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CONSTRUCTION OF HYPERBOLIC HORIKAWA SURFACES

by Yuchen LIU (*)

Abstract. — We construct a Brody hyperbolic Horikawa surface that is a double cover of $\mathbb{P}^2$ branched along a smooth curve of degree 10. We also construct Brody hyperbolic double covers of Hirzebruch surfaces with branch loci of the lowest possible bidegree.

Résumé. — Nous construisons une surface de Horikawa Brody-hyperbolique qui est un revêtement double de $\mathbb{P}^2$ ramifié le long d'une courbe lisse de degré 10. Nous construisons également des surfaces Brody-hyperboliques qui sont des revêtements doubles de surfaces de Hirzebruch, dont l'ensemble de ramification est de bidegré minimal.

1. Introduction

A complex algebraic variety X is said to be *Brody hyperbolic* if there are no non-constant holomorphic maps from $\mathbb{C}$ to X. Thanks to Brody Lemma [2], we know that a proper Brody hyperbolic variety is *Kobayashi hyperbolic*, i.e. its Kobayashi pseudometric is non-degenerate. In [14], Lang conjectured that a complex projective variety X is Brody hyperbolic if every subvariety of X is of general type. More generally, Green, Griffiths [7] and Lang [14] proposed the following conjecture:

Conjecture 1.1 (Green–Griffiths–Lang). — *If a complex projective variety X is of general type, then there exists a proper Zariski closed subset $Z \subsetneq X$ such that any non-constant holomorphic map $f : \mathbb{C} \to X$ will satisfy $f(\mathbb{C}) \subset Z$.*

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It is easy to see that Lang's conjecture follows from the Green–Griffiths–Lang conjecture by a Noetherian induction argument. Even in the case of surfaces, these conjectures are still open. Based on works of Bogomolov [1] and Lu–Yau [16], McQuillan [17] showed that Conjecture 1.1 is true for minimal surfaces of general type with $c_1^2 > c_2$. Demailly and El Goul [3] proved Conjecture 1.1 for some surfaces with $13c_1^2 > 9c_2$. In principle, minimal surfaces of general type with minimal $s_2 = c_1^2 - c_2$ should be the most difficult case for these conjectures. For example, a very general quintic surface in $\mathbb{P}^3$ ($c_1^2 = 5$, $c_2 = 55$) does not contain any rational or elliptic curve by a result of Xu [20], but we do not have a single example of quintic surface that is Brody hyperbolic.

Recall that the Chern numbers of minimal surfaces of general type satisfy the Noether inequality $c_2 \leqslant 5c_1^2 + 36$. In the extreme case, a surface that reaches the equality $c_2 = 5c_1^2 + 36$ if c_1^2 is even and $c_2 = 5c_1^2 + 30$ otherwise is called a *Horikawa surface*. A Horikawa surface with even c_1^2 is classified to be either a double cover of $\mathbb{P}^2$ or of a Hirzebruch surface (see [8]). For instance, a double cover of $\mathbb{P}^2$ branched along a smooth curve of degree 10 is Horikawa. Using orbifold techniques, Roulleau and Rousseau [18] showed that a very general member of this class of Horikawa surfaces is algebraic hyperbolic (in particular it has no rational or elliptic curve). Hence a very general member of this class of surfaces is expected to be Brody hyperbolic according to Conjecture 1.1.

Our first main result shows that there exists a Horikawa surface in this class that is Brody hyperbolic. This gives an analytic generalization of Roulleau–Rousseau's result (in particular implies [18, Theorem 3.2]) and also provides evidence supporting Conjecture 1.1.

Theorem 1.2. — *Let d be an even integer. Then there exists a smooth plane curve D of degree d such that the double cover of $\mathbb{P}^2$ branched along D is Brody hyperbolic if and only if $d \geqslant 10$.*

We remark that here that some Brody hyperbolic double covers of $\mathbb{P}^2$ have been constructed in [15, Theorem 5] with branch loci of minimal degree 30.

For an integer $N \geqslant 0$, let $\mathbb{F}_N$ be the N-th Hirzebruch surface. The surface $\mathbb{F}_N$ has a natural fibration $\mathbb{F}_N \to \mathbb{P}^1$. Denote by F a fiber, and by T a section of the fibration such that $(T^2) = N$. Any divisor D on $\mathbb{F}_N$ is linearly equivalent to $aF + bT$ for integers a and b, and we say that D is of bidegree (a, b).

In [18], Roulleau and Rousseau also showed that a very general Horikawa surface that is a double cover of $\mathbb{F}_N$ branched along a curve of bidegree $(6, 6)$

does not contain a rational curve. In general it will contain an elliptic curve, so it cannot be Brody hyperbolic. In the next theorem, we construct smooth curves of the lowest possible bidegrees in $\mathbb{F}_N$ along which the double covers of $\mathbb{F}_N$ are Brody hyperbolic.

Theorem 1.3. — *Let a, b be even integers. Then there exists a smooth curve $D \subset \mathbb{F}_N$ in the linear system $|aF + bT|$ such that the double cover of $\mathbb{F}_N$ branched along D is Brody hyperbolic if and only if one of the following is true:*

- $N = 0$ *and* $a, b \geqslant 8$*;*
- $N \geqslant 1$*,* $a \geqslant 6$ *and* $b \geqslant 8$*.*

The "only if" parts of Theorems 1.2 and 1.3 are somewhat easy which follow by showing the existence of a rational or elliptic curve on the double cover when the branch locus has a smaller (bi)degree.

Our strategy to prove the "if" parts of Theorems 1.2 and 1.3 is by using a degeneration process consisting of three steps. Denote by X the base surface $\mathbb{P}^2$ or $\mathbb{F}_N$. In step 1, we degenerate the branch locus D to a non-reduced double curve $2C$ where C is smooth. As a result, the double cover degenerates to a union of two copies of X glued along C. Using stability of intersections of entire curves, it suffices to show that both $X \setminus (C \setminus D)$ and C are Brody hyperbolic. In step 2, we degenerate C into a line arrangement $\cup_i C_i$. By a variant of Zaidenberg's method [21], it suffices to show that $X \setminus ((\cup_i C_i) \setminus D)$ is Brody hyperbolic. By classical results, we know that for $X = \mathbb{P}^2$ or $\mathbb{F}_N$, $X \setminus (\cup_i C_i)$ is Brody hyperbolic. In step 3, we apply Zaidenberg–Duval's method [5, 6, 19, 21] of degenerating D into line arrangements in order to deduce hyperbolicity of $X \setminus ((\cup_i C_i) \setminus D)$ from hyperbolicity of $X \setminus (\cup_i C_i)$ which is known by classical results.

Historical remark. — Note that Duval [5] constructed a Brody hyperbolic sextic surface in $\mathbb{P}^3$ by nicely adopting Zaidenberg's method [21], together with the hyperbolic non-percolation introduced in [19]. In this paper, we follow precisely Duval's approach [5] which was further developed in [9]. Thus we will use the term *Zaidenberg–Duval's method* for this approach in our presentation.

The paper is organized as follows. In Section 2, we recall Zaidenberg–Duval's method [5, 6, 19, 21] in constructing a smooth curve D satisfying the hyperbolicity of $X \setminus ((\cup_i C_i) \setminus D)$. We recall results in [9, Section 4] in the surface case in Lemma 2.2, and we apply this lemma to $\mathbb{P}^2$ and $\mathbb{F}_N$ in Corollaries 2.3 and 2.4. In order to deform $\cup_i C_i$ into a smooth curve C preserving the hyperbolicity of $X \setminus (C \setminus D)$, we apply Zaidenberg's method [21] in Section 4 (see Lemma 4.2). Starting with a log smooth projective surface pair (X, D) and a set of rational curves $\{C_i\}$ with $X \setminus ((\cup_i C_i) \setminus D)$ being Brody hyperbolic, we introduce the concept of *admissible deformation* (see Definition 4.1) in order to preserve the hyperbolicity of $X \setminus (C \setminus D)$ under deformation. Using the technique of smoothing of rational trees in the deformation process (e.g. [13, II.7]), we are able to translate an admissible deformation of rational curves into an admissible contraction of their dual graphs (see Lemma 4.4). In Section 3, we study dual graphs that can be admissibly contracted into singletons. Using these results, we construct a smoothing C of $\cup_i C_i$ preserving the hyperbolicity of $X \setminus (C \setminus \Delta)$ under certain assumptions on the dual graph of $\cup_i C_i$ (see Lemma 4.8). Applying this lemma to $X = \mathbb{P}^2$ or $\mathbb{F}_N$ gives smooth curves C and D with certain (bi)degrees such that $X \setminus (C \setminus D)$ is Brody hyperbolic. In Section 5, we prove Theorems 1.2 and 1.3. As an application of Theorem 1.2, we give new examples of Brody hyperbolic surfaces in $\mathbb{P}^3$ of minimal degree 10 that are cyclic covers of $\mathbb{P}^2$ under linear projections (see Theorem 5.2). This also improves [15, Theorem 25]. We mention that a Brody hyperbolic Horikawa surface of even c_1^2 has to be a double cover of $\mathbb{P}^2$ branched along a degree 10 curve (see Remark 5.3).

Notation

Throughout this paper, we work over the complex numbers $\mathbb{C}$. For a subset U of a projective variety X, we say that U is *Brody hyperbolic* if any non-constant holomorphic map $\phi : \mathbb{C} \to X$ satisfies $\phi(\mathbb{C}) \not\subset U$. A divisor D on a smooth surface X is *normal crossing* if D is reduced and has only nodal singularities. Moreover, a normal crossing divisor D is said to be *simple normal crossing* if all irreducible components of D are smooth. We say that a surface pair (X, D) is *log smooth* if X is a smooth surface and D is a simple normal crossing divisor on X. A reduced projective curve is *stable* (in the sense of Deligne–Mumford) if it has only nodal singularities and its dualizing sheaf is ample.

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2. Zaidenberg–Duval's method

We first recall the following known facts from complex analysis whose proof is a simple application of the classical Hurwitz Theorem. (See also [12, 3.6.11], [6, Stability of intersections] or [9, Section 3.1].)

LEMMA 2.1 (Stability of intersections). — *Let X be a normal proper complex analytic space. Let S be an effective Weil divisor in X, i.e. S is a sum of closed analytic subvarieties of codimension 1. Suppose that a sequence of entire curves (ϕ_n) in X converges to an entire curve ϕ. If $\phi(\mathbb{C}) \not\subset \mathrm{Supp}(S)$, then*

$$\phi(\mathbb{C}) \cap S^\circ \subset \lim_{n\to\infty} \phi_n(\mathbb{C}) \cap \mathrm{Supp}(S),$$

where $S^\circ := \{x \in \mathrm{Supp}(S) \mid S \text{ is } \mathbb{Q}\text{-Cartier in a neighborhood of } x\}$.

The following lemma was proved in [9, Section 4] (see also [6, Lemma]).

LEMMA 2.2. — *Let X be a smooth projective surface. Let $\{C_i\}_{i=1}^m$ be a set of irreducible curves on X such that $(X, \sum_{i=1}^m C_i)$ is log smooth. Let L be a globally generated line bundle on X. Assume the following holds:*

1. *$X \setminus (\cup_{i=1}^m C_i)$ is Brody hyperbolic;*
2. *For any i, $\cup_{j\neq i} C_j$ is a stable curve;*
3. *For any i, there exists an effective Cartier divisor $H_i \in |L|$ such that $\mathrm{Supp}(H_i) = \cup_{j\neq i} C_j$.*

Then there exists a smooth curve $S \in |L|$ such that $(X, S + \sum_{i=1}^m C_i)$ is log smooth and $X \setminus \big((\cup_{i=1}^m C_i) \setminus S\big)$ is Brody hyperbolic.

Proof. — See [9, Section 4]. □

The following corollary was proved in [9, Section 4] using Lemma 2.2.

COROLLARY 2.3 ([9, Section 4]). — *Let $\{C_i\}_{i=1}^m$ be a set of lines in general position in $\mathbb{P}^2$ with $m \geqslant 5$. Let $d \geqslant 4$ be an integer. Then there exists a smooth plane curve S of degree d such that $(\mathbb{P}^2, S + \sum_{i=1}^m C_i)$ is log smooth and $\mathbb{P}^2 \setminus \big((\cup_{i=1}^m C_i) \setminus S\big)$ is Brody hyperbolic.*

COROLLARY 2.4. — *Let $\{C_i\}_{i=1}^{a+b}$ be a set of curves in $\mathbb{F}_N$. Assume that C_i is a general curve in $|F|$ for any $i \leqslant a$; C_j is a general curve in $|T|$ for any $j > a$. Then there exists a smooth curve $S \in |cF + dT|$ in $\mathbb{F}_N$ such that $(\mathbb{F}_N, S + \sum_{i=1}^{a+b} C_i)$ is log smooth and $\mathbb{F}_N \setminus \big((\cup_{i=1}^{a+b} C_i) \setminus S\big)$ is Brody hyperbolic if one of the following is true:*

- *$N = 0$ and $a, b, c, d \geqslant 4$;*
- *$N \geqslant 1$, $a, c \geqslant 3$ and $b, d \geqslant 4$.*

Proof. — Firstly, let us consider special cases where a, b achieve their minima, i.e. $N = 0$, $a = b = 4$ or $N \geqslant 1$, $a = 3$, $b = 4$. Since both linear systems $|F|$ and $|T|$ are base point free, for a general choice of $\{C_i\}_{i=1}^{a+b}$ the pair $(\mathbb{F}_N, \sum_{i=1}^{a+b} C_i)$ is log smooth. Let $L := \mathcal{O}_{\mathbb{F}_N}(cF + dT)$ be a line bundle on $\mathbb{F}_N$. Then we only need to show that the assumptions (1), (2) and (3) of Lemma 2.2 are fulfilled for $(\mathbb{F}_N, \sum_{i=1}^{a+b} C_i)$ and L.

If $N = 0$ and $a = b = 4$, then $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ and $\{C_i\}_{i=1}^{8}$ consists of 4 vertical lines and 4 horizontal lines in general position. It is clear that $\mathbb{F}_0 \setminus \cup_{i=1}^{8} C_i \cong (\mathbb{P}^1 \setminus \{4 \text{ points}\}) \times (\mathbb{P}^1 \setminus \{4 \text{ points}\})$ is Brody hyperbolic, so (1) is satisfied. For (2), each C_j intersects four C_k's with $k \neq j$. So for any $i \neq j$, C_j intersects with at least three C_k's with $k \notin \{i, j\}$. Since $C_i \cap C_j \cap C_k = \emptyset$, $\cup_{j \neq i} C_j$ is stable, hence (2) is satisfied. Since $c, d \geqslant 4$, $C_i \sim F$ for $1 \leqslant i \leqslant 4$ and $C_j \sim T$ for $5 \leqslant j \leqslant 8$, it is easy to see that (3) is also satisfied.

If $N \geqslant 1$, $a = 3$ and $b = 4$, then the natural fibration $\pi : \mathbb{F}_N \to \mathbb{P}^1$ maps C_1, C_2, C_3 to three distinct points in $\mathbb{P}^1$. It is clear that $(F \cdot C_i) = (F \cdot T) = 1$ for $i = 4, \ldots, 7$. Hence for a general choice of $\{C_i\}_{i=4}^{7}$, the set $F \cap (\cup_{i=4}^{7} C_i)$ has at least three points for any fiber F of π. Since $\mathbb{P}^1 \setminus \{3 \text{ points}\}$ is Brody hyperbolic, the fiber and the base of $\pi : \mathbb{F}_N \setminus \cup_{i=1}^{7} C_i \to \mathbb{P}^1 \setminus \{\pi(C_1), \pi(C_2), \pi(C_3)\}$ are Brody hyperbolic. Hence (1) is satisfied. For (2), each C_i with $1 \leqslant i \leqslant 3$ intersects each C_k with $4 \leqslant k \leqslant 7$. Since $(T^2) = N \geqslant 1$, each C_i with $4 \leqslant i \leqslant 7$ intersects each C_k with $k \neq i$. As a result, each C_i intersects with at least four C_k's with $k \neq i$. So (2) is satisfied by the same reason as in the last paragraph. Since $c \geqslant 3$, $d \geqslant 4$, $C_i \sim F$ for $1 \leqslant i \leqslant 3$, and $C_i \sim T$ for $4 \leqslant i \leqslant 7$, it is easy to see that (3) is also satisfied.

Up to now we have shown the corollary for cases where a, b achieve their minima. More precisely, under the assumptions of N, a, b, c, d, for general choices of $\{C_i\}_{i=1}^{a_{\min}}$ and $\{C_j\}_{j=a+1}^{a+b_{\min}}$ there exists a smooth curve $S \in |cF + dT|$ in $\mathbb{F}_N$, such that $(\mathbb{F}_N, S + \sum_{i=1}^{a_{\min}} C_i + \sum_{j=a+1}^{a+b_{\min}} C_j)$ is log smooth and $\mathbb{F}_N \setminus \big(((\cup_{i=1}^{a_{\min}} C_i) \cup (\cup_{j=a+1}^{a+b_{\min}} C_j)) \setminus S\big)$ is Brody hyperbolic. If one of a, b is strictly bigger than its minimum, then

$$\mathbb{F}_N \setminus \big((\cup_{i=1}^{a+b} C_i) \setminus S\big) \subset \mathbb{F}_N \setminus \big(((\cup_{i=1}^{a_{\min}} C_i) \cup (\cup_{j=a+1}^{a+b_{\min}} C_j)) \setminus S\big)$$

where the latter set is Brody hyperbolic. Hence $\mathbb{F}_N \setminus \big((\cup_{i=1}^{a+b} C_i) \setminus S\big)$ is Brody hyperbolic. Besides, since $(\mathbb{F}_N, S + \sum_{i=1}^{a_{\min}} C_i + \sum_{j=a+1}^{a+b_{\min}} C_j)$ is log smooth, for general choices of $\{C_i\}_{i=a_{\min}+1}^{a}$ and $\{C_j\}_{j=a+b_{\min}+1}^{a+b}$ we also have that $(X, S + \sum_{i=1}^{a+b} C_i)$ is log smooth. This finishes the proof. $\square$

3. Admissible contractions of multigraphs

Definition 3.1.

(1) *A* vertex-w eighted multigraph G *is an ordered quadruple* (V, E, r, wt) *such that*
   - V *is a finite set of vertices;*
   - E *is a finite set of edges;*
   - $r : E \to \{\{v, w\} : v, w \in V, v \neq w\}$ *assigns each edge an unordered pair of endpoint vertices;*
   - $\mathrm{wt} : V \to \mathbb{Z}$ *assigns to each vertex an integer as its weight.*

(2) *For a vertex* $v \in V$*, we define the* degree *(respectively* reduced degree*) of* v *to be its number of incident edges (respectively adjacent vertices). More precisely,*

$$\deg(v) := \#\{e \in E : v \in e\},\ \mathrm{rdeg}(v) := \#\{w \in V : \{v, w\} \in r(E)\}.$$

(3) *Let* G_1, G_2 *be two vertex-weighted multigraphs. We say that* G_1 *is a* submultigraph *of* G_2 *if there exists injective maps* $\phi : V_1 \to V_2$ *and* $\psi : E_1 \to E_2$*, such that* $r_2 \circ \psi = \phi \circ r_1$ *and* $\mathrm{wt}_1 \leqslant \mathrm{wt}_2 \circ \phi$*. If, moreover,* ϕ *is bijective, then we say that* G_1 *is a* spanning submultigraph *of* G_2*.*

(4) *A vertex-weighted multigraph is* completely multipartite *if there does not exist a triple of vertices* $\{v_1, v_2, v_3\}$ *such that both* $\{v_1, v_2\}$ *and* $\{v_1, v_3\}$ *are non-adjacent, but* $\{v_2, v_3\}$ *is adjacent.*

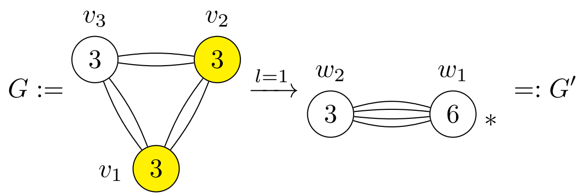
Definition 3.2.

(1) *Let* G, G' *be two vertex-weighted multigraphs. We say that* G' *is a* contraction *of* G with respect to *a pair of adjacent vertices* $\{v, w\}$ *in* G *if there exist maps* $\phi : V \to V'$ *and* $\psi : E \setminus r^{-1}(\{v, w\}) \to E'$ *such that*
   - $\phi(v) = \phi(w)$*, and* ϕ *induces a bijection between* $V \setminus \{v, w\}$ *and* $V' \setminus \{\phi(v)\}$*;*
   - ψ *is bijective, and* $r' \circ \psi = \phi \circ r$ *as maps from* $E \setminus r^{-1}(\{v, w\})$*;*
   - $\mathrm{wt}'(\phi(v)) = \mathrm{wt}(v) + \mathrm{wt}(w)$*, and* $\mathrm{wt}' \circ \phi = \mathrm{wt}$ *as maps from* $V \setminus \{v, w\}$ *to* $\mathbb{Z}$*.*

(2) *A contraction* G' *of* G *with respect to* $\{v, w\}$ *is said to be* admissible *if there exists a non-negative integer* $l < \#r^{-1}(v, w)$ *such that the following conditions hold:*
   - *For each vertex* x *other than* v *and* w*,* $\deg(x) \geqslant 3$*;*
   - $\mathrm{wt}(v) \geqslant l + 1$ *and* $\mathrm{wt}(w) \geqslant l + 2$*;*
   - $\deg(v) - \#r^{-1}(\{v, w\}) + l \geqslant 3$ *and* $\deg(w) - \#r^{-1}(\{v, w\}) + l \geqslant 3$*.*

(3) *A vertex-weighted multigraph G is said to be* admissibly contractible *if there exists a sequence of vertex-weighted multigraphs $(G_i)_{i=0}^k$ such that $G_0 = G$, G_k is a singleton, and G_i is an admissible contraction of G_{i-1} for each $1 \leqslant i \leqslant k$.*

Example 3.3. — We give an illustration of an admissible contraction of vertex-weighted multigraphs.



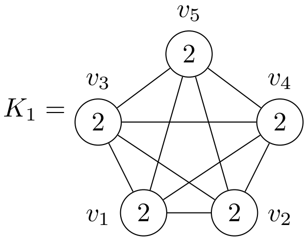
Here $V_G = \{v_1, v_2, v_3\}$, $V_{G'} = \{w_1, w_2\}$, $\mathrm{wt}_G(v_i) = 3$ for any $1 \leqslant i \leqslant 3$, $\mathrm{wt}_{G'}(w_1) = 6$ and $\mathrm{wt}_{G'}(w_2) = 3$. Each vertex is represented as a circle in the picture. The name of each vertex is marked outside the circle, and its weight is marked inside the circle. Each edge connecting two vertices is represented as an arc connecting two circles.

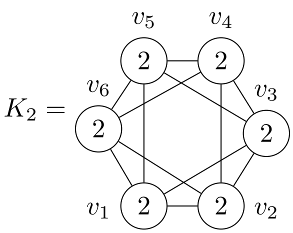
In the illustration above, we see that H is a contraction of G with respect to $\{v_1, v_2\}$, where ϕ is given by $\phi(v_1) = \phi(v_2) = w_1$ and $\phi(v_3) = w_2$. Each contraction is represented as an arrow. The two merging vertices of G are represented as yellow filled circles, and we mark $*$ outside circles representing their images under ϕ. If a contraction is admissible, we mark the corresponding value of l above the arrow. It is easy to verify that in the picture above, G' is an admissible contraction of G with respect to $\{v_1, v_2\}$.

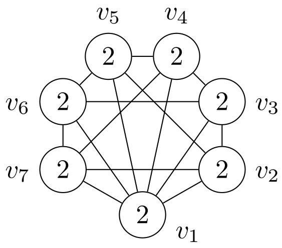
The following lemma follows easily from the definitions.

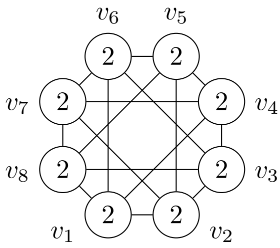
Lemma 3.4. — *Let G be a vertex-weighted multigraph. Let H be a spanning submultigraph of G. If H is admissibly contractible, then so is G.*

Proposition 3.5. — *The following vertex-weighted multigraphs K_1, K_2, K_3 and K_4 are all admissibly contractible:*

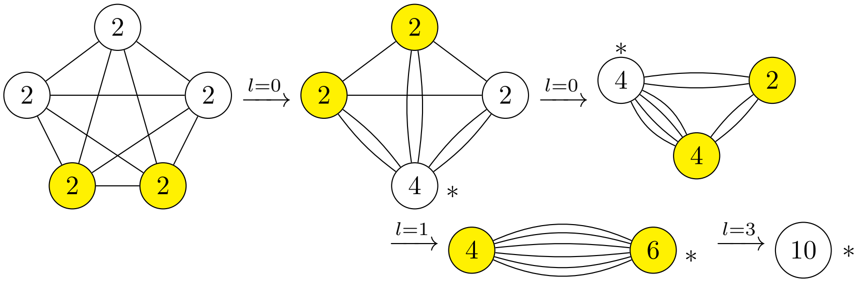




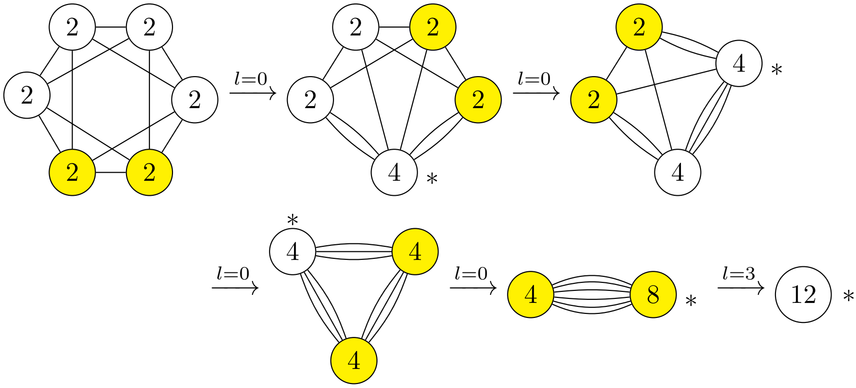
$K_3 =$ 

$K_4 =$ 

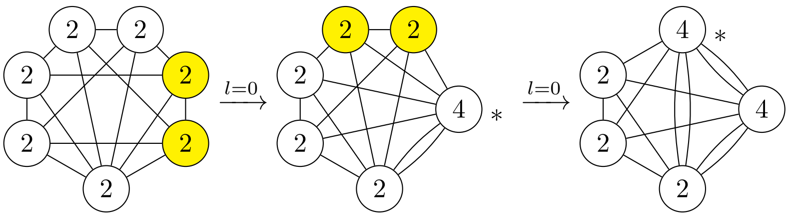
Proof. — For simplicity, we will omit the name of vertices in all pictures. A successive admissible contraction of K_1 is illustrated as below:



A successive admissible contraction of K_2 is illustrated as below:

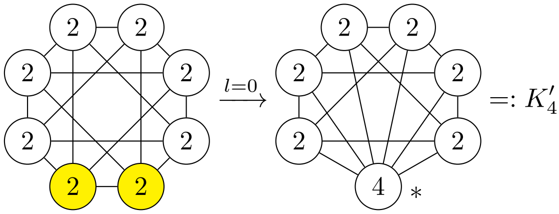


A successive admissible contraction of K_3 is illustrated as below:

 $=: K_3'$

It is clear that K_1 is a spanning submultigraph of K_3'. Since K_1 is admissibly contractible, so is K_3'. Hence K_3 is also admissibly contractible.

An admissible contraction of K_4 is illustrated as below:



It is clear that K_3 is a spanning submultigraph of K_4'. Since K_3 is admissibly contractible, so is K_4'. Hence K_4 is also admissibly contractible. $\square$

Lemma 3.6. — *Let G be a vertex-weighted multigraph. Let H be a submultigraph of G. Assume the following conditions:*

(1) *G is completely multipartite;*
(2) $\mathrm{wt}_G \geqslant 2$*;*
(3) *If $\{v_1, \ldots, v_s\} \subset V_H$ is a set of mutually non-adjacent vertices of H, then $s \leqslant \#V_H - 4$.*

Then there exists a successive admissible contraction G' of G such that H is a spanning submultigraph of G'.

Proof. — We do induction on $q := \#(V_G \setminus V_H)$. If $q = 0$, then the lemma is proved by taking $G' := G$. Assume that the lemma is proved for $q-1$. Let $w \in V_G$ be an arbitrary vertex of G. Let $\{v_1, \ldots, v_{s(w)}\}$ be the set of all vertices in V_H that are not adjacent to w in G. Since G is completely multipartite, $\{v_1, \ldots, v_{s(w)}\}$ is a set of mutually non-adjacent vertices of G (hence of H). By assumption, we have $s(w) \leqslant \#V_H - 4$. This implies that $\mathrm{rdeg}_G(w) \geqslant 4$ for any vertex w of G. Let us pick a vertex $w \in V_G \setminus V_H$, then w is adjacent to a vertex $v \in V_H$. Let G_1 be the contraction of G with respect to $\{v, w\}$. Since $\mathrm{wt}_G \geqslant 2$ and each vertex of G have reduced degree $\geqslant 4$, G_1 is an admissble contraction of G when $l = 0$. It is clear that H is a also submultigraph of G_1 with $q - 1 = \#(V_{G_1} \setminus V_H)$, $\mathrm{wt}_{G_1} \geqslant 2$, and G_1 is also a completely multipartite. By the inductive hypothesis, there exists a successive admissible contraction G_1' of G_1 such that H is a spanning submultigraph of G_1'. The proof is finished by taking $G' := G_1'$. $\square$

Remark 3.7. — It is easy to verify that $H = K_i$ satisfies assumption (3) of Lemma 3.6 for each $1 \leqslant i \leqslant 4$.

Lemma 3.8. — *Let G be a completely multipartite vertex-weighted multigraph. Assume that for any vertex v of G we have $\mathrm{rdeg}(v) \geqslant 4$ and $\mathrm{wt}(v) \geqslant 2$. Then G is admissibly contractible.*

Proof. — Since G is completely multipartite, there exists a partition of vertices $V = \cup_{i=1}^{k} V_k$ such that two vertices are non-adjacent if and only if they belong to the same V_i. Denote $a_i := \#V_i$. For simplicity we may assume that $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_k$. Then $\mathrm{rdeg}(v) \geqslant 4$ implies $\sum_{i=1}^{k-1} a_i \geqslant 4$. In particular, $k \geqslant 2$.

We divide the proof into five cases based on values of k and a_1. We will use Lemma 3.6 in all cases. Since G satisfies assumptions (1) and (2) of Lemma 3.6, we only need to verify assumption (3).

Case 1. $k \geqslant 5$. — Let us pick $v_i \in V_i$ for $1 \leqslant i \leqslant 5$. Let H be the submultigraph of G generated by $\{v_1, \ldots, v_5\}$. Since $\{v_1, \ldots, v_5\}$ are mutually adjacent in G, K_1 is a spanning submultigraph of H. Hence H satisfies condition (3). By Lemma 3.6, there exists a successive admissible contraction G' of G such that H (hence K_1) is a spanning submultgraph of G'. By Proposition 3.5, K_1 is admissibly contractible, hence G is admissibly contractible.

Case 2. $k = 4$. — Since $\sum_{i=1}^{3} a_i \geqslant 4$, we have that $a_1, a_2 \geqslant 1$ and $a_3, a_4 \geqslant 2$. Let us pick $v_1 \in V_1$, $v_4 \in V_2$, $v_2, v_5 \in V_3$ and $v_3, v_6 \in V_4$. Let H be the submultigraph of G generated by $\{v_1, \ldots, v_6\}$. It is easy to see that K_2 is a spanning submultigraph of H, hence H satisfies condition (3). By Lemma 3.6, there exists a successive admissible contraction G' of G such that H (hence K_2) is a spanning submultgraph of G'. By Proposition 3.5, K_2 is admissibly contractible, hence G is admissibly contractible.

Case 3. $k = 3$ and $a_1 \geqslant 2$. — We know that $a_2, a_3 \geqslant a_1 \geqslant 2$. Let us pick $v_1, v_4 \in V_1$, $v_2, v_5 \in V_2$ and $v_3, v_6 \in V_3$. Let H be the submultigraph of G generated by $\{v_1, \ldots, v_6\}$. It is easy to see that K_2 is a spanning submultigraph of H, hence H satisfies condition (3). By Lemma 3.6, there exists a successive admissible contraction G' of G such that H (hence K_2) is a spanning submultgraph of G'. By Proposition 3.5, K_2 is admissibly contractible, hence G is admissibly contractible.

Case 4. $k = 3$ and $a_1 = 1$. — Since $a_1 + a_2 \geqslant 4$, we have $a_2, a_3 \geqslant 3$. Let us pick $v_1 \in V_1$, $v_2, v_4, v_6 \in V_2$ and $v_3, v_5, v_7 \in V_3$. Let H be the submultigraph of G generated by $\{v_1, \ldots, v_7\}$. It is easy to see that K_3 is a spanning submultigraph of H, hence H satisfies condition (3). By Lemma 3.6, there exists a successive admissible contraction G' of G such that H (hence K_3) is a spanning submultgraph of G'. By Proposition 3.5, K_3 is admissibly contractible, hence G is admissibly contractible.

Case 5. $k = 2$. — Since $a_1 \geqslant 4$, we have $a_1, a_2 \geqslant 4$. Let us pick $v_1, v_3, v_5, v_7 \in V_1$ and $v_2, v_4, v_6, v_8 \in V_2$. Let H be the submultigraph of

G generated by $\{v_1, \ldots, v_8\}$. It is easy to see that K_4 is a spanning submultigraph of H, hence H satisfies condition (3). By Lemma 3.6, there exists a successive admissible contraction G' of G such that H (hence K_4) is a spanning submultgraph of G'. By Proposition 3.5, K_4 is admissibly contractible, hence G is admissibly contractible. □

4. Zaidenberg's method

Definition 4.1. — *Let (X, Δ) be a log smooth projective surface pair. Let C be a reduced curve in X. Let $\{\Gamma_t\}_{t\in\mathbb{D}}$ be a holomorphic flat family of reduced divisors on X. Denote by $\Gamma \subset X \times \mathbb{D}$ the development of $\{\Gamma_t\}_{t\in\mathbb{D}}$. We say that $\{\Gamma_t\}_{t\in\mathbb{D}}$ is an* admissible deformation *of C if $\Gamma_0 = C$ and the set $\Gamma_0^* := \{x \in \Gamma_0 \mid \Gamma \text{ is locally analytically irreducible at } (x, 0)\}$ is Brody hyperbolic. If, moreover, $\Delta + C$ is normal crossing, an admissible deformation $\{\Gamma_t\}_{t\in\mathbb{D}}$ of C is* nodal *if $\Delta+\Gamma_t$ is normal crossing for any $t \in \mathbb{D}$. Besides, we say that $\{\Gamma_t^{(j)}\}_{t\in\mathbb{D},1\leqslant j\leqslant k}$ is a* successive admissible deformation *of C if for each $1 \leqslant j \leqslant k$ there exists $t_j \in \mathbb{D} \setminus \{0\}$, such that $\{\Gamma_t^{(j)}\}_{t\in\mathbb{D}}$ is an admissible deformation of $\Gamma_{t_{j-1}}^{(j-1)}$ where $\Gamma_{t_0}^{(0)} := C$.*

The following lemma is a generalization of Zaidenberg's result [21, Lemma-Definition 3.2] to surface pairs.

Lemma 4.2. — *Let (X, Δ) be a log smooth projective surface pair. Let C be a reduced curve in X such that $\Delta + C$ is normal crossing. Let $\{\Gamma_t\}_{t\in\mathbb{D}}$ be an* admissible deformation *of C. If $X \setminus (C \setminus \Delta)$ is Brody hyperbolic, then $X \setminus (\Gamma_t \setminus \Delta)$ is also Brody hyperbolic for any $0 < |t| \ll 1$. (Note that $X \setminus (C \setminus \Delta)$ being Brody hyperbolic is the same as saying that $X \setminus C$ has* the property of hyperbolic non-percolation through *$C \cap \Delta$ according to* [19]*.)*

Proof. — The proof is similar to [21, Proof of Lemma-Definition 3.2]. □

Lemma 4.3. — *Let X be a smooth projective rational surface. Let C_1, C_2 be two intersecting rational nodal curves such that $C_1 + C_2$ is normal crossing. Assume that $(-K_X \cdot C_1) \geqslant l + 1$ and $(-K_X \cdot C_2) \geqslant l + 2$ for some non-negative integer $l < (C_1 \cdot C_2)$. Then for any subset $\{x_1, \ldots, x_l\} \subset C_1 \cap C_2$, there exists a holomorphic flat family of divisors $\{\Gamma_t\}$ in X such that $\Gamma_0 = C_1 + C_2$ and Γ_t is an irreducible rational nodal curve singular at x_i for any $t \neq 0$ and any $1 \leqslant i \leqslant l$.*

Proof. — Denote by $\sigma : \tilde{X} = \mathrm{Bl}_{x_1,\ldots,x_l} X \to X$ the blow up of X at $x_1,\ldots,x_l$. Let E be the reduced exceptional divisor of σ. Let $\tilde{C}_1$ and $\tilde{C}_2$ be strict transforms of C_1 and C_2 under σ. It is easy to see that $(-K_{\tilde{X}}\cdot\tilde{C}_1)\geqslant 1$, $(-K_{\tilde{X}}\cdot\tilde{C}_2)\geqslant 2$ and $(\tilde{C}_1\cdot\tilde{C}_2)>0$. It is clear that both $\tilde{C}_1$ and $\tilde{C}_2$ are irreducible rational nodal curves intersecting each other transversally. Denote by $f_i : \mathbb{P}^1 \to \tilde{X}$ the normalization of $\tilde{C}_i$. Since f_i is an immersion, we have an exact sequence $0 \to T_{\mathbb{P}^1} \to f_i^*T_{\tilde{X}} \to N_{\tilde{C}_i/\tilde{X}} \to 0$, where $\deg N_{\tilde{C}_i/\tilde{X}} = (-K_{\tilde{X}}\cdot\tilde{C}_i) - 2 \geqslant i-2$. Hence $f_1^*T_{\tilde{X}}\otimes\mathcal{O}(1)$ is nef and $f_2^*T_{\tilde{X}}\otimes\mathcal{O}(1)$ is ample. Denote by $f : \mathbb{P}^1\vee\mathbb{P}^1 \to \tilde{X}$ the gluing morphism of f_1 and f_2 at an intersection point of $\tilde{C}_1$ and $\tilde{C}_2$. Then $H^1(\mathbb{P}^1\vee\mathbb{P}^1, f^*T_{\tilde{X}})=0$ by [13, II.7.5], so the deformation of f is unobstructed. By [13, I.2.17] there exists a holomorphic flat family of divisors $\{\tilde{\Gamma}_t\}_{t\in\mathbb{D}}$ such that $\tilde{\Gamma}_0=\tilde{C}_1+\tilde{C}_2$ and $\tilde{\Gamma}_t$ is an irreducible rational nodal curve whenever $t\neq 0$. After a reparametrization of t if necessary we may also assume that $\tilde{\Gamma}_t+E$ is normal crossing for each t. The lemma is proved by taking $\Gamma_t := \sigma_*(\tilde{\Gamma}_t)$. □

LEMMA 4.4. — *Let (X,Δ) be a log smooth projective surface pair with X rational. Let $C=\sum_{i=1}^m C_i$ $(m\geqslant 2)$ be a reduced divisor on X such that each C_i is an irreducible nodal rational curve and $\Delta+C$ is normal crossing. Assume*

- $(C_1\cdot C_2)>0$;
- $(C_i\cdot(C-C_i))\geqslant 3$ *for any* $3\leqslant i\leqslant m$;
- *There exists a non-negative integer $l<(C_1\cdot C_2)$, such that $(-K_X\cdot C_i)\geqslant l+i$ and $(C_i\cdot(C-C_1-C_2))\geqslant 3-l$ for any $i\in\{1,2\}$.*

Then there exists a nodal admissible deformation $\{\Gamma_t\}_{t\in\mathbb{D}}$ of C such that $\Gamma_t = A_t+\sum_{i=3}^m C_i$ where A_t is an irreducible rational nodal curve whenever $t\neq 0$.

Proof. — Let us pick l distinct points $x_1,\ldots,x_l$ in $C_1\cap C_2$. By Lemma 4.3, there exists a holomorphic flat family $\{A_t\}_{t\in\mathbb{D}}$ of reduced divisors on X such that $A_0=C_1+C_2$ and A_t is an irreducible rational nodal curve singular at $x_1,\ldots,x_l$ for any $t\in\mathbb{D}\setminus\{0\}$. By Bertini's theorem, after a reparametrization of t we may assume that $\Delta+A_t+\sum_{i=3}^m C_i$ is normal crossing for any $t\in\mathbb{D}$. Let $\Gamma_t := A_t+\sum_{i=3}^m C_i$, then it suffices to show that Γ_0^* is hyperbolic. As a divisor in $X\times\mathbb{D}$, $\Gamma=\mathcal{A}+\sum_{i=3}^m\mathcal{C}_i$, where $\mathcal{A}$ is the development of $\{A_t\}_{t\in\mathbb{D}}$ and $\mathcal{C}_i := C_i\times\mathbb{D}$. Thus Γ is (analytically) reducible at $(x,0)$ if $x\in C_i\cap C_j$ for some $\{i,j\}\neq\{1,2\}$. By Lemma 4.3, we know that $\mathcal{A}$ is analytically reducible at $(x_1,0),\ldots,(x_l,0)$. Thus we have

$$\Gamma_0\setminus\Gamma_0^* \supset \{x_1,\ldots,x_l\}\cup\big(\cup_{\{i,j\}\neq\{1,2\}}(C_i\cap C_j)\big) =: V.$$

Since $\Gamma_0 = \sum_{i=1}^m C_i$, we only need to show that $C_i \setminus V$ is hyperbolic for any $1 \leqslant i \leqslant k$. For each $i \in \{1,2\}$, $\# C_i \cap V = \#\big(\{x_1, \ldots, x_l\} \cup (\cup_{j \geqslant 3}(C_i \cap C_j))\big) = l + (C_i \cdot (C - C_1 - C_2)) \geqslant 3$. For each $i \geqslant 3$, $\# C_i \cap V = \big(C_i \cdot (C - C_i)\big) \geqslant 3$. Hence $C_i \setminus V$ is hyperbolic for each $1 \leqslant i \leqslant k$. The lemma is proved. $\square$

Definition 4.5. — *Let X be a smooth projective surface. Let $C = \sum_{i=1}^m C_i$ be a reduced normal crossing divisor on X. The* dual graph *$\mathcal{D}(C) := (V, E, r, \mathrm{wt})$ of C is a vertex-weighted multigraph defined as follows:*

- *$V := \{v_1, \ldots, v_m\}$;*
- *$E := \cup_{1 \leqslant i < j \leqslant m}(C_i \cap C_j)$;*
- *For each $p \in E$, $r(p) := \{v_i, v_j\}$ where $\{i, j\}$ is the unique unordered pair with $p \in C_i \cap C_j$;*
- *For each $v_i \in V$, $\mathrm{wt}(v_i) := (-K_X \cdot C_i)$.*

Lemma 4.6. — *Let (X, Δ) be a log smooth projective surface pair with X rational. Let $C = \sum_{i=1}^m C_i$ be a reduced divisor such that $C + \Delta$ is normal crossing, and each C_i is an irreducible rational curve. If the dual graph $\mathcal{D}(C)$ is admissibly contractible, then there exists a successive nodal admissible deformation $\{\Gamma_t^{(j)}\}_{t \in \mathbb{D}, 1 \leqslant j \leqslant m-1}$ such that $\Gamma_{t_{m-1}}^{(m-1)}$ is an irreducible rational nodal curve. If, moreover, $X \setminus (C \setminus \Delta)$ is Brody hyperbolic, then $\{\Gamma_t^{(j)}\}_{t \in \mathbb{D}, 1 \leqslant j \leqslant m-1}$ can be chosen so that $X \setminus (\Gamma_t^{(j)} \setminus \Delta)$ is Brody hyperbolic for any $t \in \mathbb{D}$ and any $1 \leqslant j \leqslant m-1$.*

Proof. — The successive nodal admissible deformation $\{\Gamma_t^{(j)}\}_{t \in \mathbb{D}, 1 \leqslant j \leqslant m-1}$ can be constructed inductively by a successive admissible contraction of the dual graph $\mathcal{D}(C)$ using Lemma 4.4. The hyperbolicity part follows from Lemma 4.2 and taking reparametrizations of t if necessary. $\square$

Lemma 4.7. — *Let (X, Δ) be a log smooth projective surface pair with X rational. Let C be an irreducible rational nodal curve in X such that $\Delta + C$ is normal crossing. If $(-K_X \cdot C) \geqslant 8$ and $\#\,\mathrm{Sing}(C) \geqslant 4$, then there exists a successive nodal admissible deformation $\{\Gamma_t^{(j)}\}_{t \in \mathbb{D}, 1 \leqslant j \leqslant 2}$ of C such that $\Gamma_{t_2}^{(2)}$ is an irreducible smooth hyperbolic curve. If, moreover, $X \setminus (C \setminus \Delta)$ is Brody hyperbolic, then $\{\Gamma_t^{(j)}\}_{t \in \mathbb{D}, 1 \leqslant j \leqslant 2}$ can be chosen so that $X \setminus (\Gamma_t^{(j)} \setminus \Delta)$ is Brody hyperbolic for any $t \in \mathbb{D}$ and any $1 \leqslant j \leqslant 2$.*

Proof. — Let us pick two nodes p_1, p_2 of C. Denote by $\sigma : \tilde{X} = \mathrm{Bl}_{p_1, p_2} X \to X$ the blow up of X at p_1, p_2. Let $E = E_1 + E_2$ be the reduced exceptional divisor of σ. Let $\tilde{C} \subset \tilde{X}$ be the strict transform of C under σ. We claim that $\tilde{C}$ is base point free in $\tilde{X}$.

Since $\tilde{X}$ is rational, we have $H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) = 0$. Thus the claim is equivalent to saying that $\mathcal{O}_{\tilde{C}}(\tilde{C})$ is globally generated. Since $(-K_X \cdot C) \geqslant 8$, we have

$$(-K_{\tilde{X}} \cdot \tilde{C}) = (\sigma^*(-K_X) \cdot \tilde{C}) - (E \cdot \tilde{C}) = (-K_X \cdot C) - 4 \geqslant 4.$$

By adjunction we have $(-K_{\tilde{X}} \cdot \tilde{C}) = (\tilde{C}^2) - 2\# \operatorname{Sing}(\tilde{C}) + 2$, so we have $\deg \nu^* \mathcal{O}_{\tilde{C}}(\tilde{C}) = (\tilde{C}^2) \geqslant 2\# \operatorname{Sing}(\tilde{C}) + 2$, where $\nu : \mathbb{P}^1 \to \tilde{C}$ is the normalization of $\tilde{C}$. Hence the global sections of $\nu^* \mathcal{O}_{\tilde{C}}(\tilde{C})$ separate any $2\# \operatorname{Sing}(\tilde{C}) + 1$ points on $\mathbb{P}^1$. In particular, this implies that $\mathcal{O}_{\tilde{C}}(\tilde{C})$ is globally generated.

Now we have shown that $\tilde{C}$ is base point free on $\tilde{X}$. By Bertini's theorem, there exists a holomorphic flat family of irreducible divisors $\{\tilde{\Gamma}_t^{(1)}\}_{t \in \mathbb{D}}$ on $\tilde{X}$ such that $\tilde{\Gamma}_0^{(1)} = \tilde{C}$ and $(\tilde{X}, \tilde{\Gamma}_t^{(1)} + E + \sigma^*\Delta)$ is log smooth for any $t \in \mathbb{D} \setminus \{0\}$. Let $\Gamma_t^{(1)} := \sigma_* \tilde{\Gamma}_t^{(1)}$. Since $\tilde{\Gamma}_0^{(1)} = \tilde{C}$ intersects E_i transversally at two points for any $i \in \{1, 2\}$, it is clear that $\tilde{\Gamma}$ has two analytic branches intersecting $E_i \times \{0\}$ in different points. Thus $\Gamma^{(1)}$ has two analytic branches at $(p_i, 0)$ for each $i \in \{1, 2\}$ which implies that $\Gamma_0^{(1),*} \subset C \setminus \{p_1, p_2\}$ is hyperbolic. Besides, $(\tilde{X}, \tilde{\Gamma}_t^{(1)} + E + \sigma^*\Delta)$ being log smooth implies that $\Gamma_t^{(1)}$ is nodal at p_1, p_2, smooth elsewhere and intersects transversally with Δ for any $t \in \mathbb{D} \setminus \{0\}$. Hence $\{\Gamma_t^{(1)}\}_{t \in \mathbb{D}}$ is a nodal admissible deformation of C with $\Delta + \Gamma_t^{(1)}$ being normal crossing for each $t \in \mathbb{D}$.

Now let us fix an arbitrary $t_1 \in \mathbb{D} \setminus \{0\}$. Since $p_a(\tilde{C}) = \# \operatorname{Sing}(\tilde{C}) = \# \operatorname{Sing}(C) - 2 \geqslant 2$, we know that $\Gamma_t^{(1)}$ is hyperbolic for any $t \in \mathbb{D} \setminus \{0\}$. As we argued before in showing the base-point-freeness of $\tilde{C}$, $(-K_X \cdot C) \geqslant 8 \geqslant 4$ also implies that C is base point free on X. Hence by Bertini's theorem there exists a holomorphic flat family of irreducible divisors $\{\Gamma_t^{(2)}\}_{t \in \mathbb{D}}$ on X such that $\Gamma_0^{(2)} = \Gamma_{t_1}^{(1)}$ and $(X, \Gamma_t^{(2)} + \Delta)$ is log smooth for any $t \in \mathbb{D} \setminus \{0\}$. Besides, $\Gamma_0^{(2),*} \subset \Gamma_0^{(2)} = \Gamma_{t_1}^{(1)}$ is hyperbolic. Hence $\{\Gamma_t^{(2)}\}_{t \in \mathbb{D}}$ is a nodal admissible deformation of $\Gamma_{t_1}^{(1)}$ such that $\Delta + \Gamma_t^{(2)}$ is normal crossing for any $t \in \mathbb{D}$. Besides, $g(\Gamma_t^{(2)}) = p_a(\Gamma_0^{(2)}) \geqslant p_g(\Gamma_0^{(2)}) \geqslant 2$ for any $t \in \mathbb{D} \setminus \{0\}$, hence $\Gamma_t^{(2)}$ is hyperbolic for any $t \in \mathbb{D} \setminus \{0\}$. The lemma is proved by taking arbitrary $t_2 \neq 0$. $\square$

LEMMA 4.8. — *Let (X, Δ) be a log smooth projective surface pair with X rational. Let $C = \sum_{i=1}^m C_i$ be a reduced divisor on X such that $C + \Delta$ is normal crossing. Assume that each C_i is a base-point-free irreducible rational curve with $(-K_X \cdot C_i) \geqslant 2$, and it intersects with at least four other C_j's. If $X \setminus \big((\cup_{i=1}^m C_i) \setminus \Delta\big)$ is Brody hyperbolic, then there exists an irreducible smooth curve C' linearly equivalent to $\sum_{i=1}^m C_i$ such that both C' and $X \setminus (C' \setminus \Delta)$ are Brody hyperbolic.*

Proof. — Let $G := \mathcal{D}(C)$ be the dual graph of C. Since each C_i is base-point-free, G is completely multipartite. By assumptions, for each vertex v of G we have $\mathrm{rdeg}(v) \geqslant 4$ and $\mathrm{wt}(v) \geqslant 2$. Hence Lemma 3.8 implies that $\mathcal{D}(C)$ is admissibly contractible. By Lemma 4.6, there exists a successive nodal admissible deformation $\{\Gamma_t^{(j)}\}_{t\in\mathbb{D},1\leqslant j\leqslant m-1}$ of C such that $\Gamma_{t_{m-1}}^{(m-1)}$ is an irreducible rational curve and $X \setminus (\Gamma_{t_{m-1}}^{(m-1)} \setminus \Delta)$ is Brody hyperbolic. Since each C_i intersects with at least four other C_j's, we have $m \geqslant 5$, hence $(-K_X \cdot \Gamma_{t_{m-1}}^{(m-1)}) = \sum_{i=1}^m(-K_X \cdot C_i) \geqslant 10$. Since $2\sum_{1\leqslant i<j\leqslant m}(C_i \cdot C_j) = \sum_{i=1}^m \sum_{j\neq i}(C_i \cdot C_j) \geqslant 4m$, we have

$$\begin{aligned}\#\operatorname{Sing}(\Gamma_{t_{m-1}}^{(m-1)}) = p_a(\Gamma_{t_{m-1}}^{(m-1)}) = p_a(C) = \#\operatorname{Sing}(C) - (m-1)\\ \geqslant \sum_{1\leqslant i<j\leqslant m}(C_i \cdot C_j) - (m-1) \geqslant m+1 \geqslant 6.\end{aligned}$$

By applying Lemma 4.7 to $C := \Gamma_{t_{m-1}}^{(m-1)}$, we know that there exists a successive nodal admissible deformation $\{\Gamma_t^{(j)}\}_{t\in\mathbb{D},m\leqslant j\leqslant m+1}$ of $\Gamma_{t_{m-1}}^{(m-1)}$ such that $\Gamma_{t_{m+1}}^{(m+1)}$ is an irreducible smooth hyperbolic curve and $X \setminus (\Gamma_{t_{m+1}}^{(m+1)} \setminus \Delta)$ is Brody hyperbolic. It is clear that $\Gamma_{t_{m+1}}^{(m+1)}$ is numerically equivalent to C, hence they are linearly equivalent since X is rational. The lemma is proved by taking $C' := \Gamma_{t_{m+1}}^{(m+1)}$. □

The following corollary is a generalization of [21, Theorem 3.1] which says that there exists a smooth plane curve of degree m whose complement is Brody hyperbolic for $m \geqslant 5$.

Corollary 4.9. — *Let $m \geqslant 5$ and $d \geqslant 4$ be integers. Then there exists smooth plane curves C and S of degree m and d respectively, such that $(\mathbb{P}^2, S + C)$ is log smooth and $\mathbb{P}^2 \setminus \big(C \setminus S\big)$ is Brody hyperbolic.*

Proof. — Let $\{C_i\}_{i=1}^m$ be a set of lines in general position in $\mathbb{P}^2$. By Corollary 2.3, there exists a smooth plane curve S of degree d such that $(\mathbb{P}^2, \sum_{i=1}^m C_i + S)$ is log smooth and $\mathbb{P}^2 \setminus \big((\cup_{i=1}^m C_i) \setminus S\big)$ is Brody hyperbolic. We know that $(C_i^2) = 1$ and each C_i intersects all C_j's whenever $j \neq i$. Since $m - 1 \geqslant 4$, the corollary is proved by applying Lemma 4.8 to $(X, \Delta, C_i) := (\mathbb{P}^2, S, C_i)$. □

The following corollary is related to [10, 1.2] where they studied hyperbolic imbeddedness of $\mathbb{F}_0 \setminus C$.

Corollary 4.10. — *Let a, b, c, d be integers. Then there exists smooth curves C and S in $\mathbb{F}_N$ of bidegree (a, b) and (c, d) respectively, such that $(\mathbb{F}_N, S + C)$ is log smooth and $\mathbb{F}_N \setminus \big(C \setminus S\big)$ is Brody hyperbolic if one of the following is true:*

- *$N = 0$ and $a, b, c, d \geqslant 4$;*
- *$N \geqslant 1$, $a, c \geqslant 3$ and $b, d \geqslant 4$.*

Proof. — Let $\{C_i\}_{i=1}^{a+b}$ be a set of curves in $\mathbb{F}_N$, such that C_i is a general curve in $|F|$ for any $i \leqslant a$, and C_j is a general curve in $|T|$ for any $j > a$. By Corollary 2.4, there exists a smooth curve S of bidegree (c, d) such that $(\mathbb{F}_N, \sum_{i=1}^{a+b} C_i + S)$ is log smooth and $\mathbb{F}_N \setminus \big((\cup_{i=1}^{a+b} C_i) \setminus S\big)$ is Brody hyperbolic. We know that $(C_i^2) = 0$ for each $i \leqslant a$ and $(C_j^2) = N \geqslant 0$ for each $j > a$. From the proof of Corollary 2.4 we know that C_i intersects with at least four C_j's for each $1 \leqslant i \leqslant a + b$. Hence the corollary is proved by applying Lemma 4.8 to $(X, \Delta, C_i) := (\mathbb{F}_N, S, C_i)$. □

5. Proofs

Lemma 5.1. — *Let X be a smooth projective surface. Let L be a line bundle on X. Let $n \geqslant 2$ be an integer. Assume that there exists irreducible divisors $C \in |L|$ and $S \in |L^{\otimes n}|$ satisfying that $(X, S + C)$ is log smooth, and both C and $X \setminus (C \setminus S)$ are Brody hyperbolic. Then there exists a smooth curve $D \in |L^{\otimes n}|$ such that the degree n cyclic cover of X branched along D is Brody hyperbolic.*

Proof. — Let $\{S_t\}_{t \in \mathbb{P}^1}$ be the linear pencil of divisors on X spanned by $S_0 := nC$ and $S_\infty := S$. Then the development of $\{S_t\}$ is an effective Cartier divisor $\mathcal{S}$ of $X \times \mathbb{P}^1$. Since S and C intersect transversally, it is not hard to check in local charts that $\mathcal{S}$ is smooth away from the finite set $(C \cap S) \times \{0\}$. Let $\pi : \mathcal{Y} \to X \times \mathbb{P}^1$ be the degree n cyclic cover of $X \times \mathbb{P}^1$ branched along $\mathcal{S}$. Then $\mathcal{Y}$ is smooth away from $\pi^{-1}((C \cap S) \times \{0\})$. From the construction it is clear that each fiber Y_t of $pr_2 \circ \pi : \mathcal{Y} \to \mathbb{P}^1$ is a degree n cyclic cover of X branched along S_t. Since $S_0 = nC$, Y_0 is the union of n irreducible components $\{Y_{0,i}\}_{i=1}^n$ such that $Y_{0,i} \cap Y_{0,j} = \pi^{-1}(C \times \{0\})$ for any $i \neq j$, and $\pi : (Y_{0,i}, \pi^{-1}(C \times \{0\})) \to (X, C) \times \{0\}$ is an isomorphism for any i.

Assume to the contrary that Y_{t_n} is not Brody hyperbolic for a sequence of non-zero complex numbers (t_n) converging to 0. Let $\phi_n : \mathbb{C} \to Y_{t_n}$ be the sequence of entire curves. We may assume that $||\phi_n'(0)||$ tends to infinity after coordinate changes. By Brody Lemma (e.g. [6]), after choosing a subsequence if necessary, there exists a sequence of reparametrizations $r_n : \mathbb{D}_{R_n} \to \mathbb{D}$ where $\lim_{n\to\infty} R_n = +\infty$ such that $(\phi_n \circ r_n)$ converges to an entire curve $\phi_\infty : \mathbb{C} \to Y_0$ as $n \to \infty$. Notice that $Y_{0,i}$ is Cartier away from $\pi^{-1}((C \cap S) \times \{0\})$, so Lemma 2.1 implies that $\phi_\infty(\mathbb{C})$ is contained in at least one of the $(n+1)$ subsets $\{Y_0^{(i)}\}_{i=0}^n$ of Y_0, where

$$Y_0^{(0)} := \pi^{-1}(C \times \{0\}),$$
$$Y_0^{(i)} := Y_{0,i} \setminus \pi^{-1}((C \setminus S) \times \{0\}) \quad \text{for any } 1 \leqslant i \leqslant n.$$

In particular, at least one of the subsets $\{Y_0^{(i)}\}_{i=0}^n$ is not Brody hyperbolic. Under the projection π, it is not hard to see that $Y_0^{(0)} \cong C$ and $(Y_{0,i}, Y_0^{(i)}) \cong (X, X \setminus (C \setminus S))$ for any $1 \leqslant i \leqslant n$. Thus $Y_0^{(i)}$ is Brody hyperbolic for any $0 \leqslant i \leqslant n$, we get a contradiction.

As a result, Y_t is Brody hyperbolic for any $t \neq 0$ sufficiently small. Since S_t is smooth for general t, the lemma is proved by choosing $D := S_t$ for $t \neq 0$ sufficiently small. □

Proof of Theorem 1.2. — Let $\pi : Y \to \mathbb{P}^2$ be the double cover of $\mathbb{P}^2$ branched along D.

For the "only if" part, if $d \leqslant 4$ then Y is a rational surface; if $d = 6$ then Y is a K3 surface. In both cases Y is not Brody hyperbolic. If $d = 8$, since Brody hyperbolicity is preserved under small deformation, we may deform D a bit to ensure that there exists a bitangent line ℓ of D that meets D transversally in four further points. Hence by Riemann–Hurwitz formula, $\pi^{-1}(\ell)$ is an elliptic curve. Thus Y is never Brody hyperbolic when $d \leqslant 8$.

For the "if" part, Corollary 4.9 implies that there exist plane curves C and S of degree $d/2$ and d respectively, such that $(\mathbb{P}^2, C+S)$ is log smooth and $\mathbb{P}^2 \setminus (C \setminus S)$ is Brody hyperbolic. Since $d/2 \geqslant 5$, C is a smooth curve of genus at least 6, so it is Brody hyperbolic. Thus applying Lemma 5.1 to $(X, L, n, C, S) := (\mathbb{P}^2, \mathcal{O}(d/2), 2, C, S)$ finishes the proof. □

The following theorem is an application of Corollary 4.9 and Lemma 5.1. It also improves [15, Theorem 25].

Theorem 5.2. — *Let $d \geqslant 10$ be a composite number. Then there exists a smooth Brody hyperbolic surface of degree d in $\mathbb{P}^3$ that is a cyclic cover of $\mathbb{P}^2$ under some linear projection.*

Proof. — By assumption, $d = d_1 d_2$ for some integers $d_1 \geqslant 2$, $d_2 \geqslant 5$. Corollary 4.9 implies that there exist plane curves C and S of degree d_2 and d respectively, such that $(\mathbb{P}^2, C+S)$ is log smooth and $\mathbb{P}^2 \setminus (C \setminus S)$ is Brody hyperbolic. Since $d_2 \geqslant 5$, C is a smooth curve of genus at least 6, so it is Brody hyperbolic. Applying Lemma 5.1 to $(X, L, n, C, S) :=$

$(\mathbb{P}^2, \mathcal{O}(d_2), d_1, C, S)$ yields that there exists a smooth plane curve D of degree d such that the degree d_1 cyclic cover Y of $\mathbb{P}^2$ branched along D is Brody hyperbolic. Let W be the degree d cyclic cover of $\mathbb{P}^2$ branched along D, then there is a natural finite surjective morphism $W \to Y$. Since Y is Brody hyperbolic, W is also Brody hyperbolic. □

Proof of Theorem 1.3. — Let $\pi : Y \to \mathbb{F}_N$ be the double cover of $\mathbb{F}_N$ branched along D.

For the "only if" part, assume to the contrary that $b \leqslant 6$, then $(D \cdot F) = b \leqslant 6$. Since $\dim |F| = 1$, there exists a curve $F_0 \in |F|$ such that F_0 is tangent to D at some point. As a result, $\pi^{-1}(F_0)$ is a double cover of $\mathbb{P}^1$ branched along a non-reduced divisor of degree $\leqslant 6$. This implies that each irreducible component of $\pi^{-1}(F_0)$ is either a rational curve or an elliptic curve, so Y is not Brody hyperbolic. We get a contradiction. Hence we must have $b \geqslant 8$. If $N = 0$, then $a \geqslant 8$ by the symmetry between F and T. If $N \geqslant 1$, assume to the contrary that $a \leqslant 4$, then $(D \cdot (T - NF)) = a \leqslant 4$. Let $T' \subset \mathbb{F}_N$ be the unique curve with negative self-intersection number, then $T' \sim T - NF$. Hence $(D \cdot T') \leqslant 4$. This implies that each irreducible component of $\pi^{-1}(T')$ is either a rational curve or an elliptic curve, so Y is not Brody hyperbolic. We get a contradiction. Therefore, the proof of the "only if" part is completed.

For the "if" part, Corollary 4.10 implies that there exist plane curves C and S of bidegree $(a/2, b/2)$ and (a, b) respectively, such that $(\mathbb{F}_N, C + S)$ is log smooth and $\mathbb{F}_N \setminus (C \setminus S)$ is Brody hyperbolic. If $N = 0$, then $a, b \geqslant 8$ implies that C is a smooth curve of genus at least 9; if $N \geqslant 1$, then $a \geqslant 6$ and $b \geqslant 8$ implies that C is a smooth curve of genus at least $6N + 6$. So C is Brody hyperbolic for every $N \geqslant 0$. Thus applying Lemma 5.1 to $(X, L, n, C, S) := (\mathbb{F}_N, \mathcal{O}_{\mathbb{F}_N}((a/2)F + (b/2)T), 2, C, S)$ finishes the proof. □

Remarks 5.3.

(1) According to [8], the canonical model of a Horikawa surface with even c_1^2 is either a double cover of $\mathbb{P}^2$ branched along a degree 8 or 10 curve, or a minimal resolution of a double cover of $\mathbb{F}_N$ branched along a bidegree $(a, 6)$ curve where a has finite choices depending on N. Hence the "only if" parts of Theorems 1.2 and 1.3 imply that a Brody hyperbolic Horikawa surface with even c_1^2 has to be a double cover of $\mathbb{P}^2$ branched along a degree 10 curve (in fact one only needs to check algebraic hyperbolicity). However, our deformation method cannot be applied to exhibit other Brody quasi-hyperbolic Horikawa surfaces (i.e. satisfying the Green–Griffiths–Lang conjecture).

(2) Smooth quintic surfaces in $\mathbb{P}^3$ are natural examples of Horikawa surfaces with odd c_1^2. It was shown by Xu [20] that a very general quintic surface does not contain any rational or elliptic curve. However, no examples of Brody hyperbolic (even algebraic hyperbolic) quintic surfaces are known so far. Notice that the case of a (very) general quintic surface in $\mathbb{P}^3$ corresponds to the case $d = 2n - 1$ in the Kobayashi Conjecture (cf. [11, 12]).

(3) Since Brody hyperbolicity is open in the Euclidean topology (see e.g. [12, 3.11.1]), Theorems 1.2 and 1.3 imply that there exist nonempty open subsets of certain moduli spaces of double covers of $\mathbb{P}^2$ or $\mathbb{F}_N$ that parametrize Brody hyperbolic ones. Besides, we know that Brody hyperbolicity implies algebraic hyperbolicity, and algebraic hyperbolicity is a very generic property in families. Hence Theorem 1.2 gives an alternative proof of [18, Theorem 3.2].

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