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An introduction to quantum sheaf cohomology

AN INTRODUCTION TO
QUANTUM SHEAF COHOMOLOGY

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ABSTRACT. — In this note we review "quantum sheaf cohomology," a deformation of sheaf cohomology that arises in a fashion closely akin to (and sometimes generalizing) ordinary quantum cohomology. Quantum sheaf cohomology arises in the study of (0,2) mirror symmetry, which we review. We then review standard topological field theories and the A/2, B/2 models, in which quantum sheaf cohomology arises, and outline basic definitions and computations. We then discuss (2,2) and (0,2) supersymmetric Landau-Ginzburg models, and quantum sheaf cohomology in that context.

RÉSUMÉ. — Dans ces notes nous passons en revue la "cohomologie quantique des faisceaux", une déformation de la cohomologie des faisceaux qui apparaît d'une façon similaire à la cohomologie quantique ordinaire (tout en la généralisant parfois). La cohomologie quantique des faisceaux apparaît dans l'étude de la symétrie miroir (0,2), ce qui est passé en revue. Après ça nous passons en revue la théorie standard des champs topologique et les modèles A/2, B/2, dans lesquels la cohomologie quantique des faisceaux apparaît, et esquissons les définitions basiques et les calculs. Ensuite nous discutons dans ce contexte les modèles de supersymétrie Landau-Ginzburg (2,2) et (0,2) ainsi que la cohomologie quantique des faisceaux.

1. Introduction

The development of mirror symmetry, a duality between quantum field theories describing (typically) topologically distinct spaces, led to profound advances in the mathematics of enumerative geometry. Since its original development in the early 1990s there have been a number of offshoots, perhaps the most famous of which is homological mirror symmetry [18].

In this note we will review progress towards understanding another off-shoot of mirror symmetry, known as "(0,2) mirror symmetry," and in particular its analogue of enumerative geometry, "quantum sheaf cohomology." After giving a brief introduction to (0,2) mirror symmetry, we discuss quantum sheaf cohomology in some detail. After setting it up formally and discussing some examples, we then discuss Landau-Ginzburg models, and how they also give some perspective on quantum sheaf cohomology.

2. (0,2) mirror symmetry

Ordinary mirror symmetry, in its most basic incarnation, exchanges pairs of Calabi-Yau's X_1, X_2 of matching dimension in such a way as to flip their Hodge diamonds. For example, a quintic hypersurface in $\mathbf{P}^4$ is mirror to a (resolution of a) quintic hypersurface in $\mathbf{P}^4/(\mathbf{Z}_5)^3$, which have the Hodge diamonds below:

Quintic in $\mathbf{P}^4$					Mirror in $\mathbf{P}^4/(\mathbf{Z}_5)^3$		

		1				1		
	0		0		0		0	
0		1		0	0		101		0
1	101		101	1	1	1		1		1
0		1		0	0		101		0
	0		0		0		0	
		1				1		

(0,2) mirror symmetry is a conjectured generalization arising in heterotic string theories that, in its simplest formulation, exchanges pairs

$$(X_1, \mathcal{E}_1) \leftrightarrow (X_2, \mathcal{E}_2)$$

where the X_i are Calabi-Yau manifolds of matching dimension and the $\mathcal{E}_i \to X_i$ are stable holomorphic vector bundles of matching rank such that

$$\mathrm{ch}_2(\mathcal{E}) = \mathrm{ch}_2(TX).$$

(There exists a generalization in which non-Calabi-Yau spaces are mirror to Landau-Ginzburg models, though for simplicity we shall not speak to that here.) It reduces to ordinary mirror symmetry in the special case that $\mathcal{E}_i \cong TX_i$. That said, in general if $(X_1, \mathcal{E}_1)$ is (0,2) mirror to $(X_2, \mathcal{E}_2)$, the underlying spaces X_1 and X_2 need not be mirror to one another in the older sense of mirror symmetry.

Instead of exchanging (p, q) forms, (0,2) mirror symmetry exchanges sheaf cohomology groups:

$$H^j\left(X_1, \Lambda^i \mathcal{E}_1\right) \leftrightarrow H^j\left(X_2, \left(\Lambda^i \mathcal{E}_2\right)^*\right).$$

Note that when $\mathcal{E}_i \cong TX_i$, this reduces to

$$H^{n-i,j}(X_1) \leftrightarrow H^{i,j}(X_2)$$

(for X_i Calabi-Yau, $n = \dim X_i$), and so this generalizes the exchange of Hodge numbers of ordinary mirror symmetry.

One example of a (0,2) mirror pair is as follows [5]. The complete intersection $\mathbf{P}^5_{[1,1,1,1,2,2]}[4, 4]$, with bundle $\mathcal{E}$ given by

$$0 \longrightarrow \mathcal{E} \longrightarrow \bigoplus_1^5 \mathcal{O}(1) \longrightarrow \mathcal{O}(5) \longrightarrow 0$$

is (0,2) mirror to the complete intersection $\mathbf{P}^5_{[3,4,4,5,8,8]}[16, 16]$ with bundle $\mathcal{E}'$ given by

$$0 \longrightarrow \mathcal{E}' \longrightarrow \mathcal{O}(3) \oplus \mathcal{O}(4)^{\oplus 3} \oplus \mathcal{O}(5) \longrightarrow \mathcal{O}(20) \longrightarrow 0.$$

Not much is currently known about (0,2) mirror symmetry, though the basics are known, and more is rapidly developing.

In one of the first checks of the existence of (0,2) mirror symmetry [4], a computer was programmed to tabulate the dimensions of sheaf cohomology groups in a large number of examples. Within that list, most examples had partners on the list with dual sheaf cohomology dimensions, as expected if (0,2) mirror symmetry exists.

In addition, an analogue of the Greene-Plesser construction [11] (in which mirrors close to Fermat points are constructed via orbifolds) is known [5], as is an analogue [1] of the Hori-Vafa-Morrison-Plesser mirror construction [15, 26]. An analogue of quantum cohomology has been known since 2004, see *e.g.* [2, 6, 7, 12, 17, 24, 20, 21, 22, 27, 28]. Very recently, for deformations of the tangent bundle, there now exists a (0,2) monomial-divisor mirror map [23]. Put briefly, progress in this area seems to be speeding up.

In these notes we will outline one aspect of (0,2) mirrors, namely, quantum sheaf cohomology (the (0,2) analogue of quantum cohomology), and then discuss (2,2) and (0,2) Landau-Ginzburg models and some related issues.

3. Topological field theories and the A/2, B/2 models

Before discussing topological field theories, let us pause for a moment to review some language. The worldsheet theory for a heterotic string with $\mathcal{E} = TX$ has a total of 4 supersymmetries split into 2 pairs, labelled "(2,2)," and so the worldsheet theory is called a "(2,2) supersymmetric nonlinear sigma model," or simply a "(2,2) model." The worldsheet theory for a more general heterotic string has half as much supersymmetry, technically denoted "(0,2)," and hence the worldsheet theory is called a "(0,2) model."

Ordinary quantum cohomology is computed by the A model topological field theory [30], which is a theory with (2,2) supersymmetry, a twisted version of a supersymmetric nonlinear sigma model. There is a (0,2) supersymmetric analogue of the A model, responsible for quantum sheaf cohomology. This (0,2) analogue is commonly denoted the "A/2 model." Similarly, there is a (0,2) analogue of the B model, denoted the "B/2 model." We shall review these theories in this section.

Briefly, the A/2 and B/2 models are no longer strictly topological field theories, though they become honest topological field theories in the special case that $\mathcal{E} = TX$ (the "(2,2) locus"). Nevertheless, despite the fact that they are not honest topological field theories, some correlation functions still have a mathematical understanding. Also, the A/2 and B/2 models have some new symmetries not possessed by ordinary topological field theories: the A/2 model on $(X, \mathcal{E})$ is the same as the B/2 model on $(X, \mathcal{E}^*)$.

Next, we shall start working through a more detailed description of these models. First, recall the ordinary A model. This is defined by the Lagrangian

$$g_{i\bar{j}}\overline{\partial}\phi^i \partial \phi^{\bar{j}} \; + \; ig_{i\bar{j}}\psi^{\bar{j}}_- D_z \psi^i_- \; + \; ig_{i\bar{j}}\psi^{\bar{j}}_+ D_{\overline{z}}\psi^i_+ \; + \; R_{i\bar{j}k\bar{l}}\psi^i_+\psi^{\bar{j}}_+\psi^k_-\psi^{\bar{l}}_-$$

where $\phi : \Sigma \to X$ is a map from the two-dimensional worldsheet Σ into the space X, and $\psi^{i,\bar{i}}_{\pm}$ are fermions, coupling to the following bundles:

$$\psi^i_-(\equiv \chi^i) \in \Gamma\left((\phi^*T^{0,1}X)^*\right), \qquad \psi^i_+(\equiv \psi^i_z) \in \Gamma\left(K \otimes \phi^*T^{1,0}X\right),$$
$$\psi^{\bar{i}}_-(\equiv \psi^{\bar{i}}_{\overline{z}}) \in \Gamma\left(\overline{K} \otimes \phi^*T^{0,1}X\right), \qquad \psi^{\bar{i}}_+(\equiv \chi^{\bar{i}}) \in \Gamma\left((\phi^*T^{1,0}X)^*\right)$$

(where Γ denotes smooth sections and $\overline{K}$ the antiholomorphic version of the canonical bundle). This theory possesses a symmetry known as the "BRST symmetry." Infinitesimally, this symmetry group acts as

$$\begin{aligned}
\delta\phi^i &\propto \chi^i, & \delta\phi^{\bar{i}} &\propto \chi^{\bar{i}}, \\
\delta\chi^i &= 0, & \delta\chi^{\bar{i}} &= 0, \\
\delta\psi^i_z &\neq 0, & \delta\psi^{\bar{i}}_{\overline{z}} &\neq 0,
\end{aligned}$$

so the pertinent states (namely, BRST-closed, modulo BRST-exact) are

$$\mathcal{O} \sim b_{i_1 \cdots i_p \bar{\imath}_1 \cdots \bar{\imath}_q} \chi^{\bar{\imath}_1} \cdots \chi^{\bar{\imath}_q} \chi^{i_1} \cdots \chi^{i_p} \quad \leftrightarrow \quad H^{p,q}(X),$$
$$Q \quad \leftrightarrow \quad d$$

where Q denotes the "BRST operator" that generates infinitesimal BRST transformations.

In the A/2 model, the Lagrangian is

$$g_{i\bar{\jmath}} \bar{\partial} \phi^i \partial \phi^{\bar{\jmath}} \;+\; i h_{a\bar{b}} \lambda_-^{\bar{b}} D_z \lambda_-^a \;+\; i g_{i\bar{\jmath}} \psi_+^{\bar{\jmath}} D_{\bar{z}} \psi_+^i \;+\; F_{i\bar{\jmath}a\bar{b}} \psi_+^i \psi_+^{\bar{\jmath}} \lambda_-^a \lambda_-^{\bar{b}}$$

where $\lambda_-^{a,\bar{a}}$, $\psi_+^{i,\bar{\imath}}$ are fermions coupling to the following bundles:

$$\lambda_-^a \;\in\; \Gamma\left((\phi^*\overline{\mathcal{E}})^*\right), \quad \psi_+^i \;\in\; \Gamma\left(K \otimes \phi^* T^{1,0} X\right),$$
$$\lambda_-^{\bar{b}} \;\in\; \Gamma\left(\overline{K} \otimes \phi^*\overline{\mathcal{E}}\right), \quad \psi_+^{\bar{\imath}} \;\in\; \Gamma\left((\phi^* T^{1,0} X)^*\right).$$

Not any holomorphic bundle $\mathcal{E} \to X$ is allowed; only those satisfying the following two constraints. The first consistency condition is known as the "Green-Schwarz condition," and says

$$\mathrm{ch}_2(\mathcal{E}) \;=\; \mathrm{ch}_2(TX).$$

The second condition is specific to the A/2 model:

$$\Lambda^{\mathrm{top}} \mathcal{E}^* \;\cong\; K_X$$

where K_X is the canonical bundle of X. (This second constraint is a close analogue of the condition in the B model that X must be such that $K_X^{\otimes 2} \cong \mathcal{O}_X$ [28, 30].) The pertinent states of the A/2 model are of the form

$$\mathcal{O} \sim b_{\bar{\imath}_1 \cdots \bar{\imath}_n a_1 \cdots a_p} \psi_+^{\bar{\imath}_1} \cdots \psi_+^{\bar{\imath}_n} \lambda_-^{a_1} \cdots \lambda_-^{a_p} \;\leftrightarrow\; H^n(X, \Lambda^p \mathcal{E}^*).$$

In the special case that $\mathcal{E} = TX$, the A/2 model reduces to the A model. In this case, both of the conditions on $\mathcal{E}$ are satisfied automatically, and moreover the states match, as

$$H^q(X, \Lambda^p(TX)^*) \;=\; H^{p,q}(X).$$

Now, let us turn to correlation function computations. We will first give a formal overview of computations in both the A model and the A/2 model, making the simplifying assumption that all moduli spaces are smooth, compact, and have universal instantons. In practice, this will never be true, but it will serve for the purposes of a short overview. Furthermore, for those readers acquainted with Gromov-Witten theory, we will not couple to topological gravity in these notes, again for simplicity.

Let us first remind ourselves how classical contributions to A model correlation functions come about. Assume the space X is compact of dimension n, then classically (*i.e.* in degree zero), correlation functions have the form

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_m \rangle = \int_X \omega_1 \wedge \cdots \wedge \omega_m$$

where $\omega_i \in H^{p_i, q_i}(X)$ is a differential form corresponding to $\mathcal{O}_i$. Physically, in the quantum field theory there is a constraint (technically, arising as a 'selection rule' from left and right $U(1)_R$ symmetries), which implies that for the correlation function to be nonzero,

$$\sum_i p_i = \sum_i q_i = n = \dim X.$$

Thus, classical contributions to A model correlation functions have the form

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_m \rangle \sim \int_X (\text{top-form}).$$

In the A/2 model there is an analogous story. Assume that X is compact of dimension n, and that $\mathcal{E}$ has rank r, then in degree zero correlation functions have the form

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_m \rangle = \int_X \omega_1 \wedge \cdots \wedge \omega_m$$

where $\omega_i \in H^{q_i}(X, \Lambda^{p_i} \mathcal{E}^*)$. The same selection rules as before imply that in order for this correlation function to be nonzero, we must have

$$\sum_i q_i = n, \quad \sum_i p_i = r,$$

and so the correlation function has the form of an integral over X of an element of $H^{\text{top}}(X, \Lambda^{\text{top}} \mathcal{E}^*)$. The constraint $\Lambda^{\text{top}} \mathcal{E}^* \cong K_X$ makes the integrand a top-form.

Now, let us return to the A model and consider contributions from worldsheet instantons. In a sector of nonzero degree, the space of bosonic zero modes is given by some (compactified) moduli space of worldsheet instantons $\mathcal{M}$. If the sheaf

$$R^1 \pi_* \alpha^* TX \equiv 0,$$

where $\alpha : \Sigma \times \mathcal{M} \to X$ is the universal instanton (Σ the worldsheet), and $\pi : \Sigma \times \mathcal{M} \to \mathcal{M}$, then

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_m \rangle \sim \int_{\mathcal{M}} \omega_1 \wedge \cdots \wedge \omega_m$$

where $\omega_i \in H^{p_i,q_i}(\mathcal{M})$. More generally,

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_m \rangle \sim \int_{\mathcal{M}} \omega_1 \wedge \cdots \wedge \omega_m \wedge \mathrm{Eul(Obs)}.$$

where

$$\mathrm{Obs} \equiv R^1 \pi_* \alpha^* TX.$$

(Physically, the factor above arises from the effects of four-fermi terms in the action, and zero modes of ψ_z^i, $\psi_{\bar{z}}^i$.) In all cases,

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_m \rangle \sim \int_{\mathcal{M}} (\text{top-form}).$$

Next, let us consider worldsheet instantons in the A/2 model. The bundle $\mathcal{E}$ on X induces a sheaf $\mathcal{F}$ (physically, of λ zero modes) on $\mathcal{M}$:

$$\mathcal{F} \equiv R^0 \pi_* \alpha^* \mathcal{E}$$

where π and α are as above. On the (2,2) locus, where $\mathcal{E} = TX$, we have that $\mathcal{F} = T\mathcal{M}$. (Experts are reminded that we are not coupling to topological gravity.)

When

$$R^1 \pi_* \alpha^* TX = 0 = R^1 \pi_* \alpha^* \mathcal{E}$$

correlation functions look like an integral over $\mathcal{M}$ of an element of

$$H^{\mathrm{top}}\left(\mathcal{M}, \Lambda^{\mathrm{top}} \mathcal{F}^*\right).$$

By itself, this would be a problem – to get a number, we need to integrate a top-form. However, if we apply the conditions

$$\Lambda^{\mathrm{top}} \mathcal{E}^* \cong K_X,$$
$$\mathrm{ch}_2(\mathcal{E}) = \mathrm{ch}_2(TX),$$

then Grothendieck-Riemann-Roch implies that $\Lambda^{\mathrm{top}} \mathcal{F}^* \cong K_{\mathcal{M}}$, and so the integrand is a top-form, as needed.

More generally, we get a more complicated expression. Define

$$\mathrm{Obs} \equiv R^1 \pi_* \alpha^* TX, \quad \mathcal{F}_1 \equiv R^1 \pi_* \alpha^* \mathcal{E},$$

then if the ranks of Obs and $\mathcal{F}_1$ match, it can be shown there is a contribution to correlation functions given by an integral over $\mathcal{M}$ of an element of

$$H^{\sum q_i}\left(\mathcal{M}, \Lambda^{\sum p_i} \mathcal{F}^*\right) \wedge H^n\left(\mathcal{M}, \Lambda^n \mathcal{F}^* \otimes \Lambda^n \mathcal{F}_1 \otimes \Lambda^n (\mathrm{Obs})^*\right)$$

where n is the rank of Obs.

This reduces to the usual A model result on the (2,2) locus by virtue of Atiyah classes. In the special case that $\mathcal{E} = TX$, the sheaf cohomology group

$$H^1\left(\mathcal{M}, \mathcal{F}^* \otimes \mathcal{F}_1 \otimes (\text{Obs})^*\right)$$

contains the Atiyah class of Obs, and wedging copies together builds the Euler class

$$\text{Eul}(\text{Obs}).$$

In this case, the anomaly conditions

$$\begin{aligned}
\Lambda^{\text{top}} \mathcal{E}^* &\cong K_X, \\
\text{ch}_2(\mathcal{E}) &= \text{ch}_2(TX)
\end{aligned}$$

imply via Grothendieck-Riemann-Roch that

$$\Lambda^{\text{top}} \mathcal{F}^* \otimes \Lambda^{\text{top}} \mathcal{F}_1 \otimes \Lambda^{\text{top}} (\text{Obs})^* \cong K_{\mathcal{M}}$$

so again the integrand is a top-form.

So far, for simplicity of presentation, we have made the unreasonable assumptions that $\mathcal{M}$ is compact, smooth, and has a universal instanton. To do any computations, we need explicit expressions for the space $\mathcal{M}$ and sheaf $\mathcal{F}$, which will not have all of the properties above. We will use "linear sigma model" moduli spaces. This has the advantage of being closely connected to physics, though the disadvantage that there is no universal instanton $\alpha : \Sigma \times \mathcal{M} \to X$. As a result, the previous discussion was merely formal, we will need to extend the induced sheaves over the compactification divisor.

Schematically, a linear sigma model moduli space of maps $\mathbf{P}^1 \to V//G$, for V a vector space and G a reductive algebraic group, is a fine moduli space of pairs

$$\left(\text{principal } G - \text{bundle } E \text{ on } \mathbf{P}^1, G - \text{equivariant map } E \to V\right).$$

For example, a linear sigma model moduli space of maps from $\mathbf{P}^1$ into the Grassmannian $G(k, n)$ of k-planes in $\mathbf{C}^n$ (representable as $\mathbf{C}^{kn}//GL(k)$) is a Quot scheme of subsheaves of $\mathcal{O}^n$ of rank k and fixed degree on $\mathbf{P}^1$ (see *e.g.* [8] and references therein). These moduli spaces arise physically in "gauged linear sigma models," thus the name.

Over the Grassmannian $G(k, n)$, the physics construction will build all bundles from (co)kernels of short exact sequences of "special homogeneous" bundles, defined by representations of $U(k)$ (rather than $U(k) \times U(n-k)$). A simple example of a bundle built in this form is the bundle $\mathcal{E}$ defined as

the kernel of the short exact sequence below:

$$0 \longrightarrow \mathcal{E} \longrightarrow \overset{n}{\bigoplus} \mathcal{V}(\mathbf{k}) \overset{k+1}{\bigoplus} \mathrm{Alt}^2 \mathcal{V}(\mathbf{k}) \longrightarrow \overset{k-1}{\bigoplus} \mathrm{Sym}^2 \mathcal{V}(\mathbf{k}) \longrightarrow 0.$$

The vector bundle $\mathcal{V}(\mathbf{k})$ is the bundle associated to the fundamental representation $\mathbf{k}$ of $U(k)$, and so it is "special homogeneous" in the sense above, as are the other factors above.

We will describe how such bundles naturally lift to sheaves on $\mathbf{P}^1 \times \mathcal{M}$, where $\mathcal{M}$ is the linear sigma model moduli space (Quot scheme), which we can then pushforward to $\mathcal{M}$ to get the desired induced sheaves.

Corresponding to the special homogeneous bundle $\mathcal{V}(\overline{\mathbf{k}})$ is a rank k 'universal subbundle' $\mathcal{S}$ on $\mathbf{P}^1 \times \mathcal{M}$. (Existence of $\mathcal{S}$ follows from the fact that the linear sigma model moduli space is fine.) We lift all special homogeneous bundles so as to form a $U(k)$-representation homomorphism. For example,

$$\begin{aligned} \mathcal{V}(\mathbf{k}) &\mapsto \mathcal{S}^*, \\ \mathcal{V}(\mathbf{k}) \otimes \mathcal{V}(\overline{\mathbf{k}}) &\mapsto \mathcal{S}^* \otimes \mathcal{S}, \\ \mathrm{Alt}^m \mathcal{V}(\mathbf{k}) &\mapsto \mathrm{Alt}^m \mathcal{S}^*. \end{aligned}$$

Then, given sheaves on $\mathbf{P}^1 \times \mathcal{M}$, we pushforward to $\mathcal{M}$, and compute.

To give more insight into these computations, let us specialize to the projective space $\mathbf{P}^{N-1}$, defined by N homogeneous coordinates $x_0, \cdots, x_N$, each of weight 1.

For projective spaces, there is a more concrete construction of the Quot scheme $\mathcal{M}$, which we outline next. (See [25] for more details.) For degree d maps $\mathbf{P}^1 \to \mathbf{P}^{N-1}$, we expand the homogeneous coordinates as:

$$x_i = x_{i0}u^d + x_{i1}u^{d-1}v + \cdots + x_{id}v^d$$

where u, v are homogeneous coordinates on $\mathbf{P}^1$. We identify the x_{ij} with homogeneous coordinates on $\mathcal{M}$, then

$$\mathcal{M}_{\mathrm{LSM}} = \mathbf{P}^{N(d+1)-1}.$$

We can do something similar to build $\mathcal{F} \to \mathcal{M}$. For example, consider a completely reducible bundle over $\mathbf{P}^{N-1}$:

$$\mathcal{E} = \bigoplus_a \mathcal{O}(n_a).$$

Corresponding to the line bundle $\mathcal{O}(-1) \to \mathbf{P}^{N-1}$ is the bundle

$$\mathcal{S} \equiv \pi_1^* \mathcal{O}_{\mathbf{P}^1}(-d) \otimes \pi_2^* \mathcal{O}_{\mathbf{P}^{N(d+1)-1}}(-1) \longrightarrow \mathbf{P}^1 \times \mathbf{P}^{N(d+1)-1}.$$

The lift of $\mathcal{E}$ to $\mathbf{P}^1 \times \mathcal{M}$ is

$$\bigoplus_a \mathcal{S}^{-n_a} \longrightarrow \mathbf{P}^1 \times \mathbf{P}^{N(d+1)-1}$$

which pushes forward to

$$\mathcal{F} = \bigoplus_a H^0\left(\mathbf{P}^1, \mathcal{O}(n_a d)\right) \otimes_{\mathbf{C}} \mathcal{O}(n_a).$$

There is also a trivial extension of this to more general toric varieties. For example, corresponding to the completely reducible bundle

$$\mathcal{E} = \bigoplus_a \mathcal{O}(\vec{n}_a)$$

is the locally-free sheaf

$$\mathcal{F} = \bigoplus_a H^0\left(\mathbf{P}^1, \mathcal{O}(\vec{n}_a \cdot \vec{d})\right) \otimes_{\mathbf{C}} \mathcal{O}(\vec{n}_a).$$

Let us work through these details on the (2,2) locus, where $\mathcal{E} = TX$. The tangent bundle of a (compact, smooth) toric variety X can be expressed as

$$0 \longrightarrow \mathcal{O}^{\oplus k} \longrightarrow \bigoplus_i \mathcal{O}(\vec{q}_i) \longrightarrow TX \longrightarrow 0.$$

Applying the previous ansatz, we find

$$0 \longrightarrow \mathcal{O}^{\oplus k} \longrightarrow \bigoplus_i H^0\left(\mathbf{P}^1, \mathcal{O}(\vec{q}_i \cdot \vec{d})\right) \otimes_{\mathbf{C}} \mathcal{O}(\vec{q}_i) \longrightarrow \mathcal{F} \longrightarrow 0,$$

$$\mathcal{F}_1 \cong \bigoplus_i H^1\left(\mathbf{P}^1, \mathcal{O}(\vec{q}_i \cdot \vec{d})\right) \otimes_{\mathbf{C}} \mathcal{O}(\vec{q}_i).$$

The $\mathcal{F}$ above is precisely $T\mathcal{M}_{\mathrm{LSM}}$, and $\mathcal{F}_1 = \mathrm{Obs}$, as expected.

Now, let us turn to quantum cohomology. Physically, quantum cohomology is an "operator product" ring. For $\mathbf{P}^{N-1}$, quantum cohomology is defined by the correlation functions

$$\langle x^k \rangle = \begin{cases} q^m & \text{if } k = mN + N - 1, \\ 0 & \text{else.} \end{cases}$$

Physically, to get a clean result for operator products of the form above, it is sometimes claimed that one requires (2,2) supersymmetry. However, this has been shown not to be the case. Historically, [1] first conjectured examples of (0,2) analogues of quantum cohomology, which were verified in the work [17] which gave a mathematical definition of (0,2) quantum cohomology and computed some basic examples. Later, [2] found a general physical argument explaining why (0,2) quantum cohomology can exist physically. Since then, there have been a number of followup papers in the physics literature, including [12, 13, 19, 20, 21, 22, 23, 24, 27, 28].

Let us now ouline an example [1, 6, 7, 17]. Consider $X = \mathbf{P}^1 \times \mathbf{P}^1$, with bundle $\mathcal{E}$ a deformation of the tangent bundle, expressible as a cokernel

$$0 \longrightarrow \mathcal{O} \oplus \mathcal{O} \xrightarrow{\quad * \quad} \mathcal{O}(1,0)^2 \oplus \mathcal{O}(0,1)^2 \longrightarrow \mathcal{E} \longrightarrow 0 \qquad (3.1)$$

where

$$* = \begin{bmatrix} Ax & Bx \\ C\tilde{x} & D\tilde{x} \end{bmatrix}, \qquad (3.2)$$

in which A, B, C, D are 2×2 complex matrices, and

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \ \tilde{x} = \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix}$$

are the homogeneous coordinates on each copy of $\mathbf{P}^1$.

Briefly, one computes quantum (sheaf) cohomology by looking for relations in correlation functions. Let us work in degree (d, e). Then,

$$\mathcal{M} = \mathbf{P}^{2d+1} \times \mathbf{P}^{2e+1}$$

and

$$0 \longrightarrow \mathcal{O}^{\oplus 2} \longrightarrow \overset{2d+2}{\underset{1}{\bigoplus}} \mathcal{O}(1,0) \oplus \overset{2e+2}{\underset{1}{\bigoplus}} \mathcal{O}(0,1) \longrightarrow \mathcal{F} \longrightarrow 0,$$

$$\mathcal{F}_1 = 0 = \mathrm{Obs}.$$

Correlation functions are linear maps

$$\mathrm{Sym}^{2d+2e+2}\left(H^1(\mathcal{F}^*)\right) \longrightarrow H^{2d+2e+2}(\Lambda^{\mathrm{top}}\mathcal{F}^*) = \mathbf{C}.$$

It is possible to compute the kernel of this map (see [6, 7] for details); the result for the quantum sheaf cohomology ring is that

$$\det\left(A\psi + B\tilde{\psi}\right) = q_1,$$
$$\det\left(C\psi + D\tilde{\psi}\right) = q_2.$$

Let us check that in the special case $\mathcal{E} = T(\mathbf{P}^1 \times \mathbf{P}^1)$, we recover the standard quantum cohomology ring. This case is described by

$$A = D = I_{2\times 2}, \ B = C = 0,$$

and in this case, the quantum sheaf cohomology ring reduces to $\psi^2 = q_1$, $\tilde{\psi}^2 = q_2$, perfectly matching the ordinary quantum cohomology ring of $\mathbf{P}^1 \times \mathbf{P}^1$.

More generally, it has been argued [20] that for "linear" deformations of tangent bundles of toric varieties,

$$\prod_{\alpha} (\det M_\alpha)^{Q_\alpha^a} = q_a$$

generalizing Batyrev's ring

$$\prod_i \left(\sum_b Q_i^b \psi_b \right)^{Q_i^a} = q_a.$$

In addition to the A/2 model, one can also define a B/2 model, a (0,2) analogue of the ordinary B model topological field theory. The A/2 and B/2 models possess a symmetry not possessed by ordinary topological field theories. Specifically, the B/2 model on a space X with a bundle $\mathcal{E}$ is equivalent to the A/2 model on X with bundle $\mathcal{E}^*$. This duality and associated subtleties are discussed in [28].

4. (2,2) and (0,2) Landau-Ginzburg models

So far we have discussed the A/2 and B/2 models, which are "pseudo-topological-twists" of ordinary nonlinear sigma models.

A (2,2) supersymmetric Landau-Ginzburg model is a nonlinear sigma model on a space or stack X plus a "superpotential" W, a holomorphic function $W : X \to \mathbf{C}$. In physics, we describe this theory via its action:

$$
\begin{aligned}
S = \ & \int_\Sigma d^2x \, \big(g_{i\bar{\jmath}} \partial \phi^i \overline{\partial} \phi^{\bar{\jmath}} + i g_{i\bar{\jmath}} \psi_+^{\bar{\jmath}} D_{\bar{z}} \psi_+^i + i g_{i\bar{\jmath}} \psi_-^{\bar{\jmath}} D_z \psi_-^i + \cdots \\
& + g^{i\bar{\jmath}} \partial_i W \partial_{\bar{\jmath}} \overline{W} + \psi_+^i \psi_-^j D_i \partial_j W + \psi_+^{\bar{\imath}} \psi_-^{\bar{\jmath}} D_{\bar{\imath}} \partial_{\bar{\jmath}} \overline{W} \big) (4.1)
\end{aligned}
$$

There are analogues of the A and B model topological field theories for such Landau-Ginzburg models. In the math community, A-twisted Landau-Ginzburg models were discussed beginning in [9, 10]; in the physics community, they have been discussed in [14, 16]. B-twisted Landau-Ginzburg models when X is a vector space were originally discussed in [29].

Let us quickly review B-twisted Landau-Ginzburg models, over general spaces X. The states of the theory are BRST-closed (modulo BRST-exact) products of the form

$$b(\phi)^{j_1 \cdots j_m}_{\bar{\imath}_1 \cdots \bar{\imath}_n} \eta^{\bar{\imath}_1} \cdots \eta^{\bar{\imath}_n} \theta_{j_1} \cdots \theta_{j_m}$$

where the η, θ are linear combinations of the fermions $\psi_\pm^{i,\bar{\imath}}$. Under the action of the BRST operator Q,

$$Q \cdot \phi^i = 0, \quad Q \cdot \phi^{\bar{\imath}} = \eta^{\bar{\imath}}, \quad Q \cdot \eta^{\bar{\imath}} = 0, \quad Q \cdot \theta_j = \partial_j W, \quad Q^2 = 0,$$

which are almost the same as the standard B model, except for the BRST variation of θ_j, which here is nonzero, and vanishes for the ordinary B

model. If we identify

$$\eta^{\bar{i}} \leftrightarrow d\bar{z}^{\bar{i}}, \;\; \theta_j \leftrightarrow \frac{\partial}{\partial z^j}, \;\; Q \leftrightarrow \bar{\partial}$$

then we can identify the BRST cohomology with hypercohomology groups
[14]

$$\mathbf{H}^{\cdot}\left(X, \cdots \longrightarrow \Lambda^2 TX \xrightarrow{dW} TX \xrightarrow{dW} \mathcal{O}_X\right).$$

Let us perform some consistency checks of the description above.

- In the case that the superpotential W vanishes, the Landau-Ginz-burg model above should reduce to an ordinary B model on X. In this case, the hypercohomology reduces to $H^*(X, \Lambda^* TX)$, which are precisely the states of the B model [30], as expected.
- In the case that $X = \mathbf{C}^n$ and W a quasihomogeneous polynomial, the sequence above resolves the fat point $\{dW = 0\}$, so that the hypercohomology is isomorphic to the ring

$$\mathbf{C}[x_1, \cdots, x_n]/(dW)$$

matching the standard result [29] for this case.

To perform the A twist of the Landau-Ginzburg model, we need a $U(1)$ isometry on X with respect to which the superpotential is quasi-homogeneous. Technically, we then twist by the combination of that isometry and the R-symmetry of the nonlinear sigma model. In more detail, let $Q(\psi_i)$ be such that

$$W(\lambda^{Q(\psi_i)}\phi_i) = \lambda W(\phi_i)$$

then we twist by taking

$$\psi \mapsto \Gamma\left(\text{original} \otimes K_\Sigma^{-(1/2)Q_R} \otimes \overline{K}_\Sigma^{-(1/2)Q_L}\right)$$

where

$$Q_{R,L}(\psi) = Q(\psi) + \begin{cases} 1 & \psi = \psi_+^i, R, \\ 1 & \psi = \psi_-^i, L, \\ 0 & \text{else.} \end{cases}$$

For example, consider the case that $X = \mathbf{C}^n$, and W is a quasi-homogeneous polynomial. Here, to perform the A twist, we need to make sense of e.g. $K_\Sigma^{1/r}$ where $r = (2)(\text{degree})$. There are two ways to handle this issue:

- One way is to couple to topological gravity, where one can make sense of such roots of the canonical bundle. This is the procedure followed in [9, 10].
- Alternatively, one can choose not to couple to topological gravity, as was done in [14], but then one cannot twist this example.

We will discuss the second case.

Let us outline the details of a 'twistable' example. Consider a Landau-Ginzburg model on

$$X = \text{Tot}\left(\mathcal{E}^* \longrightarrow B\right)$$

with B a compact Kähler manifold, and

$$W = p\pi^*s$$

where $s \in \Gamma(B, \mathcal{E})$ and p is a fiber coordinate. In this case, we can take the $U(1)$ isometry to act as phase rotations along the fiber directions. Following the prescription above, we find that the fermions along the base B are twisted in the usual fashion:

$$\psi^i_- \left(\equiv \chi^i\right) \in \Gamma\left(\left(\phi^*T^{0,1}B\right)^*\right), \quad \psi^i_+ \left(\equiv \psi^i_z\right) \in \Gamma\left(K \otimes \phi^*T^{1,0}B\right),$$

$$\psi^{\bar{\imath}}_- \left(\equiv \psi^{\bar{\imath}}_{\bar{z}}\right) \in \Gamma\left(\overline{K} \otimes \phi^*T^{0,1}B\right), \quad \psi^{\bar{\imath}}_+ \left(\equiv \chi^{\bar{\imath}}\right) \in \Gamma\left(\left(\phi^*T^{1,0}B\right)^*\right),$$

whereas the fermionic partners of the fiber directions are twisted differently:

$$\psi^p_- \left(\equiv \psi^p_{\bar{z}}\right) \in \Gamma\left(\overline{K} \otimes \left(\phi^*T^{0,1}_\pi\right)^*\right), \qquad \psi^p_+ \left(\equiv \chi^p\right) \in \Gamma\left(\phi^*T^{1,0}_\pi\right),$$

$$\psi^{\bar{p}}_- \left(\equiv \chi^{\bar{p}}\right) \in \Gamma\left(\phi^*T^{0,1}_\pi\right), \qquad \phi^{\bar{p}}_+ \left(\equiv \psi^{\bar{p}}_z\right) \in \Gamma\left(K \otimes \left(\phi^*T^{1,0}_\pi\right)^*\right)$$

and the p field itself is also twisted.

The states we can build[1] in this theory are BRST-closed (modulo BRST-exact) products of the form

$$b(\phi)_{\bar{\imath}_1\cdots\bar{\imath}_n j_1\cdots j_m}\chi^{\bar{\imath}_1}\cdots\chi^{\bar{\imath}_n}\chi^{j_1}\cdots\chi^{j_m}$$

where $\phi \sim \{s = 0\} \subset B$, $\psi \sim TB|_{\{s=0\}}$, and the BRST operator acts as

$$Q \cdot \phi^i \propto \chi^i, \quad Q \cdot \phi^{\bar{\imath}} \propto \chi^{\bar{\imath}}, \quad Q \cdot \chi^i = Q \cdot \chi^{\bar{\imath}} = 0,$$

$$Q \cdot \chi^p \neq 0, \quad Q \cdot \chi^{\bar{p}} \neq 0, \quad Q^2 = 0.$$

We identify

$$\chi^i \leftrightarrow dz^i, \quad \chi^{\bar{\imath}} \leftrightarrow d\bar{z}^{\bar{\imath}}, \quad Q \leftrightarrow d,$$

so we see that the states above can be identified with elements of

$$H^{m,n}(B)|_{s=0}.$$

---

[1] The states 'accessible' to this theory are not precisely the same as the states available at the endpoint of renormalization group flow – we get the restriction of the cohomology of B, instead of the cohomology of a complete intersection. This is a standard technical issue in constructions of this form.

Next, consider correlation functions. In prototypical cases correlation functions can be shown to have the form

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle$$
$$= \int_{\mathcal{M}} \omega_1 \wedge \cdots \wedge \omega_n$$
$$\cdot \underbrace{\int d\chi^p d\chi^{\overline{p}} \exp\left(-|s|^2 - \chi^p dz^i D_i s - \text{c.c.} - F_{i\overline{j}} dz^i d\overline{z}^{\overline{j}} \chi^p \chi^{\overline{p}}\right)}_{\text{Mathai-Quillen form}}.$$

The Mathai-Quillen form represents a Thom class, so

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle = \int_{\mathcal{M}} \omega_1 \wedge \cdots \wedge \omega_n \wedge \text{Eul}(N_{\{s=0\}/\mathcal{M}})$$
$$= \int_{\{s=0\}} \omega_1 \wedge \cdots \wedge \omega_n$$

which is the same as the ordinary A model on the space $\{s = 0\} \subset B$. This is not a coincidence, as we shall see shortly.

Let us consider a particular example, a Landau-Ginzburg model on

$$X = \text{Tot}\left(\mathcal{O}(-5) \longrightarrow \mathbf{P}^4\right)$$

with $W = p\pi^* s$, s a quintic polynomial on $\mathbf{P}^4$. The degree zero contribution to correlation functions on a genus zero worldsheet is of the form

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle = \int_{\mathbf{P}^4} d^2\phi^i \int \prod_i d\chi^i d\chi^{\overline{i}} d\chi^p d\chi^{\overline{p}} \, \mathcal{O}_1 \cdots \mathcal{O}_n$$
$$\cdot \exp\left(-|s|^2 - \chi^i \chi^p D_i s - \chi^{\overline{p}} \chi^{\overline{i}} D_{\overline{i}} \overline{s} - R_{ip\overline{p}\overline{k}} \chi^i \chi^p \chi^{\overline{p}} \chi^{\overline{k}}\right)$$

(where the curvature term describes the curvature of $\mathcal{O}(-5)$).

In the A twist (unlike the B twist), the superpotential terms are BRST exact:

$$Q \cdot \left(\psi^i_- \partial_i W - \psi^{\overline{i}}_+ \partial_{\overline{i}} \overline{W}\right) \propto -|dW|^2 + \psi^i_+ \psi^j_- D_i \partial_j W + \text{c.c.}$$

As a result, under rescalings of the superpotential W by a constant factor λ, correlation functions are unchanged. Let us check that in more detail.

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle$$
$$= \int_{\mathbf{P}^4} d^2\phi^i \int \prod_i d\chi^i d\chi^{\overline{i}} d\chi^p d\chi^{\overline{p}} \, \mathcal{O}_1 \cdots \mathcal{O}_n$$
$$\cdot \exp\left(-\lambda^2 |s|^2 - \lambda \chi^i \chi^p D_i s - \lambda \chi^{\overline{p}} \chi^{\overline{i}} D_{\overline{i}} \overline{s} - R_{ip\overline{p}\overline{k}} \chi^i \chi^p \chi^{\overline{p}} \chi^{\overline{k}}\right).$$

From our previous remarks, the correlation function above should not depend upon the value of λ. Let us compare its values in the following two rescaling limits:

(1) $\lambda \to 0$. In this case, the exponential reduces to purely curvature terms, hence one brings down enough factors to eat up the χ^p zero modes. This is equivalent to inserting an Euler class.

(2) $\lambda \to \infty$. In this case, the theory localizes on $\{s = 0\} \subset \mathbf{P}^4$.

The results in the two limits are equivalent, as expected.

To understand how these Landau-Ginzburg models are related to nonlinear sigma models, we need to say a few words about the "renormalization group." This is a semigroup operation on the space of quantum field theories, that constructs new quantum field theories approximating previous ones valid at long distances.

The renormalization group turns out to be a very useful tool in physics for understanding long distance, low energy behavior of physical systems. One of its drawbacks, however, is that it is impracticle to follow it explicitly: usually the best one can manage is to construct an asymptotic series expansion for its tangent vector.

One of the important properties of the renormalization group is that it preserves topological field theories. If two physical theories are related by renormalization group flow, then, correlation functions in the two theories must match.

In the present case, it can be argued that a Landau-Ginzburg model on

$$X = \mathrm{Tot}\left(\mathcal{E}^* \xrightarrow{\pi} B\right)$$

with $W = p\pi^*s$, $s \in \Gamma(\mathcal{E})$, flows under the renormalization group to a nonlinear sigma model on $\{s = 0\} \subset B$. This is the physical reason for the matching correlation functions we computed earlier.

So far we have outlined (2,2) supersymmetric Landau-Ginzburg models. Next, we shall discuss (0,2) supersymmetric Landau-Ginzburg models, also known as heterotic Landau-Ginzburg models (we shall use the terms interchangeably).

A heterotic Landau-Ginzburg model is defined by an action

$$
\begin{aligned}
S \;=\; \int_\Sigma d^2x \, \Big(& g_{i\bar{j}}\partial\phi^i\overline{\partial}\phi^j \;+\; ig_{i\bar{j}}\psi^j_+ D_{\bar{z}}\psi^i_+ \;+\; ih_{a\bar{b}}\lambda^{\bar{b}}_- D_z\lambda^a_- \;+\; \cdots \\
& + h^{a\bar{b}}F_a\overline{F}_{\bar{b}} \;+\; \psi^i_+\lambda^a_- D_i F_a \;+\; \text{c.c.} \\
& + h_{a\bar{b}}E^a\overline{E}^{\bar{b}} \;+\; \psi^i_+\lambda^{\bar{a}}_- D_i E^b h_{\overline{a}b} \;+\; \text{c.c.} \Big).
\end{aligned}
$$

In the expression above, the sections

$$E^a \in \Gamma(\mathcal{E}), \quad F_a \in \Gamma(\mathcal{E}^*)$$

are constrained to obey

$$\sum_a E^a F_a \ = \ 0$$

and replace the superpotential – instead of a holomorphic function W, one has E^a, F_a.

We can recover ordinary (2,2) supersymmetric Landau-Ginzburg models as special cases of heterotic Landau-Ginzburg models by taking $\mathcal{E} = TX$, $E^a \equiv 0$, and $F_i \equiv \partial_i W$ where W is the superpotential of the (2,2) Landau-Ginzburg model. It is straightforward to check that in this case, the action above reduces to that for a (2,2) supersymmetric Landau-Ginzburg model, equation (4.1).

Heterotic Landau-Ginzburg models have analogues of the A/2 and B/2 twists:

- If $E^a \equiv 0$, then we can perform the standard B/2 twist. We take

$$\psi_+^{\bar{i}} \in \Gamma\left((\phi^* T^{1,0} X)^*\right), \quad \lambda_-^{\bar{a}} \in \Gamma\left(\phi^* \overline{\mathcal{E}}\right).$$

  We require

$$\Lambda^{\text{top}} \mathcal{E} \cong K_X, \ \text{ch}_2(\mathcal{E}) = \text{ch}_2(TX).$$

  The pertinent states are elements of the hypercohomology groups

$$\mathbf{H}^{\cdot}\left(\cdots \longrightarrow \Lambda^2 \mathcal{E} \xrightarrow{i_{F_a}} \mathcal{E} \xrightarrow{i_{F_a}} \mathcal{O}_X\right).$$

- If $F_a \equiv 0$, then we can perform the standard A/2 twist. We take

$$\psi_+^{i} \in \Gamma\left(\phi^* T^{1,0} X\right), \quad \lambda_-^{\bar{a}} \in \Gamma\left(\phi^* \overline{\mathcal{E}}\right).$$

  We require

$$\Lambda^{\text{top}} \mathcal{E}^* \cong K_X, \ \text{ch}_2(\mathcal{E}) = \text{ch}_2(TX).$$

  The pertinent states are elements of the hypercohomology groups

$$\mathbf{H}^{\cdot}\left(\cdots \longrightarrow \Lambda^2 \mathcal{E}^* \xrightarrow{i_{E^a}} \mathcal{E}^* \xrightarrow{i_{E^a}} \mathcal{O}_X\right).$$

More generally, one must combine with a $\mathbf{C}^\times$ action in order to twist.

Heterotic Landau-Ginzburg models are often related to heterotic non-linear sigma models by renormalization group flow. In general, a heterotic Landau-Ginzburg model on the space

$$X \ = \ \text{Tot}\left(\mathcal{F}_1 \oplus \mathcal{F}_3^* \xrightarrow{\pi} B\right)$$

with bundle $\mathcal{E}$ given by an extension

$$0 \longrightarrow \pi^* \mathcal{G}^* \longrightarrow \mathcal{E} \longrightarrow \pi^* \mathcal{F}_2 \longrightarrow 0$$

flows under the renormalization group to a nonlinear sigma model on $Y \equiv \{G_\mu = 0\} \subset B$, with $G_\mu \in \Gamma(\mathcal{G})$, with bundle given as the middle cohomology of the short complex

$$\mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F}_3.$$

(The maps in the short complex above are determined by the E^a, F_a in the heterotic Landau-Ginzburg theory.)

For example, a heterotic Landau-Ginzburg model on the space

$$X = \mathrm{Tot}\left(\mathcal{F}_1 \xrightarrow{\pi} B\right)$$

with bundle $\mathcal{E}' = \pi^*\mathcal{F}_2$ and $F_a \equiv 0$, $E^a = p\tilde{E}^a$ (p a fiber coordinate along $\mathcal{F}_1$, $\tilde{E}^a : \mathcal{F}_1 \to \mathcal{F}_2$) can be shown to flow, under the renormalization group, to a heterotic nonlinear sigma model on B with bundle

$$\mathcal{E} = \mathrm{coker}\left(\tilde{E}^a : \mathcal{F}_1 \longrightarrow \mathcal{F}_2\right).$$

As a check of the claim above regarding renormalization group flow, we can compare elliptic genera, which are preserved under the renormalization group. The elliptic genus of this (0,2) Landau-Ginzburg model can be computed, and is proportional to [3]

$$\int_B \mathrm{Td}(TB) \wedge \mathrm{ch}\left(\otimes S_{q^n}((TB)^{\mathbf{C}}) \otimes S_{q^n}((e^{-i\gamma}\mathcal{F}_1)^{\mathbf{C}}) \otimes \Lambda_{-q^n}((e^{-i\gamma}\mathcal{F}_2)^{\mathbf{C}})\right).$$

It is straightforward to check [3] that this matches the elliptic genus of the claimed correspoinding nonlinear sigma model. (More generally, though we shall not discuss details, there is a Thom class argument [3] that elliptic genera in (2,2) and (0,2) Landau-Ginzburg models match elliptic genera of nonlinear sigma models believed to be related by renormalization group flow.)

Let us return to the example of a nonlinear sigma model on $\mathbf{P}^1 \times \mathbf{P}^1$ with bundle a deformation of the tangent bundle, as defined in equations (3.1), (3.2). Corresponding to that heterotic nonlinear sigma model is a heterotic Landau-Ginzburg model on the space

$$X = \mathrm{Tot}\left(\mathcal{O} \oplus \mathcal{O} \xrightarrow{\pi} \mathbf{P}^1 \times \mathbf{P}^1\right)$$

with bundle

$$\pi^*\mathcal{O}(1,0)^2 \oplus \pi^*\mathcal{O}(0,1)^2$$

with $F_a \equiv 0$ and

$$(E^a) = \begin{bmatrix} Ax & Bx \\ C\tilde{x} & D\tilde{x} \end{bmatrix}\begin{bmatrix} p_1 \\ p_2 \end{bmatrix}.$$

Since $F_a \equiv 0$, we can perform the standard A/2 twist (without having to invoke a $U(1)$ isometry). Correlation functions have the form

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle =$$
$$\int_{\mathbf{P}^1 \times \mathbf{P}^1} d^2 x \int d\chi^i \int d\lambda^{\bar{a}} \, \mathcal{O}_1 \cdots \mathcal{O}_n \left(\lambda^{\bar{a}} \tilde{E}_1^{\bar{a}} \right) \left(\lambda^{\bar{b}} \tilde{E}_2^{\bar{b}} \right) f(\tilde{E}_1^{\bar{a}}, \tilde{E}_2^{\bar{a}})$$

which reproduces standard results for quantum sheaf cohomology in this example.

Let us outline computations in another example. Specifically, consider the heterotic string on a quintic hypersurface in $\mathbf{P}^4$, with bundle a deformation of the tangent bundle. This is equivalent under renormalization group flow to a Landau-Ginzburg model on the space

$$X = \text{Tot} \left(\mathcal{O}(-5) \xrightarrow{\pi} \mathbf{P}^4 \right)$$

with bundle $\mathcal{E} = TX$, $E^a \equiv 0$, and

$$F_a = (G, p(D_i G + G_i))$$

where $G \in \Gamma(\mathcal{O}(5))$ and p a fiber coordinate. This heterotic Landau-Ginzburg model flows under the renormalization group to a (0,2) theory on $\{G = 0\} \subset \mathbf{P}^4$, with bundle a deformation of the tangent bundle, as defined by the G_i.

Now, let us perform the $A/2$ twist. Correlation functions are of the form

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle =$$
$$\int d^2 \phi^i \int d\chi^i \int d\lambda^{\bar{\imath}} \int d\chi^{\bar{p}} \int d\lambda^p \, \mathcal{O}_1 \cdots \mathcal{O}_n$$
$$\cdot \exp \left(-|G|^2 - \chi^i \lambda^p D_i G - \chi^{\bar{p}} \lambda^{\bar{\imath}} \left(D_{\bar{\imath}} \overline{G} + \overline{G}_{\bar{\imath}} \right) - R_{i \bar{p} p \bar{k}} \chi^i \chi^{\bar{p}} \lambda^p \lambda^{\bar{k}} \right).$$

After performing the Grassmann integrals over $\chi^{\bar{p}}$, λ^p, we have that this correlation function is

$$\int d^2 \phi^i \int d\chi^i \int d\lambda^{\bar{\imath}} \, \mathcal{O}_1 \cdots \mathcal{O}_n$$
$$\cdot \left[\left(\chi^i D_i G \right) \left(\lambda^{\bar{\imath}} \left(D_{\bar{\imath}} \overline{G} + \overline{G}_{\bar{\imath}} \right) \right) + R_{i \bar{p} p \bar{k}} g^{p \bar{p}} \chi^i \lambda^{\bar{k}} \right] \exp \left(-|G|^2 \right).$$

The expression above encodes a (0,2) deformation of a Mathai-Quillen form.

More generally, based on computations in gauged linear sigma models, I. Melnikov and J. McOrist [21] have a formal argument that the A/2 twist should be independent of F_a's and the B/2 twist should be independent of E^a's, which generalizes the statement that the ordinary A model is independent of complex structures, and the B model is independent of Kähler structures.

5. Conclusions

Briefly, in this short note, we have given an overview of progress towards (0,2) mirror symmetry, then a more detailed discussion of quantum sheaf cohomology (part of the (0,2) mirrors story), and finally discussed (2,2) and (0,2) supersymmetric Landau-Ginzburg models over nontrivial spaces.

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