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Non locally trivializable CR line bundles over compact Lorentzian CR manifolds

NON LOCALLY TRIVIALIZABLE CR LINE BUNDLES OVER COMPACT LORENTZIAN CR MANIFOLDS

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ABSTRACT. — We consider compact CR manifolds of arbitrary CR codimension that satisfy certain geometric conditions in terms of their Levi form. Over these compact CR manifolds, we construct a deformation of the trivial CR line bundle over M which is topologically trivial over M but fails to be even locally CR trivializable over any open subset of M. In particular, our results apply to compact Lorentzian CR manifolds of hypersurface type.

RÉSUMÉ. — On considère une variété CR compacte de codimension CR quelconque qui vérifie certaines conditions géométriques en terme de sa forme de Levi. Sur ces variétés CR compactes, on construit une déformation du fibré en droites CR trivial sur M qui est topologiquement trivial sur M mais qui n'admet même pas de trivialization CR locale sur un ouvert arbitraire de M. En particulier, nos résultats s'appliquent au cas de variétés CR compactes Lorentziennes du type hypersurface.

1. Introduction

In geometry, the concept of vector bundles is quite important. In both categories: smooth real vector bundles over differentiable real manifolds, and holomorphic complex vector bundles over complex manifolds, there is a theorem which says that any such vector bundle always has a local trivialization. In the first situation, the transition functions, or matrices, are $\mathcal{C}^\infty$ smooth, and in the second situation, they are holomorphic. For this reason one usually ignores the intrinsic definition of a vector bundle and works directly with a local trivialization. It is a somewhat suprising fact, which we address here, that in the category of smooth CR vector bundles over a smooth CR manifold, the analogous theorem does not always hold.

Thus one can have a perfectly good intrinsically defined CR vector bundle for which there might not be the corresponding CR transition functions, or matrices.

Our main results are as follows:

THEOREM 1.1. — *Let M be a compact (abstract) Lorentzian CR hypersurface of CR dimension n, $n \neq 3$. Then there exists a family $(L_a)_{a>0}$ of complex CR line bundles $L_a \longrightarrow M$ converging to the trivial CR line bundle $L_0 = M \times \mathbb{C}$, as a tends to 0, such that L_a is differentiably trivial over M, but L_a is not locally CR trivializable over any open set U of M for $a > 0$.*

An easy example satisfying the hypothesis of the above theorem is Penrose's null twistor space, which is the 5-dimensional M of type (2,1) in $\mathbb{CP}^3$ given in homogeneous coordinates by

$$|z_0|^2 + |z_1|^2 = |z_2|^2 + |z_3|^3.$$

For CR manifolds of CR dimension $n = 2$ but arbitrary codimension, we obtain the following result:

THEOREM 1.2. — *Let M be a compact (abstract) CR manifold of type $(2, k)$ which is 1-pseudoconcave. Then there exists a family $(L_a)_{a>0}$ of complex CR line bundles $L_a \longrightarrow M$ converging to the trivial CR line bundle $L_0 = M \times \mathbb{C}$, as a tends to 0, such that L_a is differentiably trivial over M, but L_a is not locally CR trivializable over any open set U of M for $a > 0$.*

An example of a CR manifold of type $(2, 2)$ satisfying the hypothesis of the above theorem is the twistor space of the Fubini-Study metric on $\mathbb{CP}^2$, which is the 6-dimensional $M \subset \mathbb{CP}^2 \times \mathbb{CP}^2$ defined by the complex equation

$$z_0\overline{w}_0 + z_1\overline{w}_1 + z_2\overline{w}_2 = 0,$$

where $(z_0 : z_1 : z_2)$ are homogeneous coordinates in the first factor, and $(w_0 : w_1 : w_2)$ are homogeneous coordinates in the second factor.

More examples of CR submanifolds satisfying the hypothesis of this theorem can be found in Section 4 of [6].

In both theorems the same results apply in the setting of CR vector bundles of arbitrary rank, because it suffices to deform the vector bundle in only one fiber direction.

We would like to point out that results in the opposite direction have been obtained by Webster [8]. Namely he proved that any CR vector bundle over a strictly pseudoconvex hypersurface of CR dimension $n \geqslant 7$ is locally trivializable.

2. Definitions

Throughout this paper an abstract CR manifold of type (n, k) is a triple (M, HM, J), where M is a smooth real manifold of dimension $2n + k$, HM is a subbundle of rank $2n$ of the tangent bundle TM, and $J : HM \to HM$ is a smooth fiber preserving bundle isomorphism with $J^2 = -\mathrm{Id}$. We also require that J be formally integrable; i.e. that we have

$$[T^{0,1}M, T^{0,1}M] \subset T^{0,1}M \qquad \text{where}$$
$$T^{0,1}M = \{X + iJX \mid X \in \Gamma(M, HM)\} \subset \Gamma(M, \mathbb{C}TM),$$

with Γ denoting smooth sections. The CR dimension of M is $n \geqslant 1$ and the CR codimension is $k \geqslant 1$.

As usual, we denote by $\overline{\partial}_M$ the tangential Cauchy–Riemann operator on M. For the precise definition, we refer the reader to [6].

We denote by $H^oM = \{\xi \in T^*M \mid \langle X, \xi \rangle = 0, \forall X \in H_{\pi(\xi)}M\}$ the *characteristic conormal bundle* of M. Here $\pi : TM \longrightarrow M$ is the natural projection. To each $\xi \in H_p^oM$, we associate the Levi form at ξ :

$$\mathcal{L}_p(\xi, X) = \xi([J\tilde{X}, \tilde{X}]) = \mathrm{d}\tilde{\xi}(X, JX) \text{ for } X \in H_pM$$

which is Hermitian for the complex structure of H_pM defined by J. Here $\tilde{\xi}$ is a section of H^oM extending ξ and $\tilde{X}$ a section of HM extending X.

A *Lorentzian hypersurface* is a CR manifold M of type $(n, 1)$ such that its Levi form has one negative and $n - 1$ positive eigenvalues.

M is called 1-*pseudoconcave* if for every $p \in M$ and every characteristic conormal direction ξ the Levi form $\mathcal{L}_p(\xi, \cdot)$ has at least one positive and one negative eigenvalue.

For the definition of CR vector bundles, we follow [7]. Namely, a *complex CR vector bundle of rank r* over an abstract CR manifold M of type (n, k) is a smooth complex vector bundle $E \xrightarrow{\pi} M$ of rank r such that

(i) E has the structure of a smooth abstract CR manifold of type $(n + r, k)$,

(ii) $\pi : E \longrightarrow M$ is a CR submersion,

(iii) $E \oplus E \ni (\xi_1, \xi_2) \to \xi_1 + \xi_2 \in E$ and $\mathbb{C} \times E \ni (\lambda, \xi) \to \lambda\xi \in E$ are CR maps.

Let $E \xrightarrow{\pi} M$ be a complex CR vector bundle of rank r. A CR *trivialization* of E over an open set U of M is a CR equivalence between $E_{|U}$ and the trivial bundle $U \times \mathbb{C}^r$. We say that E is *locally trivializable* over M iff each point of M has an open neighborhood U such that $E_{|U}$ is CR equivalent to $U \times \mathbb{C}^r$.

When E is locally CR trivializable, one can arrange an open covering $\{U_\alpha\}$ of M, and obtain transition functions, or matrices, $\{g_{\alpha\beta}\}$ which are CR. Hence in that case the situation is analogous to the case of holomorphic vector bundles over complex manifolds.

Note that if E is locally CR embeddable, then E is also locally CR trivializable, by the implicit function theorem. But the local CR embeddability of the base M does not suffice to make E locally CR trivializable, which is the main point of this paper.

Another equivalent definition of CR vector bundles, which is used in [8], can be given in terms of a connection $\overline{\partial}_E$ on the complex vector bundle E; that is, we have a linear differential operator of order one,

$$\overline{\partial}_E : \mathcal{C}^\infty_{0,q}(M, E) \longrightarrow \mathcal{C}^\infty_{0,q+1}(M, E)$$

satisfying the Leibniz rule and $\overline{\partial}_E \circ \overline{\partial}_E = 0$. E is locally trivializable over $U \subset M$ iff there exists a nonvanishing section s of E over U satisfying $\overline{\partial}_E s = 0$. For more details, we refer the reader to [7].

3. Construction of a nontrivial global cohomology class

The key point in the construction of the CR line bundles is the existence of a global $\overline{\partial}_M$-closed $(0,1)$ form on M which is not $\overline{\partial}_M$-exact on any open subset of M. First we consider a single point $p \in M$, and show the existence of a global $\overline{\partial}_M$-closed form which is not $\overline{\partial}_M$-exact on any neighborhood of that point. The existence of such a form follows by similar arguments as in [2] and [3]. Therefore, in the proof of the following Theorem, we sketch only the most important ingredients.

THEOREM 3.1. — *Let M be a compact (abstract) CR manifold of type (n,k). Assume that $H^{0,2}(M)$ is Hausdorff and that there exists a point p on M at which there exists a characteristic conormal direction ξ such that $\mathcal{L}_p(\xi, \cdot)$ has one negative and $n-1$ positive eigenvalues. Then there exists a smooth $(0,1)$-form f on M satisfying $\overline{\partial}_M f = 0$ on M such that f is not $\overline{\partial}_M$-exact on any neighborhood of p in M.*

Proof. — By contradiction, we assume that any smooth $(0,1)$-form f on M satisfying $\overline{\partial}_M f = 0$ is $\overline{\partial}_M$-exact on some neighborhood U_f of p in M. Using the functional analytic arguments of [1] this implies that there exists an open neighborhood U of p in M, independent of f, such that any smooth $(0,1)$-form f on M satisfying $\overline{\partial}_M f = 0$ on M is $\overline{\partial}_M$-exact on U. Moreover, by the open mapping theorem for Fréchet spaces, we also get an a priori

estimate of the following form: For a fixed compact $K \subset U$ there exists an integer m_1 and a constant $C_1 > 0$ such that the solution u to $\overline{\partial}_M u = f$ on U can be chosen to satisfy

$$\|u\|_{K,0} \leqslant C_1 \|f\|_{m_1}.$$

Here $\| \cdot \|_m$ denotes the usual $\mathcal{C}^m$ norm on $\mathcal{C}^\infty_{\cdot,\cdot}(M)$, and $\| \cdot \|_{K,0}$ denotes the usual $\mathcal{C}^0$ norm on K.

But this implies that we have

$$(3.1) \qquad \left| \int_K f \wedge g \right| \lesssim \|f\|_{m_1} \cdot \|\overline{\partial}_M g\|_{K,0}$$

for all $f \in \mathcal{C}^\infty_{0,1}(M) \cap \mathrm{Ker}\overline{\partial}_M$ and $g \in \mathcal{C}^\infty_{n+k,n-1}(M)$ with supp $g \subset K$.

Using the geometric assumption on M at p, namely that there exists a characteristic conormal direction ξ such that $\mathcal{L}_p(\xi, \cdot)$ has one negative and $n-1$ positive eigenvalues, one can construct forms $f_\lambda \in \mathcal{C}^\infty_{0,1}(M)$, $g_\lambda \in \mathcal{C}^\infty_{n+k,n-1}(M)$ with support in K such that $\overline{\partial}_M f_\lambda$ and $\overline{\partial}_M g_\lambda$ are rapidly decreasing with respect to λ in the topology of $\mathcal{C}^\infty_{\cdot,\cdot}(M)$ as $\lambda \to \infty$. On the other hand we have

$$(3.2) \qquad \left| \int_K f_\lambda \wedge g_\lambda \right| \geqslant c\lambda^{-n-\frac{k}{2}}$$

for some constant $c > 0$. For the details of the construction of these forms, we refer the reader to Section 5 of [3].

By assumption $H^{0,2}(M)$ is Hausdorff. But this implies that we can solve the equation $\overline{\partial}_M u_\lambda = \overline{\partial}_M f_\lambda$ with an estimate

$$(3.3) \qquad \|u_\lambda\|_{m_1} \leqslant C_2 \|\overline{\partial}_M f_\lambda\|_{m_2},$$

where C_2 is a positive constant and m_2 is an integer. Hence $\|u_\lambda\|_{m_1}$ is rapidly decreasing with respect to λ. Defining $\tilde{f}_\lambda = f_\lambda - u_\lambda$, we obtain a global smooth $\overline{\partial}_M$-closed $(0,1)$-form on M.

To get a contradiction to our assumption, we use the estimate (3.1) with $\tilde{f}_\lambda$ and g_λ. Namely, (3.1) implies that $\int_M \tilde{f}_\lambda \wedge g_\lambda$ is rapidly decreasing with respect to λ, whereas (3.2) implies that $|\int_M \tilde{f}_\lambda \wedge g_\lambda|$ is bounded from below by a polynomial in $1/\lambda$. For the details of these estimates, we again refer the reader to [3]. This contradiction proves the Theorem. $\qquad \square$

THEOREM 3.2. — *Let M be a compact (abstract) CR manifold of type (n, k). Assume that $H^{0,2}(M)$ is Hausdorff and that for each $p \in M$ there exists a characteristic conormal direction ξ such that $\mathcal{L}_p(\xi, \cdot)$ has one negative and $n-1$ positive eigenvalues. Then there exists a smooth $(0,1)$-form ω on M satisfying $\overline{\partial}_M \omega = 0$ on M such that ω is not $\overline{\partial}_M$-exact on any neighborhood of any point $p \in M$.*

Proof. — We choose a countable dense set of points on M, say $S = (p_j)_{j \in \mathbb{N}}$. Assume by contradiction that for every $\overline{\partial}_M$-closed form f on M there exists $j \in \mathbb{N}$ such that f is $\overline{\partial}_M$-exact on a neighborhood U_j of p_j.

For fixed j, let $(U_m^j)_{m \in \mathbb{N}}$ denote a fundamental sequence of open neighborhoods of p_j in M. To abbreviate notations, we set $\mathcal{Z}(M) = \mathcal{C}_{0,1}^\infty(M) \cap \mathrm{Ker}\overline{\partial}_M$. Now for each $m, j \in \mathbb{N}$ we define

$$G_m^j = \{(f, u_j) \in \mathcal{Z}(M) \times \mathcal{C}^\infty(U_m^j) \mid f = \overline{\partial}_M u_j \text{ on } U_m^j\}.$$

This, as a closed subspace of $\mathcal{Z}(M) \times \mathcal{C}^\infty(U_m^j)$, is also a Fréchet space. Let $\pi_m^j : G_m^j \longrightarrow \mathcal{Z}(M)$ be the natural projection. Our assumption implies that

$$\mathcal{Z}(M) = \bigcup_{m,j \in \mathbb{N}} \pi_m^j(G_m^j).$$

By Baire's category theorem one of the spaces, $\pi_\ell^s(G_\ell^s)$, must be of second category. Then, by the Banach open mapping theorem, the linear continuous map

$$\pi_\ell^s : G_\ell^s \longrightarrow \mathcal{Z}(M)$$

is surjective. But this contradicts Theorem 3.1. $\square$

4. Construction of CR line bundles

THEOREM 4.1. — *Let M be an (abstract) CR manifold of type (n, k), and assume that there exists a smooth $(0, 1)$-form ω on M satisfying $\overline{\partial}_M \omega = 0$ on M such that ω is not $\overline{\partial}_M$-exact on any neighborhood of any point $p \in M$. Then there exists a family $(L_a)_{a>0}$ of complex CR line bundles $L_a \longrightarrow M$ converging to the trivial CR line bundle $L_0 = M \times \mathbb{C}$, as a tends to 0, such that L_a is differentiably trivial over M, but L_a is not locally CR trivializable over any open set U of M for $a > 0$.*

Proof. — The arguments in this section follow [5].

On the differentiably trivial complex line bundle $L_a = M \times \mathbb{C}_z$, we consider the CR structure whose $T^{0,1}L_a$ is defined as follows: Let U be an open set of M such that $T^{0,1}M$ is spanned over U by $\overline{L}_1, \ldots, \overline{L}_n$. We define $T^{0,1}L_a$ to be spanned over $U \times \mathbb{C}_z$ by the basis

$$(4.1) \qquad \begin{cases} \overline{X}_0 = \dfrac{\partial}{\partial \overline{z}} \\ \overline{X}_j = \overline{L}_j + a\omega(\overline{L}_j)\dfrac{\partial}{\partial z}, \quad j = 1, \ldots, n \end{cases}$$

This gives well defined CR structure on L_a . The CR line bundle L_a is differentiably trivial over M and converges to the trivial CR line bundle

over M as a tends to zero. The associated connection of L_a is defined as follows: Since L_a is smoothly trivial, every $s \in \mathcal{C}_{0,q}^\infty(M, L_a)$ is globally defined by a form $\sigma \in \mathcal{C}_{0,q}^\infty(M)$ and a smooth frame $e \in \mathcal{C}^\infty(M, \mathbb{C})$, $s = \sigma \otimes e$. Then $\overline{\partial}_{L_a} s = (\overline{\partial}_M \sigma + \omega \wedge \sigma) \otimes e$. Since $\overline{\partial}_M \omega = 0$, this connection satisfies the integrability condition $\overline{\partial}_{L_a} \circ \overline{\partial}_{L_a} = 0$.

However, for $a \neq 0$, a local CR trivialization of L_a near a point of M forces the existence of a local smooth solution u of $\overline{\partial}_M u = \omega$.

Indeed, a local nonvanishing $\overline{\partial}_{L_a}$-closed section means that we have a local nonvanishing smooth function σ on some nonempty open subset of M satisfying $\overline{\partial}_M \sigma + \omega \sigma = 0$. After shrinking U, we can assume that $u = -\log \sigma$ is well defined on U. But u satisfies $\overline{\partial}_M u = -\frac{\overline{\partial}_M \sigma}{\sigma} = \omega$, which contradicts the assumption on ω. Therefore for $a \neq 0$, L_a is not locally CR trivializable over any open subset of M. □

5. Proof of the main theorems

Proof of Theorem 1.1. — Since M is a Lorentzian hypersurface, it satisfies the classical condition $Y(q)$ for $q \neq 1, n-1$. If $n \neq 3$, then M satisfies in particular the condition $Y(2)$. Hence the $\overline{\partial}_M$-complex is $\frac{1}{2}$-subelliptic in degree $(0, 2)$ (see [4]). It follows that $H^{0,2}(M)$ is finite dimensional, thus Hausdorff. But then the statement of the theorem follows by combining Theorems 3.2 and 4.1. □

Proof of Theorem 1.2. — Since M is 1-pseudoconcave, it follows from the ε-subelliptic estimates proved in [6] that the top-degree cohomology group $H^{0,n}(M)$ is finite dimensional. But since $n = 2$ we thus have that $H^{0,2}(M)$ is Hausdorff. Therefore we may again conclude by applying Theorems 3.2 and 4.1. □

Remark. — We conjecture that Theorem 1.1 also holds for $n = 3$. Indeed, if we start with a CR embedded hypersurface, then we can find many $\overline{\partial}_M$-closed $(0, 1)$-forms defined on a neighborhood of a given point which are not $\overline{\partial}_M$-exact on any neighborhood of that point (see [1]). However it seems to be an open problem to show that $H^{0,2}(M)$ is Hausdorff for a Lorentzian hypersurface of CR dimension 3. Therefore we cannot construct a *global* $\overline{\partial}_M$-closed $(0, 1)$ form on M which is not $\overline{\partial}_M$-exact on M.

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