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Weak type operator Lipschitz and commutator estimates for commuting tuples

WEAK TYPE OPERATOR LIPSCHITZ AND COMMUTATOR ESTIMATES FOR COMMUTING TUPLES

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ABSTRACT. — Let $f : \mathbb{R}^d \to \mathbb{R}$ be a Lipschitz function. If B is a bounded self-adjoint operator and if $\{A_k\}_{k=1}^d$ are commuting bounded self-adjoint operators such that $[A_k, B] \in L_1(H)$, then

$$\|[f(A_1, \ldots, A_d), B]\|_{1,\infty} \leqslant c(d)\|\nabla(f)\|_\infty \max_{1 \leqslant k \leqslant d} \|[A_k, B]\|_1,$$

where $c(d)$ is a constant independent of f, $\mathcal{M}$ and A, B and $\|\cdot\|_{1,\infty}$ denotes the weak L_1-norm.

If $\{X_k\}_{k=1}^d$ (respectively, $\{Y_k\}_{k=1}^d$) are commuting bounded self-adjoint operators such that $X_k - Y_k \in L_1(H)$, then

$$\|f(X_1, \ldots, X_d) - f(Y_1, \ldots, Y_d)\|_{1,\infty} \leqslant c(d)\|\nabla(f)\|_\infty \max_{1 \leqslant k \leqslant d} \|X_k - Y_k\|_1.$$

RÉSUMÉ. — Soit $f : \mathbb{R}^d \to \mathbb{R}$ une fonction Lipschitzienne. Si B est un opérateur borné auto-adjoint et si $\{A_k\}_{k=1}^d$ sont des opérateurs bornés auto-adjoints qui commutent et tels que $[A_k, B] \in L_1(H)$, alors

$$\|[f(A_1, \ldots, A_d), B]\|_{1,\infty} \leqslant c(d)\|\nabla(f)\|_\infty \max_{1 \leqslant k \leqslant d} \|[A_k, B]\|_1,$$

où $c(d)$ est une constante indépendante de f, $\mathcal{M}$ et A, B et $\|\cdot\|_{1,\infty}$ désigne la norme L_1-faible.

Si $\{X_k\}_{k=1}^d$ (respectivement $\{Y_k\}_{k=1}^d$) sont des opérateurs bornés qui commutent et tels que $X_k - Y_k \in L_1(H)$, alors

$$\|f(X_1, \ldots, X_d) - f(Y_1, \ldots, Y_d)\|_{1,\infty} \leqslant c(d)\|\nabla(f)\|_\infty \max_{1 \leqslant k \leqslant d} \|X_k - Y_k\|_1.$$

1. Introduction

Let $f : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function. Let $\mathcal{M}$ be a semi-finite von Neumann algebra and let $\mathcal{M}_{sa}$ be its self-adjoint part. This paper deals with differentiability properties of (multi-dimensional versions of) the mapping

$$(1.1) \qquad\qquad \mathcal{M}_{sa} \ni A \mapsto f(A).$$

The interest in such differentiability problems comes from very diverse directions:

   (i) the mapping (1.1) relates strongly to perturbations of commutators,
   (ii) there is a prolific series of papers devoted to differentiability and Lipschitz properties of (1.1),
   (iii) the map (1.1) relates to Connes' non-commutative geometry and in particular the spectral action, see [10, 32, 34].

The roots of the results of this paper can be traced back to a problem of Krein [21] which led to a remarkable diversity of papers concerning double operator integrals and Schur multipliers. The original Krein problem asks if for a function f being Lipschitz implies that it is operator Lipschitz, meaning that (1.1) is Lipschitz for the uniform norm on $\mathcal{M}_{sa}$. Krein's question is very natural but it was shown that it has a negative answer [14], unless one imposes stricter differentiability assumptions on f (like belonging to certain Besov or Sobolev spaces), see [1, 2, 28] to name just a few. Contributions to the problem were made by various people including Davies [11], Kato [18] and Kosaki [20] who found positive and negative results (under suitable conditions) for the analogue of Krein's problem for L_p-norms.

With the development of double operator integrals (see e.g. [5, 25, 26]) significant steps forward were made on Lipschitz and differentiability properties of the mapping (1.1), which were shown to be equivalent to various commutator estimates (see [4], [12, Theorem 2.2]). In turn this led to questions on the behavior of certain Schur multipliers and related double operator integrals.

Finding estimates, even if they are non-optimal, for norms of Schur multipliers is a highly non-trivial task. The hard part is that Schur multipliers acting on L_∞-spaces (or just matrix algebras) can often be estimated using Stinespring dilations, see e.g. [29]. However, if one considers Schur multipliers on L_p-spaces this tool is inapplicable. Therefore, in order to attack Krein's problem for L_p-spaces, $p \neq 1, \infty$ we are forced to introduce new techniques.

A corner stone result was obtained in [31] (see also [16]): it was shown by D. Potapov and the second named author that the mapping (1.1) is Lipschitz continuous with respect to the L_p-norm, $1 < p < \infty$. As [31] involves an application of the vector valued Marcinkiewicz multiplier theorem (due to Bourgain) it was not clear what the optimal non-commutative Lipschitz constants are. A sharp estimate for L_p-spaces was found in [7]. However in the category of symmetric spaces the question whether the so-called weak-$(1,1)$ estimate holds remained open.

A first result in this weak-$(1,1)$ direction was obtained by Nazarov and Peller [24] who proved it in the special case that $A - B$ has rank 1. In the same paper a question concerning validity of this result for an arbitrary trace class perturbation $A - B$ was posed. A full answer for f being the absolute value map was obtained in [9] using positive definite Schur multipliers and triangular truncations. In [8] this result was extended to all Lipschitz functions. The result is ultimate for the functions of 1 variable: it is optimal within the category of symmetric spaces and it implies all other known estimates on perturbations of commutators and Lipschitz functions obtained before [7, 9, 11, 12, 13, 18, 20, 24, 31]. The key ingredient of the proof in [8] is a new connection with non-commutative Calderón–Zygmund theory and in particular with the main result from Parcet's fundamental paper [27] (see also the recent paper by Cadilhac [6] for a substantially shorter proof).

In this paper we focus on multi-dimensional (or multi-variable) Lipschitz estimates for the mapping (1.1) which naturally includes a version of the Nazarov–Peller problem for normal operators. This study is deeply connected with that of classical Fourier multipliers. In particular, the dimension dependence of classes of multipliers as Bochner–Riesz multipliers, Riesz multipliers, (directional) Hilbert transforms et cetera, has been an important theme of research (we refer to Grafakos's book [15] with ample such results). Therefore, it is natural to look at the higher dimensional behavior of (1.1). Some results were obtained in [19] and [7]. However, the results in these papers are not optimal. In this paper we obtain the following.

THEOREM 1.1. — *For every Lipschitz function $f : \mathbb{R}^d \to \mathbb{R}$ and for every collection $\mathbf{A} = \{A_k\}_{k=1}^d \subset B(H)$ of commuting self-adjoint operators such that $[A_k, B] \in L_1(H)$, we have*

$$\|[f(\mathbf{A}), B]\|_{1,\infty} \leqslant c(d)\|\nabla(f)\|_\infty \cdot \max_{1\leqslant k\leqslant d} \|[A_k, B]\|_1.$$

For every Lipschitz function $f : \mathbb{R}^d \to \mathbb{R}$ and for every collections $\mathbf{X} = \{X_k\}_{k=1}^d \subset B(H)$, $\mathbf{Y} = \{Y_k\}_{k=1}^d \subset B(H)$ of commuting self-adjoint operators such that $X_k - Y_k \in L_1(H)$, we have

$$\|f(\mathbf{X}) - f(\mathbf{Y})\|_{1,\infty} \leqslant c(d)\|\nabla(f)\|_\infty \cdot \max_{1 \leqslant k \leqslant d} \|X_k - Y_k\|_1.$$

As a corollary of Theorem 1.1 we extend our main result from [8] to normal operators, see Corollary 5.4, which substantially improves corresponding results in [2, 7] (see also [1]). This extension is based on a strengthened version of the transference principle from [8] as explained in Section 4. In the text we prove a somewhat stronger result than Theorem 1.1 in terms of double operator integrals (see the next section for the definitions), of which the main Theorem 1.1 is a corollary.

THEOREM 1.2. — *For every Lipschitz function $f : \mathbb{R}^d \to \mathbb{R}$ and for every collection $\mathbf{A} = \{A_k\}_{k=1}^d$ of commuting self-adjoint operator in a semifinite von Neumann algebra $\mathcal{M}$, we have*

$$\|T_{f_{k_0}}^{\mathbf{A},\mathbf{A}}(V)\|_{1,\infty} \leqslant c(d)\|\nabla(f)\|_\infty\|V\|_1, \quad V \in (L_1 \cap L_2)(\mathcal{M}),$$

for every $1 \leqslant k_0 \leqslant d$. Here, f_{k_0} is defined by (2.7).

Our proofs are based on weak type versions of de Leeuw theorems [22] and a delicate analysis of homogeneous Calderón–Zygmund operators.

2. Preliminaries

2.1. General notation

Throughout the paper d is an integer $\geqslant 1$. Our main result, Theorem 1.1, concerns d-tuples of commuting self-adjoint operators, whereas the proofs involve an analysis on $\mathbb{R}^{d+1}$ and $\mathbb{T}^{d+1}$. We use

$$\nabla = (\partial_1, \ldots, \partial_{d+1}) = \frac{1}{i}\left(\frac{\partial}{\partial t_1}, \ldots, \frac{\partial}{\partial t_{d+1}}\right)$$

for the gradient, which is an unbounded operator on $L_2(\mathbb{R}^{d+1})$. We use $\mathcal{F}$ for the Fourier transform $\mathcal{F}(f)(t) = (2\pi)^{-(d+1)/2} \int_{\mathbb{R}^{d+1}} f(s)e^{-i\langle s,t\rangle}\mathrm{d}s$.

Let $\mathcal{M}$ be a semifinite von Neumann algebra equipped with a faithful normal semifinite trace τ. In this paper, we always presume that $\mathcal{M}$ is represented on a separable Hilbert space.

A (closed and densely defined) operator x affiliated with $\mathcal{M}$ is called τ−measurable if $\tau(E_{|x|}(s,\infty)) < \infty$ for sufficiently large s. We denote the

set of all τ−measurable operators by $S(\mathcal{M}, \tau)$. For every $x \in S(\mathcal{M}, \tau)$, we define its singular value function $\mu(A)$ by setting

$$\mu(t, x) = \inf\{\|x(1 - p)\|_\infty : \tau(p) \leqslant t\}.$$

Equivalently, for positive self-adjoint operators $x \in S(\mathcal{M}, \tau)$, we have

$$n_x(s) = \tau(E_x(s, \infty)), \quad \mu(t, x) = \inf\{s : n_x(s) < t\}.$$

We have for $x, y \in S(\mathcal{M}, \tau)$ (see e.g. [23, Corollary 2.3.16])

$$(2.1) \qquad \mu(t + s, x + y) \leqslant \mu(t, x) + \mu(s, y), \quad t, s > 0.$$

Let $S((0, \infty) \times (0, \infty)) = S(L_\infty((0, \infty) \times (0, \infty)), \int ds)$ where the integral is the Lebesgue integral. Recall that every $x \in S(\mathcal{M}, \tau), y \in \mathcal{M}$ such $\mu(x) \otimes \mu(y) \in S((0, \infty) \times (0, \infty))$ we have (see [8, Eqn. (4.1)] for the proof),

$$(2.2) \qquad \mu(x \otimes y) = \mu(\mu(x) \otimes \mu(y)).$$

For a measurable function f on $\mathbb{R}^{d+1}$ we use $\sigma_l(f)(t) = f(l^{-1}t), l > 0$. Note that

$$(2.3) \qquad \|\sigma_l(f)\|_1 = l^{d+1}\|f\|_1, \quad \|\sigma_l(f)\|_2 = l^{(d+1)/2}\|f\|_2,$$

where the norms are with respect to the Lebesgue measure on $\mathbb{R}^{d+1}$.

2.2. Non-commutative spaces

For $1 \leqslant p < \infty$ we set,

$$L_p(\mathcal{M}) = \{x \in S(\mathcal{M}, \tau) : \tau(|x|^p) < \infty\}, \quad \|x\|_p = (\tau(|x|^p))^{\frac{1}{p}}.$$

The Banach spaces $(L_p(\mathcal{M}), \|\cdot\|_p), 1 \leqslant p < \infty$ are separable. Further, in this paper the Schatten trace class $L_1(B(H))$ is denoted by $L_1(H)$. Define the space $L_{1,\infty}(\mathcal{M})$ by setting

$$L_{1,\infty}(\mathcal{M}) = \{x \in S(\mathcal{M}, \tau) : \sup_{t>0} t\mu(t, x) < \infty\}.$$

We equip $L_{1,\infty}(\mathcal{M})$ with the functional $\|\cdot\|_{1,\infty}$ defined by the formula

$$\|x\|_{1,\infty} = \sup_{t>0} t\mu(t, x), \quad x \in L_{1,\infty}(\mathcal{M}).$$

It follows from (2.1) that

$$\|x + y\|_{1,\infty} = \sup_{t>0} t\mu(t, x + y) \leqslant \sup_{t>0} t\left(\mu\left(\frac{t}{2}, x\right) + \mu\left(\frac{t}{2}, y\right)\right)$$

$$\leqslant \sup_{t>0} t\mu\left(\frac{t}{2}, x\right) + \sup_{t>0} t\mu\left(\frac{t}{2}, y\right) = 2\|x\|_{1,\infty} + 2\|y\|_{1,\infty}.$$

In particular, $\|\cdot\|_{1,\infty}$ is a quasi-norm. The quasi-normed space $(L_{1,\infty}(\mathcal{M}),$ $\|\cdot\|_{1,\infty})$ is, in fact, quasi-Banach (see e.g. [17, Section 7] or [35]). Naturally we set $L_{1,\infty}(\mathbb{R}^{d+1}) = L_{1,\infty}(L_\infty(\mathbb{R}^{d+1}))$ and $L_{1,\infty}(\mathbb{T}^{d+1}) = L_{1,\infty}(L_\infty(\mathbb{T}^{d+1}))$.

2.3. Weak type inequalities for Calderón–Zygmund operators

Parcet [27] proved a non-commutative extension of Calderón–Zygmund theory.

Let K be a tempered distribution on $\mathbb{R}^{d+1}$ which we refer to as the *convolution kernel*. We let W_K be the associated Calderón–Zygmund operator, formally given by $f \mapsto K * f$. In what follows, we only consider tempered distributions having local values. Our kernels have local value at *every* non-zero point in $\mathbb{R}^{d+1}$ (see the proof of Theorem 2.2 below) and hence can be identified with measurable functions $K : \mathbb{R}^{d+1}\backslash\{0\} \to \mathbb{C}$.

Let $\mathcal{M}$ be a semi-finite von Neumann algebra with normal, semi-finite, faithful trace τ. The operator $1 \otimes W_K$ can, under suitable conditions, be defined as a non-commutative Calderón–Zygmund operator by letting it act on the second tensor leg of $L_1(\mathcal{M})\widehat{\otimes}L_1(\mathbb{R}^{d+1})$. The following theorem in particular gives a sufficient condition for such an operator to act from L_1 to $L_{1,\infty}$. Its proof was improved/shortened very recently by Cadilhac [6].

THEOREM 2.1 ([6, 27]). — *Let* $K : \mathbb{R}^{d+1}\backslash\{0\} \to \mathbb{C}$ *be a kernel satisfying the conditions*

$$(2.4) \qquad |K|(t) \leqslant \frac{\text{const}}{|t|^{d+1}}, \quad |\nabla K|(t) \leqslant \frac{\text{const}}{|t|^{d+2}}.$$

Let $\mathcal{M}$ *be a semi-finite von Neumann algebra. If* $W_K \in B(L_2(\mathbb{R}^{d+1}))$*, then the operator* $1 \otimes W_K$ *defines a bounded map from* $L_1(\mathcal{M} \otimes L_\infty(\mathbb{R}^{d+1}))$ *to* $L_{1,\infty}(\mathcal{M} \otimes L_\infty(\mathbb{R}^{d+1}))$.

We need a very special case of Theorem 2.1.

THEOREM 2.2. — *If* $g \in L_\infty(\mathbb{R}^{d+1})$ *is a smooth homogeneous function of degree 0, then* $1 \otimes g(\nabla)$ *defines*[1] *a bounded map from* $L_1(\mathcal{M} \otimes L_\infty(\mathbb{R}^{d+1}))$ *to* $L_{1,\infty}(\mathcal{M} \otimes L_\infty(\mathbb{R}^{d+1}))$.

Proof. — Without loss of generality, the function g is mean zero on the sphere $\mathbb{S}^d$ (this can always be achieved by subtracting a constant from g). By Theorem 6 on p. 75 in [33] and using that g has mean 0, we have

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[1] Here, $g(\nabla) = \mathcal{F}^{-1}M_g\mathcal{F}$, where M_g is the multiplication operator by the function g. In other words, it is a Fourier multiplier associated to the symbol g.

$g(\nabla) = W_K$, where $K = \mathcal{F}^{-1}(g)$ is a smooth homogeneous function of degree $-d-1$. The gradient of the function K is a smooth homogeneous function of degree $-d-2$. These conditions guarantee that (2.4) holds for K and by Theorem 2.1, the assertion follows. $\qquad\square$

In Section 3, we prove the following compact analogue of Theorem 2.2. The transference arguments in Section 4 require such a compact form. We let $\nabla_{\mathbb{T}^{d+1}}$ be the gradient operator on the $(d+1)$-torus.

THEOREM 2.3. — If g is a smooth homogeneous function on $\mathbb{R}^{d+1}\backslash\{0\}$ of degree 0 (we also set $g(0) = 0$), then the operator $1 \otimes g(\nabla_{\mathbb{T}^{d+1}}) : L_2(\mathcal{M} \otimes L_\infty(\mathbb{T}^{d+1})) \to L_2(\mathcal{M} \otimes L_\infty(\mathbb{T}^{d+1}))$ admits a bounded extension acting from $L_1(\mathcal{M} \otimes L_\infty(\mathbb{T}^{d+1}))$ to $L_{1,\infty}(\mathcal{M} \otimes L_\infty(\mathbb{T}^{d+1}))$.

Remark 2.4. — Theorem 2.3 should be understood as a de Leeuw theorem in the following sense. Assume for simplicity that $\mathcal{M} = \mathbb{C}$. The operator $g(\nabla)$ in Theorem 2.2 is a Fourier multiplier with symbol g, whereas the operator $g(\nabla_{\mathbb{T}^{d+1}})$ is the Fourier multiplier on $L_2(\mathbb{T}^{d+1})$ whose symbol is the restriction of g to $\mathbb{Z}^{d+1}$. Theorem 2.3 then shows that $g|_{\mathbb{Z}^{d+1}}$ is the symbol of a bounded multiplier $L_1(\mathbb{T}^{d+1}) \to L_{1,\infty}(\mathbb{T}^{d+1})$. This is a weak $(1,1)$ version of de Leeuw's theorem [22].

2.4. Double operator integrals

Let $\mathbf{A} = \{A_k\}_{k=1}^d$ be a collection of *commuting* self-adjoint operators affiliated with $\mathcal{M}$. Consider projection valued measures on $\mathbb{R}^d$ acting on the Hilbert space $L_2(\mathcal{M})$ by the formulae

$$x \to \left(\prod_{k=1}^d E_{A_k}(\mathcal{B}_k)\right)x, \quad x \to x\left(\prod_{k=1}^d E_{A_k}(\mathcal{C}_k)\right), \quad x \in L_2(\mathcal{M}).$$

These spectral measures commute and, hence (see Theorem V.2.6 in [3]), there exists a countably additive (in the strong operator topology) projection-valued measure ν on $\mathbb{R}^{2d}$ acting on the Hilbert space $L_2(\mathcal{M})$ by the formula

$$(2.5)\ \nu(\mathcal{B}_1 \times \cdots \times \mathcal{B}_d \times \mathcal{C}_1 \times \cdots \times \mathcal{C}_d) : x \to \left(\prod_{k=1}^d E_{A_k}(\mathcal{B}_k)\right)x\left(\prod_{k=1}^d E_{A_k}(\mathcal{C}_k)\right),$$

with $x \in L_2(\mathcal{M})$. Integrating a bounded Borel function ξ on $\mathbb{R}^{2d}$ with respect to the measure ν produces a bounded operator acting on the Hilbert space $L_2(\mathcal{M})$. In what follows, we denote the latter operator by $T_\xi^{\mathbf{A},\mathbf{A}}$ (see also [26, Remark 3.1]).

In the special case when A_k are bounded and $\text{spec}(A_k) \subset \mathbb{Z}$, we have

$$(2.6) \quad T_\xi^{\mathbf{A},\mathbf{A}}(V) = \sum_{\mathbf{i},\mathbf{j}\in\mathbb{Z}^d} \xi(\mathbf{i},\mathbf{j}) \left(\prod_{k=1}^d E_{A_k}(\{i_k\}) \right) V \left(\prod_{k=1}^d E_{A_k}(\{j_k\}) \right).$$

We are mostly interested in the case $\xi = f_k$ for a Lipschitz function f. Here, for $1 \leqslant k \leqslant d$ and $\lambda, \mu \in \mathbb{R}^d$,

$$(2.7) \qquad f_k(\lambda,\mu) = \begin{cases} \frac{(f(\lambda)-f(\mu))(\lambda_k-\mu_k)}{\langle\lambda-\mu,\lambda-\mu\rangle}, & \lambda \neq \mu \\ 0, & \lambda = \mu. \end{cases}$$

3. A de Leeuw type theorem for Calderón–Zygmund operators of convolution type

In this section we collect de Leeuw type results (cf. [22]) needed in the subsequent proofs. The main result is Theorem 2.3. This theorem should be understood as a restriction theorem for (homogeneous) Fourier multipliers, see Remark 2.4.

The strategy of the proof is as follows. One finds an asymptotic embedding of $L_1(\mathbb{T}^{d+1})$ (resp. $L_{1,\infty}(\mathbb{T}^{d+1})$) into $L_1(\mathbb{R}^{d+1})$ (resp. $L_{1,\infty}(\mathbb{R}^{d+1})$) such that this asymptotic embedding intertwines the Fourier multipliers/ Calderón–Zygmund operators and their discretizations.

In what follows,

$$G_l(t) = (l\sqrt{2\pi})^{-(d+1)}e^{-\frac{|t|^2}{2l^2}}, \quad t \in \mathbb{R}^{d+1}, \quad l > 0.$$

We have that $\|G_l\|_1 = 1$. Let $\mathcal{F}$ stand for the Fourier transform. Note that

$$(3.1) \quad \begin{aligned} (\mathcal{F}G_l)(t) &= (l\sqrt{2\pi})^{-(d+1)} \int e^{-\frac{|s|^2}{2l^2}} e^{-i\langle t,s\rangle} ds \\ &= (l\sqrt{2\pi})^{-(d+1)} \int e^{-\frac{|s|^2}{2}} e^{-i\langle lt,s\rangle} ds = G_1(lt). \end{aligned}$$

We set

$$(3.2) \qquad e_k(t) := e^{i\langle k,t\rangle}, \quad k,t \in \mathbb{R}^{d+1}.$$

Let $\alpha = (\alpha_1,\ldots,\alpha_{d+1})$ where $\alpha_k \in \mathbb{Z}_+$. The notation ∂^α is used for $h(\nabla)$, where $h(t) = \prod_{k=1}^{d+1} t_k^{\alpha_k}$, $t \in \mathbb{R}^{d+1}$. Since $|e_k| = 1$, it follows that the multiplication operator $M_{e_k} : L_2(\mathbb{R}^{d+1}) \to L_2(\mathbb{R}^{d+1})$ is unitary. We have

$$M_{e_{-k}}g(\nabla)M_{e_k} = g(M_{e_{-k}}\nabla M_{e_k}) = g(\nabla + k).$$

Remark 3.1. — The Gaussian functions G_l are needed to normalize our asymptotic embeddings given by periodizations of functions (see Lemmas 3.8 and 3.9 for exact statements). These asymptotic embeddings are closely related to the Bohr compactification of $\mathbb{R}^{d+1}$.

The following lemma is a $(d+1)-$dimensional analogue of Lemma 7 in [30]. Here, for every multi-index α, we use the notation $|\alpha| = \alpha_1 + \cdots + \alpha_{d+1}$.

LEMMA 3.2. — For every function h on $\mathbb{R}^{d+1}$ whose partial derivatives up to order $d+1$ belong to $L_2(\mathbb{R}^{d+1})$ we have

$$\|\mathcal{F}^{-1}(h)\|_1 \leqslant 2^{\frac{d+1}{2}} \sum_{|\alpha| \leqslant d+1} \|\partial^\alpha(h)\|_2.$$

Proof. — For every $\mathscr{A} \subset \{1, \ldots, d+1\}$, we define the set $O_{\mathscr{A}} \subset \mathbb{R}^{d+1}$ by setting

$$O_{\mathscr{A}} = \{t \in \mathbb{R}^{d+1} : |t_k| \geqslant 1, \ k \in \mathscr{A}, \ |t_k| \leqslant 1, \ k \notin \mathscr{A}\}.$$

We also define the function $h_{\mathscr{A}}$ on $\mathbb{R}^{d+1}$ by setting

$$h_{\mathscr{A}}(t) = \prod_{k \in \mathscr{A}} t_k, \quad t \in \mathbb{R}^{d+1}.$$

Note that the sets $O_{\mathscr{A}}$ form a partition of $\mathbb{R}^{d+1}$ and that for every choice of $\mathcal{A}$ we have $\|h_{\mathscr{A}}^{-1} \chi_{O_{\mathscr{A}}}\|_2 \leqslant 2^{\frac{d+1}{2}}$.

We have

$$\|\mathcal{F}^{-1}(h)\|_1 \leqslant \sum_{\mathscr{A} \subset \{1, \ldots, d+1\}} \|\mathcal{F}^{-1}(h)\chi_{O_{\mathscr{A}}}\|_1.$$

By the Hölder inequality

$$\|\mathcal{F}^{-1}(h)\|_1 \leqslant \sum_{\mathscr{A} \subset \{1, \ldots, d+1\}} \|h_{\mathscr{A}} \mathcal{F}^{-1}(h)\|_2 \|h_{\mathscr{A}}^{-1} \chi_{O_{\mathscr{A}}}\|_2.$$

By the previous paragraph and the Plancherel identity

$$\|\mathcal{F}^{-1}(h)\|_1 \leqslant 2^{\frac{d+1}{2}} \sum_{\mathscr{A} \subset \{1, \ldots, d+1\}} \|\mathcal{F}^{-1}(h_{\mathscr{A}}(\nabla)h)\|_2$$

$$= 2^{\frac{d+1}{2}} \sum_{\mathscr{A} \subset \{1, \ldots, d+1\}} \|h_{\mathscr{A}}(\nabla)h\|_2.$$

The proof follows as $h_{\mathscr{A}}(\nabla) = \partial^\alpha$. $\qquad\square$

For a multi-index $\alpha = (\alpha_1, \ldots, \alpha_{d+1}) \in \mathbb{Z}_+^{d+1}$ let $|\alpha| = \sum_{i=1}^{d+1} \alpha_i$. We shall without further reference use the fact that $\partial^\alpha(\sigma_l(f)) = l^{-|\alpha|}\sigma_l(\partial^\alpha(f))$ for any smooth function f on $\mathbb{R}^{d+1}$.

LEMMA 3.3. — *Let $g \in L_\infty(\mathbb{R}^{d+1})$ be a smooth function with all derivatives assumed to be uniformly bounded. If $(\partial^\alpha g)(0) = 0$ for every multiindex α with $|\alpha| \leqslant d$, then*

$$\|(g(\nabla))(G_l)\|_1 \to 0, \quad l \to \infty.$$

Proof. — We have $g(\nabla) = \mathcal{F}^{-1} M_g \mathcal{F}$, with M_g the multiplication operator with g on $L_2(\mathbb{R}^{d+1})$. Recall again that $\mathcal{F}(G_l)(t) = G_1(lt), t \in \mathbb{R}^{d+1}$. Thus, see e.g. (3.1),

$$(g(\nabla))(G_l) = \mathcal{F}^{-1} M_g \mathcal{F}(G_l) = \mathcal{F}^{-1}(gh_l),$$

where $h_l(t) = G_1(lt)$, $t \in \mathbb{R}^{d+1}$. It follows from Lemma 3.2 that

(3.3)
$$\|\mathcal{F}^{-1}(gh_l)\|_1 \leqslant 2^{\frac{d+1}{2}} \sum_{|\alpha| \leqslant d+1} \|\partial^\alpha(gh_l)\|_2$$
$$\leqslant 2^{\frac{d+1}{2}} \sum_{|\alpha|+|\beta| \leqslant d+1} \|\partial^\alpha(g)\partial^\beta(h_l)\|_2.$$

Due to the assumption that $(\partial^\beta g)(0) = 0$ for every multi-index β with $|\beta| \leqslant d$, all coefficients in the Taylor expansion of g around 0 of the terms of order $\leqslant d$ vanish. If $|\alpha| < d+1$, then all coefficients in the Taylor expansion of $\partial^\alpha g$ around 0 of the terms of order $\leqslant d - |\alpha|$ vanish. In other words, we have

$$\partial^\alpha g(t) = O(|t|^{d+1-|\alpha|}), \quad t \to 0.$$

As all derivatives of g are assumed to be uniformly bounded functions, we obtain that

$$\partial^\alpha g(t) = O(|t|^{d+1-|\alpha|}), \quad t \in \mathbb{R}^{d+1}.$$

Let $f(t) = |t|, t \in \mathbb{R}^{d+1}$. It follows from the preceding equality that $|\partial^\alpha g| \leqslant c(g) f^{d+1-|\alpha|}, |\alpha| < d+1$, for some constant $c(g)$. As all derivatives of g are assumed to be uniformly bounded functions, the latter inequality trivially holds for $|\alpha| = d+1$.

By (3.3) and the preceding paragraph, we have

$$\|\mathcal{F}^{-1}(gh_l)\|_1 \leqslant 2^{\frac{d+1}{2}} c(g) \sum_{|\alpha|+|\beta| \leqslant d+1} \|f^{d+1-|\alpha|}\partial^\beta(h_l)\|_2.$$

We have

$$\partial^\beta(h_l) = l^{|\beta|}\sigma_{\frac{1}{l}}(\partial^\beta G_1), \quad f^{d+1-|\alpha|} = l^{|\alpha|-d-1}\sigma_{\frac{1}{l}}(f^{d+1-|\alpha|}).$$

Thus,

$$\|f^{d+1-|\alpha|}\partial^\beta(h_l)\|_2 = l^{|\beta|+|\alpha|-d-1}\|\sigma_{\frac{1}{l}}(f^{d+1-|\alpha|}\partial^\beta(G_1))\|_2$$
$$= l^{|\beta|+|\alpha|-\frac{3}{2}(d+1)}\|f^{d+1-|\alpha|}\partial^\beta G_1\|_2 \to 0.$$

This concludes the proof. □

LEMMA 3.4. — If $g : \mathbb{R}^{d+1} \to \mathbb{C}$ is a Schwartz function such that $g(0) = 0$, then
$$\|(g(\nabla))(G_l)\|_1 \to 0, \quad l \to \infty.$$

Proof. — Define Schwartz functions $g_j : \mathbb{R}^{d+1} \to \mathbb{C}$, $1 \leqslant j \leqslant d+1$, by setting
$$g_j(t) = \frac{g(0,\ldots,0,t_j,\ldots,t_{d+1}) - g(0,\ldots,0,t_{j+1},\ldots,t_{d+1})}{t_j}, \quad t \in \mathbb{R}^{d+1}.$$

We have,
$$g(t) = \sum_{j=1}^{d+1} t_j g_j(t).$$
and, therefore,

(3.4) $$g(\nabla)(G_l) = \sum_{j=1}^{d+1} g_j(\nabla) \cdot \left(\partial_j G_l\right).$$

It follows from the Young inequality that
$$\|g_j(\nabla)x\|_1 = \|\mathcal{F}^{-1} M_{g_j} \mathcal{F} x\|_1 = \|\mathcal{F}^{-1}(g_j) * x\|_1 \leqslant \|\mathcal{F}^{-1}(g_j)\|_1 \|x\|_1,$$
where $x \in L_1(\mathbb{R}^{d+1})$. The proof then follows provided that for $x = \partial_j G_l, 1 \leqslant j \leqslant d+1$ we have,

(3.5) $$\|\partial_j G_l\|_1 \to 0, \quad l \to \infty.$$

Indeed, a direct computation yields,
$$\partial_j G_l = \frac{1}{l^{d+2}} \sigma_l(h_j), \quad \text{where } h_j(t) := i t_j G_1(t), \quad t \in \mathbb{R}^{d+1}.$$

So appealing to (2.3), we obtain
$$\|\partial_j (G_l)\|_1 = \frac{1}{l} \|h_j\|_1 \to 0. \qquad \square$$

LEMMA 3.5. — Let $g \in L_\infty(\mathbb{R}^{d+1})$ be a smooth function with all its derivatives assumed to be uniformly bounded. If $k \in \mathbb{R}^{d+1}$, then
$$\|(g(\nabla))(G_l e_k) - g(k)G_l e_k\|_1 \to 0, \quad l \to \infty.$$
Here e_k is given by (3.2).

Proof. — Suppose first that $k = 0$ and $g(0) = 0$. Let ψ be a Schwartz function on $\mathbb{R}^{d+1}$ such that $\psi(t) = 1$ whenever $|t| \leqslant 1$. Set
$$\phi(t) = \sum_{|\alpha| \leqslant d} \frac{i^{|\alpha|}}{\prod_{k=1}^{d+1}(\alpha_k)!} (\partial^\alpha g)(0) t^\alpha \psi(t), \quad t \in \mathbb{R}^{d+1}.$$

Clearly, ϕ is a Schwartz function with $\phi(0) = 0$. By definition of ∂^β, we have, for $\beta \neq \alpha$,

$$\partial^\beta(t^\alpha)|_{t=0} = 0 \text{ and, therefore, } \partial^\beta(t^\alpha\psi(t))|_{t=0} = 0.$$

By definition of ∂^β, we also have, for $\beta = \alpha$,

$$\partial^\beta(t^\alpha) = i^{-|\alpha|} \cdot \prod_{k=1}^{d+1}(\alpha_k)!$$

and, therefore,

$$\partial^\beta(t^\alpha\psi(t))|_{t=0} = i^{-|\alpha|} \cdot \prod_{k=1}^{d+1}(\alpha_k)!.$$

It follows that

$$(\partial^\beta g)(0) = (\partial^\beta\phi)(0), \quad |\beta| \leqslant d.$$

In other words, the function $g - \phi$ satisfies the assumptions of Lemma 3.3. Using Lemmas 3.3 and 3.4, we obtain

$$\|((g - \phi)(\nabla))(G_l)\|_1 \to 0, \quad \|(\phi(\nabla))(G_l)\|_1 \to 0, \quad l \to \infty.$$

Using triangle inequality, we obtain

$$\|(g(\nabla))(G_l)\|_1 \to 0, \quad l \to \infty.$$

This proves the assertion in our special case.

To prove the assertion in general, note that

$$(3.6) \qquad \begin{aligned} \|g(\nabla)(G_l e_k) - g(k)G_l e_k\|_1 &= \|(M_{e_{-k}}g(\nabla)M_{e_k} - g(k))(G_l)\|_1 \\ &= \|(g(\nabla + k) - g(k))(G_l)\|_1. \end{aligned}$$

Now as $t \to g(t + k) - g(k)$ is a function satisfying the assumptions of the first paragraph, we see that (3.6) goes to 0 as $l \to \infty$. $\qquad\square$

The following Lemma 3.6 is the main intertwining property as we explained in the beginning of this section.

LEMMA 3.6. — *Let $g \in L_\infty(\mathbb{R}^{d+1})$ be a smooth (except at 0) homogeneous function of degree 0. For every $0 \neq k \in \mathbb{R}^{d+1}$, we have*

$$\|(g(\nabla))(G_l e_k) - g(k)G_l e_k\|_{1,\infty} \to 0, \quad l \to \infty.$$

Proof. — Fix $0 \neq k \in \mathbb{R}^{d+1}$. Fix a Schwartz function ϕ supported on the ball $\{|t|_2 < |k|_2\}$ such that $\phi(t) = 1$ whenever $|t|_2 \leqslant \frac{1}{2}|k|_2$. Clearly, both functions ϕ and $g(1 - \phi)$ satisfy the conditions of Lemma 3.5. We obtain

$$\begin{aligned} \|((g(1 - \phi))(\nabla))&(G_l e_k) - g(k)G_l e_k\|_{1,\infty} \\ &\leqslant \|((g(1 - \phi))(\nabla))(G_l e_k) - g(k)G_l e_k\|_1 \to 0, \quad l \to \infty \end{aligned}$$

And also

$$\|(\phi(\nabla))(G_l e_k)\|_1 \to 0, \quad l \to \infty.$$

By Theorem 1 on p. 29 in [33] (see especially Step 2 on p. 30; one can also use Theorem 2.2 here), the operator $g(\nabla) : L_1(\mathbb{R}^{d+1}) \to L_{1,\infty}(\mathbb{R}^{d+1})$ is bounded. Thus, since ϕ satisfies the assumptions of Lemma 3.3,

$$\|(g\phi(\nabla))(G_l e_k)\|_{1,\infty} \leqslant \|g(\nabla)\|_{L_1 \to L_{1,\infty}} \|(\phi(\nabla))(G_l e_k)\|_1 \to 0, \quad l \to \infty.$$

The assertion follows by applying triangle inequality. $\square$

LEMMA 3.7. — Let $A \in L_1(\mathcal{M}_1)$ and let $B \in L_{1,\infty}(\mathcal{M}_2)$. We have

$$\|A \otimes B\|_{1,\infty} \leqslant \|A\|_1 \|B\|_{1,\infty}.$$

Proof. — Define the function z on $(0, \infty)$ by setting $z(t) := t^{-1}$, $t > 0$. We have

$$\mu(A \otimes B) \overset{(2.2)}{=} \mu(\mu(A) \otimes \mu(B)) \leqslant \|B\|_{1,\infty} \mu(\mu(A) \otimes z).$$

We claim that for every positive decreasing function $x \in L_1(0, \infty)$, we have $\mu(x \otimes z) = \|x\|_1 z$. Set $x_n = \sum_{k=0}^{n^2-1} \mu(\frac{k+1}{n}, x) \chi_{(\frac{k}{n}, \frac{k+1}{n})}, n > 1$. The functions $\chi_{(\frac{k}{n}, \frac{k+1}{n})} \otimes z, 0 \leqslant k < n^2$, are disjointly supported. These functions are equidistributed and have the same distribution as $\frac{1}{n} z$. Indeed, if we compare distribution functions

$$n_{\chi_{\left(\frac{k}{n}, \frac{k+1}{n}\right)} \otimes z}(u) := m\left(\left\{(s,t) : \chi_{\left(\frac{k}{n}, \frac{k+1}{n}\right)}(s) t^{-1} > u\right\}\right)$$

$$= \frac{1}{n} m(\{t : t^{-1} > u\}) = \frac{1}{n} u^{-1} = n_{\frac{1}{n} z}(u).$$

Therefore,

$$\mu(x_n \otimes z) = \mu\left(\sum_{k=0}^{n^2-1} \mu\left(\frac{k+1}{n}, x\right) \chi_{\left(\frac{k}{n}, \frac{k+1}{n}\right)} \otimes z\right)$$

$$= \mu\left(\bigoplus_{k=0}^{n^2-1} \frac{1}{n} \mu\left(\frac{k+1}{n}, x\right) z\right) = \|x_n\|_1 z.$$

It is immediate that $x_n \uparrow \mu(x)$ and, therefore, $x_n \otimes z \mu(x) \otimes z$ and $\mu(x_n \otimes z) \uparrow \mu(x \otimes z)$. This proves the claim. $\square$

Let

$$\text{per} : \mathcal{M} \otimes L_\infty(\mathbb{T}^{d+1}) \to \mathcal{M} \otimes L_\infty(\mathbb{R}^{d+1})$$

be the natural embedding by periodicity. Under the identification $\mathcal{M} \otimes L_\infty(\mathbb{R}^{d+1}) \simeq L_\infty(\mathbb{R}^{d+1}, \mathcal{M})$ (the latter being understood as weakly measurable, essentially bounded functions) and similarly for the torus, it is

defined as
$$\operatorname{per}(f)(t) = f(t \mod 2\pi), \qquad t \in \mathbb{R}^{d+1}.$$
We consider $\mathbb{T}$ with total Haar measure 2π. The next Lemma 3.8 provides the asymptotic embedding of $L_1(\mathbb{T}^{d+1})$ to $L_1(\mathbb{R}^{d+1})$.

LEMMA 3.8. — *For every* $W \in L_1(\mathcal{M} \otimes L_\infty(\mathbb{T}^{d+1}))$, *we have*
$$\lim_{l \to \infty} \| \operatorname{per}(W) \cdot (1 \otimes G_l) \|_{L_1(\mathcal{M} \otimes L_\infty(\mathbb{R}^{d+1}))} = \frac{1}{(2\pi)^{d+1}} \|W\|_{L_1(\mathcal{M} \otimes L_\infty(\mathbb{T}^{d+1}))}.$$

Proof. — For every $m \in \mathbb{Z}$, define $l(m), n(m) \in \mathbb{Z}$ by setting

$$(3.7) \qquad l(m) = \begin{cases} m & m \geqslant 0 \\ m+1 & m < 0, \end{cases}$$

$$(3.8) \qquad n(m) = \begin{cases} m+1 & m \geqslant 0 \\ m & m < 0. \end{cases}$$

Next set
$$l(m) = (l(m_1), \ldots, l(m_{d+1})), \qquad m \in \mathbb{Z}^{d+1},$$
$$n(m) = (n(m_1), \ldots, n(m_{d+1})), \qquad m \in \mathbb{Z}^{d+1}.$$
Clearly,
$$\| \operatorname{per}(W) \cdot (1 \otimes G_l) \|_{L_1(\mathcal{M} \otimes L_\infty(\mathbb{R}^{d+1}))}$$
$$= \sum_{m \in \mathbb{Z}^{d+1}} \| \operatorname{per}(W) \cdot (1 \otimes G_l) \cdot (1 \otimes \chi_{2\pi m + [0, 2\pi]^{d+1}}) \|_{L_1(\mathcal{M} \otimes L_\infty(\mathbb{R}^{d+1}))}.$$
By construction,
$$G_l(2\pi n(m)) \leqslant G_l(t) \leqslant G_l(2\pi l(m)), \quad t \in 2\pi m + [0, 2\pi]^{d+1}.$$
Hence,
$$\| \operatorname{per}(W) \cdot (1 \otimes G_l) \|_{L_1(\mathcal{M} \otimes L_\infty(\mathbb{R}^{d+1}))}$$
$$\leqslant \sum_{m \in \mathbb{Z}^{d+1}} G_l(2\pi l(m)) \| \operatorname{per}(W) \cdot (1 \otimes \chi_{2\pi m + [0, 2\pi]^{d+1}}) \|_{L_1(\mathcal{M} \otimes L_\infty(\mathbb{R}^{d+1}))}$$
$$= \|W\|_{L_1(\mathcal{M} \otimes L_\infty(\mathbb{T}^{d+1}))} \cdot \sum_{m \in \mathbb{Z}^{d+1}} G_l(2\pi l(m)).$$
Similarly,
$$\| \operatorname{per}(W) \cdot (1 \otimes G_l) \|_{L_1(\mathcal{M} \otimes L_\infty(\mathbb{R}^{d+1}))}$$
$$\geqslant \sum_{m \in \mathbb{Z}^{d+1}} G_l(2\pi n(m)) \| \operatorname{per}(W) \cdot (1 \otimes \chi_{2\pi m + [0, 2\pi]^{d+1}}) \|_{L_1(\mathcal{M} \otimes L_\infty(\mathbb{R}^{d+1}))}$$
$$= \|W\|_{L_1(\mathcal{M} \otimes L_\infty(\mathbb{T}^{d+1}))} \cdot \sum_{m \in \mathbb{Z}^{d+1}} G_l(2\pi n(m)).$$

We have

$$\sum_{m\in\mathbb{Z}^{d+1}} G_l(2\pi l(m)) = \left(\sum_{m\in\mathbb{Z}} G_l(2\pi l(m))\right)^{d+1}$$

$$= \left(\frac{1}{l\sqrt{2\pi}} + \frac{1}{l\sqrt{2\pi}}\sum_{m\in\mathbb{Z}} e^{-\frac{(2\pi m)^2}{2l^2}}\right)^{d+1} \to \frac{1}{(2\pi)^{d+1}}, \quad l\to\infty,$$

where the limit is by elementary Riemann integration. Similarly

$$\sum_{m\in\mathbb{Z}^{d+1}} G_l(2\pi n(m)) = \left(\sum_{m\in\mathbb{Z}} G_l(2\pi n(m))\right)^{d+1}$$

$$= \left(-\frac{1}{l\sqrt{2\pi}} + \frac{1}{l\sqrt{2\pi}}\sum_{m\in\mathbb{Z}} e^{-\frac{(2\pi m)^2}{2l^2}}\right)^{d+1} \to \frac{1}{(2\pi)^{d+1}}, \quad l\to\infty.$$

Combining the last 4 equations completes the proof as they show that we have estimates

$$\frac{1}{(2\pi)^{d+1}}\|W\|_{L_1(\mathcal{M}\otimes L_\infty(\mathbb{T}^{d+1}))} - \epsilon_l \leqslant \|\operatorname{per}(W)\cdot(1\otimes G_l)\|_{L_1(\mathcal{M}\otimes L_\infty(\mathbb{R}^{d+1}))}$$

$$\leqslant \frac{1}{(2\pi)^{d+1}}\|W\|_{L_1(\mathcal{M}\otimes L_\infty(\mathbb{T}^{d+1}))} + \epsilon_l.$$

for some sequences $\epsilon_l > 0$ that converges to 0. $\qquad\square$

The next lemma gives the asymptotic norm estimate of periodizations of elements of $L_{1,\infty}(\mathbb{T}^{d+1})$ with the norms of $L_{1,\infty}(\mathbb{R}^{d+1})$.

LEMMA 3.9. — For every $W \in L_{1,\infty}(\mathcal{M}\otimes L_\infty(\mathbb{T}^{d+1}))$, we have

$$\liminf_{l\to\infty} \|\operatorname{per}(W)\cdot(1\otimes G_l)\|_{L_{1,\infty}(\mathcal{M}\otimes L_\infty(\mathbb{R}^{d+1}))} \gtrsim \|W\|_{L_{1,\infty}(\mathcal{M}\otimes L_\infty(\mathbb{T}^{d+1}))}.$$

Here, $\gtrsim$ means inequality up to some constant independent of W.

Proof. — We estimate crudely,

$$G_l(t) \geqslant c(d)l^{-d-1}, \quad |t| \leqslant 4\pi l,$$

$$\chi_{\{|t|\leqslant 4\pi l\}} \geqslant \sum_{|m|\leqslant l} \chi_{2\pi m+[0,2\pi]^d}.$$

Hence,

$$\|\operatorname{per}(W)\cdot(1\otimes G_l)\|_{L_{1,\infty}(\mathcal{M}\otimes L_\infty(\mathbb{R}^{d+1}))}$$

$$\geqslant c(d)l^{-d-1}\left\|\operatorname{per}(W)\cdot\left(1\otimes\sum_{|m|\leqslant l}\chi_{2\pi m+[0,2\pi]^d}\right)\right\|_{L_{1,\infty}(\mathcal{M}\otimes L_\infty(\mathbb{R}^{d+1}))}.$$

Since the elements $\mathrm{per}(W) \cdot (1 \otimes \chi_{2\pi m + [0,2\pi]^d})$ with $|m| \leqslant l$ are pairwise orthogonal we have that

$$\mathrm{per}(W) \cdot (1 \otimes \sum_{|m| \leqslant l} \chi_{2\pi m + [0,2\pi]^d}) \in L_{1,\infty}(\mathcal{M} \otimes L_\infty(\mathbb{R}^{d+1}))$$

and

$$\bigoplus_{|m| \leqslant l} W \in L_{1,\infty}(\mathcal{M} \otimes L_\infty(\mathbb{T}^{d+1}) \otimes l_\infty)$$

are unitarily equivalent. Then

$$\| \mathrm{per}(W) \cdot (1 \otimes G_l) \|_{L_{1,\infty}(\mathcal{M} \otimes L_\infty(\mathbb{R}^{d+1}))}$$
$$\geqslant c(d) l^{-d-1} \left\| \bigoplus_{|m| \leqslant l} W \right\|_{L_{1,\infty}(\mathcal{M} \otimes L_\infty(\mathbb{T}^{d+1}) \otimes l_\infty)}.$$

Let n_l be the number of $m \in \mathbb{Z}^{d+1}$ with $|m|_2 \leqslant l$. Note that $n_l \gtrsim l^{d+1}$. Then $\mu(t, \bigoplus_{|m| \leqslant l} W) = \mu(n_l^{-1} t, W)$ from which we may continue the estimate

$$\| \mathrm{per}(W) \cdot (1 \otimes G_l) \|_{L_{1,\infty}(\mathcal{M} \otimes L_\infty(\mathbb{R}^{d+1}))}$$
$$\geqslant c(d) l^{-d-1} n_l \|W\|_{L_{1,\infty}(\mathcal{M} \otimes L_\infty(\mathbb{T}^{d+1}))}$$
$$\geqslant c(d) \|W\|_{L_{1,\infty}(\mathcal{M} \otimes L_\infty(\mathbb{T}^{d+1}))}. \qquad \square$$

We are now fully equipped to prove our main result.

Proof of Theorem 2.3. — Let $\mathscr{A} \subset \mathbb{Z}^{d+1}$ be a finite set. Let

$$W = \sum_{k \in \mathscr{A}} W_k \otimes e_k, \quad W_k \in L_1(\mathcal{M}).$$

Firstly, we prove

$$\|(1 \otimes g(\nabla))(W)\|_{1,\infty} \lesssim \|W\|_1,$$

for W as above. We have

$$W_0 \otimes e_0 = \left(1 \otimes \int\right)(W).$$

As conditional expectations are contractions on L_1, then so is the operator $1 \otimes \int$. We, therefore, have

$$\left\| \sum_{0 \neq k \in \mathscr{A}} W_k \otimes e_k \right\|_1 \leqslant \left\| \sum_{k \in \mathscr{A}} W_k \otimes e_k \right\|_1 + \|W_0 \otimes e_0\|_1 \leqslant 2\|W\|_1, \quad k \in \mathcal{A}.$$

Therefore, we may (and will) assume without loss of generality that $0 \notin \mathscr{A}$. By Theorem 2.1, we have

$$\|(1 \otimes g(\nabla))(\operatorname{per}(W) \cdot (1 \otimes G_l))\|_{L_{1,\infty}(\mathcal{M} \otimes L_\infty(\mathbb{R}^{d+1}))}$$
$$\leqslant \|\operatorname{per}(W) \cdot (1 \otimes G_l)\|_{L_1(\mathcal{M} \otimes L_\infty(\mathbb{R}^{d+1}))}.$$

By respectively Lemma 3.7 and Lemma 3.6 we have for each $k \in \mathcal{A}$ as $l \to \infty$,

$$\|(1 \otimes g(\nabla))(W_k \otimes G_l e_k) - g(k)(W_k \otimes G(l)e_k)\|_{1,\infty}$$
$$\leqslant \|W_k\|_1 \|(g(\nabla))(G_l e_k) - g(k)G_l e_k\|_{1,\infty} \to 0.$$

The quasi-triangle inequality gives for sums of arbitrary operators x_α that

$$\left\| \sum_{\alpha \in \mathcal{A}} x_\alpha \right\|_{1,\infty} \leqslant 2^{|\mathcal{A}|} \sum_{\alpha \in \mathcal{A}} \|x_\alpha\|_{1,\infty}.$$

So it follows that as $l \to \infty$

$$\left\| \sum_{k \in \mathcal{A}} (1 \otimes g(\nabla))(W_k \otimes G_l e_k) - \sum_{k \in \mathcal{A}} g(k)(W_k \otimes G(l)e_k) \right\|_{1,\infty} \to 0.$$

In other words we have as $l \to \infty$

$$\|(1 \otimes g(\nabla))(\operatorname{per}(W) \cdot (1 \otimes G_l)) - \operatorname{per}((1 \otimes g(\nabla))(W))(1 \otimes G_l)\|_{1,\infty} \to 0.$$

Thus,

$$(3.9) \quad \liminf_{l \to \infty} \|\operatorname{per}((1 \otimes g(\nabla))(W)) \cdot (1 \otimes G_l)\|_{L_{1,\infty}(\mathcal{M} \otimes L_\infty(\mathbb{R}^{d+1}))}$$
$$\leqslant \liminf_{l \to \infty} \|\operatorname{per}(W) \cdot (1 \otimes G_l)\|_{L_1(\mathcal{M} \otimes L_\infty(\mathbb{R}^{d+1}))}.$$

It follows now from Lemma 3.9, (3.9) and Lemma 3.8 that

$$(3.10) \quad \|(1 \otimes g(\nabla_{\mathbb{T}^{d+1}}))(W)\|_{L_{1,\infty}(\mathcal{M} \otimes L_\infty(\mathbb{T}^{d+1}))}$$
$$\lesssim \liminf_{l \to \infty} \|\operatorname{per}((1 \otimes g(\nabla_{\mathbb{T}^{d+1}}))W) \cdot (1 \otimes G_l)\|_{L_{1,\infty}(\mathcal{M} \otimes L_\infty(\mathbb{R}^{d+1}))}$$
$$\leqslant \liminf_{l \to \infty} \|\operatorname{per}(W) \cdot (1 \otimes G_l)\|_{L_1(\mathcal{M} \otimes L_\infty(\mathbb{R}^{d+1}))}$$
$$\lesssim \|W\|_{L_1(\mathcal{M} \otimes L_\infty(\mathbb{T}^{d+1}))}.$$

This proves the assertion for our specific W.

To see the assertion in general, fix an arbitrary $W \in L_1(\mathcal{M} \otimes L_\infty(\mathbb{T}^{d+1}))$ and choose W^m as above such that $W^m \to W$ in $L_1(\mathcal{M} \otimes L_\infty(\mathbb{T}^{d+1}))$ as $m \to \infty$ (see Lemma A.6). In particular, the sequence $\{W^m\}_{m \geqslant 1} \subset L_1(\mathcal{M} \otimes L_\infty(\mathbb{T}^{d+1}))$ is Cauchy. By (3.10), the sequence $\{(1 \otimes g(\nabla)) (W^m)\}_{m \geqslant 1} \subset L_{1,\infty}(\mathcal{M} \otimes L_\infty(\mathbb{T}^{d+1}))$ is also Cauchy. Denote the limit by $T(W)$. If also $W \in L_2(\mathcal{M} \otimes L_\infty(\mathbb{T}^{d+1}))$, then the sequence $\{W^m\}_{m \geqslant 1}$

can be chosen such that also $W^m \to W$ in $L_2(\mathcal{M} \otimes L_\infty(\mathbb{T}^{d+1}))$ (see Remark A.5). Thus, $T(W) = (1 \otimes g(\nabla))(W)$ for $W \in (L_1 \cap L_2)(\mathcal{M} \otimes L_\infty(\mathbb{T}^{d+1}))$. This completes the proof. $\qquad\square$

4. Proof of Theorem 1.2 for the case of integral spectra

The next Theorem 4.1 provides the crucial connection between Calderón–Zygmund operators and commutator estimates. The equality (4.1) should be understood as a transference to Schur multipliers argument. Note that here we have an exact equality (4.1), which we did not yet obtain in [8].

THEOREM 4.1. — *For every contraction*[2] $f : \mathbb{Z}^d \to \mathbb{Z}$ *and for every collection of commuting self-adjoint operators* $\mathbf{A} = \{A_k\}_{k=1}^d \subset \mathcal{M}$ *with* $\operatorname{spec}(A_k) \subset \mathbb{Z}$, *we have*

$$\|T_{f_{k_0}}^{\mathbf{A},\mathbf{A}}(V)\|_{1,\infty} \leqslant c(d)\|V\|_1, \quad V \in L_1(\mathcal{M}), \quad 1 \leqslant k_0 \leqslant d.$$

Here, f_{k_0} *is given by* (2.7).

Proof. — Fix $1 \leqslant k_0 \leqslant d$. The idea is to construct a bounded linear operator $S : L_1(\mathcal{M} \otimes L_\infty(\mathbb{T}^{d+1})) \to L_{1,\infty}(\mathcal{M} \otimes L_\infty(\mathbb{T}^{d+1}))$ (independent of f) and an isometric embedding $I : L_1(\mathcal{M}) \to L_1(\mathcal{M} \otimes L_\infty(\mathbb{T}^{d+1}))$, $I : L_{1,\infty}(\mathcal{M}) \to L_{1,\infty}(\mathcal{M} \otimes L_\infty(\mathbb{T}^{d+1}))$ (dependent on f) such that

$$(4.1) \qquad\qquad S \circ I = I \circ T_{f_{k_0}}^{\mathbf{A},\mathbf{A}}.$$

Fix a smooth function $\varsigma : [0,1] \to \mathbb{R}$ such that $\varsigma(u) = u$, $u \in [\frac{1}{2},1]$ and $\varsigma(u) \geqslant \frac{1}{3}$, $u \in [0, \frac{1}{2}]$. Define a smooth function $g : \mathbb{S}^d \to \mathbb{R}$ by setting

$$g(t) = \frac{t_{k_0} t_{d+1}}{\varsigma(\sum_{k=1}^d t_k^2)}, \quad |t|_2 = 1.$$

Extend g to a smooth homogeneous function $g : \mathbb{R}^{d+1} \setminus \{0\} \to \mathbb{R}$ (of degree 0) by setting $g(t) = g(\frac{t}{|t|_2})$, $0 \neq t \in \mathbb{R}^{d+1}$, and $g(0) = 0$. For $|t|_2 = 1$, the conditions $\sum_{k=1}^d t_k^2 \geqslant \frac{1}{2}$ and $|t_{d+1}| \leqslant (\sum_{k=1}^d t_k^2)^{\frac{1}{2}}$ are equivalent. Hence,

$$(4.2) \qquad g(t) = \frac{t_{k_0} t_{d+1}}{\sum_{k=1}^d t_k^2}, \quad |t_{d+1}| \leqslant \left(\sum_{k=1}^d t_k^2\right)^{\frac{1}{2}}, \quad 0 \neq t \in \mathbb{R}^{d+1}.$$

By assumption, $A_k = \sum_{i_k \in \mathbb{Z}} i_k p_{k,i_k}$, where $\{p_{k,i_k}\}_{i_k \in \mathbb{Z}}$ are pairwise orthogonal projections such that $\sum_{i_k \in \mathbb{Z}} p_{k,i_k} = 1$. Since A is bounded, it follows that $p_{k,i_k} = 0$ for all but finitely many $i_k \in \mathbb{Z}$. Hence, these sums are, in

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[2] Here, contraction means a Lipschitz function with constant 1. That is, $|f(\mathbf{i}) - f(\mathbf{j})| \leqslant |\mathbf{i} - \mathbf{j}|_2$ for all $\mathbf{i}, \mathbf{j} \in \mathbb{Z}^d$.

fact, finite. For every $\mathbf{i} = (i_1, \ldots, i_d) \in \mathbb{Z}^d$, set $p_{\mathbf{i}} = p_{1,i_1} \ldots p_{d,i_d}$. It is immediate that $\{p_{\mathbf{i}}\}_{\mathbf{i} \in \mathbb{Z}^d}$ are pairwise orthogonal projections and $\sum_{\mathbf{i} \in \mathbb{Z}^d} p_{\mathbf{i}} = 1$. Consider a unitary operator

$$U_f = \sum_{\mathbf{i} \in \mathbb{Z}^d} p_{\mathbf{i}} \otimes e_{(\mathbf{i}, f(\mathbf{i}))},$$

where $e_{(\mathbf{i}, f(\mathbf{i}))}$ is given in (3.2).

We are now ready to define the operators S and I. Set

$$S(W) = (1 \otimes g(\nabla_{\mathbb{T}^{d+1}})) \left(\sum_{\substack{\mathbf{i}, \mathbf{j} \in \mathbb{Z}^d \\ \mathbf{i} \neq \mathbf{j}}} (p_{\mathbf{i}} \otimes 1) W (p_{\mathbf{j}} \otimes 1) \right),$$

$$W \in L_1(\mathcal{M} \otimes L_\infty(\mathbb{R}^{d+1})),$$

$$I(V) = U_f(V \otimes 1)U_f^*, \quad V \in L_{1,\infty}(\mathcal{M}).$$

Since f is a contraction we have that $|f(\mathbf{i}) - f(\mathbf{j})| \leqslant |\mathbf{i} - \mathbf{j}|_2$ and therefore by (4.2) we obtain

$$g(\mathbf{i} - \mathbf{j}, f(\mathbf{i}) - f(\mathbf{j})) = f_{k_0}(\mathbf{i}, \mathbf{j}), \quad \mathbf{i}, \mathbf{j} \in \mathbb{Z}^d.$$

In particular

$$g(\nabla_{\mathbb{T}^{d+1}})(e_{(\mathbf{i} - \mathbf{j}, f(\mathbf{i}) - f(\mathbf{j}))}) = f_{k_0}(\mathbf{i}, \mathbf{j}) e_{(\mathbf{i} - \mathbf{j}, f(\mathbf{i}) - f(\mathbf{j}))}, \quad \mathbf{i}, \mathbf{j} \in \mathbb{Z}^d.$$

Recall also that $f_{k_0}(\mathbf{i}, \mathbf{i}) = 0, \mathbf{i} \in \mathbb{Z}^d$. We now prove the transference equality (4.1):

$S(I(V))$

$$= S \left(\left(\sum_{\mathbf{i} \in \mathbb{Z}^d} p_{\mathbf{i}} \otimes e_{(\mathbf{i}, f(\mathbf{i}))} \right) \cdot \left(\sum_{\mathbf{i}, \mathbf{j} \in \mathbb{Z}^d} p_{\mathbf{i}} V p_{\mathbf{j}} \otimes 1 \right) \cdot \left(\sum_{\mathbf{i} \in \mathbb{Z}^d} p_{\mathbf{i}} \otimes e_{(\mathbf{i}, f(\mathbf{i}))} \right)^* \right)$$

$$= S \left(\sum_{\mathbf{i}, \mathbf{j} \in \mathbb{Z}^d} p_{\mathbf{i}} V p_{\mathbf{j}} \otimes e_{(\mathbf{i} - \mathbf{j}, f(\mathbf{i}) - f(\mathbf{j}))} \right)$$

$$\overset{(4.2)}{=} \sum_{\substack{\mathbf{i}, \mathbf{j} \in \mathbb{Z}^d \\ \mathbf{i} \neq \mathbf{j}}} p_{\mathbf{i}} V p_{\mathbf{j}} \otimes f_{k_0}(\mathbf{i}, \mathbf{j}) e_{(\mathbf{i} - \mathbf{j}, f(\mathbf{i}) - f(\mathbf{j}))}$$

$$= \left(\sum_{\mathbf{i} \in \mathbb{Z}^d} p_{\mathbf{i}} \otimes e_{(\mathbf{i}, f(\mathbf{i}))} \right) \cdot \left(\sum_{\mathbf{i}, \mathbf{j} \in \mathbb{Z}^d} p_{\mathbf{i}} T^{\mathbf{A}, \mathbf{A}}_{f_{k_0}}(V) p_{\mathbf{j}} \otimes 1 \right) \cdot \left(\sum_{\mathbf{i} \in \mathbb{Z}^d} p_{\mathbf{i}} \otimes e_{(\mathbf{i}, f(\mathbf{i}))} \right)^*$$

$$= U_f \cdot (T^{\mathbf{A}, \mathbf{A}}_{f_{k_0}}(V) \otimes 1) \cdot U_f^* = I(T^{\mathbf{A}, \mathbf{A}}_{f_{k_0}}(V)).$$

By Theorem 2.3, the mapping

$$1 \otimes g(\nabla_{\mathbb{T}^{d+1}}) : L_1(\mathcal{M} \otimes L_\infty(\mathbb{T}^{d+1})) \to L_{1,\infty}(\mathcal{M} \otimes L_\infty(\mathbb{T}^{d+1})).$$

is bounded. Therefore,

$$
\begin{aligned}
\|T_{f_{k_0}}^{\mathbf{A},\mathbf{A}}(V)\|_{L_{1,\infty}(\mathcal{M})} &= \|I(T_{f_{k_0}}^{\mathbf{A},\mathbf{A}}(V))\|_{L_{1,\infty}(\mathcal{M} \otimes L_\infty(\mathbb{T}^{d+1}))} \\
&= \|S(I(V))\|_{L_{1,\infty}(\mathcal{M} \otimes L_\infty(\mathbb{T}^{d+1}))} \\
&\leqslant \|S\|_{L_1(\mathcal{M} \otimes L_\infty(\mathbb{T}^{d+1})) \to L_{1,\infty}(\mathcal{M} \otimes L_\infty(\mathbb{T}^{d+1}))} \|I(V)\|_{L_1(\mathcal{M} \otimes L_\infty(\mathbb{T}^{d+1}))} \\
&\lesssim \|1 \otimes g(\nabla)\|_{L_1(\mathcal{M} \otimes L_\infty(\mathbb{T}^{d+1})) \to L_{1,\infty}(\mathcal{M} \otimes L_\infty(\mathbb{T}^{d+1}))} \|V\|_{L_1(\mathcal{M})}.
\end{aligned}
$$

This completes the proof. □

5. Proof of the main results

In this section we collect the results announced in the abstract and its corollaries.

LEMMA 5.1. — *Let $\mathbf{A} = \{A_k\}_{k=1}^d \subset \mathcal{M}$ be an arbitrary collection of commuting self-adjoint operators. If $\{\xi_n\}_{n \geqslant 0}$ is a uniformly bounded sequence of Borel functions on $\mathbb{R}^{2d}$ such that $\xi_n \to \xi$ everywhere, then*

$$(5.1) \qquad T_{\xi_n}^{\mathbf{A},\mathbf{A}}(V) \to T_\xi^{\mathbf{A},\mathbf{A}}(V), \quad V \in L_2(\mathcal{M})$$

in $L_2(\mathcal{M})$ as $n \to \infty$.

Proof. — Let ν be a projection valued measure on $\mathbb{R}^{2d}$ considered in Subsection 2.4 (see (2.5)). Let $\gamma : \mathbb{R} \to \mathbb{R}^{2d}$ be a Borel measurable bijection. Clearly, $\nu \circ \gamma$ is a countably additive projection valued measure on $\mathbb{R}$. Hence, there exists a self-adjoint operator B acting on the Hilbert space $L_2(\mathcal{M})$ such that $E_B = \nu \circ \gamma$.

Set $\eta_n = \xi_n \circ \gamma$ and $\eta = \xi \circ \gamma$. We have $\eta_n \to \eta$ everywhere on $\mathbb{R}$. Thus,

$$
\begin{aligned}
T_{\xi_n}^{\mathbf{A},\mathbf{A}} &= \int_{\mathbb{R}^{2d}} \xi_n \mathrm{d}\nu = \int_{\mathbb{R}} \eta_n(\lambda) \mathrm{d}E_B(\lambda) = \eta_n(B) \to \eta(B) \\
&= \int_{\mathbb{R}} \eta(\lambda) \mathrm{d}E_B(\lambda) = \int_{\mathbb{R}^{2d}} \xi \mathrm{d}\nu = T_\xi^{\mathbf{A},\mathbf{A}}.
\end{aligned}
$$

Here, the convergence is understood with respect to the strong operator topology on the space $B(L_2(\mathcal{M}))$. In particular, (5.1) follows. □

In the next proof let $\lfloor x \rfloor$ be the largest integer smaller than x and let $\{x\} = x - \lfloor x \rfloor$ be the fractional part.

Proof of Theorem 1.2.

Step 1. — Let $f : \mathbb{R}^d \to \mathbb{R}$ be a contraction. We claim that the mapping $f^n : \mathbb{Z}^d \to \mathbb{Z}$ defined by the formula

$$f^n(\mathbf{i}) = \left\lfloor \frac{n}{2} f\left(\frac{\mathbf{i}}{n}\right) \right\rfloor, \quad \mathbf{i} \in \mathbb{Z}^d,$$

is also a contraction.

Indeed, we have

$$f^n(\mathbf{i}) - f^n(\mathbf{j}) = \frac{n}{2}\left(f\left(\frac{\mathbf{i}}{n}\right) - f\left(\frac{\mathbf{j}}{n}\right)\right) + \left(\left\{\frac{n}{2} f\left(\frac{\mathbf{j}}{n}\right)\right\} - \left\{\frac{n}{2} f\left(\frac{\mathbf{i}}{n}\right)\right\}\right).$$

By assumption, we have that

$$\frac{n}{2}\left|f\left(\frac{\mathbf{i}}{n}\right) - f\left(\frac{\mathbf{j}}{n}\right)\right| \leqslant \frac{n}{2}\left|\frac{\mathbf{i}}{n} - \frac{\mathbf{j}}{n}\right| \leqslant \frac{1}{2}|\mathbf{i} - \mathbf{j}|.$$

It is immediate that

$$\left\{\frac{n}{2} f\left(\frac{\mathbf{j}}{n}\right)\right\} - \left\{\frac{n}{2} f\left(\frac{\mathbf{i}}{n}\right)\right\} \in (-1, 1).$$

Thus,

$$|f^n(\mathbf{i}) - f^n(\mathbf{j})| < \frac{1}{2}|\mathbf{i} - \mathbf{j}| + 1.$$

If $|\mathbf{i} - \mathbf{j}| \geqslant 2$, then

$$|f^n(\mathbf{i}) - f^n(\mathbf{j})| < \frac{1}{2}|\mathbf{i} - \mathbf{j}| + 1 \leqslant |\mathbf{i} - \mathbf{j}|$$

and the claim follows. If $|\mathbf{i} - \mathbf{j}| < 2$, then

$$|f^n(\mathbf{i}) - f^n(\mathbf{j})| < \frac{1}{2}|\mathbf{i} - \mathbf{j}| + 1 < 2.$$

Since $|f^n(\mathbf{i}) - f^n(\mathbf{j})| \in \mathbb{N}$, it follows that

$$|f^n(\mathbf{i}) - f^n(\mathbf{j})| \leqslant 1 \leqslant |\mathbf{i} - \mathbf{j}|$$

provided that $\mathbf{i} \neq \mathbf{j}$. This proves the claim for $|\mathbf{i} - \mathbf{j}| < 2$.

Step 2. — Let $f : \mathbb{R}^d \to \mathbb{R}$ be a contraction. For every $n \geqslant 1$, set

$$A_{k,n} \overset{\text{def}}{=} \sum_{i_k \in \mathbb{Z}} i_k E_{A_k}\left(\left[\frac{i_k}{n}, \frac{i_k + 1}{n}\right)\right), \quad \mathbf{A}_n = \{A_{k,n}\}_{k=1}^d.$$

Fix $1 \leqslant k_0 \leqslant d$. Then

$$\xi_n(t, s) = (f^n)_{k_0}(\mathbf{i}, \mathbf{j}), \quad t_k \in \left[\frac{i_k}{n}, \frac{i_k + 1}{n}\right), s_k \in \left[\frac{j_k}{n}, \frac{j_k + 1}{n}\right), \quad \mathbf{i}, \mathbf{j} \in \mathbb{Z}^d.$$

It is immediate that (see e.g. Lemma 8 in [30] for a much stronger assertion)

$$T_{\xi_n}^{\mathbf{A},\mathbf{A}}(V) = T_{(f^n)_{k_0}}^{\mathbf{A}_n, \mathbf{A}_n}(V).$$

It follows from Theorem 4.1 that

$$\|T^{\mathbf{A},\mathbf{A}}_{\xi_n}(V)\|_{1,\infty} \leqslant c(d)\|V\|_1.$$

Note that $\xi_n \to \frac{1}{2}f_{k_0}$ everywhere. It follows from Lemma 5.1 that

$$T^{\mathbf{A},\mathbf{A}}_{\xi_n}(V) \to T^{\mathbf{A},\mathbf{A}}_{\frac{1}{2}f_{k_0}}(V), \quad V \in L_2(\mathcal{M})$$

in $L_2(\mathcal{M})$ (and, hence, in measure; see e.g [26]) as $n \to \infty$. Since the quasi-norm in $L_{1,\infty}(\mathcal{M})$ is a Fatou quasi-norm [23], it follows that

$$\|T^{\mathbf{A},\mathbf{A}}_{f_{k_0}}(V)\|_{1,\infty} \leqslant c(d)\|V\|_1, \quad V \in (L_1 \cap L_2)(\mathcal{M}). \qquad \square$$

COROLLARY 5.2. — *For every Lipschitz function $f : \mathbb{R}^d \to \mathbb{R}$ and for every collection $\mathbf{A} = \{A_k\}_{k=1}^d$ of bounded commuting self-adjoint operators, the operator $T^{\mathbf{A},\mathbf{A}}_{f_k}$ extends to a bounded operator from $L_p(\mathcal{M})$ to $L_p(\mathcal{M})$, $1 < p < \infty$.*

Proof. — By Theorem 1.2, $T^{\mathbf{A},\mathbf{A}}_{f_k}$ extends to a bounded operator from $L_1(\mathcal{M})$ to $L_{1,\infty}(\mathcal{M})$ for every $1 \leqslant k \leqslant d$. Since also $T^{\mathbf{A},\mathbf{A}}_{f_k} : L_2(\mathcal{M}) \to L_2(\mathcal{M})$, it follows from real interpolation that $T^{\mathbf{A},\mathbf{A}}_{f_k} : L_p(\mathcal{M}) \to L_p(\mathcal{M})$, $1 < p < 2$. Thus, $(T^{\mathbf{A},\mathbf{A}}_{f_k})^* : L_{\frac{p}{p-1}}(\mathcal{M}) \to L_{\frac{p}{p-1}}(\mathcal{M})$, $1 < p < 2$. Since $f_k(s,t) = f_k(t,s)$, $s,t \in \mathbb{R}^d$, it follows that $(T^{\mathbf{A},\mathbf{A}}_{f_k})^* = T^{\mathbf{A},\mathbf{A}}_{f_k}$. In particular, $T^{\mathbf{A},\mathbf{A}}_{f_k} : L_{\frac{p}{p-1}}(\mathcal{M}) \to L_{\frac{p}{p-1}}(\mathcal{M})$, $1 < p < 2$. This concludes the proof. $\square$

LEMMA 5.3. — *If $A_k, B \in B(H)$, $1 \leqslant k \leqslant d$, are self-adjoint operators such that $[A_k, B] \in L_2(H)$, $1 \leqslant k \leqslant d$, then, for every Lipschitz function f, we have*

$$\sum_{k=1}^d T^{\mathbf{A},\mathbf{A}}_{f_k}([A_k, B]) = [f(\mathbf{A}), B].$$

Here f_k is given by (2.7).

Proof. — By definition of double operator integral given in Subsection 2.4, we have for any bounded Borel function on $\mathbb{R}^{2d}$,

$$(5.2) \qquad T^{\mathbf{A},\mathbf{A}}_{\xi_1} T^{\mathbf{A},\mathbf{A}}_{\xi_2} = T^{\mathbf{A},\mathbf{A}}_{\xi_1\xi_2}.$$

Let $\xi_{1,k} = f_k$ and let $\xi_{2,k}(\lambda, \mu) = \lambda_k - \mu_k$ when $|\lambda|_2, |\mu|_2 \leqslant \sup_{1 \leqslant k \leqslant d} \|A_k\|_\infty$, $\xi_{2,k}(\lambda, \mu) = 0$ when $|\lambda|_2 > \sup_{1 \leqslant k \leqslant d} \|A_k\|_\infty$ or $|\mu|_2 > \sup_{1 \leqslant k \leqslant d} \|A_k\|_\infty$. It is immediate that

$$\left(\sum_{k=1}^d \xi_{1,k}\xi_{2,k}\right)(\lambda, \mu) = f(\lambda) - f(\mu),$$

$$\lambda, \mu \in \mathbb{R}^d, \text{ s.t. } |\lambda|_2, |\mu|_2 \leqslant \sup_{1 \leqslant k \leqslant d} \|A_k\|_\infty.$$

If p is a finite rank projection, then $pB \in L_2(H)$ and

$$T^{\mathbf{A},\mathbf{A}}_{\sum_{k=1}^d \xi_{1,k}\xi_{2,k}}(pB) = f(\mathbf{A})pB - pBf(\mathbf{A}), \quad T^{\mathbf{A},\mathbf{A}}_{\xi_{2,k}}(pB) = A_k pB - pBA_k,$$

Applying (5.2) to the operator $pB \in L_2(H)$, we obtain

$$(5.3) \qquad \sum_{k=1}^d T^{\mathbf{A},\mathbf{A}}_{f_k}(A_k pB - pBA_k) = f(\mathbf{A})pB - pBf(\mathbf{A}).$$

By Theorem 4.2 in [36], there exists a sequence p_l of finite rank projections such that $p_l \to 1$ strongly and such that, for every $1 \leqslant k \leqslant d$, $[A_k, p_l] \to 0$ as $l \to \infty$ in $L_d(H)$ for $d > 1$ and in $L_2(H)$ if $d = 1$. In particular,

$$A_k p_l B - p_l B A_k = p_l[A_k, B] + [A_k, p_l]B \to [A_k, B], \quad l \to \infty,$$

in $L_d(H)$.

By the preceding paragraph and Corollary 5.2, we have

$$(5.4) \qquad T^{\mathbf{A},\mathbf{A}}_{f_k}(A_k p_l B - p_l B A_k) \to T^{\mathbf{A},\mathbf{A}}_{f_k}([A_k, B]), \quad l \to \infty,$$

in $L_d(H)$. On the other hand,

$$(5.5) \qquad f(\mathbf{A})p_l B - p_l B f(\mathbf{A}) \to f(\mathbf{A})B - Bf(\mathbf{A}), \quad l \to \infty,$$

in the strong operator topology. Substituting (5.4) and (5.5) into (5.3), we conclude the proof. □

Proof of Theorem 1.1. — By assumption, $[A_k, B] \in L_1(H) \subset L_2(H)$. The first assertion follows by combining Lemma 5.3 and Theorem 1.2. Applying the first assertion to the operators

$$A_k = \begin{pmatrix} X_k & 0 \\ 0 & Y_k \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

we obtain the second assertion. □

COROLLARY 5.4. — *For every Lipschitz function $f : \mathbb{C} \to \mathbb{R}$ and for every normal operator $A \in B(H)$ and every $B \in B(H)$ such that $[A, B] \in L_1(H)$, we have*

$$\|[f(A), B]\|_{1,\infty} \leqslant c(d)\|\nabla(f)\|_\infty\|[A, B]\|_1.$$

For every Lipschitz function $f : \mathbb{C} \to \mathbb{R}$ and for every pair $X, Y \in B(H)$ of normal operators such that $X - Y \in L_1(H)$, we have

$$\|f(X) - f(Y)\|_{1,\infty} \leqslant c(d)\|\nabla(f)\|_\infty\|X - Y\|_1.$$

Proof. — An operator A is normal if and only if it can be written as $A = A_1 + iA_2$ with A_1 and A_2 commuting self-adjoint operators. Identifying $\mathbb{C} \simeq \mathbb{R}^2$ we may see f as a 2 real variable Lipschitz function, say $\widetilde{f}$, and this identification is compatible with spectral calculus, i.e. $f(A) = \widetilde{f}(A_1, A_2)$. Then the corollary is a direct consequence of the statements in Theorem 1.1. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Appendix. Fejér's lemma

In our proof we use a von Neumann-valued Fejér's lemma. As we could not find a reference to this type of vector valued case we prove it here for convenience of the reader.

We let $e_l, l \in \mathbb{Z}$ denote the standard trigonometric functions on the torus. Let $\mathcal{E}$ be the conditional expectation $\mathcal{M} \otimes L^\infty(\mathbb{T}^{d+1}) \to \mathcal{M} \otimes 1$. For $k \in \mathbb{Z}_+^{d+1}$, let

$$S_k(x) = \sum_{\substack{l \in \mathbb{Z}^{d+1} \\ -k \leqslant l \leqslant k}} \mathcal{E}(x(1 \otimes e_l)^*)(1 \otimes e_l).$$

For $n \in \mathbb{Z}_+$, we set

$$A_n(x) = (n+1)^{-d-1} \sum_{\substack{k \in \mathbb{Z}_+^{d+1} \\ k \leqslant (n,\dots,n)}} S_k(x).$$

Here, the order on $\mathbb{Z}_+^{d+1}$ is defined by $m \leqslant n$ if $m_j \leqslant n_j$ for all $1 \leqslant j \leqslant d+1$.

Remark A.5. — It follows directly that for $x \in L_2(\mathcal{M} \otimes \mathbb{T}^{d+1})$ we have $\|A_n x - x\|_2 \to 0$ as $n \to \infty$.

The assertion below is known as Fejér's lemma.

LEMMA A.6. — *We have $\|A_n(x) - x\|_1 \to 0$ for all $x \in L_1(\mathcal{M} \otimes \mathbb{T}^{d+1})$ as $n \to \infty$.*

Proof. — We split the proof in steps.

Step 1. — We claim that

$$\|A_n x\|_1 \leqslant \|x\|_1, \quad x \in L_1(\mathcal{M} \otimes \mathbb{T}^{d+1}), \quad n \geqslant 0.$$

To see this fact, we identify the space $L_1(\mathcal{M} \otimes \mathbb{T}^{d+1})$ with the space of vector-valued functions $L_1(\mathbb{T}^{d+1}, L_1(\mathcal{M}))$. We now write a pointwise equality

$$(A_n(x))(t) = \int_{\mathbb{T}^{d+1}} x(t+s)\Phi_n(s)\mathrm{d}s, \quad s \in \mathbb{T}^{d+1}.$$

Here, $\Phi_n : \mathbb{T}^{d+1} \to \mathbb{R}$ is the Fejér kernel possessing the following properties.

$$\Phi_n(s) \geqslant 0, \quad \int_{\mathbb{T}^{d+1}} \Phi_n(s)\mathrm{d}s = 1.$$

Thus,

$$\|A_n x\|_1 \leqslant \int_{\mathbb{T}^{d+1}} \|x(\,\cdot\,+s)\|_1 \Phi_n(s)\mathrm{d}s = \|x\|_1.$$

Step 2. — Fix $\epsilon > 0$ and choose a projection $p \in \mathcal{M}$ such that $\tau(p) < \infty$ and such that $\|x'\|_1 < \epsilon$, where

$$x' := x - (p \otimes 1)x(p \otimes 1).$$

Choose $y \in L_2(p\mathcal{M}p \otimes \mathbb{T}^{d+1})$ such that

$$\|y - (p \otimes 1)x(p \otimes 1)\|_1 < \epsilon.$$

In particular, we have that $\|y - x\|_1 < 2\epsilon$.

We clearly have $A_n y \to y$ in $L_2(p\mathcal{M}p \otimes \mathbb{T}^{d+1})$. Since $\tau(p) < \infty$, it follows that $A_n y \to y$ in $L_1(p\mathcal{M}p \otimes \mathbb{T}^{d+1})$. Thus, $A_n y \to y$ in $L_1(\mathcal{M} \otimes \mathbb{T}^{d+1})$. Choose N so large that $\|A_n y - y\|_1 < \epsilon$ for $n > N$. It follows from Step 1 that

$$\|A_n x - x\|_1 \leqslant \|A_n(x-y)\|_1 + \|A_n y - y\|_1 + \|x - y\|_1$$
$$\leqslant 2\|x - y\|_1 + \|A_n y - y\|_1 \leqslant 4\epsilon + \|A_n y - y\|_1 < 5\epsilon, \quad n > N.$$

Since $\epsilon > 0$ is arbitrarily small, the assertion follows. $\qquad\square$

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