroups we are interested in are actually finite. Then the theorem of Geisser and Levine and the theorem of Merkurjev and Suslin relate the motivic cohomology groups modulo their uniquely divisible parts to the étale cohomology and cohomology of de Rham–Witt complexes. We use the arguments that appear in [41], [9], and [20] to compute such cohomology groups (see Section 1.3 in which we explain our general strategy). The computation of the exact orders of torsion may be new.

One geometric property of an elliptic surface that makes this explicit calculation possible is that the abelian fundamental group is isomorphic to that of the base curve. This follows from a theorem in [55] for the prime-to-p part.

We also use class field theory of Kato and Saito for surfaces over finite fields [29]. We combine their results and Lemma 6.11 to show that the groups of zero-cycles on the elliptic surface and on the base curve are isomorphic.

Let us give a brief outline. Recall in Section 6.3 that the main part of the zeta function of $\mathcal{E}$ is the L-function of E. Note that there are contributions from the L-function of the base curve C and also of the singular fibers of $\mathcal{E}$ (Lemma 6.9, Corollary 6.16). We relate the G-groups of these singular fibers to the motivic cohomology groups, then they are in turn related to the values of the L-function (Lemmas 6.20 and 6.21). The technical input for these lemmas originate from our results in [32] and the classification of the singular fibers of an elliptic fibration.

6.1. Notation

Let k, E, S_0, S_2, r, C, and $\mathcal{E}$ be as defined in Section 1. We also let S_1 denote the set of primes of k at which E has multiplicative reduction, thus we have $S_0 \subset S_1 \subset S_2$. Let p denote the characteristic of k. The closure of the origin of E in $\mathcal{E}$ gives a section to $\mathcal{E} \to C$, which we denote by $\iota : C \to \mathcal{E}$. Throughout this section, we assume that the structure morphism $f : \mathcal{E} \to C$ is not smooth, which in particular implies that it is not isotrivial. For any scheme X over C, let $\mathcal{E}_X$ denote the base change $\mathcal{E} \times_C X$. For any non-empty open subscheme $U \subset C$, we use $\mathcal{E}^U$ to denote the complement $\mathcal{E} \setminus \mathcal{E}_U$ with the reduced scheme structure.

Let $\mathbb{F}_q$ denote the field of constants of C. We take an algebraic closure $\overline{\mathbb{F}}_q$ of $\mathbb{F}_q$. Let $\mathrm{Frob} \in G_{\mathbb{F}_q} = \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ denote the geometric Frobenius. For a scheme X over $\mathrm{Spec}\, \mathbb{F}_q$, we use $\overline{X}$ to denote its base change $\overline{X} = X \times_{\mathrm{Spec}\, \mathbb{F}_q} \mathrm{Spec}\, \overline{\mathbb{F}}_q$ to $\overline{\mathbb{F}}_q$. We often regard the set $\mathrm{Irr}(X)$ of irreducible components of X as a finite étale scheme over $\mathrm{Spec}\, \mathbb{F}_q$ corresponding to the $G_{\mathbb{F}_q}$-set $\mathrm{Irr}(\overline{X})$.

6.2. Results

We set $T = E(k \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_q)_{\mathrm{tors}}$. For each integer $j \in \mathbb{Z}$, we let $T'_{(j)} = \bigoplus_{\ell \neq p} (T \otimes_{\mathbb{Z}} \mathbb{Z}_\ell(j))^{G_{\mathbb{F}_q}}$.

Theorem 6.1. — *Let the notations and assumptions be as above. Let $L(E,s)$ denote the L-function of the elliptic curve E over global field k (see Section 6.3).*

(1) *The $\mathbb{Q}$-vector space $(K_2(E)^{\mathrm{red}})_{\mathbb{Q}}$ is of dimension r.*
(2) *The cokernel of the boundary map $\partial_2 : K_2(E) \to \bigoplus_{\wp \in C_0} G_1(\mathcal{E}_\wp)$ is a finite group of order*
$$\frac{(q-1)^2 |L(h^0(\mathrm{Irr}(\mathcal{E}_{S_2})), -1)|}{|T'_{(1)}| \cdot |L(h^0(S_2), -1)|}.$$
(3) *The group $K_1(E)_{\mathrm{div}}$ is uniquely divisible.*
(4) *The kernel of the boundary map $\partial_1 : K_1(E)^{\mathrm{red}} \to \bigoplus_{\wp \in C_0} G_0(\mathcal{E}_\wp)$ is a finite group of order $(q-1)^2 |T'_{(1)}| \cdot |L(E,0)|$. The cokernel of ∂_1 is a finitely generated abelian group of rank $2 + |\mathrm{Irr}(\mathcal{E}_{S_2})| - |S_2|$ whose torsion subgroup is isomorphic to $\mathrm{Jac}(C)(\mathbb{F}_q)^{\oplus 2}$, where $\mathrm{Jac}(C)$ denotes the Jacobian of C (when the genus of C is 0, we understand it to be a point).*

Let X be a scheme of finite type over $\mathrm{Spec}\, \mathbb{F}_q$. For an integer $i \in \mathbb{Z}$ and a prime number $\ell \neq p$, we set $L(h^i(X), s) = \det(1 - \mathrm{Frob} \cdot q^{-s}; H^i_{\mathrm{et}}(\overline{X}, \mathbb{Q}_\ell))$. In all cases considered in our paper, $L(h^i(X), s)$ does not depend on the choice of ℓ.

For each non-empty open subscheme $U \subset C$, let T_U denote the torsion subgroup of the group $\mathrm{Div}(\overline{\mathcal{E}}_U)/{\sim_{\mathrm{alg}}}$ of divisors on $\overline{\mathcal{E}}_U$ modulo algebraic equivalence. For each integer $j \in \mathbb{Z}$, we set $T'_{U,(j)} = \bigoplus_{\ell \neq p} (T_U \otimes_{\mathbb{Z}} \mathbb{Z}_\ell(j))^{G_{\mathbb{F}_q}}$. We deduce from [55, Theorem 1.3] that the canonical homomorphism $\varinjlim_{U'} T_{U'} \to T$, where the limit is taken over the open subschemes of C, is an isomorphism. We easily verify that the canonical homomorphism $T_U \to \varinjlim_{U'} T_{U'}$ is injective and is an isomorphism if $\mathcal{E}_U \to U$ is smooth. In particular, we have an injection $T'_{U,(j)} \hookrightarrow T'_{(j)}$, which is an isomorphism if $\mathcal{E}_U \to U$ is smooth.

In Section 6.7, we deduce Theorem 6.1 from the following two theorems.

Theorem 6.2. — *Let the notations and assumptions be as above. Let $\partial^i_{\mathcal{M},j} : H^i_{\mathcal{M}}(E, \mathbb{Z}(j))^{\mathrm{red}} \to \bigoplus_{\wp \in C_0} H^{i-1}_{\mathcal{M}}(\mathcal{E}_\wp, \mathbb{Z}(j-1))$ denote the homomorphism induced by the boundary map of the localization sequence established in* [6, Corollary (0.2)].

(1) *For any $i \in \mathbb{Z}$, the group $H^i_{\mathcal{M}}(E, \mathbb{Z}(2))_{\mathrm{div}}$ is uniquely divisible.*
(2) *For $i \leqslant 0$, the cohomology group $H^i_{\mathcal{M}}(E, \mathbb{Z}(2))$ is uniquely divisible. $H^1_{\mathcal{M}}(E, \mathbb{Z}(2))$ is finite modulo a uniquely divisible subgroup and $H^1_{\mathcal{M}}(E, \mathbb{Z}(2))_{\mathrm{tors}}$ is cyclic of order $q^2 - 1$.*
(3) *The kernel (resp. cokernel) of the homomorphism $\partial^2_{\mathcal{M},2}$ is a finite group of order $|L(h^1(C), -1)|$ (resp. of order*
$$\frac{(q-1)|L(h^0(\mathrm{Irr}(\mathcal{E}_{S_2})), -1)|}{|T'_{(1)}| \cdot |L(h^0(S_2), -1)|}).$$
(4) *The kernel (resp. cokernel) of the homomorphism $\partial^3_{\mathcal{M},2}$ is a finite group of order $(q-1)|T'_{(1)}| \cdot |L(E, 0)|$ (resp. is isomorphic to* $\mathrm{Pic}(C)$*).*
(5) *For $i \geqslant 4$, the group $H^i_{\mathcal{M}}(E, \mathbb{Z}(2))$ is zero.*
(6) *$H^4_{\mathcal{M}}(E, \mathbb{Z}(3))$ is a torsion group, and the cokernel of the homomorphism $\partial^4_{\mathcal{M},3}$ is a finite cyclic group of order $q - 1$.*

Theorem 6.3. — *Let $U \subset C$ be a non-empty open subscheme. Then, we have the following.*

(1) *For any $i \in \mathbb{Z}$, the group $H^i_{\mathcal{M}}(\mathcal{E}_U, \mathbb{Z}(2))$ is finitely generated modulo a uniquely divisible subgroup.*
(2) *For $i \leqslant 0$, the cohomology group $H^i_{\mathcal{M}}(\mathcal{E}_U, \mathbb{Z}(2))$ is uniquely divisible, $H^1_{\mathcal{M}}(\mathcal{E}_U, \mathbb{Z}(2))$ is finite modulo a uniquely divisible subgroup and $H^1_{\mathcal{M}}(\mathcal{E}_U, \mathbb{Z}(2))_{\mathrm{tors}}$ is cyclic of order $q^2 - 1$.*
(3) *The rank of $H^2_{\mathcal{M}}(\mathcal{E}_U, \mathbb{Z}(2))^{\mathrm{red}}$ is $|S_0 \setminus U|$. If $U = C$ (resp. $U \neq C$), the torsion subgroup of $H^2_{\mathcal{M}}(\mathcal{E}_U, \mathbb{Z}(2))^{\mathrm{red}}$ is of order $|L(h^1(C), -1)|$ (resp. of order $|T'_{U,(1)}| \cdot |L(h^1(C), -1) L(h^0(C \setminus U), -1)|/(q-1)$).*

(4) *If* $U = C$ *(resp.* $U \neq C$*), the cokernel of the boundary homomorphism* $H^2_{\mathcal{M}}(\mathcal{E}_U, \mathbb{Z}(2)) \to H^1_{\mathcal{M}}(\mathcal{E}^U, \mathbb{Z}(1))$ *is zero (resp. is finite of order*
$$\frac{(q-1)|L(h^0(\mathrm{Irr}(\mathcal{E}^U)), -1)|}{|T'_{U,(1)}| \cdot |L(h^0(C \setminus U), -1)|}).$$

(5) *The rank of* $H^3_{\mathcal{M}}(\mathcal{E}_U, \mathbb{Z}(2))^{\mathrm{red}}$ *is* $\max(|C \setminus U| - 1, 0)$*. If* $U = C$ *(resp.* $U \neq C$*), the torsion subgroup* $H^3_{\mathcal{M}}(\mathcal{E}_U, \mathbb{Z}(2))_{\mathrm{tors}}$ *is finite of order* $|L(h^2(\mathcal{E}), 0)|$ *(resp. of order*
$$\frac{|T'_{U,(1)}| \cdot |L(h^2(\mathcal{E}), 0) L^*(h^1(\mathcal{E}^U), 0) L(h^0(C \setminus U), -1)|}{(q-1)|L(h^0(\mathrm{Irr}(\mathcal{E}^U)), -1)|}).$$
Here,
$$L^*(h^1(\mathcal{E}^U), 0) = \lim_{s \to 0} (s \log q)^{-|S_0 \setminus U|} L(h^1(\mathcal{E}^U), s)$$
is the leading coefficient of $L(h^1(\mathcal{E}^U), s)$ *at* $s = 0$*.*

(6) $H^4_{\mathcal{M}}(\mathcal{E}_U, \mathbb{Z}(2))$ *is canonically isomorphic to* $\mathrm{Pic}(U)$*. For* $i \geqslant 5$*, the group* $H^i_{\mathcal{M}}(\mathcal{E}_U, \mathbb{Z}(2))$ *is zero.*

6.3. Relation between $L(E,s)$ and the congruence zeta function of $\mathcal{E}$.

Let $\ell \neq p$ be a prime number. Take an open subset $j : U \to C$ such that the restriction $f_U : \mathcal{E}_U \to U$ is (proper) smooth.

By the Grothendieck–Lefschetz trace formula, we have
$$L(E, s) = \prod_{i=0}^{2} \det(1 - \mathrm{Frob} \cdot q^{-s}; H^i_{\mathrm{et}}(\overline{C}, j_* R^1 f_{U*} \mathbb{Q}_\ell))^{(-1)^{i-1}}.$$

Note that by the adjunction $\mathrm{id} \to j_* j^*$, we have a canonical morphism
$$R^1 f_* \mathbb{Q}_\ell \longrightarrow j_* j^* R^1 f_* \mathbb{Q}_\ell \cong j_* R^1 f_{U*} \mathbb{Q}_\ell.$$
We will prove the following lemma.

Lemma 6.4. — *The map above is an isomorphism.*

Proof. — Let $\wp$ be a closed point of C, and let S be strict henselization of C at $\wp$. Let s be the closed point of S and let $\overline{\eta}$ denote the spectrum of a separable closure of the function field of S.

We have the specialization map $H^1(\mathcal{E}_s, \mathbb{Q}_\ell) \to H^1(\mathcal{E}_{\overline{\eta}}, \mathbb{Q}_\ell)$. This is identified with $H^1(\pi_1(\mathcal{E}_{\overline{\eta}}), \mathbb{Q}_\ell) \to H^1(\pi_1(\mathcal{E}_{\overline{s}}), \mathbb{Q}_\ell)$, and is injective, since $\pi_1(\mathcal{E}_{\overline{s}}) \to \pi_1(\mathcal{E}_{\overline{\eta}})$ is surjective ([22, Exp. X, Corollaire 2.4]).

Set $V = H^1(\mathcal{E}_{\overline{\eta}}, \mathbb{Q}_\ell)$. For surjectivity, we prove that the image of the specialization map is equal to the invariant part V^{I_s} under the action of the inertia group I_s at s. It suffices to show that $H^1(\mathcal{E}_s, \mathbb{Q}_\ell)$ and V^{I_s} have the same dimension.

We use the Kodaira–Néron–Tate classification of bad fibers (cf. [38, 10.2.1]) to compute that $H^1(\mathcal{E}_s, \mathbb{Q}_\ell)$ is of dimension 2, 1, 0 if E has good reduction, semistable bad reduction, and additive reduction at $\wp$, respectively. It is well known that V^{I_s} is of dimension 2 if and only if E has good reduction at p. If E has semistable bad reduction, then the action of I_s on V is nontrivial and unipotent. Hence V^{I_s} is of dimension 1 in this case. Suppose that E has additive reduction and $V^{I_s} \neq 0$. Then V^{I_s} is of dimension one. Poincare duality implies that I_s acts trivially on V/V^{I_s}, hence the action of I_s on V is unipotent. Therefore, E must have either good or semistable reduction, which leads to a contradiction. □

Corollary 6.5. — *We have*

$$L(E,s) = \prod_{i=0}^{2} \det(1 - \mathrm{Frob} \cdot q^{-s}; H^i_{\mathrm{et}}(\overline{C}, R^1 f_* \mathbb{Q}_\ell))^{(-1)^{i-1}}.$$

Below, we will work with this expression.

Lemma 6.6. — *Let D be a scheme of dimension $\leqslant 1$ that is proper over $\operatorname{Spec} \mathbb{F}_q$. Let $\ell \neq p$ be an integer. Then, the group $H^i_{\mathrm{et}}(\overline{D}, \mathbb{Z}_\ell)$ is torsion free for any $i \in \mathbb{Z}$ and is zero for $i \neq 0, 1, 2$. The group $H^i_{\mathrm{et}}(\overline{D}, \mathbb{Q}_\ell)$ is pure of weight i for $i \neq 1$ and is mixed of weight $\{0,1\}$ for $i = 1$. The group $H^1_{\mathrm{et}}(\overline{D}, \mathbb{Q}_\ell)$ is pure of weight one (resp. pure of weight zero) if D is smooth (resp. every irreducible component of $\overline{D}$ is rational).*

Proof. — We may assume that D is reduced. Let D' be the normalization of D. Let $\pi : D' \to D$ denote the canonical morphism. Let $\mathcal{F}_n$ denote the cokernel of the homomorphism $\mathbb{Z}/\ell^n \to \pi_*(\mathbb{Z}/\ell^n)$ of étale sheaves. The sheaf $\mathcal{F}_n$ is supported on the singular locus D_{sing} of D and is isomorphic to $i_*(\operatorname{Coker}[\mathbb{Z}/\ell^n \to \pi_{\mathrm{sing}*}(\mathbb{Z}/\ell^n)])$, where $i : D_{\mathrm{sing}} \hookrightarrow D$ is the canonical inclusion and $\pi_{\mathrm{sing}} : D' \times_D D_{\mathrm{sing}} \to D_{\mathrm{sing}}$ is the base change of π. Then, the claim follows from the long exact sequence

$$\cdots \longrightarrow H^i_{\mathrm{et}}(\overline{D}, \mathbb{Z}/\ell^n) \longrightarrow H^i_{\mathrm{et}}(\overline{D}', \mathbb{Z}/\ell^n) \longrightarrow H^i_{\mathrm{et}}(\overline{D}, \mathcal{F}_n) \longrightarrow \cdots. \quad \square$$

Lemma 6.7. — *If $i \neq 1$, then $H^i_{\mathrm{et}}(\overline{C}, R^1 f_* \mathbb{Q}_\ell) = 0$.*

Proof. — For any point $x \in \overline{C}(\overline{\mathbb{F}}_q)$ lying over a closed point $\wp \in C$, the canonical homomorphism $H^0_{\mathrm{et}}(\overline{C}, R^1 f_* \mathbb{Q}_\ell) \to H^1_{\mathrm{et}}(\mathcal{E}_x, \mathbb{Q}_\ell)$ is injective since $H^0_{\mathrm{et},c}(\overline{C} \setminus \{x\}, R^1 f_* \mathbb{Q}_\ell) = 0$. By Lemma 6.6, the module $H^1_{\mathrm{et}}(\mathcal{E}_x, \mathbb{Q}_\ell)$ is pure of weight one (resp. of weight zero) if $\mathcal{E}_x$ is smooth (resp. is not smooth). Since we have assumed that $f : \mathcal{E} \to C$ is not smooth, $H^0_{\mathrm{et}}(\overline{C}, R^1 f_* \mathbb{Q}_\ell) = 0$.

Consider a non-empty open subscheme $U \subset C$ such that $f_U : \mathcal{E}_U \to U$ is smooth. The group $H^2_{\mathrm{et}}(\overline{C}, R^1 f_* \mathbb{Q}_\ell) \cong H^2_{\mathrm{et},c}(\overline{U}, R^1 f_{U*} \mathbb{Q}_\ell)$ is the dual of $H^0_{\mathrm{et}}(\overline{U}, R^1 f_{U*} \mathbb{Q}_\ell(1))$ by Poincaré duality. Next, assume $H^0_{\mathrm{et}}(\overline{U}, R^1 f_{U*} \mathbb{Q}_\ell(1))$ is not null. Let $T_\ell(E)$ denote the ℓ-adic Tate module of E. The étale fundamental group $\pi_1(U)$ acts on $T_\ell(E)$. By the given assumption, the $\pi_1(\overline{U})$-invariant part $V = (T_\ell(E) \otimes \mathbb{Q}_\ell)^{\pi_1(\overline{U})}$ is nonzero. Since f is not smooth, V is one-dimensional, hence we have a nonzero homomorphism $\pi_1(\overline{U})^{\mathrm{ab}} \to \mathrm{Hom}(T_\ell(E) \otimes \mathbb{Q}_\ell / V, V)$ of $G_{\mathbb{F}_q}$-modules. By the weight argument, we observe that this is impossible, hence $H^2_{\mathrm{et}}(\overline{C}, R^1 f_* \mathbb{Q}_\ell(1)) = 0$. □

As an immediate consequence, we obtain the following corollary.

Corollary 6.8. — *The spectral sequence*

$$E_2^{i,j} = H^i_{\mathrm{et}}(\overline{C}, R^j f_* \mathbb{Q}_\ell) \Rightarrow H^{i+j}_{\mathrm{et}}(\overline{\mathcal{E}}, \mathbb{Q}_\ell)$$

is E_2-degenerate.

Lemma 6.9. — *Let $U \subset C$ be a non-empty open subscheme such that $f_U : \mathcal{E}_U \to U$ is smooth. Let $\mathrm{Irr}^0(\mathcal{E}^U) \subset \mathrm{Irr}(\mathcal{E}^U)$ denote the subset of irreducible components of $\mathcal{E}^U$ that do not intersect $\iota(C)$. We regard $\mathrm{Irr}^0(\mathcal{E}^U)$ as a closed subscheme of $\mathrm{Irr}(\mathcal{E}^U)$ (recall the convention in Section 6.1 on $\mathrm{Irr}(\mathcal{E}^U)$). Then,*

$$L(h^i(\mathcal{E}), s) = \begin{cases} (1 - q^{-s}), & \text{if } i = 0, \\ L(h^1(C), s), & \text{if } i = 1, \\ (1 - q^{1-s})^2 L(E, s) L(h^0(\mathrm{Irr}^0(\mathcal{E}^U)), s-1), & \text{if } i = 2, \\ L(h^1(C), s-1), & \text{if } i = 3, \\ (1 - q^{2-s}), & \text{if } i = 4. \end{cases}$$

Proof. — We prove the lemma for $i = 2$; the other cases are straightforward. Since $R^2 f_{U*} \mathbb{Q}_\ell \cong \mathbb{Q}_\ell(-1)$, there exists an exact sequence

$$0 \longrightarrow H^0_{\mathrm{et}}(\overline{C}, R^2 f_* \mathbb{Q}_\ell) \longrightarrow H^2_{\mathrm{et}}(\overline{\mathcal{E}^U}, \mathbb{Q}_\ell) \longrightarrow H^1_{\mathrm{et},c}(\overline{U}, \mathbb{Q}_\ell(-1)).$$

The map $H^2_{\mathrm{et}}(\overline{\mathcal{E}^U}, \mathbb{Q}_\ell) \to H^1_{\mathrm{et},c}(\overline{U}, \mathbb{Q}_\ell(-1))$ decomposes as

$$H^2_{\mathrm{et}}(\overline{\mathcal{E}^U}, \mathbb{Q}_\ell) \longrightarrow H^0_{\mathrm{et}}(\overline{C} \setminus \overline{U}, \mathbb{Q}_\ell(-1)) \longrightarrow H^1_{\mathrm{et},c}(\overline{U}, \mathbb{Q}_\ell(-1)).$$

Given the above, $H^0_{\mathrm{et}}(\overline{C}, R^2 f_* \mathbb{Q}_\ell)$ is isomorphic to the inverse image of the image of the homomorphism $H^0_{\mathrm{et}}(\overline{C}, \mathbb{Q}_\ell(-1)) \to H^0_{\mathrm{et}}(\overline{C} \setminus \overline{U}, \mathbb{Q}_\ell(-1))$ under the surjective homomorphism $H^2_{\mathrm{et}}(\overline{\mathcal{E}^U}, \mathbb{Q}_\ell) \to H^0_{\mathrm{et}}(\overline{C} \setminus \overline{U}, \mathbb{Q}_\ell(-1))$. This proves the claim. □

6.4. The fundamental group of $\mathcal{E}$

Lemma 6.10. — *For $i=0,1$, the pull-back $H^i(C,\mathcal{O}_C)\to H^i(\mathcal{E},\mathcal{O}_{\mathcal{E}})$ is an isomorphism.*

Proof. — The claim for $i=0$ is clear, thus we prove the claim for $i=1$. Let us write $\mathcal{L}=R^1f_*\mathcal{O}_{\mathcal{E}}$. It suffices to prove $H^0(C,\mathcal{L})=0$. We note that $\mathcal{L}$ is an invertible $\mathcal{O}_C$-module since the morphism $\mathcal{E}\to C$ has no multiple fiber. The Leray spectral sequence $E_2^{i,j}=H^i(C,R^jf_*\mathcal{O}_{\mathcal{E}})\Rightarrow H^{i+j}(\mathcal{E},\mathcal{O}_{\mathcal{E}})$ shows that the Euler–Poincaré characteristic $\chi(\mathcal{O}_{\mathcal{E}})$ equals $\chi(\mathcal{O}_C)-\chi(\mathcal{L})=-\deg\mathcal{L}$. By the well-known inequality $\chi(\mathcal{O}_{\mathcal{E}})>0$ (cf. [49], [12], or [50, Theorem 2]), we obtain $\deg\mathcal{L}<0$, which proves $H^0(C,\mathcal{L})=0$. ☐

Lemma 6.11.

(1) *The canonical homomorphism $\pi_1^{\mathrm{ab}}(\mathcal{E})\to\pi_1^{\mathrm{ab}}(C)$ between the abelian (étale) fundamental groups is an isomorphism.*
(2) *The canonical morphism $\mathrm{Pic}^o_{C/\mathbb{F}_q}\to\mathrm{Pic}^o_{\mathcal{E}/\mathbb{F}_q}$ between the identity components of the Picard schemes is an isomorphism.*

Proof. — The homomorphism $\mathrm{Pic}^o_{C/\mathbb{F}_q}\to\mathrm{Pic}^o_{\mathcal{E}/\mathbb{F}_q,\mathrm{red}}$ is an isomorphism by [55, Theorem 4.1]. This, combined with the cohomology long exact sequence of the Kummer sequence, implies that if $p\nmid m$, then $H^1_{\mathrm{et}}(C,\mathbb{Z}/m)\to H^1_{\mathrm{et}}(\mathcal{E},\mathbb{Z}/m)$ is an isomorphism, hence to prove (1), we are reduced to showing that $H^1_{\mathrm{et}}(C,\mathbb{Z}/p^n)\to H^1_{\mathrm{et}}(\mathcal{E},\mathbb{Z}/p^n)$ is an isomorphism for all $n\geqslant 1$. For any scheme X that is proper over $\operatorname{Spec}\mathbb{F}_q$, there exists an exact sequence

$$0\longrightarrow\mathbb{Z}/p^n\mathbb{Z}\longrightarrow W_n\mathcal{O}_X\xrightarrow{1-\sigma}W_n\mathcal{O}_X\longrightarrow 0$$

of étale sheaves, where $W_n\mathcal{O}_X$ is the sheaf of Witt vectors and $\sigma:W_n\mathcal{O}_X\to W_n\mathcal{O}_X$ is the Frobenius endomorphism. This gives rise to the following commutative diagram with exact rows

$$\begin{tikzcd}
\cdots \arrow[r, "1-\sigma"] & H^0(C,W_n\mathcal{O}_C) \arrow[r] \arrow[d] & H^1_{\mathrm{et}}(C,\mathbb{Z}/p^n) \arrow[r] \arrow[d] & H^1(C,W_n\mathcal{O}_C) \arrow[r, "1-\sigma"] \arrow[d] & \cdots \\
\cdots \arrow[r, "1-\sigma"] & H^0(\mathcal{E},W_n\mathcal{O}_{\mathcal{E}}) \arrow[r] & H^1_{\mathrm{et}}(\mathcal{E},\mathbb{Z}/p^n) \arrow[r] & H^1(\mathcal{E},W_n\mathcal{O}_{\mathcal{E}}) \arrow[r, "1-\sigma"] & \cdots .
\end{tikzcd}$$

By Lemma 6.10 and induction on n, we observe that the homomorphism $H^i(C,W_n\mathcal{O}_C)\to H^i(\mathcal{E},W_n\mathcal{O}_{\mathcal{E}})$ is an isomorphism for $i=0,1$, thus the map $H^1_{\mathrm{et}}(C,\mathbb{Z}/p^n)\to H^1_{\mathrm{et}}(\mathcal{E},\mathbb{Z}/p^n)$ is an isomorphism. This proves claim (1).

For (2), it suffices to prove that homomorphism $\mathrm{Lie}\ \mathrm{Pic}_{C/\mathbb{F}_q}\to\mathrm{Lie}\ \mathrm{Pic}_{\mathcal{E}/\mathbb{F}_q}$ between the tangent spaces is an isomorphism. Since this homomorphism is identified with homomorphism $H^1(C, \mathcal{O}_C) \to H^1(\mathcal{E}, \mathcal{O}_{\mathcal{E}})$, claim (2) follows from Lemma 6.10. □

Remark 6.12. — Using Lemma 6.11 (1), we can prove that the homomorphism $\pi_1(\mathcal{E}) \to \pi_1(C)$ is an isomorphism. Since it is not used in this paper, we only sketch the proof below.

Let $x \to C$ be a geometric point. Since the morphism $f : \mathcal{E} \to C$ has a section, fiber $\mathcal{E}_x$ of f at x has a reduced irreducible component. This, together with the regularity of $\mathcal{E}$ and C, shows that the canonical ring homomorphism $H^0(x, \mathcal{O}_x) \to H^0(Y, \mathcal{O}_Y)$ is an isomorphism for any connected finite étale covering Y of $\mathcal{E}_x$, hence by the same argument as in the proof of [22, Exp. X, Proposition 1.2, Théorème 1.3], we have an exact sequence

$$\pi_1(\mathcal{E}_x) \longrightarrow \pi_1(\mathcal{E}) \longrightarrow \pi_1(C) \longrightarrow 1.$$

In particular here, the kernel of $\pi_1(\mathcal{E}) \to \pi_1(C)$ is abelian. Applying Lemma 6.11 (1) to $\mathcal{E} \times_C C' \to C'$ for each finite connected étale cover $C' \to C$, we obtain the bijectivity of $\pi_1(\mathcal{E}) \to \pi_1(C)$.

More generally, the statements in Lemma 6.11 and the statement above that the fundamental groups are isomorphic are also valid if $\mathcal{E}$ is a regular minimal elliptic fibration that is proper flat non-smooth over C with a section, where C is a proper smooth curve over an arbitrary perfect base field.

Corollary 6.13. — *For any prime number $\ell \neq p$ and for any $i \in \mathbb{Z}$, the group $H^i_{\mathrm{et}}(\overline{\mathcal{E}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$ is divisible.*

Proof. — The claim for $i \neq 1, 2$ is obvious. By Lemma 6.11, we have $H^1_{\mathrm{et}}(\overline{\mathcal{E}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \cong H^1_{\mathrm{et}}(\overline{C}, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$, hence $H^1_{\mathrm{et}}(\overline{\mathcal{E}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$ is divisible. The group $H^2_{\mathrm{et}}(\overline{\mathcal{E}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$ is divisible since $H^2_{\mathrm{et}}(\overline{\mathcal{E}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell)^{\mathrm{red}}$ is isomorphic to the Pontryagin dual of $H^1_{\mathrm{et}}(\overline{\mathcal{E}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))^{\mathrm{red}}$. □

Corollary 6.14. — *For $i \in \mathbb{Z}$, we set $M^i_j = \bigoplus_{\ell \neq p} H^i_{\mathrm{et}}(\mathcal{E}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(j))$. For a rational number a, we write $|a|^{(p')} = |a| \cdot |a|_p$.*

1. *For $i \leqslant -1$ or $i \geqslant 6$, the group M^i_j is zero.*
2. *For $j \neq 2$ (resp. $j = 2$), the group M^5_j is zero (resp. is isomorphic to $\bigoplus_{\ell \neq p} \mathbb{Q}_\ell/\mathbb{Z}_\ell$).*
3. *For $j \neq 0$, the group M^0_j is cyclic of order $q^{|j|} - 1$. The group M^0_0 is isomorphic to $\bigoplus_{\ell \neq p} \mathbb{Q}_\ell/\mathbb{Z}_\ell$.*
4. *For $j \neq 0$, the group M^1_j is finite of order $|L(h^1(C), 1-j)|^{(p')}$.*
5. *For $j \neq 1$, the group M^2_j is finite of order $|L(h^2(\mathcal{E}), 2-j)|^{(p')}$.*
6. *For $j \neq 1$, group M^3_j is finite of order $|L(h^1(C), 2-j)|^{(p')}$.*

(7) *For $j \neq 2$, the group M_j^4 is cyclic of order $q^{|2-j|}-1$. The group M_2^4 is isomorphic to $\bigoplus_{\ell \neq p} \mathbb{Q}_\ell/\mathbb{Z}_\ell$.*

Proof. — By Corollary 6.13, if $i \neq 2j+1$ and $\ell \neq p$, the group $H^i_{\mathrm{et}}(\mathcal{E}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(j))$ is isomorphic to $H^i_{\mathrm{et}}(\overline{\mathcal{E}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(j))^{G_{\mathbb{F}_q}}$. Then, we have

$$|H^i_{\mathrm{et}}(\mathcal{E}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(j))| = |L(h^{4-i}(\mathcal{E}), 2-j)|_\ell^{-1}$$

by Poincaré duality for $i \neq 2j, 2j+1$, hence the claim follows from Lemma 6.9. □

6.5. Torsion in the étale cohomology of open elliptic surfaces

We first fix a non-empty open subscheme $U \subset C$.

LEMMA 6.15. — *Let $\ell \neq p$ be a prime number. For $i \in \mathbb{Z}$, let γ_i denote the pull-back $\gamma_i : H^i_{\mathrm{et}}(\overline{\mathcal{E}}, \mathbb{Z}_\ell) \to H^i_{\mathrm{et}}(\overline{\mathcal{E}^U}, \mathbb{Z}_\ell)$.*

(1) *For $i \neq 0, 2$, the homomorphism γ_i is zero.*
(2) *The cokernel $(\mathrm{Coker}\ \gamma_2)_{\mathbb{Q}_\ell}$ is isomorphic to the kernel of $H^0_{\mathrm{et}}(\overline{C} \setminus \overline{U}, \mathbb{Q}_\ell(-1)) \to H^0_{\mathrm{et}}(\mathrm{Spec}\ \overline{\mathbb{F}}_q, \mathbb{Q}_\ell(-1))$.*
(3) *There exists a canonical isomorphism*

$$\mathrm{Hom}_{\mathbb{Z}}(T_U, \mathbb{Q}_\ell/\mathbb{Z}_\ell(-1)) \cong (\mathrm{Coker}\ \gamma_2)_{\mathrm{tors}}.$$

Proof. — By Lemma 6.11, the pull-back $H^1_{\mathrm{et}}(\overline{C}, \mathbb{Z}_\ell) \to H^1_{\mathrm{et}}(\overline{\mathcal{E}}, \mathbb{Z}_\ell)$ is an isomorphism, hence the homomorphism $H^1_{\mathrm{et}}(\overline{\mathcal{E}}, \mathbb{Z}_\ell) \to H^1_{\mathrm{et}}(\overline{\mathcal{E}^U}, \mathbb{Z}_\ell)$ is zero. Claim (1) follows.

Let $\mathrm{NS}(\overline{\mathcal{E}}) = \mathrm{Pic}_{\mathcal{E}/\mathbb{F}_q}(\overline{\mathbb{F}}_q)/\mathrm{Pic}^o_{\mathcal{E}/\mathbb{F}_q}(\overline{\mathbb{F}}_q)$ denote the Néron–Severi group of $\overline{\mathcal{E}}$. For a prime number ℓ, we set $T_\ell M = \mathrm{Hom}(\mathbb{Q}_\ell/\mathbb{Z}_\ell, M)$.

We have the following exact sequence from Kummer theory:

$$0 \longrightarrow \mathrm{NS}(\overline{\mathcal{E}}) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \xrightarrow{\mathrm{cl}_\ell} H^2_{\mathrm{et}}(\overline{\mathcal{E}}, \mathbb{Z}_\ell(1)) \longrightarrow T_\ell H^2_{\mathrm{et}}(\overline{\mathcal{E}}, \mathbb{G}_m) \longrightarrow 0.$$

We note that $T_\ell H^2_{\mathrm{et}}(\overline{\mathcal{E}}, \mathbb{G}_m)$ is torsion free. For $D \in \mathrm{Irr}(\overline{\mathcal{E}^U})$, let $[D] \in \mathrm{NS}(\overline{\mathcal{E}})$ denote the class of the Weil divisor D_{red} on $\overline{\mathcal{E}}$. By [10, Cycle, Definition 2.3.2], the D-component of the homomorphism $\gamma_2 : H^2_{\mathrm{et}}(\overline{\mathcal{E}}, \mathbb{Z}_\ell) \to H^2_{\mathrm{et}}(\overline{\mathcal{E}^U}, \mathbb{Z}_\ell) \cong \mathrm{Map}(\mathrm{Irr}(\overline{\mathcal{E}^U}), \mathbb{Z}_\ell(-1))$ is identified with the homomorphism

$$H^2_{\mathrm{et}}(\overline{\mathcal{E}}, \mathbb{Z}_\ell) \xrightarrow{\cup \mathrm{cl}_\ell([D])} H^4_{\mathrm{et}}(\overline{\mathcal{E}}, \mathbb{Z}_\ell(1)) \cong \mathbb{Z}_\ell(-1).$$

Let $M \subset \mathrm{NS}(\overline{\mathcal{E}})$ denote the subgroup generated by $\{[D] \mid D \in \mathrm{Irr}(\overline{\mathcal{E}^U})\}$. By Corollary 6.13, the cup-product

$$H^2_{\mathrm{et}}(\overline{\mathcal{E}}, \mathbb{Z}_\ell(1)) \times H^2_{\mathrm{et}}(\overline{\mathcal{E}}, \mathbb{Z}_\ell(1)) \longrightarrow H^4_{\mathrm{et}}(\overline{\mathcal{E}}, \mathbb{Z}_\ell(2)) \cong \mathbb{Z}_\ell$$

is a perfect pairing, hence the image of γ_2 is identified with the image of the composition

$$\begin{aligned}\mathrm{Hom}_{\mathbb{Z}_\ell}(H^2_{\mathrm{et}}(\overline{\mathcal{E}},\mathbb{Z}_\ell(1)),\mathbb{Z}_\ell(-1)) &\xrightarrow{\alpha^*} \mathrm{Hom}_{\mathbb{Z}}(M,\mathbb{Z}_\ell(-1)) \\ &\xrightarrow{\beta^*} \mathrm{Map}(\mathrm{Irr}(\overline{\mathcal{E}^U}),\mathbb{Z}_\ell(-1)) \cong H^2_{\mathrm{et}}(\overline{\mathcal{E}^U},\mathbb{Z}_\ell),\end{aligned}$$

where α^* is the homomorphism induced by the restriction $\alpha : M\otimes_{\mathbb{Z}}\mathbb{Z}_\ell \hookrightarrow H^2_{\mathrm{et}}(\overline{\mathcal{E}},\mathbb{Z}_\ell(1))$ of the cycle class map cl_ℓ to M, and the homomorphism β^* is induced by the canonical surjection $\beta : \bigoplus_{D\in\mathrm{Irr}(\overline{\mathcal{E}^U})}\mathbb{Z} \twoheadrightarrow M$. Since β is surjective, the homomorphism β^* is injective and we have an exact sequence

$$0 \longrightarrow \mathrm{Coker}\ \alpha^* \longrightarrow \mathrm{Coker}\ \gamma_2 \longrightarrow \mathrm{Coker}\ \beta^* \longrightarrow 0.$$

Since α is a homomorphism of finitely generated $\mathbb{Z}_\ell$-modules that is injective, the cokernel Coker α^* is a finite group. Further, the group M is a free abelian group with basis $\mathrm{Irr}^0(\overline{\mathcal{E}^U})\cup\{D'\}$, where D' is an arbitrary element in $\mathrm{Irr}(\overline{\mathcal{E}^U})\setminus\mathrm{Irr}^0(\overline{\mathcal{E}^U})$, hence Coker β^* is isomorphic to the group $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{Ker}\ \beta,\mathbb{Z}_\ell(-1))$. This proves (2).

The torsion part of Coker γ_2 is identified with the group Coker α^*. The homomorphism α^* is the composition of the two homomorphisms

$$\begin{aligned}&\mathrm{Hom}_{\mathbb{Z}_\ell}(H^2_{\mathrm{et}}(\overline{\mathcal{E}},\mathbb{Z}_\ell(1)),\mathbb{Z}_\ell(-1)) \\ &\qquad\xrightarrow{\mathrm{cl}_\ell^*} \mathrm{Hom}_{\mathbb{Z}}(\mathrm{NS}(\overline{\mathcal{E}}),\mathbb{Z}_\ell(-1)) \xrightarrow{\iota^*} \mathrm{Hom}_{\mathbb{Z}}(M,\mathbb{Z}_\ell(-1)),\end{aligned}$$

where the first (resp. second) homomorphism is that induced by the cycle class map $\mathrm{cl}_\ell : \mathrm{NS}(\overline{\mathcal{E}})\otimes_{\mathbb{Z}}\mathbb{Z}_\ell \hookrightarrow H^2_{\mathrm{et}}(\overline{\mathcal{E}},\mathbb{Z}_\ell(1))$ (resp. the inclusion $M\otimes_{\mathbb{Z}}\mathbb{Z}_\ell \hookrightarrow \mathrm{NS}(\overline{\mathcal{E}})$). Since Coker cl_ℓ is torsion free, as noted above, the homomorphism cl_ℓ^* is surjective, hence we have isomorphisms

$$\begin{aligned}&\mathrm{Coker}\ \alpha^* \cong \mathrm{Coker}\ \iota^* \cong \mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{NS}(\overline{\mathcal{E}})/M,\mathbb{Z}_\ell(-1)) \\ &\cong \mathrm{Hom}_{\mathbb{Z}}(\mathrm{NS}(\overline{\mathcal{E}})/M,\mathbb{Q}_\ell/\mathbb{Z}_\ell(-1)) = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Div}(\overline{\mathcal{E}}_U)/\sim_{\mathrm{alg}},\mathbb{Q}_\ell/\mathbb{Z}_\ell(-1)).\end{aligned}$$

Given the above, we have claim (3). □

Corollary 6.16. — *For $i\neq 3$, the group $H^i_{c,\mathrm{et}}(\overline{\mathcal{E}}_U,\mathbb{Z}_\ell)$ is torsion free and the group $H^3_{c,\mathrm{et}}(\overline{\mathcal{E}}_U,\mathbb{Z}_\ell)_{\mathrm{tors}}$ is canonically isomorphic to the group $\mathrm{Hom}_{\mathbb{Z}}(T_U,\mathbb{Q}_\ell/\mathbb{Z}_\ell(-1))$. We set*

$$L(h^i_{c,\ell}(\mathcal{E}_U),s) = \det(1-\mathrm{Frob}\cdot q^{-s}; H^i_{c,\mathrm{et}}(\overline{\mathcal{E}}_U,\mathbb{Q}_\ell)).$$

Then if $U \neq C$, we have

$$L(h^i_{c,\ell}(\mathcal{E}_U), s) = \begin{cases} 1, & \text{if } i \leqslant 0 \text{ or } i \geqslant 5, \\ \frac{L(h^1(C),s)L(h^0(C\setminus U),s)}{1-q^{-s}} & \text{if } i = 1, \\ \frac{L(h^2(\mathcal{E}),s)L(h^1(\mathcal{E}^U),s)L(h^0(C\setminus U),s-1)}{(1-q^{1-s})L(h^2(\mathcal{E}^U),s)} & \text{if } i = 2, \\ \frac{L(h^1(C),s-1)L(h^0(C\setminus U),s-1)}{1-q^{1-s}} & \text{if } i = 3, \\ 1-q^{2-s}, & \text{if } i = 4. \end{cases}$$

Proof. — The above follows from Lemmas 6.9 and 6.15, as well as the long exact sequence

$$\cdots \longrightarrow H^i_{\mathrm{et},c}(\overline{\mathcal{E}}_U, \mathbb{Z}_\ell) \longrightarrow H^i_{\mathrm{et}}(\overline{\mathcal{E}}, \mathbb{Z}_\ell) \longrightarrow H^i_{\mathrm{et}}(\overline{\mathcal{E}^U}, \mathbb{Z}_\ell) \longrightarrow \cdots . \qquad \square$$

Remark 6.17. — Corollary 6.16 in particular shows that the function $L(h^i_{c,\ell}(\mathcal{E}_U), s)$ is independent of $\ell \neq p$. We can show the ℓ-independence of $L(h^i_{c,\ell}(X), s) = \det(1 - \mathrm{Frob} \cdot q^{-s}; H^i_{c,\mathrm{et}}(X, \mathbb{Q}_\ell))$ for any normal surface X over $\mathbb{F}_q$, which is not necessarily proper. Since we will not need this further, we only show a sketch here. There is a proper smooth surface X' and a closed subset $D \subset X'$ of pure codimension one such that $X = X' \setminus D$. We can express the cokernel and kernel of the restriction map $H^1_{\mathrm{et}}(\overline{X}', \mathbb{Q}_l) \to H^1_{\mathrm{et}}(\overline{D}, \mathbb{Q}_l)$ in terms of $\mathrm{Pic}_{X'/\mathbb{F}_q}$ and the Jacobian of the normalization of each irreducible component of D. Then, we apply the same method as above to obtain the result.

Corollary 6.18. — *Suppose that $U \neq C$. Then, we have the following.*

(1) *$H^i_{\mathrm{et}}(\mathcal{E}_U, \mathbb{Q}_\ell/\mathbb{Z}_\ell(j))$ is zero for $i \leqslant -1$ or $i \geqslant 5$.*

(2) *For $j \neq 0$, the group $H^0_{\mathrm{et}}(\mathcal{E}_U, \mathbb{Q}_\ell/\mathbb{Z}_\ell(j))$ is isomorphic to $\mathbb{Z}_\ell/(q^j - 1)$, and $H^0_{\mathrm{et}}(\mathcal{E}_U, \mathbb{Q}_\ell/\mathbb{Z}_\ell(0)) = \mathbb{Q}_\ell/\mathbb{Z}_\ell$.*

(3) *For $j \neq 0, 1$, the group $H^1_{\mathrm{et}}(\mathcal{E}_U, \mathbb{Q}_\ell/\mathbb{Z}_\ell(j))$ is finite of order*

$$\frac{|T'_{U,(j-1)}|_\ell^{-1} \cdot |L(h^1(C), 1-j)L(h^0(C \setminus U), 1-j)|_\ell^{-1}}{|q^{j-1} - 1|_\ell^{-1}}.$$

The group $H^1_{\mathrm{et}}(\mathcal{E}_U, \mathbb{Q}_\ell/\mathbb{Z}_\ell(0))$ is isomorphic to the direct sum of $\mathbb{Q}_\ell/\mathbb{Z}_\ell$ and a finite group of order

$$\frac{|T'_{U,(-1)}|_\ell^{-1} \cdot |L(h^1(C), 1)L(h^0(C \setminus U), 1)|_\ell^{-1}}{|q - 1|_\ell^{-1}}.$$

(4) *For $j \neq 1, 2$, the cohomology group $H^2_{\mathrm{et}}(\mathcal{E}_U, \mathbb{Q}_\ell/\mathbb{Z}_\ell(j))$ is finite of order*

$$\frac{|T'_{U,(j-1)}|_\ell^{-1} \cdot |L(h^2(\mathcal{E}), 2-j)L(h^1(\mathcal{E}^U), 2-j)L(h^0(C \setminus U), 1-j)|_\ell^{-1}}{|(q^{j-1} - 1)L(h^2(\mathcal{E}^U), 2-j)|_\ell^{-1}}.$$

(5) *For $j \neq 1, 2$, the group $H^3_{\mathrm{et}}(\mathcal{E}_U, \mathbb{Q}_\ell/\mathbb{Z}_\ell(j))$ is finite of order*

$$\frac{|L(h^1(C), 2-j)L(h^0(C \setminus U), 2-j)|_\ell^{-1}}{|q^{j-2}-1|_\ell^{-1}}.$$

The cohomology group $H^3_{\mathrm{et}}(\mathcal{E}_U, \mathbb{Q}_\ell/\mathbb{Z}_\ell(1))$ is isomorphic to the direct sum of $(\mathbb{Q}_\ell/\mathbb{Z}_\ell)^{\oplus|C\setminus U|-1}$ and a finite group of order

$$\frac{|L(h^1(C), 1)L(h^0(C \setminus U), 1)|_\ell^{-1}}{|q-1|_\ell^{-1}}.$$

(6) *For $j \neq 2$ (resp. $j = 2$), the group $H^4_{\mathrm{et}}(\mathcal{E}_U, \mathbb{Q}_\ell/\mathbb{Z}_\ell(j))$ is zero (resp. is isomorphic to $(\mathbb{Q}_\ell/\mathbb{Z}_\ell)^{\oplus|C\setminus U|-1}$).*

Proof. — The cohomology group $H^i_{\mathrm{et}}(\mathcal{E}_U, \mathbb{Q}_\ell/\mathbb{Z}_\ell(j))$ is the Pontryagin dual of the group $H^{5-i}_{\mathrm{et},c}(\mathcal{E}_U, \mathbb{Z}_\ell(2-j))$. The claims follow from Corollary 6.16 and the short exact sequence

$$0 \longrightarrow H^{4-i}_{c,\mathrm{et}}(\overline{\mathcal{E}}_U, \mathbb{Z}_\ell(2-j))_{G_{\mathbb{F}_q}} \longrightarrow H^{5-i}_{c,\mathrm{et}}(\mathcal{E}_U, \mathbb{Z}_\ell(2-j)) \longrightarrow H^{5-i}_{c,\mathrm{et}}(\overline{\mathcal{E}}_U, \mathbb{Z}_\ell(2-j))^{G_{\mathbb{F}_q}} \longrightarrow 0. \quad \square$$

Lemma 6.19. — *Suppose that $U \neq C$. Then, $H^2_{\mathrm{et}}(\mathcal{E}_U, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))^{\mathrm{red}}$ is finite of order*

$$\frac{|T'_{U,(1)}|_\ell^{-1} \cdot |L(h^2(\mathcal{E}), 0)L^*(h^1(\mathcal{E}^U), 0)L(h^0(C \setminus U), -1)|_\ell^{-1}}{|(q-1)L(h^0(\mathrm{Irr}(\mathcal{E}^U)), -1)|_\ell^{-1}}.$$

Proof. — We note that the group $H^2_{\mathrm{et}}(\mathcal{E}_U, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))^{\mathrm{red}}$ is canonically isomorphic to the group $H^3_{\mathrm{et}}(\mathcal{E}_U, \mathbb{Z}_\ell(2))_{\mathrm{tors}}$. Let us consider the long exact sequence

$$\cdots \longrightarrow H^i_{\mathcal{E}^U,\mathrm{et}}(\mathcal{E}, \mathbb{Z}_\ell(2)) \xrightarrow{\mu_i} H^i_{\mathrm{et}}(\mathcal{E}, \mathbb{Z}_\ell(2)) \longrightarrow H^i_{\mathrm{et}}(\mathcal{E}_U, \mathbb{Z}_\ell(2)) \longrightarrow \cdots.$$

The group Ker μ_4 is isomorphic to the Pontryagin dual of the cokernel of the homomorphism $H^1_{\mathrm{et}}(\mathcal{E}, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \to H^1_{\mathrm{et}}(\mathcal{E}^U, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$. By Lemma 6.11, this homomorphism factors through $H^1_{\mathrm{et}}(C \setminus U, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \to H^1_{\mathrm{et}}(\mathcal{E}^U, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$. In particular, the group (Ker $\mu_4)_{\mathrm{tors}}$ is isomorphic to the Pontryagin dual of $(H^1_{\mathrm{et}}(\overline{\mathcal{E}^U}, \mathbb{Q}_\ell/\mathbb{Z}_\ell)^{G_{\mathbb{F}_q}})^{\mathrm{red}}$. By the weight argument, we observe that Coker μ_3 is a finite group. It then follows that

$$|H^3_{\mathrm{et}}(\mathcal{E}_U, \mathbb{Z}_\ell(2))_{\mathrm{tors}}| = |L^*(h^1(\mathcal{E}^U), 0)| \cdot |\mathrm{Coker}\,\mu_3|.$$

Let μ' denote the homomorphism $H^2_{\overline{\mathcal{E}^U},\mathrm{et}}(\overline{\mathcal{E}}, \mathbb{Z}_\ell(2)) \to H^2_{\mathrm{et}}(\overline{\mathcal{E}}, \mathbb{Z}_\ell(2))$. We have an exact sequence

$$\text{(6.1)} \quad \mathrm{Ker}\,\mu_3 \longrightarrow H^3_{\overline{\mathcal{E}^U},\mathrm{et}}(\overline{\mathcal{E}}, \mathbb{Z}_\ell(2))^{G_{\mathbb{F}_q}} \longrightarrow (\mathrm{Coker}\,\mu')_{G_{\mathbb{F}_q}} \longrightarrow \mathrm{Coker}\,\mu_3 \longrightarrow 0.$$

Since $\mathrm{Ker}\ \mu_3 \cong \mathrm{Coker}[H^2_{\mathrm{et}}(\mathcal{E}, \mathbb{Z}_\ell(2)) \to H^2_{\mathrm{et}}(\mathcal{E}_U, \mathbb{Z}_\ell(2))]$, the cokernel of $\mathrm{Ker}\ \mu_3 \to H^3_{\overline{\mathcal{E}^U},\mathrm{et}}(\overline{\mathcal{E}}, \mathbb{Z}_\ell(2))^{G_{\mathbb{F}_q}}$ is isomorphic to the cokernel of the homomorphism

$$\nu' : H^2_{\mathrm{et}}(\overline{\mathcal{E}}_U, \mathbb{Z}_\ell(2))^{G_{\mathbb{F}_q}} \longrightarrow H^3_{\overline{\mathcal{E}^U},\mathrm{et}}(\overline{\mathcal{E}}, \mathbb{Z}_\ell(2))^{G_{\mathbb{F}_q}}.$$

Consider the following diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Coker}\,\mu' \arrow[r] \arrow[d, "1-\mathrm{Frob}"'] & H^2_{\mathrm{et}}(\overline{\mathcal{E}}_U, \mathbb{Z}_\ell(2)) \arrow[r, "\nu"] \arrow[d, "1-\mathrm{Frob}"'] & H^3_{\overline{\mathcal{E}^U},\mathrm{et}}(\overline{\mathcal{E}}, \mathbb{Z}_\ell(2)) \arrow[d, "1-\mathrm{Frob}"'] \\
0 \arrow[r] & \mathrm{Coker}\ \mu' \arrow[r] & H^2_{\mathrm{et}}(\overline{\mathcal{E}}_U, \mathbb{Z}_\ell(2)) \arrow[r, "\nu"] & H^3_{\overline{\mathcal{E}^U},\mathrm{et}}(\overline{\mathcal{E}}, \mathbb{Z}_\ell(2)).
\end{tikzcd}$$

Since $(\mathrm{Coker}\ \nu)^{G_{\mathbb{F}_q}} \subset H^3_{\mathrm{et}}(\overline{\mathcal{E}}, \mathbb{Z}_\ell(2))^{G_{\mathbb{F}_q}} = 0$, Coker ν' is isomorphic to the kernel of $(\mathrm{Coker}\ \mu')_{G_{\mathbb{F}_q}} \to H^2_{\mathrm{et}}(\overline{\mathcal{E}}_U, \mathbb{Z}_\ell(2))_{G_{\mathbb{F}_q}}$, hence by (6.1), $|\mathrm{Coker}\,\mu_3|$ equals the order of

$$\begin{aligned} M'' &= \mathrm{Im}[(\mathrm{Coker}\ \mu')_{G_{\mathbb{F}_q}} \longrightarrow H^2_{\mathrm{et}}(\overline{\mathcal{E}}_U, \mathbb{Z}_\ell(2))_{G_{\mathbb{F}_q}}] \\ &= \mathrm{Im}[H^2_{\mathrm{et}}(\overline{\mathcal{E}}, \mathbb{Z}_\ell(2))_{G_{\mathbb{F}_q}} \longrightarrow H^2_{\mathrm{et}}(\overline{\mathcal{E}}_U, \mathbb{Z}_\ell(2))_{G_{\mathbb{F}_q}}]. \end{aligned}$$

Next, we set $M' = \mathrm{Im}[H^2_{\mathrm{et}}(\overline{\mathcal{E}}, \mathbb{Z}_\ell(2)) \to H^2_{\mathrm{et}}(\overline{\mathcal{E}}_U, \mathbb{Z}_\ell(2))]$. From the commutative diagram with exact rows
(6.2)

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{NS}(\overline{\mathcal{E}}) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \arrow[r] \arrow[d] & H^2_{\mathrm{et}}(\overline{\mathcal{E}}, \mathbb{Z}_\ell(1)) \arrow[r] \arrow[d] & T_\ell H^2_{\mathrm{et}}(\overline{\mathcal{E}}, \mathbb{G}_m) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (\mathrm{Div}(\overline{\mathcal{E}}_U)/{\sim_{\mathrm{alg}}}) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \arrow[r] & H^2_{\mathrm{et}}(\overline{\mathcal{E}}_U, \mathbb{Z}_\ell(1)) \arrow[r] & T_\ell H^2_{\mathrm{et}}(\overline{\mathcal{E}}_U, \mathbb{G}_m) \arrow[r] & 0
\end{tikzcd}$$

and the exact sequence

$$0 \longrightarrow H^2_{\mathrm{et}}(\overline{\mathcal{E}}, \mathbb{G}_m) \longrightarrow H^2_{\mathrm{et}}(\overline{\mathcal{E}}_U, \mathbb{G}_m) \longrightarrow H^1_{\mathrm{et}}(\overline{\mathcal{E}^U}, \mathbb{Q}/\mathbb{Z}),$$

we obtain an exact sequence

$$0 \longrightarrow M' \longrightarrow H^2_{\mathrm{et}}(\overline{\mathcal{E}}_U, \mathbb{Z}_\ell(2)) \longrightarrow T_\ell H^1_{\mathrm{et}}(\overline{\mathcal{E}^U}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(1)).$$

By the weight argument, we obtain $(T_\ell H^1_{\mathrm{et}}(\overline{\mathcal{E}^U}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(1)))^{G_{\mathbb{F}_q}} = 0$, hence the canonical surjection $M'_{G_{\mathbb{F}_q}} \to M''$ is an isomorphism. From (6.2), we have an exact sequence

$$0 \longrightarrow (\mathrm{Div}(\overline{\mathcal{E}}_U)/{\sim_{\mathrm{alg}}}) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell(1) \longrightarrow M' \longrightarrow T_\ell H^2_{\mathrm{et}}(\overline{\mathcal{E}}, \mathbb{G}_m)(1) \longrightarrow 0.$$

By the weight argument, we yield $(T_\ell H^2_{\mathrm{et}}(\overline{\mathcal{E}}, \mathbb{G}_m)(1))^{G_{\mathbb{F}_q}} = 0$, hence

$$0 \longrightarrow ((\mathrm{Div}(\overline{\mathcal{E}}_U)/\sim_{\mathrm{alg}}) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell(1))_{G_{\mathbb{F}_q}} \longrightarrow M'_{G_{\mathbb{F}_q}} \longrightarrow (T_\ell H^2_{\mathrm{et}}(\overline{\mathcal{E}}, \mathbb{G}_m)(1))_{G_{\mathbb{F}_q}} \longrightarrow 0$$

is exact. Therefore, $|\mathrm{Coker}\,\mu_3| = |M'_{G_{\mathbb{F}_q}}|$ equals

$$\frac{|(T_U \otimes_{\mathbb{Z}} \mathbb{Z}_\ell(1))_{G_{\mathbb{F}_q}}| \cdot |\det(1 - \mathrm{Frob};\, H^2_{\mathrm{et}}(\overline{\mathcal{E}}, \mathbb{Q}_\ell(2)))|_\ell^{-1}}{|\det(1 - \mathrm{Frob};\, \mathrm{Ker}[\mathrm{NS}(\overline{\mathcal{E}}) \longrightarrow \mathrm{Div}(\overline{\mathcal{E}}_U)/\sim_{\mathrm{alg}}] \otimes_{\mathbb{Z}} \mathbb{Q}_\ell(1))|_\ell^{-1}}.$$

This proves the claim. □

6.6. Cohomology of the fibers

Fix a non-empty open subscheme $U \subset C$. Let $f^U : \mathcal{E}^U \to C \setminus U$ denote the structure morphism and let $\iota^U : C \setminus U \to \mathcal{E}^U$ denote the morphism induced from $\iota : C \to \mathcal{E}$.

Lemma 6.20. — *The homomorphism*

$$(\mathrm{ch}'_{1,1}, f^U_*) : G_1(\mathcal{E}^U) \longrightarrow H^1_{\mathcal{M}}(\mathcal{E}^U, \mathbb{Z}(1)) \oplus K_1(C \setminus U)$$

is an isomorphism.

Proof. — The morphism $f^U : \mathcal{E}^U \to C \setminus U$ has connected fibers, hence the claim follows from Proposition 4.3 and the construction of $\mathrm{ch}'_{1,2}$. □

Lemma 6.21. — *The group $H^2_{\mathcal{M}}(\mathcal{E}^U, \mathbb{Z}(1))$ is finitely generated of rank $|C \setminus U|$. Moreover, $H^2_{\mathcal{M}}(\mathcal{E}^U, \mathbb{Z}(1))_{\mathrm{tors}}$ is of order $|L^*(h^1(\mathcal{E}^U), 0)|$.*

Proof. — It suffices to prove the following claim: if E has good reduction (resp. non-split multiplicative reduction, resp. split multiplicative or additive reduction) at $\wp \in C_0$, then $H^2_{\mathcal{M}}(\mathcal{E}_\wp, \mathbb{Z}(1))$ is a finitely generated abelian group of rank one, and $|H^2_{\mathcal{M}}(\mathcal{E}_\wp, \mathbb{Z}(1))_{\mathrm{tors}}|$ equals $|\mathcal{E}_\wp(\kappa(\wp))|$ (resp. rank two, resp. rank one). We set $\mathcal{E}_{\wp,(0)} = (\mathcal{E}_{\wp,\mathrm{red}})_{\mathrm{sm}} \setminus \iota(\wp)$ and $\mathcal{E}_{\wp,(1)} = \mathcal{E}^U \setminus \mathcal{E}_{\wp,(0)}$. We then have an exact sequence

$$H^1_{\mathcal{M}}(\mathcal{E}_{\wp,(0)}, \mathbb{Z}(1)) \longrightarrow H^0_{\mathcal{M}}(\mathcal{E}_{\wp,(1)}, \mathbb{Z}(0)) \longrightarrow H^2_{\mathcal{M}}(\mathcal{E}_\wp, \mathbb{Z}(1)) \longrightarrow \mathrm{Pic}(\mathcal{E}_{\wp,(0)}) \longrightarrow 0.$$

First, suppose that E does not have non-split multiplicative reduction at $\wp$ or that E has non-split multiplicative reduction at $\wp$ and $\mathcal{E}_\wp \otimes_{\kappa(\wp)} \overline{\mathbb{F}}_q$ has an even number of irreducible components. Then, using the classification of Kodaira, Néron, and Tate (cf. [38, 10.2.1]) of singular fibers of $\mathcal{E} \to C$, we can verify the equality

$$\begin{aligned}&\mathrm{Im}[H^0_{\mathcal{M}}(\mathcal{E}_{\wp,(1)}, \mathbb{Z}(0)) \longrightarrow H^2_{\mathcal{M}}(\mathcal{E}_{\wp}, \mathbb{Z}(1))] \\ &\qquad = \mathrm{Im}[\iota_* : H^0_{\mathcal{M}}(\mathrm{Spec}\ \kappa(\wp), \mathbb{Z}(0)) \longrightarrow H^2_{\mathcal{M}}(\mathcal{E}_{\wp}, \mathbb{Z}(1))].\end{aligned}$$

This shows that the group $H^2_{\mathcal{M}}(\mathcal{E}_{\wp}, \mathbb{Z}(1))$ is isomorphic to the direct sum of Picard group $\mathrm{Pic}(\mathcal{E}_{\wp,(0)})$ and $H^0_{\mathcal{M}}(\mathrm{Spec}\ \kappa(\wp), \mathbb{Z}(0)) \cong \mathbb{Z}$. In particular, we have $H^2_{\mathcal{M}}(\mathcal{E}_{\wp}, \mathbb{Z}(1))_{\mathrm{tors}} \cong \mathrm{Pic}(\mathcal{E}_{\wp,(0)})$, from which we easily deduce the claim.

Next, suppose that E has non-split multiplicative reduction at $\wp$ and $\mathcal{E}_{\wp} \otimes_{\kappa(\wp)} \overline{\mathbb{F}}_q$ has an odd number of irreducible components. In this case, we can directly verify that the image of $H^0_{\mathcal{M}}(\mathcal{E}_{\wp,(1)}, \mathbb{Z}(0)) \to H^2_{\mathcal{M}}(\mathcal{E}_{\wp}, \mathbb{Z}(1))$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}/2$ and $\mathrm{Pic}(\mathcal{E}_{\wp,(0)}) = 0$. The claim in this case follows. □

Lemma 6.22. — *The diagram*

$$\begin{tikzcd} K_1(E) \arrow[r] \arrow[d, "\iota^*"'] & G_0(\mathcal{E}^U) \arrow[d, "\iota^{U*}"'] \\ K_1(k) \arrow[r] & K_0(C \setminus U) \end{tikzcd}$$

is commutative.

Proof. — The group $K_1(E)$ is generated by the image of $f^* : K_1(k) \to K_1(E)$ and the image of $\bigoplus_{x \in E_0} K_1(\kappa(x)) \to K_1(E)$. The claim follows from the fact that the localization sequence in G-theory commutes with flat pull-backs and finite push-forwards. □

6.7. Proofs of Theorems 6.1, 6.2, and 6.3

Lemma 6.23. — *For any non-empty open subscheme $U \subset C$, the cokernel of the boundary map $\partial_U : H^2_{\mathcal{M}}(\mathcal{E}_U, \mathbb{Z}(2)) \to H^1_{\mathcal{M}}(\mathcal{E}^U, \mathbb{Z}(1))$ is finite.*

Proof. — If suffices to prove the claim for sufficiently small U, hence we assume that $\mathcal{E}_U \to U$ is smooth. Since $K_2(\mathcal{E}_U)_{\mathbb{Q}} \to K_2(E)_{\mathbb{Q}}$ is an isomorphism in this case, the claim follows from Theorem 1.3 and Lemma 5.2. □

Proof of Theorem 6.3. — Claims (1) and (2) follow from Theorem 2.3, Proposition 2.8, and Lemma 6.23. Proposition 2.8 gives an exact sequence

$$\begin{aligned} 0 \longrightarrow H^2_{\mathcal{M}}(\mathcal{E}, \mathbb{Z}(2))_{\mathrm{tors}} \longrightarrow H^2_{\mathcal{M}}(\mathcal{E}_U, \mathbb{Z}(2))^{\mathrm{red}} \\ \xrightarrow{\partial^2_U} H^1_{\mathcal{M}}(\mathcal{E}^U, \mathbb{Z}(1)) \longrightarrow H^3_{\mathcal{M}}(\mathcal{E}, \mathbb{Z}(2))_{\mathrm{tors}} \longrightarrow H^3_{\mathcal{M}}(\mathcal{E}_U, \mathbb{Z}(2))^{\mathrm{red}} \\ \xrightarrow{\partial^3_U} H^2_{\mathcal{M}}(\mathcal{E}^U, \mathbb{Z}(1)) \longrightarrow \mathrm{CH}_0(\mathcal{E}) \longrightarrow \mathrm{CH}_0(\mathcal{E}_U) \longrightarrow 0. \end{aligned}$$

From Lemma 6.23, it follows that Coker ∂^2_U is a finite group, which implies that the group $H^2_{\mathcal{M}}(\mathcal{E}_U, \mathbb{Z}(2))^{\mathrm{red}}$ is of rank $|S_0 \setminus U|$. By Theorem 2.3, $|H^2_{\mathcal{M}}(\mathcal{E}_U, \mathbb{Z}(2))_{\mathrm{tors}}|$ equals

$$\prod_{\ell \neq p} |H^1_{\mathrm{et}}(\mathcal{E}_U, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))|.$$

By Corollaries 6.14 and 6.18, this equals

$$|T'_{U,(1)}| \cdot |L(h^1(C), -1)L(h^0(C \setminus U, -1)/(q-1)|.$$

The above proves claim (3).

As noted in the proof of Theorem 2.3 (1), the group $\mathrm{CH}_0(\mathcal{E})$ is a finitely generated abelian group of rank one and $\mathrm{CH}_0(\mathcal{E}_U)$ is finite if $U \neq C$. By Lemma 6.21, $H^2_{\mathcal{M}}(\mathcal{E}^U, \mathbb{Z}(1))$ is a finitely generated abelian group of rank $|C \setminus U|$, hence the rank of $H^3_{\mathcal{M}}(\mathcal{E}_U, \mathbb{Z}(2))^{\mathrm{red}}$ equals $\max(|C \setminus U| - 1, 0)$.

From the class field theory of varieties over finite fields [29, Theorem 1] (see also [8, p. 283–284]) and Lemma 6.11, it follows that the push-forward map $\mathrm{CH}_0(\mathcal{E}) \to \mathrm{Pic}(C)$ is an isomorphism, hence the homomorphism $H^2_{\mathcal{M}}(\mathcal{E}^U, \mathbb{Z}(1)) \to \mathrm{CH}_0(\mathcal{E}) \cong \mathrm{Pic}(C)$ factors through the push-forward map $f^U_* : H^2_{\mathcal{M}}(\mathcal{E}^U, \mathbb{Z}(1)) \to H^0_{\mathcal{M}}(C \setminus U, \mathbb{Z}(0))$. By the surjectivity of f^U_*, we have isomorphisms

$$\mathrm{CH}_0(\mathcal{E}^U) \cong \mathrm{Coker}[H^0_{\mathcal{M}}(C \setminus U, \mathbb{Z}(0)) \longrightarrow \mathrm{Pic}(C)] \cong \mathrm{Pic}(U),$$

which proves claim (6). Since the group $H^0_{\mathcal{M}}(C \setminus U, \mathbb{Z}(0))$ is torsion free, the image of $H^2_{\mathcal{M}}(\mathcal{E}^U, \mathbb{Z}(1))_{\mathrm{tors}}$ in $\mathrm{CH}_0(\mathcal{E})$ is zero, thus we have an exact sequence

$$\begin{aligned} 0 \longrightarrow \mathrm{Coker}\ \partial^2_U \longrightarrow H^3_{\mathcal{M}}(\mathcal{E}, \mathbb{Z}(2))_{\mathrm{tors}} \\ \longrightarrow H^3_{\mathcal{M}}(\mathcal{E}_U, \mathbb{Z}(2))_{\mathrm{tors}} \longrightarrow H^2_{\mathcal{M}}(\mathcal{E}^U, \mathbb{Z}(1))_{\mathrm{tors}} \longrightarrow 0. \end{aligned}$$

By Proposition 2.8 and Lemma 6.19, the group $H^3_{\mathcal{M}}(\mathcal{E}_U, \mathbb{Z}(2))_{\mathrm{tors}}$ is finite of order

$$\frac{p^m |T'_{U,(1)}| \cdot |L(h^2(\mathcal{E}), 0) L^*(h^1(\mathcal{E}^U), 0) L(h^0(C \setminus U), -1)|}{(q-1)|L(h^0(\mathrm{Irr}(\mathcal{E}^U)), -1)|}$$

for some $m \in \mathbb{Z}$. By Lemma 6.21, the group $H^2_{\mathcal{M}}(\mathcal{E}^U, \mathbb{Z}(1))_{\mathrm{tors}}$ is finite of order $|L^*(h^1(\mathcal{E}^U), 0)|$. By Lemma 6.11, the Picard scheme $\mathrm{Pic}^o_{\mathcal{E}/\mathbb{F}_q}$ is an abelian variety and, in particular, $\mathrm{Hom}(\mathrm{Pic}^o_{\mathcal{E}/\mathbb{F}_q}, \mathbb{G}_m) = 0$, hence by Theorem 2.3 and Corollary 6.14, the group $H^3_{\mathcal{M}}(\mathcal{E}, \mathbb{Z}(2))_{\mathrm{tors}}$ is of order $|L(h^2(\mathcal{E}), 0)|$. Therefore,

$$\begin{aligned} |\mathrm{Coker}\, \partial^2_U| &= \frac{|H^3_{\mathcal{M}}(\mathcal{E}, \mathbb{Z}(2))_{\mathrm{tors}}| \cdot |H^2_{\mathcal{M}}(\mathcal{E}^U, \mathbb{Z}(1))_{\mathrm{tors}}|}{|H^3_{\mathcal{M}}(\mathcal{E}_U, \mathbb{Z}(2))_{\mathrm{tors}}|} \\ &= \frac{p^{-m}(q-1)|L(h^0(\mathrm{Irr}(\mathcal{E}^U)), -1)|}{|T'_{U,(1)}| \cdot |L(h^0(C \setminus U), -1)|}. \end{aligned}$$

Since $|\mathrm{Coker}\, \partial^2_U|$ is prime to p, we have $m = 0$. This proves claims (4) and (5) and completes the proof. $\square$

Proof of Theorem 6.2. — Claim (5) is clear. Claim (1) follows from Corollary 2.9 and Theorem 1.3. We easily verify that $H^i_{\mathcal{M}}(\mathcal{E}^U, \mathbb{Z}(1))$ is zero for $i \leqslant 0$. By the localization sequence of higher Chow groups (cf. [6, Corollary (0.2)]), we have $H^i_{\mathcal{M}}(\mathcal{E}, \mathbb{Z}(2)) \cong H^i_{\mathcal{M}}(\mathcal{E}_U, \mathbb{Z}(2))$ for $i \leqslant 1$. Taking the inductive limit with respect to U, we obtain claim (2).

By Corollary 2.9, we have an exact sequence

$$\begin{aligned} (6.3)\quad & 0 \longrightarrow H^2_{\mathcal{M}}(\mathcal{E}, \mathbb{Z}(2))_{\mathrm{tors}} \xrightarrow{\alpha} H^2_{\mathcal{M}}(E, \mathbb{Z}(2))^{\mathrm{red}} \\ & \xrightarrow{\partial^2_{\mathcal{M},2}} \bigoplus_{\wp \in C_0} H^1_{\mathcal{M}}(\mathcal{E}_\wp, \mathbb{Z}(1)) \longrightarrow H^3_{\mathcal{M}}(\mathcal{E}, \mathbb{Z}(2))_{\mathrm{tors}} \longrightarrow H^3_{\mathcal{M}}(E, \mathbb{Z}(2))^{\mathrm{red}} \\ & \qquad\qquad \xrightarrow{\partial^3_{\mathcal{M},2}} \bigoplus_{\wp \in C_0} H^2_{\mathcal{M}}(\mathcal{E}_\wp, \mathbb{Z}(1)) \longrightarrow \mathrm{Pic}(C) \longrightarrow 0. \end{aligned}$$

Hence by Theorem 2.3 and Corollary 6.14, the group $\mathrm{Ker}\, \partial^2_{\mathcal{M},2}$ is finite of order $|L(h^1(C), -1)|$. For a non-empty open subscheme $U \subset C$, consider the group $\mathrm{Coker}\, \partial^2_U$ in the proof of Theorem 6.3. For two non-empty open subschemes $U', U \subset C$ with $U' \subset U$, the homomorphism $\mathrm{Coker}\, \partial^2_U \to \mathrm{Coker}\, \partial^2_{U'}$ is injective since both $\mathrm{Coker}\, \partial^2_U$ and $\mathrm{Coker}\, \partial^2_{U'}$ canonically inject into $H^3_{\mathcal{M}}(\mathcal{E}, \mathbb{Z}(2))_{\mathrm{tors}}$. Claim (3) follows from claim (4) of Theorem 6.3 by taking the inductive limit. Claim (4) follows from exact sequence (6.3) and Lemma 6.9.

From the localization sequence, it follows that the push-forward homomorphism $\bigoplus_{x \in E_0} H^2_{\mathcal{M}}(\mathrm{Spec}\, \kappa(x), \mathbb{Z}(2)) \to H^4_{\mathcal{M}}(E, \mathbb{Z}(3))$ is surjective, hence $H^4_{\mathcal{M}}(E, \mathbb{Z}(3))$ is a torsion group and claim (6) follows from Lemma 5.4. This completes the proof. $\square$

Proof of Theorem 6.1. — Consider the restriction $\gamma : \mathrm{Ker}\, c_{2,3} \to H^2_{\mathcal{M}}(E, \mathbb{Z}(2))$ of $c_{2,2}$ to $\mathrm{Ker}\, c_{2,3}$. By Lemma 5.1, both $\mathrm{Ker}\, \gamma$ and $\mathrm{Coker}\, \gamma$

are annihilated by the multiplication-by-2 map, which implies that the image of γ contains $H^2_{\mathcal{M}}(E,\mathbb{Z}(2))_{\mathrm{div}}$ and that $\mathrm{Ext}^1_{\mathbb{Z}}(H^2_{\mathcal{M}}(E,\mathbb{Z}(2))_{\mathrm{div}}, \mathrm{Ker}\ \gamma)$ is zero. From this, it follows that the homomorphism γ induces an isomorphism $(\mathrm{Ker}\ c_{2,3})_{\mathrm{div}} \xrightarrow{\cong} H^2_{\mathcal{M}}(E,\mathbb{Z}(2))_{\mathrm{div}}$, which shows that the homomorphism $K_2(E)^{\mathrm{red}} \to H^2_{\mathcal{M}}(E,\mathbb{Z}(2))^{\mathrm{red}}$ induced by $c_{2,2}$ is surjective with torsion kernel, thus claim (1) follows from Theorem 6.2 (3).

Claim (3) follows from Theorem 6.2 (1) and Lemma 5.1.

For $\wp \in C_0$, let $\iota_\wp : \mathrm{Spec}\ \kappa(\wp) \to \mathcal{E}_\wp$ denote the reduction at $\wp$ of the morphism $\iota : C \to \mathcal{E}$. Diagram (5.2) gives an exact sequence

$$\mathrm{Coker}\ \partial^4_{\mathcal{M},3} \longrightarrow \mathrm{Coker}\ \partial_2 \longrightarrow \mathrm{Coker}\ \partial^2_{\mathcal{M},2} \longrightarrow 0.$$

By Lemma 5.4, we have an isomorphism $\mathrm{Coker}\ \partial^4_{\mathcal{M},3} \cong \mathbb{F}_q^\times$. By the construction of this isomorphism, we see that the composition

$$\mathbb{F}_q^\times \cong \mathrm{Coker}\ \partial^4_{\mathcal{M},3} \longrightarrow \mathrm{Coker}\ \partial_2 \hookrightarrow K_1(\mathcal{E}) \longrightarrow K_1(\mathrm{Spec}\ \mathbb{F}_q) \cong \mathbb{F}_q^\times$$

equals the identity, hence the map $\mathrm{Coker}\ \partial^4_{\mathcal{M},3} \to \mathrm{Coker}\ \partial_2$ is injective. Then, claim (2) follows from Theorem 6.2 (3).

From Proposition 4.3 and Lemmas 5.1, 5.2, and 6.22, it follows that the homomorphism $\partial_1 : K_1(E)^{\mathrm{red}} \to \bigoplus_{\wp\in C_0} G_0(\mathcal{E}_\wp)$ is identified with the direct sum of the map

$$\partial'_1 : k^\times \to \bigoplus_{\wp\in C_0} H^0_{\mathcal{M}}(\mathrm{Spec}\ \kappa(\wp), \mathbb{Z}(0)) \to \bigoplus_{\wp} H^0_{\mathcal{M}}(\mathcal{E}_\wp, \mathbb{Z}(0))$$

and the map

$$\partial^3_{\mathcal{M},2} : H^3_{\mathcal{M}}(E,\mathbb{Z}(2))^{\mathrm{red}} \to \bigoplus_{\wp} H^2_{\mathcal{M}}(\mathcal{E}_\wp, \mathbb{Z}(1)).$$

We then have isomorphisms

$$\mathrm{Ker}\ \partial'_1 \cong \mathbb{F}_q^\times, \quad \mathrm{Coker}\ \partial'_1 \cong \mathrm{Pic}(C) \oplus \bigoplus_{\wp} \mathbb{Z}^{|\mathrm{Irr}(\mathcal{E}_\wp)|-1},$$

$$\mathrm{Ker}\ \partial^3_{\mathcal{M},2} \cong H^3_{\mathcal{M}}(\mathcal{E},\mathbb{Z}(2))_{\mathrm{tors}} / \mathrm{Coker}\ \partial^2_{\mathcal{M},2}, \quad \mathrm{Coker}\ \partial^3_{\mathcal{M},2} \cong \mathrm{Pic}(C).$$

Claim (4) follows, which completes the proof of Theorem 6.1. □

7. Results for $j \geqslant 3$

In this section, we obtain results for $j \geqslant 3$, generalizing the theorems of Section 6. The proofs here are simpler than those of Section 6 in that we do not use tools such as the class field theory of Kato–Saito [29] or Theorem 1.3. We also refer the reader to Section 1.7 for remarks concerning the contents of this section. Finally, we note that the notation we use is as in Section 6.

7.1. Statements

For integers i, j, consider the boundary map

$$\partial^i_{\mathcal{M},j} : H^i_{\mathcal{M}}(E, \mathbb{Z}(j))^{\mathrm{red}} \longrightarrow \bigoplus_{\wp \in C_0} H^{i-1}_{\mathcal{M}}(\mathcal{E}_{\wp}, \mathbb{Z}(j-1)).$$

THEOREM 7.1. — *Let $j \geqslant 3$ be an integer.*

(1) *For any $i \in \mathbb{Z}$, both* Ker $\partial^i_{\mathcal{M},j}$ *and* Coker $\partial^i_{\mathcal{M},j}$ *are finite groups.*

(2) *We have*

$$|\mathrm{Ker}\,\partial^i_{\mathcal{M},j}| = \begin{cases} 1, & \text{if } i \leqslant 0 \text{ or } i \geqslant 5, \\ q^j - 1, & \text{if } i = 1, \\ |L(h^1(C), 1-j)|, & \text{if } i = 2, \\ \frac{|T'_{(j-1)}| \cdot |L(h^2(\mathcal{E}), 2-j)|}{q^{j-1}-1}, & \text{if } i = 3, \\ |L(h^1(C), 2-j)|, & \text{if } i = 4. \end{cases}$$

Further, the group Ker $\partial^1_{\mathcal{M},j}$ *is cyclic of order $q^j - 1$.*

(3) *We have*

$$|\mathrm{Coker}\,\partial^i_{\mathcal{M},j}| = \begin{cases} 1, & \text{if } i \leqslant 1,\ i = 3,\ \text{or } i \geqslant 5, \\ \frac{q^{j-1}-1}{|T'_{(j-1)}|}, & \text{if } i = 2, \\ q^{j-2} - 1 & \text{if } i = 4. \end{cases}$$

(4) *Let $U \subset C$ be a non-empty open subscheme. Then, the group $H^i_{\mathcal{M}}(\mathcal{E}_U, \mathbb{Z}(j))$ is finite modulo a uniquely divisible subgroup for any $i \in \mathbb{Z}$. The group $H^i_{\mathcal{M}}(\mathcal{E}_U, \mathbb{Z}(j))$ is zero if $i \geqslant \max(6, j)$ and is finite for $(i, j) = (4, 3), (5, 3), (4, 4), (5, 4),$ or $(5, 5)$.*

(5) *$H^i_{\mathcal{M}}(\mathcal{E}_U, \mathbb{Z}(j))$ is uniquely divisible for $i \leqslant 0$ or $6 \leqslant i \leqslant j$, and $H^1_{\mathcal{M}}(\mathcal{E}_U, \mathbb{Z}(j))_{\mathrm{tors}}$ is cyclic of order $q^j - 1$.*

(6) *Suppose that $U = C$ (resp. $U \neq C$). The group $H^2_{\mathcal{M}}(\mathcal{E}_U, \mathbb{Z}(j))_{\mathrm{tors}}$ is of order $|L(h^1(C), 1-j)|$ (resp. of order*

$$\frac{|T'_{U,(j-1)}| \cdot |L(h^1(C), 1-j)L(h^0(C \setminus U), 1-j)|}{q^{j-1} - 1}).$$

The group $H^3_{\mathcal{M}}(\mathcal{E}_U, \mathbb{Z}(j))_{\mathrm{tors}}$ is of order $|L(h^2(\mathcal{E}), 2-j)|$ (resp. of order

$$\frac{|T'_{U,(j-1)}| \cdot |L(h^2(\mathcal{E}), 2-j)L(h^1(\mathcal{E}^U), 2-j)L(h^0(C \setminus U), 1-j)|}{(q^{j-1} - 1)|L(h^0(\mathrm{Irr}(\mathcal{E}^U)), 1-j)|}).$$

The group $H^4_{\mathcal{M}}(\mathcal{E}_U, \mathbb{Z}(j))_{\mathrm{tors}}$ is of order $|L(h^1(C), 2-j)|$ (resp. of order
$$\frac{|L(h^1(C), 2-j)L(h^0(C \setminus U), 2-j)|}{q^{j-2}-1}).$$
The group $H^5_{\mathcal{M}}(\mathcal{E}_U, \mathbb{Z}(j))_{\mathrm{tors}}$ is cyclic of order $q^{j-2}-1$ (resp. is zero).

THEOREM 7.2. — *The following statements hold.*

(1) *The group $K_2(E)_{\mathrm{div}}$ is uniquely divisible and $c_{2,2}$ induces an isomorphism $K_2(E)_{\mathrm{div}} \cong H^2_{\mathcal{M}}(E, \mathbb{Z}(2))_{\mathrm{div}}$.*
(2) *The kernel of the boundary map $\partial : K_2(E)^{\mathrm{red}} \to \bigoplus_{\wp \in C_0} G_1(\mathcal{E}_\wp)$ is a finite group of order $|L(h^1(C), -1)|^2$.*

7.2. Lemmas

LEMMA 7.3. — *Let X be a smooth projective geometrically connected curve over a global field k'. Let $k'(X)$ denote the function field of X. Then, the Milnor K-group $K^M_n(k'(X))$ is torsion for $n \geqslant 2 + \mathrm{gon}(X)$, and is of exponent two (resp. zero) for $n \geqslant 3 + \mathrm{gon}(X)$ if $\mathrm{char}(k') = 0$ (resp. $\mathrm{char}(k') > 0$). Here, $\mathrm{gon}(X)$ denotes the gonality of X, i.e., the minimal degree of morphisms from X to $\mathbb{P}^1_{k'}$.*

Proof. — The field $k'(X)$ is an extension of degree $\mathrm{gon}(X)$ of a subfield K of the form $K = k'(t)$. From the split exact sequence
$$0 \longrightarrow K^M_n(k') \longrightarrow K^M_n(K) \longrightarrow \bigoplus_P K^M_{n-1}(k'[t]/P) \longrightarrow 0$$
in [43, Theorem 2.3], where P runs over the irreducible monic polynomials in $k'[t]$, and using [3, Chapter II, (2.1)], we observe that $K^M_n(K)$ is torsion for $n \geqslant 3$ and is of exponent two (resp. zero) for $n \geqslant 4$ if $\mathrm{char}(k') = 0$ (resp. $\mathrm{char}(k') > 0$).

Next, consider a flag $K = V_1 \subset V_2 \subset \cdots \subset V_{\mathrm{gon}(X)} = k'(X)$ of K-subspaces of $k'(X)$ with $\dim_K V_i = i$. For each i, we set $V_i^* = V_i \setminus \{0\}$. Suppose $i \geqslant 2$ and consider two elements $\alpha, \beta \in V_i \setminus V_{i-1}$. Then, there exist $a, b \in K^\times$ such that $\gamma = a\alpha + b\beta \in V_{i-1}$. If $\gamma = 0$ (resp. $\gamma \neq 0$), then $\{a\alpha, b\beta\} = 0$ (resp. $\{a\alpha/\gamma, b\beta/\gamma\} = 0$) in $K^M_2(k'(X))$. Expanding this equality, we obtain an expression for $\{\alpha, \beta\}$. We observe that $\{\beta, \gamma\}$ belongs to the subgroup of $K^M_2(k'(X))$ generated by $\{V_i^*, V_{i-1}^*\}$, hence for $n \geqslant \mathrm{gon}(X) - 1$, the group $K^M_n(k'(X))$ is generated by the image of $\{V^*_{\mathrm{gon}(X)}, \ldots, V_2^*\} \times K^M_{n-\mathrm{gon}(X)+1}(K)$. This proves the claim. $\square$

Lemma 7.4. — *The push-forward homomorphism* $H^2_{\mathcal{M}}(\mathcal{E}^U, \mathbb{Z}(j-1)) \to H^4_{\mathcal{M}}(\mathcal{E}, \mathbb{Z}(j))$ *is zero.*

Proof. — Consider the composition

$$H^2_{\mathcal{M}}(\mathcal{E}^U, \mathbb{Z}(j-1)) \longrightarrow H^4_{\mathcal{M}}(\mathcal{E}, \mathbb{Z}(j)) \xrightarrow{f_*} H^2_{\mathcal{M}}(C, \mathbb{Z}(j-1))$$

of push-forwards. This is the zero map since this factors through the group $H^0_{\mathcal{M}}(C \setminus U, \mathbb{Z}(j-2))$, which is zero by the theorem presented by Geisser and Levine [16, Corollary 1.2]. By Lemma 2.6, the group $H^2_{\mathcal{M}}(\mathcal{E}^U, \mathbb{Z}(j-1))$ is torsion, hence it suffices to show that the homomorphism $f_{*,\mathrm{tors}} : H^4_{\mathcal{M}}(\mathcal{E}, \mathbb{Z}(j))_{\mathrm{tors}} \to H^2_{\mathcal{M}}(C, \mathbb{Z}(j-1))_{\mathrm{tors}}$ induced by f_* is an isomorphism. Next, consider the commutative diagram

$$\begin{tikzcd}
H^4_{\mathcal{M}}(\mathcal{E}, \mathbb{Z}(j))_{\mathrm{tors}} \arrow[r, "f_{*,\mathrm{tors}}"] & H^2_{\mathcal{M}}(C, \mathbb{Z}(j-1))_{\mathrm{tors}} \\
H^3_{\mathcal{M}}(\mathcal{E}, \mathbb{Q}/\mathbb{Z}(j)) \arrow[r] \arrow[u, "\cong"] \arrow[d, "\cong"'] & H^1_{\mathcal{M}}(C, \mathbb{Q}/\mathbb{Z}(j-1)) \arrow[u, "\cong"'] \arrow[d, "\cong"] \\
\bigoplus_{\ell \neq p} H^3_{\mathrm{et}}(\overline{\mathcal{E}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(j))^{G_{\mathbb{F}_q}} \arrow[r] & \bigoplus_{\ell \neq p} H^1_{\mathrm{et}}(\overline{C}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(j-1))^{G_{\mathbb{F}_q}}.
\end{tikzcd}$$

Here, the horizontal arrows are push-forward maps, the upper vertical arrows are boundary maps obtained from the exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$, and the lower vertical arrows are those obtained from Theorem 2.3 (2) (b) (the same argument also applies to curves) further using the weight argument. The homomorphism at the bottom is an isomorphism by Lemma 6.11, hence $f_{*,\mathrm{tors}}$ is an isomorphism, as desired. □

7.3. Proofs of theorems

Proof of Theorem 7.1. — From Theorem 2.3 (2) and Lemma 7.3, claims (4) and (5) follow. Claim (6) follows from Theorem 2.3 (2) and Corollary 6.18. Using an approach similar to that of the proof of Corollary 2.9, we are able to show that the pull-back map induces an isomorphism $H^i_{\mathcal{M}}(\mathcal{E}, \mathbb{Z}(j))_{\mathrm{div}} \cong H^i_{\mathcal{M}}(E, \mathbb{Z}(j))_{\mathrm{div}}$ for all $i \in \mathbb{Z}$, and that the localization sequence induces a long exact sequence

$$(7.1)\quad \cdots \longrightarrow \bigoplus_{\wp\in C_0} H^{i-2}_{\mathcal{M}}(\mathcal{E}_\wp, \mathbb{Z}(j-1)) \\ \longrightarrow H^i_{\mathcal{M}}(\mathcal{E}, \mathbb{Z}(j))_{\mathrm{tors}} \longrightarrow H^i_{\mathcal{M}}(E, \mathbb{Z}(j))_{\mathrm{tors}} \longrightarrow \cdots .$$

Using the theorem introduced by Rost and Voevodsky (i.e., Theorem 2.1 above), we observe that for any $\wp \in C_0$, even if $\mathcal{E}_\wp$ is singular, the group $H^i_{\mathcal{M}}(\mathcal{E}_\wp, \mathbb{Z}(j-1))$ is finite for all i, is zero for $i \leqslant 0$ or $i \geqslant 4$, and is cyclic of order $q_\wp^{j-1} - 1$, where $q_\wp = |\kappa(\wp)|$ is the cardinality of the residue field at $\wp$, for $i = 1$. From the exact sequence (7.1) and Lemma 7.4, we can deduce claims (1), (2), and (3) from claims (4), (5), and (6), thus completing the proof. □

Proof of Theorem 7.2. — Let $U \neq C$. Then, by Lemmas 5.4 and 7.4, the following sequence is exact:

$$0 \longrightarrow H^4_{\mathcal{M}}(\mathcal{E}, \mathbb{Z}(3)) \longrightarrow H^4_{\mathcal{M}}(\mathcal{E}_U, \mathbb{Z}(3)) \xrightarrow{\partial} H^3_{\mathcal{M}}(\mathcal{E}^U, \mathbb{Z}(2)) \xrightarrow{\alpha} \mathbb{F}_q^\times \longrightarrow 1.$$

Here, ∂ and α are as in Lemma 5.4, and the second map is the pull-back. By taking the inductive limit, we obtain an exact sequence

$$(7.2)\quad 0 \longrightarrow H^4_{\mathcal{M}}(\mathcal{E}, \mathbb{Z}(3)) \longrightarrow H^4_{\mathcal{M}}(E, \mathbb{Z}(3)) \\ \xrightarrow{\partial^4_{\mathcal{M},3}} \bigoplus_{\wp\in C_0} H^3_{\mathcal{M}}(\mathcal{E}_\wp, \mathbb{Z}(2)) \longrightarrow \mathbb{F}_q^\times \longrightarrow 1.$$

By Theorem 2.3 and Corollary 2.9, the group $H^4_{\mathcal{M}}(E, \mathbb{Z}(3))_{\mathrm{div}}$ is zero, hence using Lemma 5.1, we obtain $K_2(E)_{\mathrm{div}} \subset \mathrm{Ker}\ c_{2,3}$. From the proof of Theorem 6.1, we saw that map $c_{2,2}$ induces an isomorphism $(\mathrm{Ker}\ c_{2,3})_{\mathrm{div}} \xrightarrow{\cong} H^2_{\mathcal{M}}(E, \mathbb{Z}(2))_{\mathrm{div}}$, hence $c_{2,2}$ induces an isomorphism $K_2(E)_{\mathrm{div}} \cong H^2_{\mathcal{M}}(E, \mathbb{Z}(2))_{\mathrm{div}}$, which proves claim (1). Claim (2) follows from Theorems 6.1 and 6.2, the commutative diagram (5.2), and exact sequence (7.2). This completes the proof. □

Appendix. A proposition on the p-part

The aim of this Appendix is to provide a proof of Proposition A.5 below, which we used in the proof of Theorem 2.3. Nothing in this Appendix is new except for the definition of the Frobenius map on the inductive limit (not on the inverse limit) given in Section A.3. A similar presentation has already been provided in the work of Milne [41] and Nygaard [48].

Proposition A.5. — *Let X be a smooth projective geometrically connected surface over a finite field $\mathbb{F}_q$ of cardinal q of characteristic p. Let $W_n\Omega^i_{X,\log}$ denote the logarithmic de Rham–Witt sheaf (cf.* [25, I, 5.7]*). Then, the inductive limit $\varinjlim_n H^0_{\mathrm{et}}(X, W_n\Omega^2_{X,\log})$ with respect to multiplication-by-p is finite of order $|\operatorname{Hom}(\mathrm{Pic}^o_{X/\mathbb{F}_q}, \mathbb{G}_m)|_p^{-1} \cdot |L(h^2(X), 0)|_p^{-1}$. Here, $\operatorname{Hom}(\mathrm{Pic}^o_{X/\mathbb{F}_q}, \mathbb{G}_m)$ denotes the set of homomorphisms $\mathrm{Pic}^o_{X/\mathbb{F}_q} \to \mathbb{G}_m$ of $\mathbb{F}_q$-group schemes, and $L(h^2(X), s)$ is the (Hasse–Weil) L-function of $h^2(X)$.*

A.1. The de Rham–Witt complex

In this Appendix, let k be a perfect field of characteristic p. Let X be a scheme of dimension δ that is proper over Spec k. For $i, n \in \mathbb{Z}$, let $W_n\Omega^\bullet_X$ denote the de Rham–Witt complex (cf. [25, I, 1.12]) of the ringed topos of schemes over X with Zariski topology. We let $R : W_n\Omega^i_X \to W_{n-1}\Omega^i_X$, $F : W_n\Omega^i_X \to W_{n-1}\Omega^i_X$, and $V : W_n\Omega^i_X \to W_{n+1}\Omega^i_X$ denote the restriction, the Frobenius, and the Verschiebung, respectively. For each $i \in \mathbb{Z}$, the sheaf $W_n\Omega^i_X$ has a canonical structure of coherent $W_n\mathcal{O}_X$-module, which enables us to regard $W_n\Omega^i_X$ as an étale sheaf. Therefore, in this Appendix, we focus on the category of étale sheaves on schemes over X.

A.2. Logarithmic de Rham–Witt sheaves

For $n \in \mathbb{Z}$, let $W_n\Omega^i_{X,\log} \subset W_n\Omega^i_X$ denote the logarithmic de Rham–Witt sheaf (cf. [25, I, 5.7]).

Lemma A.6. — *The homomorphism $V : W_n\Omega^i_X \to W_{n+1}\Omega^i_X$ sends $W_n\Omega^i_{X,\log}$ into $W_{n+1}\Omega^i_{X,\log}$.*

Proof. — Let $x \in W_n\Omega^i_{X,\log}$ be an étale local section. By the definition of $W_n\Omega^i_{X,\log}$, there exists an étale local section $y \in W_{n+1}\Omega^i_{X,\log}$ such that $x = Ry$. We observe that $Ry = Fy$, hence $Vx = VRy = VFy = py \in W_{n+1}\Omega^i_{X,\log}$. □

Let $CW\Omega^i_X$ denote the inductive limit $CW\Omega^i_X = \varinjlim_{n,V} W_n\Omega^i_X$ with respect to V. The above lemma enables us to define the inductive limit $CW\Omega^i_{X,\log} = \varinjlim_{n,V} W_n\Omega^i_{X,\log}$.

A.3. Modified Frobenius operator

In this subsection we define an operator $F' : CW\Omega^i_X \to CW\Omega^i_X$ such that the sequence

$$0 \longrightarrow CW\Omega^i_{X,\log} \longrightarrow CW\Omega^i_X \xrightarrow{1-F'} CW\Omega^i_X \longrightarrow 0 \tag{A.3}$$

is exact.

For $n \geqslant 0$, let $\widetilde{W}_n\Omega^i_X$ denote the cokernel of the homomorphism $V^n : \Omega^i_X = W_1\Omega^i_X \to W_{n+1}\Omega^i_X$. The homomorphisms R, F and V on $W_{n+1}\Omega^i_X$ induce homomorphisms on $\widetilde{W}_n\Omega^i_X$, which we denote using the same notation. If $n \geqslant 1$, the homomorphisms $R, F : W_{n+1}\Omega^i_X \to W_n\Omega^i_X$ factor through the canonical surjection $W_{n+1}\Omega^i_X \to \widetilde{W}_n\Omega^i_X$. We let $\widetilde{R}, \widetilde{F} : \widetilde{W}_n\Omega^i_X \to W_n\Omega^i_X$ denote the induced homomorphisms. Then, both $\widetilde{R}$ and $\widetilde{F}$ commute with R, F and V. For $n \geqslant 0$, we let $\widetilde{W}_n\Omega^i_{X,\log}$ denote the image of $W_{n+1}\Omega^i_{X,\log}$ by the canonical surjection $W_{n+1}\Omega^i_X \to \widetilde{W}_n\Omega^i_X$. The restriction of $\widetilde{R} : \widetilde{W}_n\Omega^i_X \to W_n\Omega^i_X$ to $\widetilde{W}_n\Omega^i_{X,\log}$ gives a surjective homomorphism $\widetilde{R}_{\log} : \widetilde{W}_n\Omega^i_{X,\log} \to W_n\Omega^i_{X,\log}$.

Lemma A.7. — *The homomorphisms $\widetilde{R}$, $\widetilde{R}_{\log}$ induce isomorphisms*

$$\varinjlim_{n\geqslant 0,\, V} \widetilde{W}_n\Omega^i_X \cong CW\Omega^i_X, \qquad \varinjlim_{n\geqslant 0,\, V} \widetilde{W}_n\Omega^i_{X,\log} \cong CW\Omega^i_{X,\log}.$$

Proof. — Surjectivity here is clear. From [25, I, Proposition 3.2], it follows that the kernel of $\widetilde{R}$ equals the image of the composition $W_1\Omega^i_X \xrightarrow{dV^n} W_{n+1}\Omega^i_X \twoheadrightarrow \widetilde{W}_n\Omega^i_X$. Since $Vd = pdV$, we have $V(\operatorname{Ker} \widetilde{R}) = 0$. This proves the injectivity. $\square$

We observe that $\widetilde{W}_n\Omega^i_{X,\log}$ is contained in the kernel of $\widetilde{R}-\widetilde{F} : \widetilde{W}_n\Omega^i_X \to W_n\Omega^i_X$, hence

$$0 \longrightarrow \widetilde{W}_n\Omega^i_{X,\log} \longrightarrow \widetilde{W}_n\Omega^i_X \xrightarrow{\widetilde{R}-\widetilde{F}} W_n\Omega^i_X \longrightarrow 0 \tag{A.4}$$

is a complex.

Lemma A.8. — *The inductive limit*

$$0 \longrightarrow \varinjlim_{n\geqslant 0,\, V} \widetilde{W}_n\Omega^i_{X,\log} \longrightarrow \varinjlim_{n\geqslant 0,\, V} \widetilde{W}_n\Omega^i_X \longrightarrow CW\Omega^i_X \longrightarrow 0$$

of (A.4) *with respect to V is exact.*

Proof. — The argument in the proof of [25, I, Théorème 5.7.2] shows that the kernel of $R - F : W_{n+1}\Omega^i_X \to W_n\Omega^i_X$ is contained in $W_{n+1}\Omega^i_{X,\log} + \operatorname{Ker} R$, hence the claim follows from Lemma A.7. $\square$

The inductive limit of $\widetilde{F} : \widetilde{W}_n\Omega^i_X \to \widetilde{W}_{n+1}\Omega^i_X$ gives the endomorphism $F' : CW\Omega^i_X \cong \varprojlim_{n\geqslant 1,\, V} \widetilde{W}_n\Omega^i_X \to CW\Omega^i_X$. By Lemmas A.7 and A.8, we have a canonical exact sequence (A.3).

A.4. The duality

Let $H^*(X, W_n\Omega^i_X)$ denote the cohomology groups of $W_n\Omega^i_X$ with respect to the Zariski topology.

The trace map $\mathrm{Tr} : H^\delta(X, W_n\Omega^\delta_X) \cong W_n(\mathbb{F}_q)$ is defined in [26, 4.1.3]. This commutes with homomorphisms R, F and V. For $0 \leqslant i, j \leqslant \delta$, the product $m : W_n\Omega^i_X \times W_n\Omega^{\delta-i}_X \to W_n\Omega^\delta_X$ gives a $W_n(k)$-bilinear paring

$$(\ ,\ ) : H^j(X, W_n\Omega^i_X) \times H^{\delta-j}(X, W_n\Omega^{\delta-i}_X) \longrightarrow H^\delta(X, W_n\Omega^\delta_X) \xrightarrow{\mathrm{Tr}} W_n(k).$$

By [26, Corollary 4.2.2], this pairing is perfect.

Since $m \circ (\mathrm{id} \otimes V) = V \circ m \circ (F \otimes \mathrm{id})$, the diagram

$$\begin{tikzcd}
W_{n+1}\Omega^i_X \arrow[d, "F"'] & \times & W_{n+1}\Omega^{\delta-i}_X \arrow[r] & W_{n+1}(k) \\
W_n\Omega^i_X & \times & W_n\Omega^{\delta-i}_X \arrow[u, "V"] \arrow[r] & W_n(k) \arrow[u, "V"']
\end{tikzcd}$$

is commutative, hence this induces an isomorphism

$$H^{\delta-j}(X, CW\Omega^{\delta-i}_X) \cong \varinjlim_n \mathrm{Hom}_{W_n(k)}(H^j(X, W_n\Omega^i_X), W_n(k)), \tag{A.5}$$

where the transition map in the inductive limit of the right hand side is given by $f \mapsto V \circ f \circ F$. We endow each $H^j(X, W_n\Omega^i_X)$ with the discrete topology. We set $H^j(X, W'\Omega^i_X) = \varprojlim_{n,\, F} H^j(X, W_n\Omega^i_X)$ and endow it with the induced topology. Next, we turn $H^j(X, W'\Omega^i_X)$ into a $W(k)$-module by letting $a \cdot (b_n) = (\sigma^{-n}(a)b_n)$ for $a \in W(k)$, $b_n \in H^j(X, W_n\Omega^i_X)$. We set $D = \varinjlim_{n,\, V} W_n(k)$ and endow it with the discrete topology. We turn D into a $W(k)$-module by letting $a \cdot c_n = \sigma^{-n}(a)c_n$ for $a \in W(k)$, $c_n \in W_n(k)$. Then, the right hand side of (A.5) equals $\mathrm{Hom}_{W(k),\mathrm{cont}}(H^j(X, W'\Omega^i_X), D)$. The homomorphism $R : H^j(X, W_n\Omega^i_X) \to H^j(X, W_{n-1}\Omega^i_X)$ induces an endomorphism $R' : H^j(X, W'\Omega^i_X) \to H^j(X, W'\Omega^i_X)$. The Frobenius endomorphism $\sigma : W_n(k) \to W_n(k)$ induces an endomorphism $\sigma : D \to D$.

Lemma A.9. — *Under the isomorphism* (A.5), *the endomorphism*

$$F' : H^{\delta-j}(X, CW\Omega^{\delta-i}_X) \longrightarrow H^{\delta-j}(X, CW\Omega^{\delta-i}_X)$$

is identified with the endomorphism of $\mathrm{Hom}_{W(k),\mathrm{cont}}(H^j(X, W'\Omega^i_X), D)$, *that sends a homomorphism* $f : H^j(X, W'\Omega^i_X) \to D$ *to the homomorphism* $\sigma \circ f \circ R'$.

Proof. — The proof here is immediate from the definition of the isomorphism (A.5) and the module D. □

A.5. The degree zero case

We are primarily concerned with the case in which $i = 0$. We denote $H^j(X, W'\Omega^0_X)$ by $H^j(X, W'\mathcal{O}_X)$. Recall that $F : W_n\Omega^0_X \to W_{n-1}\Omega^0_X$ equals the composition $W_n\mathcal{O}_X \xrightarrow{\sigma} W_n\mathcal{O}_X \xrightarrow{R} W_{n-1}\mathcal{O}_X$. From [25, II, Proposition 2.1], it follows that $H^j(X, W\Omega^i_X) \to \varprojlim_{n,\,R} H^j(X, W_n\Omega^i_X)$ is an isomorphism, hence $H^j(X, W'\mathcal{O}_X)$ is isomorphic to the projective limit

$$\widetilde{H}^j(X, W\mathcal{O}_X) = \varprojlim[\cdots \xrightarrow{\sigma} H^j(X, W\mathcal{O}_X) \xrightarrow{\sigma} H^j(X, W\mathcal{O}_X)].$$

The endomorphism $\sigma : H^j(X, W\mathcal{O}_X) \to H^j(X, W\mathcal{O}_X)$ induces an automorphism $\sigma : \widetilde{H}^j(X, W\mathcal{O}_X) \xrightarrow{\cong} \widetilde{H}^j(X, W\mathcal{O}_X)$. We observe here that the endomorphism R' on $H^j(X, W'\mathcal{O}_X)$ corresponds to the endomorphism σ^{-1} on $\widetilde{H}^j(X, W\mathcal{O}_X)$.

Let $K = \mathrm{Frac}\, W(k)$ denote the field of fractions of $W(k)$. The homomorphism $\sigma^n/p^n : W_n(k) \to K/W(k)$ for each $n \geqslant 1$ induces a canonical isomorphism $D \cong K/W(k)$ of $W(k)$-modules that commutes with the action of σ.

A.6. Proof of Proposition A.5

Suppose that $k = \mathbb{F}_q$. Then by Lemma A.9, $H^0(X, CW\Omega^d_X)$ is isomorphic to the Pontryagin dual of $\widetilde{H}^\delta(X, W\mathcal{O}_X)$, hence the group

$$H^0(X, CW\Omega^\delta_{X,\log}) \cong \mathrm{Ker}[H^0(X, CW\Omega^\delta_X) \xrightarrow{1-F'} H^0(X, CW\Omega^\delta_X)]$$

is isomorphic to the Pontryagin dual of the cokernel of $1 - \sigma^{-1}$ on $\widetilde{H}^\delta(X, W\mathcal{O}_X)$.

Proposition A.10. — *Let $k = \mathbb{F}_q$ be a finite field and X be a scheme of dimension δ that is smooth and projective over* $\mathrm{Spec}\ k$. *Suppose that the V-torsion part T of $H^\delta(X, W\mathcal{O}_X)$ is finite. Then, $H^0(X, CW\Omega^\delta_X)$ is a finite group of order $|T^\sigma| \cdot |L(h^\delta(X), 0)|_p^{-1}$. Here, T^σ denotes the σ-invariant part of T.*

Proof. — By the argument above, the order of $H^0(X, CW\Omega_X^\delta)$ equals the order of the cokernel of $1-\sigma$ on $\widetilde{H}^\delta(X, W\mathcal{O}_X)$ if it is finite. The torsion subgroup of $\widetilde{H}^\delta(X, W\mathcal{O}_X)$ is finite since it injects into T. By [25, II, Corollaire 3.5], $\widetilde{H}^\delta(X, W\mathcal{O}_X)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ is isomorphic to the slope-zero part of $H^\delta_{\mathrm{crys}}(X/W(k))\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$, hence the claim follows. □

Proof of Proposition A.5. — Let the notation be as above and suppose that $\delta = 2$. Then, by [25, II, Remarque 6.4], the module T in the above proposition is canonically isomorphic to the group

$$\mathrm{Hom}_{W(\mathbb{F}_q)}(M(\mathrm{Pic}^o_{X/\mathbb{F}_q}/\mathrm{Pic}^o_{X/\mathbb{F}_q,\mathrm{red}}), K/W(\mathbb{F}_q)),$$

where $M(\ )$ denotes the contravariant Dieudonné module functor. In particular, T is a finite group. Let T_σ denote the σ-coinvariants of T. Then, by Dieudonné theory (cf. [11]), $\mathrm{Hom}_{W(\mathbb{F}_q)}(T_\sigma, K/W(\mathbb{F}_q))$ is canonically isomorphic to $\mathrm{Hom}(\mathrm{Pic}^o_{X/\mathbb{F}_q}, \mathbb{G}_m)$, hence the claim follows from Proposition A.10. □

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